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# LIMITS OF ON/OFF HIERARCHICAL PRODUCT MODELS FOR DATA TRANSMISSION[1]

BY SIDNEY RESNICK AND GENNADY SAMORODNITSKY

*Cornell University*

A hierarchical product model seeks to model network traffic as a product of independent on/off processes. Previous studies have assumed a Markovian structure for component processes amounting to assuming that exponential distributions govern *on* and *off* periods, but this is not in good agreement with traffic measurements. However, if the number of factor processes grows and input rates are stabilized by allowing the *on* period distribution to change suitably, a limiting on/off process can be obtained which has exponentially distributed *on* periods and whose *off* periods are equal in distribution to the busy period of an $M/G/\infty$ queue. We give a fairly complete study of the possible limits of the product process as the number of factors grows and offer various characterizations of the approximating processes. We also study the dependence structure of the approximations.

**1. Introduction.** A hierarchical product model seeks to model network traffic as a product of independent processes. The idea is that network dynamics depend on various mechanical and software processes and controls which operate at different protocol layers and time scales. The consideration of such models is motivated by the need for explanations of both large time scale long-range dependence and self-similarity in measured network traffic as well as perception of small time scale multifractality. See Kulkarni, Marron and Smith (2001), Misra and Gong (1998), Mannersalo, Norros and Riedi (1999), Carlsson and Fiedler (2000) and Riedi and Willinger (2000).

The usual scheme is to consider a process $\{Z^{(n)}(t) = \prod_{j=1}^{n} I_j^{(n)}(t),\ \ t \ge 0\}$ where $I_j^{(n)}(\cdot)$, $j = 1, \dots, n$, are i.i.d. on/off processes or perhaps the i.i.d. structure is varied by allowing a progressive scaling of time. An on/off process is an alternating renewal processes with states $\{0, 1\}$. [For background on the role of on/off models in traffic modeling see Heath, Resnick and Samorodnitsky, (1997, 1998), Leland, Taqqu, Willinger and Wilson (1994), Willinger, Taqqu and Erramilli (1996), Willinger, Taqqu, Leland and Wilson (1995a, b) and Taqqu, Willinger and Sherman (1997).] The key idea is that transmission can proceed iff all the component processes are in the *on* state. This gives an idealized representation of different layers and time scales though it does not fully represent

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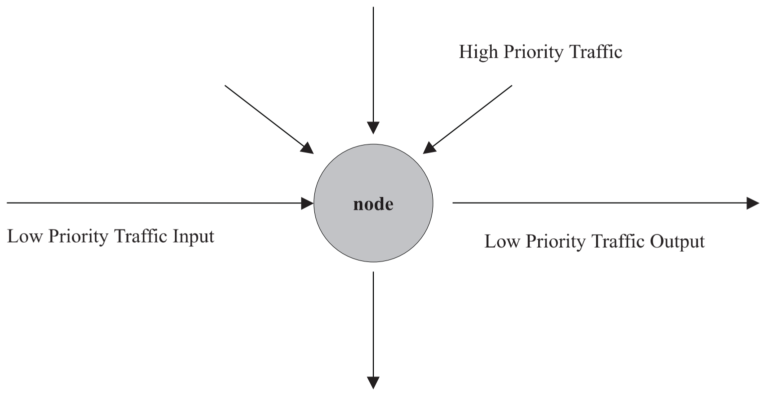


FIG. 1. *Low priority traffic encounters blocking from higher priority streams.*

in a detailed manner the network dynamics, protocols and controls. This model is a proposed balance between realism and statistical and mathematical tractability. The *off* periods of the factor processes provide a way to model spacings between packet arrivals due to hardware and software, TCP windowing and congestion control, server delays and effects of switches and routers.

The model has the added attractive feature that it can represent other scenarios. Consider low priority traffic which must traverse a node subject to cross traffic of $n$ higher priority on/off streams. See Figure 1. The low priority traffic can traverse the node only when each of the $n$ higher priority streams is in the *off* state (from the point of view of the low priority stream this is the *on*, or transmission enabling, state) and thus traversal is controlled by a product model. Similarly, if a low priority stream must traverse $n$ nodes, each subject to blocking by a higher priority on/off cross stream, then the channel will be open to the low priority stream only when each node is free of the *on* state of the cross stream. See Figure 2.

Several authors [e.g., Carlsson and Fiedler (2000), Misra and Gong (1998) and Kulkarni, Marron and Smith (2001)] assume, in the interest of greater tractability, that the individual on/off processes are Markovian, which amounts to assuming

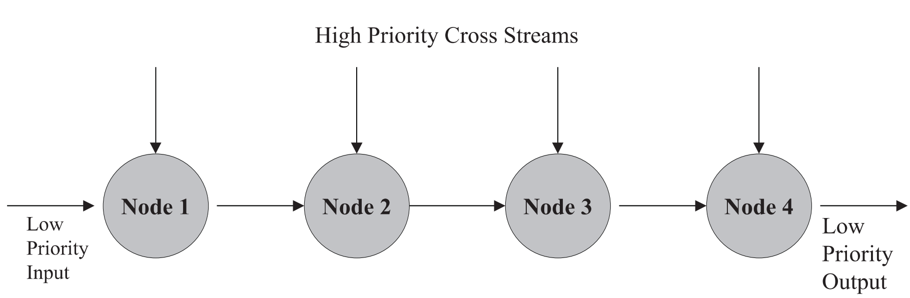


FIG. 2. *Low priority traffic encounters blocking at several nodes.*

that both the *on* and *off* length distributions $F_{\text{on}}$ and $F_{\text{off}}$ are exponential. This permits fairly explicit calculation of moments and some queueing characteristics. In particular, Kulkarni, Marron and Smith (2001) provide a compelling and stimulating account of this model applied to analysis of TCP traffic traces at the packet level. The following problems are evident with the use of Markovian factors.

1. Measured *on* periods as given in Kulkarni, Marron and Smith (2001) are demonstrably not exponentially distributed and the factor on/off processes are probably not Markovian.
2. A model with exponential distributions for both *on* and *off* periods cannot exhibit long-range dependence, a property usually observed in network traffic rates. In fact, exponential distributions imply correlations will decrease exponentially fast in the lag [see Kulkarni, Marron and Smith (2001), Misra and Gong (1998) and Section 6].
3. The hierarchical model already tends toward the black box philosophy and the assumption of exponential distributions in the interests of both mathematical and statistical tractability emphasizes the black box aspect. Reliance on statistical goodness of fit by visual impression of simulated traces also emphasizes that there is minimal structural modeling.

Despite the last itemized problem, the statistical analysis presented by Kulkarni, Marron and Smith (2001) is very attractive for its skill and level of detail and we wondered how important was the feature that the measured *on* periods were not exponentially distributed. Thus, we sought approximations to $\{Z^{(n)}(t), t \geq 0\}$ as the number of factors $n$ satisfies $n \to \infty$. It is evident that if just one factor process is in the *off* state, then so is the product, so *off* periods of $Z^{(n)}(\cdot)$ tend to grow with $n$. Thus, to get a sensible approximation as $n$ grows, one must stabilize the overall rate. This can be done by either letting the input rate grow with $n$ (instead of being constantly 1) or by letting the *on* periods grow with $n$. This paper concentrates on the latter mechanism. We find in Section 3 that under quite general conditions, a suitable approximation for the $n$-factor hierarchical model $Z^{(n)}(\cdot)$ is an on/off model where the *on* periods are exponential and the *off* periods are $M/G/\infty$ queueing model busy periods. For such an approximation, it is easy to make sensible assumptions which guarantee long-range dependence.

Section 2 discusses more formally the mathematical set-up and Section 3 provides a warm-up to the general theory which discusses the on/off approximation mentioned in the previous paragraph. Subsequent sections deal with the general asymptotic theory of product models. Intuitively our set-up can be viewed as follows. As the number of factors $n$ in the product model grows, individual *on* periods become long. We will, however, allow occasional short *on* periods. Those can be viewed as breakdowns or other imperfections in the system. Our general theory describes, in particular, the effect of such "imperfections" on the limiting approximating model.

Sections 4 and 5 give the necessary and sufficient conditions for $Z^{(n)}(\cdot)$ to converge to a limiting approximation $Z^{(\infty)}(\cdot)$ in the sense of convergence of finite dimensional distributions and we also provide various interpretations for the limiting process $Z^{(\infty)}(\cdot)$. The last Section 6, gives information about the dependence structure of the limiting process $Z^{(\infty)}(\cdot)$. In particular, we give various facts about the decay of the correlation function.

**2. Preliminaries.** We now review necessary constructions and notations. A single channel stationary on/off process is constructed with the following ingredients. Let $\{X_{\text{on}}, X_n, n = \pm 1, \pm 2, \dots\}$ be i.i.d. nonnegative random variables representing *on* periods and similarly let $\{Y_{\text{off}}, Y_n,\ n = \pm 1, \pm 2, \dots\}$ be i.i.d. nonnegative random variables representing *off* periods. The $X$'s are assumed independent of the $Y$'s and the common distribution of *on* periods is $F_{\text{on}}$ and the distribution of *off* periods is $F_{\text{off}}$. We assume both $F_{\text{on}}$ and $F_{\text{off}}$ have finite means $\mu_{\text{on}}$ and $\mu_{\text{off}}$ and we set $\mu = \mu_{\text{on}} + \mu_{\text{off}}$.

Define

$$S_n^{(\pm,X)} = \sum_{i=1}^{n} X_{\pm i}, \qquad S_n^{(\pm,Y)} = \sum_{i=1}^{n} Y_{\pm i}.$$

Consider the doubly infinite pure renewal sequence that begins with an *on* period at time 0:

$$\left\{ \left( \sum_{i=1}^{n} (X_{-i} + Y_{-i}) \right)_{n=\dots,-2,-1}, 0, \left( \sum_{i=1}^{n} (X_i + Y_i) \right)_{n=1,2,\dots} \right\}.$$

The interarrival distribution is $F_{\text{on}} * F_{\text{off}}$ and the mean interarrival time is $\mu$. This pure renewal process has a stationary version [Resnick (1992), page 224 ff.].

$$\left\{ \left(\forall\, D_- - S_n^{(-,X)} - S_n^{(-,Y)}\right)_{n=1,2,\dots}, \forall\, D_-, D_+, \left(D_+ + S_n^{(+,X)} + S_n^{(+,Y)}\right)_{n=1,2,\dots} \right\},$$

where $(D_-, D_+)$ is a random vector independent of $\{X_n, Y_n\}$ with distribution

$$\begin{aligned} P[D_- > x, D_+ > y] &= \int_{x+y}^{\infty} \frac{P[X_{\text{on}} + Y_{\text{off}} > s]}{\mu}\, ds \\ &= \int_{x+y}^{\infty} \frac{1 - F_{\text{on}} * F_{\text{off}}(s)}{\mu}\, ds. \end{aligned} \tag{2.1}$$

Here is an explicit construction of the stationary on/off process [see Heath, Resnick and Samorodnitsky (1998) for a one-sided version]. Define three independent random vectors $B, (X_{\text{on}}^{(-,0)}, X_{\text{on}}^{(+,0)}), (Y_{\text{off}}^{(-,0)}, Y_{\text{off}}^{(+,0)})$ which are independent of $\{X_{\text{on}}, Y_{\text{off}}, X_n, Y_n, n \geq 1\}$ as follows: $B$ is a Bernoulli random variable with values $\{0, 1\}$ and mass function

$$P[B = 1] = \frac{\mu_{\text{on}}}{\mu} = 1 - P[B = 0]$$

and $(x > 0, y > 0)$

$$P[X_{\mathrm{on}}^{(-,0)} > x, X_{\mathrm{on}}^{(+,0)} > y] = \int_{x+y}^{\infty} \frac{1 - F_{\mathrm{on}}(s)}{\mu_{\mathrm{on}}} \, ds =: 1 - F_{\mathrm{on}}^{(0)}(x + y),$$

$$P[Y_{\mathrm{off}}^{(-,0)} > x, Y_{\mathrm{off}}^{(+,0)} > y] = \int_{x+y}^{\infty} \frac{1 - F_{\mathrm{off}}(s)}{\mu_{\mathrm{off}}} \, ds =: 1 - F_{\mathrm{off}}^{(0)}(x + y).$$

Let

$$D_{+}^{(0)} = B(X_{\mathrm{on}}^{(+,0)} + Y_{\mathrm{off}}) + (1 - B)Y_{\mathrm{off}}^{(+,0)},$$
$$D_{-}^{(0)} = BX_{\mathrm{on}}^{(-,0)} + (1 - B)(Y_{\mathrm{off}}^{(-,0)} + X_{\mathrm{on}}),$$

and define a delayed renewal sequence by

$$\begin{aligned}\{S_n, n = \dots, -1, 0, 1, 2, \dots\} \\ := \Big\{ &(-D_{-}^{(0)} - S_n^{(-,X)} - S_n^{(-,Y)})_{n=1,2,\dots}, -D_{-}^{(0)}, \\ &D_{+}^{(0)}, (D_{+}^{(0)} + S_n^{(+,X)} + S_n^{(+,Y)})_{n=1,2,\dots} \Big\}\end{aligned}$$

and this delayed renewal sequence is stationary. (In our terminology $S_0 = D_{+}^{(0)}$.)

We now define the indicator process of *on* periods $I(t)$ to be 1 if $t$ falls in an *on* period and $I(t) = 0$ if $t$ is in an *off* period. More precisely, the process $\{I(t), t \in \mathbb{R}\}$ is defined in terms of $\{S_n, n = \dots, -1, 0, 1, 2, \dots\}$ as follows:

$$\begin{aligned} I(t) = {} & B\mathbf{1}_{[-X_{\mathrm{on}}^{(-,0)}, X_{\mathrm{on}}^{(+,0)})}(t) + (1 - B)\mathbf{1}_{[-Y_{\mathrm{off}}^{(-,0)} - X_{\mathrm{on}}, -Y_{\mathrm{off}}^{(-,0)})}(t) \\ & + \sum_{n \neq -1} \mathbf{1}_{[S_n, S_n + X_{n+1})}(t). \end{aligned} \tag{2.2}$$

REMARK 2.1. A useful fact [Heath, Resnick and Samorodnitsky (1998), Corollary 2.2] is that for any $t \geq 0$, conditional on $I(t) = 1$, the subsequent sequence of on/off periods is the same as seen from time 0 in the stationary process with $B = 1$. In particular, conditionally on $I(0) = 1$, looking forward into the future produces an *on* period with distribution $F_{\mathrm{on}}^{(0)}$ and then a sequence of *off* and *on* periods with distributions $F_{\mathrm{off}}$ and $F_{\mathrm{on}}$. The situation is similar looking backwards, or looking both ways.

Our hierarchical product model is formed by taking products of $n$ on/off indicator processes. As $n \to \infty$, we need to keep the overall *on* rate roughly constant to get useful limits. There are two ways to do this. As $n$ increases, one can either lengthen the *on* periods or one can increase the individual line *on* rates rather than keeping them fixed at 1. We take the former approach in this paper and investigate the latter elsewhere. So we suppose we have $n$ i.i.d. stationary on/off indicator processes $I_1^{(n)}, \dots, I_n^{(n)}$. The *on* period distribution of each factor process depends on $n$ and is denoted by $F_{\mathrm{on}}^{(n)}$ and has mean $\mu_{\mathrm{on}}^{(n)}$. The *off* period distribution is supposed independent of $n$ and as usual is $F_{\mathrm{off}}$ which has mean $\mu_{\mathrm{off}}$. We will always assume that $F_{\mathrm{off}}(0)=0$; this can always be assured by replacing $F_{\mathrm{on}}^{(n)}$ by its appropriate geometric convolution power. The corresponding complementary cumulative distributions are denoted $F_{\mathrm{on}}^{(n,0)}$ and $F_{\mathrm{off}}^{(0)}$,

$$F_{\mathrm{on}}^{(n,0)}(x) := \int_0^x \frac{1-F_{\mathrm{on}}^{(n)}(s)}{\mu_{\mathrm{on}}^{(n)}}\,ds, \qquad F_{\mathrm{off}}^{(0)}(x) := \int_0^x \frac{1-F_{\mathrm{off}}(s)}{\mu_{\mathrm{off}}}\,ds \tag{2.3}$$

and $B_i^{(n)}$ is a Bernoulli random variable for the $i$th process independent of the *on* and *off* periods with distribution

$$P\big[B_i^{(n)}=1\big] = \frac{\mu_{\mathrm{on}}^{(n)}}{\mu_{\mathrm{on}}^{(n)}+\mu_{\mathrm{off}}}.$$

The product model is

$$Z^{(n)}(t) = \prod_{i=1}^{n} I_i^{(n)}(t). \tag{2.4}$$

Since $Z^{(n)}(t)=1$ iff $I_i^{(n)}(t)=1$ for $i=1,\dots,n$, we have

$$\begin{aligned} P\big[Z^{(n)}(t)=1\big] &= \left(\frac{\mu_{\mathrm{on}}^{(n)}}{\mu_{\mathrm{on}}^{(n)}+\mu_{\mathrm{off}}}\right)^{n} \\ &= \left(1+\frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}}^{(n)}}\right)^{-n} \\ &= \left(1+\frac{n\mu_{\mathrm{off}}/\mu_{\mathrm{on}}^{(n)}}{n}\right)^{-n}, \end{aligned}$$

and hence, we stabilize the input rates and get

$$\lim_{n\to\infty} P\big[Z^{(n)}(t)=1\big] = e^{-\mu_{\mathrm{off}}/\mu_{\mathrm{on}}} \tag{2.5}$$

iff

$$\lim_{n\to\infty} \frac{\mu_{\mathrm{on}}^{(n)}}{n} = \mu_{\mathrm{on}}. \tag{2.6}$$

Condition (2.6) will be our standing assumption in the rest of the paper and serves as the mechanism for stabilizing the input rate of $Z^{(n)}(\cdot)$ as $n\to\infty$.

In subsequent sections we will:

1. Examine conditions under which a limit process $Z^{(\infty)}(\cdot)$ exists such that

$$Z^{(n)}(\cdot) \Rightarrow Z^{(\infty)}(\cdot), \tag{2.7}$$

in the sense of convergence of finite dimensional distributions.

2. Provide interpretations of the limit processes $Z^{(\infty)}(\cdot)$. Only in certain cases can the limit process $Z^{(\infty)}(\cdot)$ be constructed from independent *on/off* cycles.
3. Examine the dependence structure of the limit process $Z^{(\infty)}(\cdot)$.

To show convergence of the finite dimensional distributions of $Z^{(n)}(\cdot)$ to a limit $Z^{(\infty)}(\cdot)$, it suffices to show, since $Z^{(n)}(\cdot)$ has range $\{0, 1\}$, that for any $k$ and time points $0 = h_0 \le h_1 \le \cdots \le h_k$ that

$$P\big[Z^{(n)}(h_i) = 1,\ i = 1, \dots, k\big] \to P\big[Z^{(\infty)}(h_i) = 1,\ i = 1, \dots, k\big].$$

The reason for restricting attention to the range point 1 is that

$$P\big[Z^{(n)}(h_i) = 1,\ i = 1, \dots, k\big] = \big(P\big[I_i^{(n)}(h_i) = 1,\ i = 1, \dots, k\big]\big)^n. \tag{2.8}$$

**3. An illuminating special case.** In the single channel *on/off* construction we replace $X_i$ by $nX_i$ and $X_i^{(0)}$ by $nX_i^{(0)}$ so that $F_{\mathrm{on}}(x)$ is replaced by $F_{\mathrm{on}}(x/n) = F_{\mathrm{on}}^{(n)}(x)$ and $F_{\mathrm{on}}^{(0)}(x)$ is replaced by $F_{\mathrm{on}}^{(0)}(x/n) =: F_{\mathrm{on}}^{(n,0)}(x)$. This means that

$$\mu_{\mathrm{on}}^{(n)} = n\mu_{\mathrm{on}} = n\int_0^\infty x F_{\mathrm{on}}(dx)$$

in accordance with (2.6). Since (2.6) holds, we have (2.5) holding as well.

We will assume in this section that $F_{\mathrm{on}}(0) = 0$. The reader will find it easy to see what changes in our calculations if this assumption does not hold. Alternatively, one can see what happens in that case from our general discussion in Section 4.

With these assumptions that $F_{\mathrm{on}}^{(n)}(x) = F_{\mathrm{on}}(x/n)$ it is not hard to see that $Z^{(n)}(\cdot) \Rightarrow Z^{(\infty)}(\cdot)$. We illustrate the proof by showing that for any $h > 0$,

$$P\big[Z^{(n)}(0) = Z^{(n)}(h) = 1\big] \to P\big[Z^{(\infty)}(0) = Z^{(\infty)}(h) = 1\big]$$

and defer the proof for an arbitrary finite collection of time points until the general discussion. Understanding the bivariate distributions of the limit process will already allow us to identify the limit process.

Define the ordinary renewal function

$$U^{(n)} = \sum_{n=0}^{\infty} \big(F_{\mathrm{on}}^{(n)} * F_{\mathrm{off}}\big)^{n*} \tag{3.1}$$

and the delayed renewal function

$$V^{(n)} = F_{\mathrm{on}}^{(n,0)} * F_{\mathrm{off}} * U^{(n)}. \tag{3.2}$$

Conditional on $I_i^{(n)}(0) = 1$, we have $I_i^{(n)}(h) = 1$ if either the initial *on* period extends past $h$ [which occurs with probability $1 - F_{\mathrm{on}}^{(n,0)}(h)$] or if the initial *on* period plus an *off* period terminate before $h$ and then there is a last *off* period before $h$ followed by an *on* period which covers $h$. Thus, we see that as $n \to \infty$,

$$P[Z^{(n)}(0) = Z^{(n)}(h) = 1]$$

$$\sim e^{-\mu_{\mathrm{off}}/\mu_{\mathrm{on}}}\left(1 - F_{\mathrm{on}}^{(0)}(h/n) + \int_0^h \left(1 - F_{\mathrm{on}}\left(\frac{h-u}{n}\right)\right) V^{(n)}(du)\right)^n \tag{3.3}$$

$$= e^{-\mu_{\mathrm{off}}/\mu_{\mathrm{on}}}\left(1 - \frac{nF_{\mathrm{on}}^{(0)}(h/n) - \int_0^h (1 - F_{\mathrm{on}}(\frac{h-u}{n}))nV^{(n)}(du)}{n}\right)^n. \tag{3.4}$$

To get a limit, we obviously need to show that

$$nF_{\mathrm{on}}^{(0)}(h/n) - \int_0^h \left(1 - F_{\mathrm{on}}\left(\frac{h-u}{n}\right)\right)nV^{(n)}(du) \tag{3.5}$$

converges as $n \to \infty$. Now first, as $n \to \infty$, since $F_{\mathrm{on}}(0) = 0$,

$$nF_{\mathrm{on}}^{(0)}(h/n) = n\int_0^{h/n} \frac{1 - F_{\mathrm{on}}(s)}{\mu_{\mathrm{on}}}\, ds \sim \frac{n \cdot (h/n)}{\mu_{\mathrm{on}}} = h/\mu_{\mathrm{on}}. \tag{3.6}$$

This is an expression of the regular variation of $F_{\mathrm{on}}^{(0)}(\cdot)$ at 0 and is equivalent to the weak convergence of the minimum of the $n$ initial *on* times. Of course, it is this minimum which determines the end of the *on* time initiated at 0 by conditioning on $Z^{(n)}(0) = 1$.

For the integral term in (3.5), we have, as $n \to \infty$,

$$\int_0^h \left(1 - F_{\mathrm{on}}\left(\frac{h-u}{n}\right)\right)nV^{(n)}(du) \sim nV^{(n)}(h), \tag{3.7}$$

so it suffices to understand the limit of $nV^{(n)}(h)$. This quantity certainly remains bounded as $n$ varies since

$$nV^{(n)}(h) = nE\left(\sum_{j=0}^{\infty} \mathbf{1}_{[nX^{(0)}+Y+nS_j^{(X)}+S_j^{(Y)} \le h]}(u)\right) \le nE\left(\sum_{j=0}^{\infty} \mathbf{1}_{[nX^{(0)}+nS_j^{(X)} \le h]}(u)\right)$$

and by stationarity, this is $h/\mu_{\mathrm{on}}$.

The Laplace transform of $nV^{(n)}$ is, for any $\theta > 0$,

$$\begin{aligned} n\widehat{V^{(n)}}(\theta) &= \int_0^\infty e^{-\theta x} nV^{(n)}(dx) = \frac{n\widehat{F_{\mathrm{on}}^{(0)}}(n\theta)\widehat{F_{\mathrm{off}}}(\theta)}{1 - \widehat{F_{\mathrm{on}}}(n\theta)\widehat{F_{\mathrm{off}}}(\theta)} \\ &= \frac{n((1 - \widehat{F_{\mathrm{on}}}(n\theta))/(n\theta\mu_{\mathrm{on}}))\widehat{F_{\mathrm{off}}}(\theta)}{1 - \widehat{F_{\mathrm{on}}}(n\theta)\widehat{F_{\mathrm{off}}}(\theta)} \\ &\to \frac{\widehat{F_{\mathrm{off}}}(\theta)}{\theta\mu_{\mathrm{on}}} = \int_0^\infty e^{-\theta x}\frac{F_{\mathrm{off}}(x)}{\mu_{\mathrm{on}}}\, dx, \end{aligned}$$

which implies as $n \to \infty$,

$$nV^{(n)}(h) \to \int_0^h \frac{F_{\mathrm{off}}(s)}{\mu_{\mathrm{on}}}\,ds. \tag{3.8}$$

This leads to the following result.

PROPOSITION 3.1. *If $F_{\mathrm{on}}^{(n)}(x) = F_{\mathrm{on}}(x/n)$, then*

$$Z^{(n)}(\cdot) \Rightarrow Z^{(\infty)}(\cdot),$$

*in the sense of convergence of finite dimensional distributions. The bivariate distributions of the limit process $Z^{(\infty)}(\cdot)$ satisfy*

$$P\big[Z^{(\infty)}(0) = Z^{(\infty)}(h) = 1\big] = \exp\left\{-\frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}}}\big[1 + F_{\mathrm{off}}^{(0)}(h)\big]\right\}.$$

PROOF. We only verify bivariate distributions converge here; the general case follows in Section 4.5. From (3.5)–(3.8) we conclude

$$\begin{aligned}
&P\big[Z^{(n)}(0) = Z^{(n)}(h) = 1\big] \\
&\quad \sim \exp\left\{-\frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}}}\right\}\exp\left\{-\left[h/\mu_{\mathrm{on}} - \int_0^h F_{\mathrm{off}}(s)\,ds/\mu_{\mathrm{on}}\right]\right\} \\
&\quad = \exp\left\{-\left[\frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}}} + \frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}}}\int_0^h \big(1 - F_{\mathrm{off}}(s)\big)/\mu_{\mathrm{off}}\,ds\right]\right\} \\
&\quad = \exp\left\{-\frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}}}\big[1 + F_{\mathrm{off}}^{(0)}(h)\big]\right\}.
\end{aligned}$$

□

3.1. *Identifying the limit process.* Since $F_{\mathrm{on}}^{(0)}$ is regularly varying with index 1 at 0 as a consequence of having a density which is nonzero at 0, we get (3.6), a familiar condition from extreme value theory. Condition (3.6) implies that suitably normalized minima of $n$ forward recurrence times converge weakly to an exponential distribution [Resnick (1987)]. Conditional on $Z^{(n)}(0) = 1$, this minimum is the time the initial *on* period ends. The subsequent *off* period extends as long as any of $n$ lines is in the *off* state. It is also instructive to remember that condition (3.6) is the condition that ensures that the $n$-initial *on* periods arrange themselves asymptotically as Poisson points and we may think of the *off* periods as delays in a queueing system. The first *off* period of $Z^{(n)}(\cdot)$, conditional on $Z^{(n)}(0) = 1$, should then be related to the busy period of an $M/G/\infty$ queue.

PROPOSITION 3.2. *Suppose $F_{\mathrm{on}}^{(n)}(x) = F_{\mathrm{on}}(x/n)$. Then $Z^{(\infty)}(\cdot)$ is the indicator process of a stationary on/off process where the* on *distribution is exponential with parameter $1/\mu_{\mathrm{on}}$ and the* off *distribution is the busy period distribution of an $M/G/\infty$ queue whose input is a Poisson process with rate $1/\mu_{\mathrm{on}}$ and whose service length distribution is $F_{\mathrm{off}}$.*

PROOF. Let $C$ be the busy period length distribution of a stationary $M/G/\infty$ queue described in the statement of the proposition. The Laplace transform of $C$ is given in Takács (1962) and Hall (1988). For $\theta > 0$,

$$\int_0^\infty e^{-\theta s} C(ds) = 1 + \theta\mu_{\text{on}} - \frac{\mu_{\text{on}}}{\int_0^\infty \exp\{-\theta t - (\mu_{\text{off}}/\mu_{\text{on}}) F_{\text{off}}^{(0)}(t)\}\, dt}$$

and the mean is

$$\int_0^\infty x C(dx) = \mu_{\text{on}}(e^{\mu_{\text{off}}/\mu_{\text{on}}} - 1).$$

Let $Z^*(\cdot)$ be the indicator process of a stationary on/off process generated by an *off* distribution $C$ and an *on* period distribution $E(\cdot)$ which is exponential with parameter $1/\mu_{\text{on}}$. Then for any $t > 0$,

$$\begin{aligned} P[Z^*(t) = 1] &= \frac{\int_0^\infty x E(dx)}{\int_0^\infty x E(dx) + \int_0^\infty x C(dx)} \\ &= \frac{\mu_{\text{on}}}{\mu_{\text{on}}(1 + \exp\{\mu_{\text{off}}/\mu_{\text{on}}\} - 1)} \\ &= e^{-\mu_{\text{off}}/\mu_{\text{on}}}, \end{aligned}$$

as desired. Furthermore, for any $h > 0$, since $E(x) = E^{(0)}(x)$,

$$\begin{aligned} P[Z^*(0) = Z^*(h) = 1] &= e^{-\mu_{\text{off}}/\mu_{\text{on}}} P[Z^*(h) = 1 | Z^*(0) = 1] \\ &= e^{-\mu_{\text{off}}/\mu_{\text{on}}} \int_0^h e^{-(h-s)/\mu_{\text{on}}} U(ds), \end{aligned}$$

where the renewal function $U$ is given by

$$U = \sum_{n=0}^\infty (E * C)^{n*}.$$

To evaluate this using transforms, we get, for any $\theta > 0$,

$$\begin{aligned} &\int_0^\infty e^{-\theta h}\left[\int_{s=0}^h e^{-(h-s)/\mu_{\text{on}}} U(ds)\right] dh \\ &\quad = \int_{s=0}^\infty e^{-\theta s}\left[\int_{h=s}^\infty e^{-(h-s)/\mu_{\text{on}}} e^{-\theta(h-s)}\, dh\right] U(ds). \end{aligned}$$

The inner integral evaluates to $(\theta + 1/\mu_{\text{on}})^{-1}$ and, with "^" denoting transform, we get

$$\begin{aligned} &= \frac{1}{\theta + 1/\mu_{\text{on}}} \hat{U}(\theta) \\ &= \frac{1}{\theta + 1/\mu_{\text{on}}} \frac{1}{1 - \hat{C}(\theta)\mu_{\text{on}}^{-1}/(\theta + \mu_{\text{on}}^{-1})} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\theta + \mu_{\mathrm{on}}^{-1} - \hat{C}(\theta)/\mu_{\mathrm{on}}} \\
&= \frac{1}{\theta + \mu_{\mathrm{on}}^{-1} - [\mu_{\mathrm{on}}^{-1} + \theta - 1/\int_0^\infty \exp\{-\theta t - (\mu_{\mathrm{off}}/\mu_{\mathrm{on}}) F_{\mathrm{off}}^{(0)}(t)\}\,dt]} \\
&= \int_0^\infty e^{-\theta t} \exp\left\{-\frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}}} F_{\mathrm{off}}^{(0)}(t)\right\} dt.
\end{aligned}
$$

We conclude

$$
\int_0^h e^{-(h-s)\mu_{\mathrm{on}}} U(ds) = \exp\left\{-\frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}}} F_{\mathrm{off}}^{(0)}(h)\right\}
$$

and therefore

$$
P[Z^*(0) = Z^*(h) = 1] = \exp\left\{-\frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}}}[1 + F_{\mathrm{off}}^{(0)}(h)]\right\},
$$

which matches the bivariate distributions of $Z^{(\infty)}(\cdot)$.

Showing that $Z^*$ and $Z^{(\infty)}$ have all multivariate distributions equal is omitted here. A more general statement follows from a representation theorem proved in Section 5 and is given in part (i) of Corollary 5.7. $\square$

## 4. General approximating limits.

Here we make use of the lessons learned from consideration of the special case in Section 3. We assume that the *on* periods composing the $n$-component on/off processes whose product yields $Z^{(n)}(\cdot)$ all have common distribution $F_{\mathrm{on}}^{(n)}$ and that *off* periods as usual have distribution $F_{\mathrm{off}}$. We have seen that as $n \to \infty$, the overall input rate must remain stable in order to obtain useful approximations and this is achieved by imposing condition (2.6). It is also apparent that in order to get convergence of bivariate distributions, it is required that the minimum of the $n$-i.i.d. forward recurrence times each having distribution $F_{\mathrm{on}}^{(n,0)}$ should converge weakly. Conditions guaranteeing this weak convergence of minima are discussed next. We continue to use notation for the complementary cumulative distributions given in (2.3).

THEOREM 4.1. *The following are equivalent*:

(i) *There exists a proper distribution function* $F_{\mathrm{on}}^{(\infty)}$ *and a number* $q$, *with* $0 \le q \le 1$ *such that for all points of continuity* $x$ *of* $F_{\mathrm{on}}^{(\infty)}$,

$$
F_{\mathrm{on}}^{(n)}(x) \to q F_{\mathrm{on}}^{(\infty)}(x). \tag{4.1}
$$

(ii) *There exists a measure* $\nu$ *which is Radon on* $[0, \infty)$ *such that in* $[0, \infty)$,

$$
n F_{\mathrm{on}}^{(n,0)} \xrightarrow{v} \nu, \tag{4.2}
$$

*where "$\xrightarrow{v}$" denotes vague convergence. In this case, with $p = 1 - q$ and $\mathbb{L}$ being Lebesgue measure, and $m^{(\infty)}[0, x] = m^{(\infty)}(x) = \int_0^x (1 - F_{\mathrm{on}}^{(\infty)}(s))\, ds$,*

$$\nu = \frac{p}{\mu_{\mathrm{on}}}\mathbb{L} + \frac{q}{\mu_{\mathrm{on}}} m^{(\infty)}. \tag{4.3}$$

(iii) *Let $\{X_i^{(n,0)}, 1 \le i \le n\}$ be i.i.d. random variables with common distribution $F_{\mathrm{on}}^{(n,0)}$. Then for all $x \ge 0$ which are continuity points of the limit distribution*

$$P\left[\bigwedge_{i=1}^{n} X_i^{(n,0)} > x\right] \to e^{-\nu[0,x]}, \tag{4.4}$$

*as $n \to \infty$.*

(iv) *Let $\{X_i^{(n,0)}, 1 \le i \le n\}$ be i.i.d. random variables with common distribution $F_{\mathrm{on}}^{(n,0)}$. Then the sequence of random point processes whose $n$th point process has points $\{X_i^{(n,0)}, 1 \le i \le n\}$ converges weakly in the space of Radon point measures on $[0, \infty)$ to a limit Poisson point process, equivalently to a Poisson random measure with mean measure $\nu$ [PRM($\nu$)],*

$$\sum_{i=1}^{n} \varepsilon_{X_i^{(n,0)}} \Rightarrow \mathrm{PRM}(\nu). \tag{4.5}$$

*The limiting Poisson process with mean measure $\nu$ is the superposition of a homogeneous Poisson process with rate $p/\mu_{\mathrm{on}}$ and a nonhomogeneous Poisson process with mean measure $qm^{(\infty)}/\mu_{\mathrm{on}}$.*

*Define $U^{(n)}, V^{(n)}$ as in (3.1), (3.2) and*

$$U^{(\infty)} = \sum_{n=0}^{\infty} \left(qF_{\mathrm{on}}^{(\infty)} * F_{\mathrm{off}}\right)^{n*}, \qquad V^{(\infty)} = \nu * F_{\mathrm{off}} * U^{(\infty)}. \tag{4.6}$$

*Then, any of the previous conditions (4.1)–(4.5) imply pointwise convergence*

$$nV^{(n)} \to V^{(\infty)}. \tag{4.7}$$

PROOF. The equivalence of (4.2), (4.4) and (4.5) is well known from extreme value theory [e.g., Resnick (1987)]. Focus on why (4.1) and (4.2) are equivalent. Given (4.1) we have as $n \to \infty$,

$$nF_{\mathrm{on}}^{(n,0)}(x) = n\int_0^x \frac{(1 - F_{\mathrm{on}}^{(n)}(u))}{\mu_{\mathrm{on}}^{(n)}}\, du \sim \int_0^x \frac{(1 - F_{\mathrm{on}}^{(n)}(u))}{\mu_{\mathrm{on}}}\, du$$

and by dominated convergence this converges to

$$\to \int_0^x \frac{(1 - qF_{\mathrm{on}}^{(\infty)}(u))}{\mu_{\mathrm{on}}}\, du = \frac{px}{\mu_{\mathrm{on}}} + \frac{q}{\mu_{\mathrm{on}}} m^{(\infty)}(x)$$

and hence (4.2) follows.

Conversely, if (4.2) holds, then taking Laplace transforms yields for $\theta > 0$,

$$n\widehat{F_{\text{on}}^{(n,0)}}(\theta) = n\left(\frac{1-\widehat{F_{\text{on}}^{(n)}}(\theta)}{\theta\mu_{\text{on}}^{(n)}}\right) \sim \frac{1-\widehat{F_{\text{on}}^{(n)}}(\theta)}{\theta\mu_{\text{on}}}.$$

For any fixed $\theta \ge 0$, $\{(1-\widehat{F_{\text{on}}^{(n)}}(\theta))/(\theta\mu_{\text{on}}), n \ge 1\}$ is bounded in $n$. Hence [Feller (1971), page 433],

$$n\widehat{F_{\text{on}}^{(n,0)}}(\theta) \to \hat{\nu}(\theta),$$

where $\hat{\nu}(\theta)$ is the Laplace transform of the measure $\nu$ on $[0,\infty)$. Thus

$$\lim_{n\to\infty} \widehat{F_{\text{on}}^{(n)}}(\theta) = 1 - \theta\mu_{\text{on}}\hat{\nu}(\theta)$$

and therefore $F_{\text{on}}^{(n)}(x) \to qF_{\text{on}}^{(\infty)}(x)$ at points of continuity for some proper distribution $F_{\text{on}}^{(\infty)}$ and some $0 \le q \le 1$.

Finally, we show why (4.7) is true. Taking Laplace transforms we have

$$\begin{aligned} n\widehat{V^{(n)}}(\theta) &= n\frac{\widehat{F_{\text{on}}^{(n,0)}}(\theta)\widehat{F_{\text{off}}}(\theta)}{1-\widehat{F_{\text{on}}^{(n)}}(\theta)\widehat{F_{\text{off}}}(\theta)} \\ &= \frac{((1-\widehat{F_{\text{on}}^{(n)}}(\theta))/(\theta\mu_{\text{on}}^{(n)}/n))\widehat{F_{\text{off}}}(\theta)}{1-\widehat{F_{\text{on}}^{(n)}}(\theta)\widehat{F_{\text{off}}}(\theta)} \\ &\to \frac{((1-q\widehat{F_{\text{on}}^{(\infty)}}(\theta))/(\theta\mu_{\text{on}}))\widehat{F_{\text{off}}}(\theta)}{1-q\widehat{F_{\text{on}}^{(\infty)}}(\theta)\widehat{F_{\text{off}}}(\theta)} \\ &= \frac{(p/(\theta\mu_{\text{on}}) + (q/\mu_{\text{on}})((1-\widehat{F_{\text{on}}^{(\infty)}}(\theta))/\theta)\widehat{F_{\text{off}}}(\theta)}{1-q\widehat{F_{\text{on}}^{(\infty)}}(\theta)\widehat{F_{\text{off}}}(\theta)}, \end{aligned}$$

which is the Laplace transform of $V^{(\infty)} = \nu * F_{\text{off}} * \sum_{n=0}^{\infty}(qF_{\text{on}}^{(\infty)} * F_{\text{off}})^{n*}$. $\square$

As an example, let

$$f_n(x) = n^2\mathbf{1}_{[0,1/n)}(x).$$

Suppose $\{U_n, n \ge 1\}$ are i.i.d. $U(0,1)$ random variables independent of the nonnegative i.i.d. random variables $\{\xi_n, n \ge 1\}$ assumed to have finite mean. Define $X_i^{(n)} = f_n(U_i) + \xi_i$. Then $f_n(U_i) \Rightarrow 0$, so that $X_1^{(n)} \Rightarrow \xi_1$. Also

$$Ef_n(U_i) = n^2P[U_i < 1/n] = n,$$

so

$$\mu_{\text{on}}^{(n)} = E(X_1^{(n)}) = Ef_n(U_1) + E(\xi_1) \sim n.$$

Hence $q = 1$, $\mu_{\mathrm{on}} = 1$, $F_{\mathrm{on}}^{(\infty)}(x) = P[\xi_1 \le x]$ and $\nu$ is

$$\nu[0,x] = \int_0^x P[\xi_1 > u]\,du = m^{(\infty)}(x).$$

4.1. *General bivariate limits.* We now show under any of the equivalent conditions in Theorem 4.1 that bivariate distributions of $Z^{(n)}(\cdot)$ converge to those of a limit process $Z^{(\infty)}(\cdot)$. However, unlike the case in Section 3, the limiting process will not, in general, be composed of alternating *independent* on/off periods.

Using the reasoning that led to (3.3) and (3.4), we get

$$\begin{aligned} &P\big[Z^{(n)}(0) = Z^{(n)}(h) = 1\big] \\ &\qquad \sim e^{-\mu_{\mathrm{off}}/\mu_{\mathrm{on}}}\left(1 - \frac{nF_{\mathrm{on}}^{(n,0)}(h) - \int_0^h (1 - F_{\mathrm{on}}^{(n)}(h-u))nV^{(n)}(du)}{n}\right)^n. \end{aligned}$$

Now from (4.2) we have $nF_{\mathrm{on}}^{(n,0)}(h) \to \nu[0,h]$, and from (4.7) it follows that $nV^{(n)}(x) \to V^{(\infty)}(x)$ for all $x \ge 0$. Also $1 - F_{\mathrm{on}}^{(n)}(x) \to 1 - qF_{\mathrm{on}}^{(\infty)}(x)$. Thus

$$\begin{aligned} &nF_{\mathrm{on}}^{(n,0)}(h) - \int_0^h \big(1 - F_{\mathrm{on}}^{(n)}(h-u)\big)nV^{(n)}(du) \\ &\quad \to \nu[0,h] - \int_0^h \big(1 - qF_{\mathrm{on}}^{(\infty)}(h-u)\big)V^{(\infty)}(du) \\ &\quad = \nu[0,h] - V^{(\infty)}(h) + qF_{\mathrm{on}}^{(\infty)} * V^{(\infty)}(h) \\ &\quad = \nu[0,h] - \nu * F_{\mathrm{off}} * U^{(\infty)}(h) + qF_{\mathrm{on}}^{(\infty)} * F_{\mathrm{off}} * U^{(\infty)} * \nu(h) \\ &\quad = \nu[0,h] - \nu * F_{\mathrm{off}} * U^{(\infty)}(h) + \big(U^{(\infty)} - \delta_0\big) * \nu(h) \\ &\quad = \nu * U^{(\infty)} * (\delta_0 - F_{\mathrm{off}})(h), \end{aligned}$$

where $\delta_0$ is the measure putting mass 1 at 0. We have proved the following result.

PROPOSITION 4.2. *Suppose* (2.6) *and* (4.1) *or one of its equivalents hold. Then for any* $h > 0$,

$$\begin{aligned} &\lim_{n\to\infty} P\big[Z^{(n)}(0) = Z^{(n)}(h) = 1\big] \\ &\qquad = \exp\left\{-\left[\frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}}} + \nu * U^{(\infty)} * (\delta_0 - F_{\mathrm{off}})(h)\right]\right\}, \end{aligned} \tag{4.8}$$

*where* $\nu$ *is given by* (4.3) *and* $U^{(\infty)}$ *is given by* (4.6).

4.2. *A special case.* Consider the special case where $q = 0$ and $p = 1$. Therefore, $\nu = \mu_{\text{on}}^{-1}\mathbb{L}$ and $U^{(\infty)} = \delta_0$. Thus the limit in (4.8) is

$$\begin{aligned}
&\exp\biggl\{-\biggl[\frac{\mu_{\text{off}}}{\mu_{\text{on}}} + \nu * U^{(\infty)} * (\delta_0 - F_{\text{off}})(h)\biggr]\biggr\} \\
&\qquad = \exp\biggl\{-\biggl[\frac{\mu_{\text{off}}}{\mu_{\text{on}}} + \frac{1}{\mu_{\text{on}}}\mathbb{L} * \delta_0 * (\delta_0 - F_{\text{off}})(h)\biggr]\biggr\} \\
&\qquad = \exp\biggl\{-\biggl[\frac{\mu_{\text{off}}}{\mu_{\text{on}}} + \frac{1}{\mu_{\text{on}}}\int_0^h \bigl(1 - F_{\text{off}}(s)\bigr)\,ds\biggr]\biggr\} \\
&\qquad = \exp\biggl\{-\frac{\mu_{\text{off}}}{\mu_{\text{on}}}\bigl[1 + F_{\text{off}}^{(0)}(h)\bigr]\biggr\},
\end{aligned} \tag{4.9}$$

which is the same limit found in Section 3.

4.3. *A converse.* It turns out that (2.6) and (4.2) are the exact conditions for bivariate convergence.

PROPOSITION 4.3. *Suppose*:

(i) $\lim_{n\to\infty} P[Z^{(n)}(0) = 1] = l_1 \in (0, 1)$.

(ii) *For all* $h \ge 0$, $\lim_{n\to\infty} P[Z^{(n)}(0) = 1, Z^{(n)}(h) = 1] = l_2(h) \in (0, 1)$.

*Then* (2.6) *holds and there exists a Radon measure* $\nu(\cdot)$ *on* $[0, \infty)$ *such that*

$$nF_{\text{on}}^{(n,0)}(\cdot) \xrightarrow{v} \nu(\cdot),$$

*so* (4.2) *holds as well.*

PROOF. Condition (i) implies, as $n \to \infty$ that $(\mu_{\text{on}}^{(n)}/(\mu_{\text{on}}^{(n)} + \mu_{\text{off}}))^n \to l_1$. This gives (2.6).

Next, we have

$$P\bigl[Z^{(n)}(h) = 1\big|Z^{(n)}(0) = 1\bigr] \to \frac{l_2(h)}{l_1},$$

and the left-hand side equals

$$\biggl(1 - \frac{[nF_{\text{on}}^{(n,0)}(h) - \int_0^h (1 - F_{\text{on}}^{(n)}(h-s))nV^{(n)}(ds)]}{n}\biggr)^n.$$

This implies

$$\begin{aligned}
G_n'(h) &= \biggl[nF_{\text{on}}^{(n,0)}(h) - \int_0^h \bigl(1 - F_{\text{on}}^{(n)}(h-s)\bigr)nV^{(n)}(ds)\biggr] \\
&\to -\log\frac{l_2(h)}{l_1} =: G_\infty'(h).
\end{aligned}$$

After some manipulation involving the renewal function $V^{(n)}$, we get

$$G'_n(h) = nF_{\text{on}}^{(n,0)} * U^{(n)} * (\delta_0 - F_{\text{off}})(h).$$

Note for $x \geq 0$,

$$\begin{aligned} G'_n(x) &\leq nF_{\text{on}}^{(n,0)} * U^{(n)}(x) \\ &= n \sum_{j=0}^{\infty} F_{\text{on}}^{(n,0)} * \left(F_{\text{on}}^{(n)} * F_{\text{off}}\right)^{j*}(x) \\ &\leq n \sum_{j=0}^{\infty} F_{\text{on}}^{(n,0)} * \left(F_{\text{on}}^{(n)}\right)^{j*}(x) \end{aligned}$$

and because $\sum_{j=0}^{\infty} F_{\text{on}}^{(n,0)} * (F_{\text{on}}^{(n)})^{j*}$ is a stationary renewal measure, it is Lebesgue measure divided by the mean renewal time so that we get

$$= n \cdot \frac{x}{\mu_{\text{on}}^{(n)}} = \frac{x}{\mu_{\text{on}}^{(n)}/n} \leq \frac{2x}{\mu_{\text{on}}}$$

for all large $n$. This bounding function is locally integrable and hence $G'_n \to G'_\infty$ implies

$$\begin{aligned} G_n(x) := \int_0^x G'_n(h)\,dh &\to \int_0^x G'_\infty(h)\,dh \\ &= \int_0^x -\log \frac{l_2(h)}{l_1}\,dh =: G_\infty(x). \end{aligned}$$

Now take Laplace transforms. For $\theta > 0$,

$$\begin{aligned} \hat{G}_n(\theta) &= \int_0^\infty e^{-\theta u} nF_{\text{on}}^{(n,0)} * U^{(n)} * (\delta_0 - F_{\text{off}})(u)\,du \\ &= \frac{1 - \widehat{F_{\text{off}}}(\theta)}{\theta} \frac{n\widehat{F_{\text{on}}^{(n,0)}}(\theta)}{1 - \widehat{F_{\text{on}}^{(n)}}(\theta)\widehat{F_{\text{off}}}(\theta)} \end{aligned}$$

and since

$$\widehat{F_{\text{on}}^{(n,0)}}(\theta) = \frac{1 - \widehat{F_{\text{on}}^{(n)}}(\theta)}{\mu_{\text{on}}^{(n)}\theta},$$

we get

$$\hat{G}_n(\theta) = \frac{1 - \widehat{F_{\text{off}}}(\theta)}{\theta} \frac{\hat{\nu}_n(\theta)}{1 - \widehat{F_{\text{off}}}(\theta)[1 - \theta(\mu_{\text{on}}^{(n)}/n)\hat{\nu}_n(\theta)]}, \tag{4.10}$$

where we set

$$\nu_n = nF_{\text{on}}^{(n,0)} \quad \text{and} \quad \hat{\nu}_n(\theta) = \int_0^\infty e^{-\theta u} \nu_n(du).$$

Note $\{\hat{G}_n(\theta), n \geq 1\}$ is a bounded sequence for each $\theta$ since

$$\begin{aligned} \hat{G}_n(\theta) &= \frac{1 - \widehat{F_{\text{off}}}(\theta)}{\theta} \frac{(1 - \widehat{F_{\text{on}}^{(n)}}(\theta))/(\theta\mu_{\text{on}}^{(n)}/n)}{1 - \widehat{F_{\text{on}}^{(n)}}(\theta)\widehat{F_{\text{off}}}(\theta)} \\ &\leq \frac{1 - \widehat{F_{\text{off}}}(\theta)}{\theta} \frac{2/(\theta\mu_{\text{on}})}{1 - \widehat{F_{\text{off}}}(\theta)} = \frac{2}{\theta^2\mu_{\text{on}}}. \end{aligned}$$

We conclude [Feller (1971), Theorem 2a, page 433] that since $G_n \to G_\infty$ and $\{\hat{G}_n(\theta),\ n \geq 1\}$ is bounded, that $\hat{G}_n(\theta) \to \hat{G}_\infty(\theta)$. Referring to (4.10), we conclude $\hat{\nu}_n(\theta) \to \hat{\nu}_\infty(\theta)$ where

$$\frac{1 - \widehat{F_{\text{off}}}(\theta)}{\theta} \frac{\hat{\nu}_\infty(\theta)}{1 - \widehat{F_{\text{off}}}(\theta)[1 - \theta\mu_{\text{on}}\hat{\nu}_\infty(\theta)]} = \hat{G}_\infty(\theta).$$

Again applying Theorem 2a, page 433 of Feller (1971) we conclude

$$\nu_n = nF_{\text{on}}^{(n,0)} \to \nu_\infty$$

and the statement of the proposition is proved with $\nu = \nu_\infty$. $\square$

4.4. *Asymptotic independence.* In Section 4.5, we will show that the convergence of bivariate limits in Proposition 4.2 can be extended to higher dimensional convergence. The following Section 5 will show various representations of processes $Z^{(\infty)}(\cdot)$ which have the limiting multivariate distributions.

We say that *asymptotic independence holds* if $Z^{(n)} \Rightarrow Z^{(\infty)}$ and

$$\begin{aligned} &\lim_{h\to\infty} P\big[Z^{(\infty)}(0) = Z^{(\infty)}(h) = 1\big] \\ &\quad = \exp\left\{-2\frac{\mu_{\text{off}}}{\mu_{\text{on}}}\right\} = P\big[Z^{(\infty)}(0) = 1\big]P\big[Z^{(\infty)}(h) = 1\big]. \end{aligned} \tag{4.11}$$

Here are some special cases where it is relatively easy to resolve whether asymptotic independence holds or not.

*The case $q=0$.* In this case, using (4.9) we have

$$\lim_{h\to\infty} P\big[Z^{(\infty)}(0)=Z^{(\infty)}(h)=1\big] = \lim_{h\to\infty}\exp\left\{-\frac{\mu_{\text{off}}}{\mu_{\text{on}}}\big[1+F_{\text{off}}^{(0)}(h)\big]\right\} = \exp\left\{-2\frac{\mu_{\text{off}}}{\mu_{\text{on}}}\right\}$$

and hence asymptotic independence holds. Note $\nu=\mu_{\text{on}}^{-1}\mathbb{L}$ so $\nu[0,\infty)=\infty$.

*The case $q=1$ and $m^{(\infty)}(\infty)<\infty$.* Set

$$F_{\text{on}}^{(\infty,0)}(x)=\frac{m^{(\infty)}(x)}{m^{(\infty)}(\infty)}=\frac{\int_0^x(1-F_{\text{on}}^{(\infty)}(s))\,ds}{\int_0^\infty(1-F_{\text{on}}^{(\infty)}(s))\,ds}. \tag{4.12}$$

So $\nu=\frac{m^{(\infty)}(\infty)}{\mu_{\text{on}}}F_{\text{on}}^{(\infty,0)}$ and therefore

$$\nu * U^{(\infty)} * (\delta_0-F_{\text{off}})=\frac{m^{(\infty)}(\infty)}{\mu_{\text{on}}}\big(F_{\text{on}}^{(\infty,0)} * U^{(\infty)}\big) * (\delta_0-F_{\text{off}}).$$

Note that $F_{\text{on}}^{(\infty,0)} * U^{(\infty)}$ is a (delayed) renewal function corresponding to mean interrenewal time $m^{(\infty)}(\infty)+\mu_{\text{off}}$ and hence by the key renewal theorem, the right-hand side of the above display converges to

$$\frac{m^{(\infty)}(\infty)}{\mu_{\text{on}}}\frac{\int_0^\infty(1-F_{\text{off}}(s))\,ds}{m^{(\infty)}(\infty)+\mu_{\text{off}}}=\frac{m^{(\infty)}(\infty)}{\mu_{\text{on}}}\left(\frac{\mu_{\text{off}}}{m^{(\infty)}(\infty)+\mu_{\text{off}}}\right).$$

Consequently,

$$\begin{aligned}&\lim_{h\to\infty} P\big[Z^{(\infty)}(0)=Z^{(\infty)}(h)=1\big]\\ &\qquad=\exp\left\{-\left[\frac{\mu_{\text{off}}}{\mu_{\text{on}}}+\frac{m^{(\infty)}(\infty)}{\mu_{\text{on}}}\left(\frac{\mu_{\text{off}}}{m^{(\infty)}(\infty)+\mu_{\text{off}}}\right)\right]\right\}.\end{aligned}\tag{4.13}$$

Asymptotic independence does not hold. Note $\nu[0,\infty)<\infty$ in this case.

A particularization is obtained by assuming $F_{\text{on}}^{(\infty)}=\delta_0$ in which case $\nu=0$ and the entire bivariate distribution is $\exp\{-\mu_{\text{off}}/\mu_{\text{on}}\}$.

*The case $q=1$ and $m^{(\infty)}(\infty)=\infty$.* Here $\nu=\mu_{\text{on}}^{-1}m^{(\infty)}$ and we must evaluate the limit of

$$\mu_{\text{on}}^{-1}m^{(\infty)} * U^{(\infty)} * (\delta_0-F_{\text{off}})(h)$$

as $h\to\infty$. Because $m^{(\infty)} * U^{(\infty)}$ has a density, we can write

$$\begin{aligned}m^{(\infty)} * U^{(\infty)}(x)&=\int_0^x\left(\int_{y=0}^s(1-F_{\text{on}}^{(\infty)}(s-y))U^{(\infty)}(dy)\right)ds\\ &=\int_0^x p(s)\,ds.\end{aligned}\tag{4.14}$$

[Note that this formula is valid whether or not $m_\infty(\infty)$ is finite.] Considering an on/off process with *on* distribution $F_{\text{on}}^{(\infty)}$ and *off* distribution $F_{\text{off}}$ and starting with an *on* period, we see that $p(s)$ is the probability of being in the *on* state at time $s$. Since $m^{(\infty)}(\infty) = \infty$, we have $p(s) \to 1$ as $s \to \infty$. Note that

$$\begin{aligned}
&\mu_{\text{on}}^{-1} m^{(\infty)} * U^{(\infty)} * (\delta_0 - F_{\text{off}})(h) \\
&\quad = \frac{1}{\mu_{\text{on}}} \int_0^h \bigl(1 - F_{\text{off}}(h-s)\bigr) p(s)\, ds \\
&\quad = \frac{1}{\mu_{\text{on}}} \int_0^\infty \bigl(\mathbf{1}(s \le h) p(h-s)\bigr)\bigl(1 - F_{\text{off}}(s)\bigr)\, ds \\
&\quad \to \frac{1}{\mu_{\text{on}}} \int_0^\infty \bigl(1 - F_{\text{off}}(s)\bigr)\, ds \\
&\quad = \mu_{\text{off}}/\mu_{\text{on}}
\end{aligned}$$

by the dominated convergence theorem. Consequently, asymptotic independence holds and

$$\lim_{h\to\infty} P\bigl[Z^{(\infty)}(0) = Z^{(\infty)}(h) = 1\bigr] = \exp\biggl\{-2\frac{\mu_{\text{off}}}{\mu_{\text{on}}}\biggr\}.$$

With the experience built up in the previous cases, we now state a general result.

PROPOSITION 4.4. *Under the assumptions of Proposition* 4.2, *we have asymptotic independence iff* $\nu[0,\infty) = \infty$.

REMARK 4.5. The dichotomy between asymptotic independence holding and not holding corresponds to whether the limiting Poisson process in (4.5) contains infinitely or finitely many points. This dichotomy will be better understood once we discuss representations of the limit process in Section 4.

PROOF. Due to the consideration of special cases above, we need only consider the case when $0 < q < 1$ and show asymptotic independence. First,

$$\begin{aligned}
&\nu * (\delta_0 - F_{\text{off}})(h) \\
&\quad = \frac{p}{\mu_{\text{on}}} \int_0^h \bigl(1 - F_{\text{off}}(s)\bigr)\, ds + \frac{q}{\mu_{\text{on}}} \int_0^h \bigl(1 - F_{\text{off}}(s)\bigr)\bigl(1 - F_{\text{on}}^{(\infty)}(h-s)\bigr)\, ds \\
&\quad = A + B.
\end{aligned}$$

Now $A \to p\mu_{\text{off}}/\mu_{\text{on}}$ and

$$\begin{aligned}
&\int_0^h \bigl(1 - F_{\text{off}}(s)\bigr)\bigl(1 - F_{\text{on}}^{(\infty)}(h-s)\bigr)\, ds \\
&\qquad = \int_0^\infty \bigl(1 - F_{\text{off}}(s)\bigr)\bigl(\mathbf{1}(s \le h)\bigl(1 - F_{\text{on}}^{(\infty)}(h-s)\bigr)\bigr)\, ds \to 0
\end{aligned}$$

by the dominated convergence theorem. We conclude that

$$v(h) := \nu * (\delta_0 - F_{\text{off}})(h) \to p\frac{\mu_{\text{off}}}{\mu_{\text{on}}}. \tag{4.15}$$

Since $U(x) \uparrow U(\infty) = 1/(1-q)$ we have [e.g., see Resnick (1992), page 253]

$$U * \nu * (\delta_0 - F_{\text{off}})(h) \to U(\infty) p\frac{\mu_{\text{off}}}{\mu_{\text{on}}} = \frac{\mu_{\text{off}}}{\mu_{\text{on}}}$$

and therefore asymptotic independence holds. $\square$

4.5. *Convergence of finite-dimensional distributions.* We now show why finite-dimensional distributions of $Z^{(n)}(\cdot)$ converge.

For this section we need the quantity

$$p_j^{(n)}(h_1, \dots, h_j) = P\big[I_1^{(n)}(h_i) = 1,\ i = 1, \dots, j \big| I_1^{(n)}(0) = 0\big], \tag{4.16}$$

where we understand the conditioning to mean that the on/off process is initiated by an *off* period with distribution $F_{\text{off}}$. Note, for example, that

$$\begin{aligned} p_1^{(n)}(h) &= P\big[I_1^{(n)}(h) = 1 \big| I_1^{(n)}(0) = 0\big] \\ &= F_{\text{off}} * U^{(n)} * \big(\delta_0 - F_{\text{on}}^{(n)}\big)(h) \\ &\to F_{\text{off}} * U^{(\infty)} * \big(\delta_0 - qF_{\text{on}}^{(\infty)}\big)(h) \end{aligned} \tag{4.17}$$

as $n \to \infty$.

As before [see (2.8)] we have for any $k \geq 1$ and $0 = h_0 < h_1 < \ldots < h_k$,

$$\begin{aligned} &P\big[Z^{(n)}(h_i) = 1,\ i = 0, \dots, k\big] \\ &\quad = \big(P\big[I_1^{(n)}(h_i) = 1,\ i = 0, \dots, k\big]\big)^n \\ &\quad \sim e^{-\mu_{\text{off}}/\mu_{\text{on}}}\big(P\big[I_1^{(n)}(h_i) = 1,\ i = 1, \dots, k \big| I_1^{(n)}(0) = 1\big]\big)^n, \end{aligned}$$

where now the conditioning at time 0 to be in state 1 indicates an on/off process is started in the *on* state with distribution $F_{\text{on}}^{(n,0)}$. The event $\bigcap_{i=1}^k [I_1^{(n)}(h_i) = 1]$ can be realized if the initial *on* period with distribution $F_{\text{on}}^{(n,0)}$ extends beyond $h_k$ or if the initial *on* period ends at time $u$ in some $(h_{j-1}, h_j]$ and then a system starting with an *off* period is in the *on* state at times $h_i - u$, $i = j, \dots, k$. Thus

$$\begin{aligned} &\big(P\big[I_1^{(n)}(h_i) = 1,\ i = 1, \dots, k \big| I_1^{(n)}(0) = 1\big]\big)^n \\ &\quad = \left(1 - F_{\text{on}}^{(n,0)}(h_k) + \sum_{j=1}^k \int_{h_{j-1}}^{h_j} F_{\text{on}}^{(n,0)}(du)\, p_{k-j+1}^{(n)}(h_j - u, \dots, h_k - u)\right)^n \\ &\quad = \left(1 - \frac{1}{n}\left[nF_{\text{on}}^{(n,0)}(h_k)\right.\right. \\ &\qquad\qquad \left.\left. - \sum_{j=1}^k \int_{h_{j-1}}^{h_j} nF_{\text{on}}^{(n,0)}(du)\, p_{k-j+1}^{(n)}(h_j - u, \dots, h_k - u)\right]\right)^n. \end{aligned}$$

We claim that for every $j \geq 1$ and $h_1, \dots, h_j$ (except possibly countably many),

$$p_j^{(n)}(h_1, \dots, h_j) \to p_j^{(\infty)}(h_1, \dots, h_j), \tag{4.18}$$

where $p_j^{(\infty)}(h_1, \dots, h_j)$ is the probability that in a (possibly terminating) on/off process with *off* distribution $F_{\mathrm{off}}$ and *on* distribution $qF_{\mathrm{on}}^{(\infty)}$, starting with an *off* period of distribution $F_{\mathrm{off}}$, the times $h_1, \dots, h_j$ are in the *on* state.

Assuming this claim to be true, we will have

$$\begin{aligned}
&\lim_{n\to\infty} P\bigl[Z^{(n)}(h_j) = 1,\ j = 0, \dots, k\bigr] \\
&\qquad =: P\bigl[Z^{(\infty)}(h_j) = 1,\ j = 0, \dots, k\bigr] \\
&\qquad = \exp\Biggl\{-\Biggl[\frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}}} + \nu[0, h_k] \\
&\qquad\qquad - \sum_{j=1}^{k} \int_{h_{j-1}}^{h_j} \nu(du)\, p_{k-j+1}^{(\infty)}(h_j - u, \dots, h_k - u)\Biggr]\Biggr\}.
\end{aligned} \tag{4.19}$$

We may rewrite (4.19) as follows. Write $q^{(\infty)} = 1 - p^{(\infty)}$ and then

$$\begin{aligned}
&\nu[0, h_k] - \sum_{j=1}^{k} \int_{h_{j-1}}^{h_j} \nu(du)\, p_{k-j+1}^{(\infty)}(h_j - u, \dots, h_k - u) \\
&\quad = \sum_{j=1}^{k} \int_{h_{j-1}}^{h_j} \nu(du)\bigl(1 - p_{k-j+1}^{(\infty)}(h_j - u, \dots, h_k - u)\bigr) \\
&\quad = \sum_{j=1}^{k} \int_{h_{j-1}}^{h_j} \nu(du)\, q_{k-j+1}^{(\infty)}(h_j - u, \dots, h_k - u) \\
&\quad = \frac{1}{\mu_{\mathrm{on}}} \sum_{j=1}^{k} \int_{h_{j-1}}^{h_j} \bigl(p + q(1 - F_{\mathrm{on}}^{(\infty)}(u)\bigr)\, du\, q_{k-j+1}^{(\infty)}(h_j - u, \dots, h_k - u) \\
&\quad = \frac{1}{\mu_{\mathrm{on}}} \sum_{j=1}^{k} \int_{h_{j-1}}^{h_j} \bigl(1 - qF_{\mathrm{on}}^{(\infty)}(u)\bigr) q_{k-j+1}^{(\infty)}(h_j - u, \dots, h_k - u)\, du.
\end{aligned} \tag{4.20}$$

Note that

$$q_l^{(\infty)}(h_1, \dots, h_l) = P\Biggl[\bigvee_{i=1}^{l} I_1^{(\infty)}(h_i) = 0 \Bigg| I_1^{(\infty)}(0) = 0\Biggr]$$

is the conditional probability that at some time point the system is *off* and where the conditioning means start the on/off process with an *off* period with distribution $F_{\mathrm{off}}$ and *on* periods have distribution $qF_{\mathrm{on}}^{(\infty)}$. Thus we have the following theorem.

THEOREM 4.6. *For any* $k = 0, 1, \dots$ *and* $0 = h_0 < h_1 < \cdots < h_k$,

$$
\begin{aligned}
&P[Z^{(\infty)}(h_j) = 1,\ j = 0, \dots, k] \\
&\quad = \exp\Biggl\{-\Biggl[\frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}}} + \frac{1}{\mu_{\mathrm{on}}}\sum_{j=0}^{k-1}\int_{h_j}^{h_{j+1}} \bigl(1 - qF_{\mathrm{on}}^{(\infty)}(u)\bigr) \\
&\qquad\qquad \times q_{k-j}^{(\infty)}(h_{j+1}-u, \dots, h_k - u)\,du\Biggr]\Biggr\}, \qquad (4.21)
\end{aligned}
$$

$$
\quad = \exp\Biggl\{-\Biggl[\frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}}} + \sum_{l=1}^{k}\int_{h_{l-1}}^{h_l} q_{k-l+1}^{(\infty)}(h_0, \dots, h_k; x)\nu(dx)\Biggr]\Biggr\}. \qquad (4.22)
$$

One sees from (4.21) that the limiting process $Z^{(\infty)}$ is characterized by a quadruple $(\mu_{\mathrm{on}}, q, F_{\mathrm{on}}^{(\infty)}, F_{\mathrm{off}})$.

It remains to prove (4.18), which we do by induction. We have already verified the case $j = 1$ in (4.17), so make the induction hypothesis that (4.18) holds for all $j' < j$. Conditioning on where the first off/on cycle ends, we decompose $p_j^{(n)}(h_1, \dots, h_j)$ as

$$
\begin{aligned}
&p_j^{(n)}(h_1, \dots, h_j) \\
&\quad = \int_0^{h_1} p_j^{(n)}(h_1 - u, \dots, h_j - u)F_{\mathrm{off}} * F_{\mathrm{on}}^{(n)}(du) \\
&\qquad + \sum_{i=1}^{j-1}\int_0^{h_1} F_{\mathrm{off}}(du) \\
&\qquad\qquad \times \int_{h_i - u}^{h_{i+1}-u} F_{\mathrm{on}}^{(n)}(dw)\,p_{j-i}^{(n)}(h_{i+1} - u - w, \dots, h_j - u - w) \\
&\qquad + \int_0^{h_1} \bigl(1 - F_{\mathrm{on}}^{(n)}(h_j - u)\bigr)F_{\mathrm{off}}(du) \\
&\quad =: f_j^{(n)}(h_1, \dots, h_j) + \int_0^{h_1} p_j^{(n)}(h_1 - u, \dots, h_j - u)F_{\mathrm{off}} * F_{\mathrm{on}}^{(n)}(du).
\end{aligned}
$$

From the induction hypothesis

$$
f_j^{(n)}(h_1, \dots, h_j) \to f_j^{(\infty)}(h_1, \dots, h_j)
$$

as $n \to \infty$, for all except, perhaps, countably many $h_1, \dots, h_j$, with $f_j^{(\infty)}(h_1, \dots, h_j)$ defined in the obvious way. Therefore,

$$
\begin{aligned}
p_j^{(n)}(h_1, \dots, h_j) &= \int_0^{h_1} f_j^{(n)}(h_1 - u, \dots, h_j - u)U^{(n)}(du) \\
&\to \int_0^{h_1} f_j^{(\infty)}(h_1 - u, \dots, h_j - u)U^{(\infty)}(du) = p_j^{(\infty)}(h_1, \dots, h_j)
\end{aligned}
$$

and so we conclude that (4.18) holds for all, except perhaps, countably many $h_1, \dots, h_j$ as required.

REMARK 4.7. Here is an immediate and, perhaps, not surprising conclusion from (4.21). The limiting process $Z^{(\infty)}$ is *product infinitely divisible*. Indeed, let $(\mu_{\mathrm{on}}, q, F_{\mathrm{on}}^{(\infty)}, F_{\mathrm{off}})$ be the quadruple corresponding to $Z^{(\infty)}$. For $k = 1, 2, \dots$ let $Z_j^{(\infty)}$, $j = 1, \dots, k$, be i.i.d. processes corresponding to the quadruple $(\mu_{\mathrm{on}}/k, q, F_{\mathrm{on}}^{(\infty)}, F_{\mathrm{off}})$. Then

$$\{Z^{(\infty)}(t),\, t \in \mathbb{R}\} \stackrel{d}{=} \left\{ \prod_{j=1}^{k} Z_j^{(\infty)}(t),\, t \in \mathbb{R} \right\},$$

in terms of equality of finite dimensional distributions, which is what we mean by infinite divisibility of $Z^{(\infty)}$.

**5. Representations of the limiting process.** If we think of *off* periods as representing interruptions of service, then we saw in Proposition 3.2 that in case $q = 0$, the limiting process $Z^{(\infty)}$ has a very simple representation: i.i.d. interruptions distributed according to $F_{\mathrm{off}}$ arrive according to a time homogeneous Poisson process with intensity $1/\mu_{\mathrm{on}}$, and $Z^{(\infty)}(t)$ is simply the indicator function of the event that at time $t$ there are no interruptions present in the system. The duration of the interruptions form a busy period in the $M/G/\infty$ queue and $Z^{(\infty)}(\cdot)$ is an *on/off* process. In this section we develop various representations of this type, valid in more general cases.

It is possible to generate representations of the limiting process $Z^{(\infty)}(\cdot)$ on $[0, \infty)$ but because $Z^{(\infty)}(\cdot)$ is stationary, it is also natural and illuminating to consider representations on $\mathbb{R}$. We study each type of representation in turn in the next two subsections.

5.1. *Representations of $Z^{(\infty)}$ on* $[0, \infty)$. Here is an outline of how to develop a general representation for $\{Z^{(\infty)}(t), t \ge 0\}$. We regard this outline as suggestive of the result and do not justify all the steps. Once the representation $Z^*(\cdot)$ suggested by this outline is in place, we show it has the same finite dimensional distributions as $Z^{(\infty)}(\cdot)$ given in (4.20).

Let $\mathbf{X}^{(n,0)} = \{X_j^{(n,0)}, 1 \le j \le n\}$ be i.i.d. with distribution $F_{\mathrm{on}}^{(n,0)}$, $\mathbf{Y}^{(0)} = \{Y_j^{(0)}, 1 \le j \le n\}$ be i.i.d. with distribution $F_{\mathrm{off}}^{(0)}$ and suppose that conditionally on $\mathbf{X}^{(n,0)}, \mathbf{Y}^{(0)}$, we have $\{W_j^{(n,+,\cdot)}(\cdot), j \ge 1\}$, $\{W_j^{(n,-,\cdot)}(\cdot), j \ge 1\}$ independent with the following descriptions: $W_j^{(n,+,x)}(\cdot)$ is a nonstationary *on/off* indicator process with *on* period distribution $F_{\mathrm{on}}^{(n)}$ and *off* period distribution $F_{\mathrm{off}}$ starting with an initial *on* period of length $x$. Similarly, $W_j^{(n,-,y)}(\cdot)$ is an *on/off* indicator process with *on* period distribution $F_{\mathrm{on}}^{(n)}$ and *off* period distribution $F_{\mathrm{off}}$ starting with an initial *off* period of length $y$. We will also need $W_j^{(+,\cdot)}(\cdot)$, $j \geq 1$, $W_j^{(-,\cdot)}(\cdot)$, $j \geq 1$ which are independent, possibly terminating, *on/off* processes with $W_j^{(+,x)}(\cdot)$ starting with an *on* period of length $x$ and $W_j^{(-,y)}(\cdot)$ starting with an *off* period of length $y$. The *on* period distribution is $qF_{\text{on}}^{(\infty)}$ and the *off* period distribution is $F_{\text{off}}$.

Consider the $j$th indicator process $I_j^{(n)}$ in (2.4). Observing $I_j^{(n)}$ is equivalent to observing

$$W_j^{(n,+,X_j^{(n,0)})} \qquad \text{with probability } \frac{\mu_{\text{on}}^{(n)}}{\mu_{\text{on}}^{(n)} + \mu_{\text{off}}}$$

and observing

$$W_j^{(n,-,Y_j^{(0)})} \qquad \text{with probability } \frac{\mu_{\text{off}}}{\mu_{\text{on}}^{(n)} + \mu_{\text{off}}}.$$

Think of $I_j^{(n)}(\cdot), 1 \leq j \leq n$, as points in the space $D[0,\infty) \cap \{0,1\}^{[0,\infty)}$ of $\{0,1\}$-valued cadlag functions. We thus get a sequence of point processes

$$\left\{ \sum_{j=1}^{n} \varepsilon_{I_j^{(n)}},\ n \geq 1 \right\}$$

and Proposition 3.2.1, page 154 of Resnick (1987) suggests this sequence of point processes converges weakly to a Poisson limit provided $nP[I_1^{(n)} \in \cdot]$ converges vaguely. However, note for a measurable set $A \subset D[0,\infty) \cap \{0,1\}^{[0,\infty)}$, we have

$$\begin{aligned}
&nP[I_1^{(n)} \in A] \\
&\quad = nP\Big[W_1^{(n,+,X_1^{(n,0)})}(\cdot) \in A\Big] \frac{\mu_{\text{on}}^{(n)}}{\mu_{\text{on}}^{(n)} + \mu_{\text{off}}} \\
&\qquad + nP\Big[W_1^{(n,-,Y_1^{(0)})}(\cdot) \in A\Big] \frac{\mu_{\text{off}}}{\mu_{\text{on}}^{(n)} + \mu_{\text{off}}} \\
&\quad = \int_0^\infty P\big[W_1^{(n,+,x)}(\cdot) \in A\big] nF_{\text{on}}^{(n,0)}(dx)\big(1 + o(1)\big) \qquad (5.1) \\
&\qquad + \int_0^\infty P\big[W_1^{(n,-,y)}(\cdot) \in A\big] F_{\text{off}}^{(0)}(dy) \frac{\mu_{\text{off}}}{\mu_{\text{on}}^{(n)}/n + \mu_{\text{off}}/n} \\
&\quad \to \int_0^\infty P\big[W_1^{(+,x)}(\cdot) \in A\big] \nu(dx) \\
&\qquad + \int_0^\infty P\big[W_1^{(-,y)}(\cdot) \in A\big] F_{\text{off}}^{(0)}(dy) \frac{\mu_{\text{off}}}{\mu_{\text{on}}}.
\end{aligned}$$

Now let $M_1 = \sum_j \varepsilon_{\chi_j}$ be PRM($\nu$), that is, a Poisson process with mean measure $\nu$, and similarly let $M_2 = \sum_j \varepsilon_{\gamma_j}$ be PRM($(\mu_{\mathrm{off}}/\mu_{\mathrm{on}})F_{\mathrm{off}}^{(0)}$) on $[0,\infty)$ with $M_1, M_2$ independent and independent of $W_j^{(+,\cdot)}$, $j \geq 1$; $W_j^{(-,\cdot)}$, $j \geq 1$. The previous convergence (5.1) suggests [Resnick (1987), page 154]

$$\sum_{j=1}^{n} \varepsilon_{I_j^{(n)}} \Rightarrow \sum_{j=1}^{\infty} \varepsilon_{I_j^{(\infty)}},$$

where the limit is PRM on $D[0,\infty) \cap \{0,1\}^{[0,\infty)}$ with mean measure given by (5.1), and this further suggests

$$Z^{(n)}(\cdot) \Rightarrow Z^*(\cdot),$$

where

$$Z^*(\cdot) = \prod_j I_j^{(\infty)}(\cdot).$$

Because of the structure of the mean measure in (5.1), we have

$$Z^*(\cdot) = \prod_{\chi_j \in M_1} W_j^{(+,\chi_j)}(\cdot) \prod_{\gamma_j \in M_2} W_j^{(-,\gamma_j)}(\cdot). \tag{5.2}$$

THEOREM 5.1. *The process $Z^*(\cdot)$ given in* (5.2) *has the same finite-dimensional distributions as $Z^{(\infty)}(\cdot)$ given in* (4.20) *or* (4.22).

PROOF. Consider time points

$$0 = h_0 < h_1 < \cdots < h_k$$

and define for any $j \leq k$,

$$p_j^+(h_1, \ldots, h_j; x) = P\big[W_j^{(+,x)}(h_l) = 1,\ l = 0, 1, \ldots, j\big],$$

and set $q_j^+ = 1 - p_j^+$. Then using (5.2) and conditioning on $M_1$ and $M_2$, we get

$$\begin{aligned}
&P[Z^*(h_l) = 1,\ l = 0, \ldots, k] \\
&\quad = E\left(\prod_j p_{k+1}^+(h_0, \ldots, h_k; \chi_j)\right) P\big(M_2([0,\infty)) = 0\big) \\
&\quad = E\left(\exp\left\{-\int_{[0,\infty)} \big(-\log p_{k+1}^+(h_0, \ldots, h_k; x) M_1(dx)\big)\right\}\right) P\big(M_2([0,\infty)) = 0\big)
\end{aligned}$$

and using the standard form for the Laplace functional of a PRM [e.g., Resnick (1987), page 129], we get

$$
\begin{aligned}
&= \exp\left\{-\left[\int_{[0,\infty)} (1 - p^+_{k+1}(h_0, \dots, h_k; x))\nu(dx) + \frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}}}\right]\right\} \\
&= \exp\left\{-\left[\int_{[0,\infty)} q^+_{k+1}(h_0, \dots, h_k; x)\nu(dx) + \frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}}}\right]\right\} \\
&= \exp\left\{-\left[\sum_{l=1}^{k} \int_{h_{l-1}}^{h_l} q^{(\infty)}_{k-l+1}(h_l - x, \dots, h_k - x)\nu(dx) + \frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}}}\right]\right\},
\end{aligned}
$$

which agrees with the specification of the finite-dimensional distributions of $Z^{(\infty)}(\cdot)$ given by (4.20). □

The following is an immediate consequence of Theorem 5.1 or Theorem 5.3 below. We consider the case $q = 1$ so that

$$
\nu(dx) = \frac{m^{(\infty)}(dx)}{\mu_{\mathrm{on}}} = \frac{(1 - F^{(\infty)}_{\mathrm{on}}(x))\,dx}{\mu_{\mathrm{on}}}
$$

and also assume that

$$
\mu^{(\infty)}_{\mathrm{on}} = \int_0^\infty \big(1 - F^{(\infty)}_{\mathrm{on}}(x)\big)\,dx < \infty. \tag{5.3}
$$

(Note that $\mu^{(\infty)}_{\mathrm{on}}$ is *not* the limit of $\mu^{(n)}_{\mathrm{on}}$ as $n \to \infty$.) Thus

$$
\nu = \frac{\mu^{(\infty)}_{\mathrm{on}}}{\mu_{\mathrm{on}}} F^{(\infty,0)}_{\mathrm{on}},
$$

where

$$
F^{(\infty,0)}_{\mathrm{on}}(y) = \int_0^y \frac{(1 - F^{(\infty)}_{\mathrm{on}}(s))}{\mu^{(\infty)}_{\mathrm{on}}}\,ds.
$$

COROLLARY 5.2. *In the case $q = 1$ and $\mu^{(\infty)}_{\mathrm{on}} < \infty$ the limiting process $Z^{(\infty)}$ can be represented in law as*

$$
Z^{(\infty)}(t) = \prod_{j=1}^{N} W_j(t), \qquad t \ge 0, \tag{5.4}
$$

*where $N$ is a Poisson random variable with mean $\mu^{-1}_{\mathrm{on}}(\mu^{(\infty)}_{\mathrm{on}} + \mu_{\mathrm{off}})$, independent of a sequence $\{W_j(\cdot),\ j \ge 1\}$ of i.i.d. stationary* on/off *processes with* on *distribution $F^{(\infty)}_{\mathrm{on}}$ and* off *distribution $F_{\mathrm{off}}$.*

PROOF. Since $\nu[0,\infty) < \infty$, we have $N_1 := M_1[0,\infty)$ is a Poisson random variable with parameter $\nu[0,\infty) = \mu_{\text{on}}^{(\infty)}/\mu_{\text{on}}$. Likewise, $N_2 := M_2[0,\infty)$ is a Poisson random variable with parameter $\mu_{\text{off}}/\mu_{\text{on}}$. Furthermore, we have the representations

$$M_1 = \sum_{j=1}^{N_1} \varepsilon_{X_j^{(\infty,0)}}, \qquad M_2 = \sum_{j=1}^{N_2} \varepsilon_{Y_j^{(0)}},$$

where $\{X_j^{(\infty,0)}\}$ is i.i.d. with common distribution $F_{\text{on}}^{(\infty,0)}$, independent of $N_1$ and $\{Y_j^{(0)}\}$ is i.i.d. with common distribution $F_{\text{off}}^{(0)}$ independent of $N_2$. Let $\{B_j^{(\infty)}, j \ge 1\}$ be i.i.d. Bernoulli random variables independent of $M_1, M_2$ and assume

$$P[B_j^{(\infty)} = 1] = \frac{\mu_{\text{on}}^{(\infty)}}{\mu_{\text{on}}^{(\infty)} + \mu_{\text{off}}}.$$

By a simple thinning argument,

$$Z^*(\cdot) \stackrel{d}{=} \prod_{j=1}^{N_1+N_2} \Big[ B_j^{(\infty)} W_j^{(+,X_j^{(\infty,0)})}(\cdot) + (1 - B_j^{(\infty)}) W_j^{(-,Y_j^{(0)})} \Big].$$

Since what is inside the square brackets is a stationary *on/off* process, the result follows. $\square$

Corollary 5.2 confirms what we saw in Proposition 4.4: in the case $q = 1$ and $\mu_{\text{on}}^{(\infty)} < \infty$ the measure $\nu$ is finite and the limiting process $Z^{(\infty)}$ is nonergodic. Moreover, there is a positive probability

$$\exp\left\{ -\frac{\mu_{\text{off}} + \mu_{\text{on}}^{(\infty)}}{\mu_{\text{on}}} \right\} = P[N = 0]$$

that $Z^{(\infty)} \equiv 1$; that is, there are no interruptions in the limit. The corresponding probability when $\nu[0,\infty) = \infty$ is equal to zero. This is somewhat surprising because the case $q = 1$ and $\mu_{\text{on}}^{(\infty)} < \infty$ corresponds, intuitively, to "more interruptions" than the other two cases. Furthermore, the representation (5.4) offers an alternative explanation of the limiting dependence results obtained in Section 4.4. Specifically, for all $k = 0, 1, \ldots,$

$$\varphi_k := \lim_{\Delta_k \to \infty} \cdots \lim_{\Delta_1 \to \infty} P\left[ Z^{(\infty)}\left( \sum_{i=1}^{j} \Delta_i \right) = 1,\ j = 0, 1, \ldots, k \right]$$

$$= \sum_{n=0}^{\infty} P[N = n] \tag{5.5}$$

$$\times \left( \lim_{\Delta_k \to \infty} \cdots \lim_{\Delta_1 \to \infty} P\left[ W_1\left( \sum_{i=1}^{j} \Delta_i \right) = 1,\ j = 0, 1, \dots, k \right] \right)^n$$

$$= \exp\left\{ -\left[ \frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}}} + \frac{\mu_{\mathrm{on}}^{(\infty)}}{\mu_{\mathrm{on}}} \left( 1 - \left( \frac{\mu_{\mathrm{on}}^{(\infty)}}{\mu_{\mathrm{on}}^{(\infty)} + \mu_{\mathrm{off}}} \right)^k \right) \right] \right\},$$

and then

$$\varphi_* := \lim_{k \to \infty} \varphi_k = \exp\left\{ -\frac{\mu_{\mathrm{on}}^{(\infty)} + \mu_{\mathrm{off}}}{\mu_{\mathrm{on}}} \right\}$$

is simply the probability $P[N = 0]$ that there are no interruptions in the limiting process. A similar computation can be performed for the probability $\psi_k$ that the limiting process is in the *off* state at $k + 1$ points very far from each other.

5.2. *Representations of* $Z^{(\infty)}$ *on* $\mathbb{R}$. Now we give a construction of the limit process on $\mathbb{R}$.

We start with the case $q = 1$. Let $\mathbb{D} = \{(x, y) : 0 \le x \le y\}$ and $T : \mathbb{D} \to \mathbb{R}_+ \times \mathbb{R}_+$ be a mapping defined by $T(x, y) = (x, y - x)$. Let $M_1$ and $M_2$ be independent Poisson random measures on $\mathbb{R}_+ \times \mathbb{R}_+$ with mean measures $m_1$ and $m_2$, respectively, where

$$m_1 = \frac{1}{\mu_{\mathrm{on}}} \bigl( \mathbb{L} \times F_{\mathrm{on}}^{(\infty)} \bigr) \circ T^{-1} \tag{5.6}$$

and

$$m_2 = \frac{1}{\mu_{\mathrm{on}}} (\mathbb{L} \times F_{\mathrm{off}}) \circ T^{-1}. \tag{5.7}$$

Note that $m_2$ is a finite measure with total mass $\mu_{\mathrm{off}}/\mu_{\mathrm{on}}$, while $m_1$ is finite if and only if the mean $\mu_{\mathrm{on}}^{(\infty)}$ of the distribution $F_{\mathrm{on}}^{(\infty)}$ is finite in which case the total mass is $\mu_{\mathrm{on}}^{(\infty)}/\mu_{\mathrm{on}}$.

Let

$$Z^*(t) = Z^{+,*}(t) Z^{-,*}(t), \qquad t \in \mathbb{R}, \tag{5.8}$$

where

$$Z^{+,*}(t) = \prod_{(X_j^+, Y_j^+) \in M_1} W_j^{+,(X_j^+, Y_j^+)}(t), \qquad t \in \mathbb{R}, \tag{5.9}$$

and

$$Z^{-,*}(t) = \prod_{(X_j^-, Y_j^-) \in M_2} W_j^{-,(X_j^-, Y_j^-)}(t), \qquad t \in \mathbb{R}. \tag{5.10}$$

Furthermore, conditionally on $M_1$ and $M_2$ the stochastic processes $W_j^{+,\cdot}$, $j \ge 1$, and $W_j^{-,\cdot}$, $j \geq 1$, are independent, with the following distribution. For $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$, $W_j^{+,(x,y)}$ is an on/off process with *on* distribution $F_{\text{on}}^{(\infty)}$ and *off* distribution $F_{\text{off}}$, that has a special *on* period containing the origin, equal to $(-y, x)$. Similarly, for $(x, y) \in R_+ \times R_+$, $W_j^{-,(x,y)}$ is an on/off process with *on* distribution $F_{\text{on}}^{(\infty)}$ and *off* distribution $F_{\text{off}}$, that has a special *off* period containing the origin, equal to $(-y, x)$.

A straightforward marking of a Poisson random measure argument shows that the process $Z^*$ in (5.8) has an alternative representation,

$$Z^*(t) = \prod_{p \in M} p(t), \qquad t \in \mathbb{R}, \tag{5.11}$$

where $M$ is a Poisson random measure on $\mathbb{E} = D(\mathbb{R}) \cap \{0, 1\}^{\mathbb{R}}$, endowed with the cylindrical $\sigma$-field $\mathcal{F}$, whose mean measure is given by

$$\begin{aligned} m(A) &= \int_0^\infty \int_0^\infty P\bigl(W_1^{+,(x,y)} \in A\bigr) m_1(dx, dy) \\ &\quad + \int_0^\infty \int_0^\infty P\bigl(W_1^{-,(x,y)} \in A\bigr) m_2(dx, dy), \qquad A \in \mathcal{F}. \end{aligned} \tag{5.12}$$

THEOREM 5.3. *The stochastic process $Z^*$ defined by either* (5.8) *or* (5.11) *is a version of the limiting process $Z^{(\infty)}$ in the case $q = 1$.*

PROOF. We work with the representation (5.11). The first step is to establish stationarity of the process $Z^*$. This will follow once we show that the mean measure $m$ in (5.12) is shift invariant; that is,

$$m(A) = m(A_s) \qquad \text{for all } A \in \mathcal{F} \text{ and } s \geq 0, \tag{5.13}$$

where

$$A_s = \{p \in \mathbb{E} : p(\cdot + s) \in A\}.$$

This fact is clear if $\mu_{\text{on}}^{(\infty)} < \infty$. Indeed, in this case one can write

$$\begin{aligned} \frac{\mu_{\text{on}}}{\mu_{\text{on}}^{(\infty)} + \mu_{\text{off}}} m(A) &= \frac{\mu_{\text{on}}^{(\infty)}}{\mu_{\text{on}}^{(\infty)} + \mu_{\text{off}}} \int_0^\infty \int_0^\infty P\bigl(W_1^{+,(x,y)} \in A\bigr) \frac{\mu_{\text{on}}}{\mu_{\text{on}}^{(\infty)}} m_1(dx, dy) \\ &\quad + \frac{\mu_{\text{off}}}{\mu_{\text{on}}^{(\infty)} + \mu_{\text{off}}} \int_0^\infty \int_0^\infty P\bigl(W_1^{-,(x,y)} \in A\bigr) \frac{\mu_{\text{on}}}{\mu_{\text{off}}} m_2(dx, dy), \end{aligned}$$

and the right-hand side of the latter expression is the probability law of the stationary on/off process on $\mathbb{R}$ with *on* distribution $F_{\text{on}}^{(\infty)}$ and *off* distribution $F_{\text{off}}$; see Section 2. Since any distribution $F_{\text{on}}^{(\infty)}$ is the weak limit of a sequence of probability laws with finite means, it follows immediately that the measure $m$ in (5.12) is shift invariant in all cases.

We have, for any $m = 0, 1, \dots$ and $0 = h_0 < h_1 < \cdots < h_m$,

$$
\begin{aligned}
&P[Z^*(h_i) = 1,\ i = 0, 1, \dots, m] \\
&\quad = P[M_2(R_+ \times R_+) = 0]P[Z^{+,*}(h_i) = 1,\ i = 0, 1, \dots, m] \\
&\quad = e^{-\mu_{\text{off}}/\mu_{\text{on}}} \\
&\qquad \times \exp\biggl\{-\frac{1}{\mu_{\text{on}}}\int_0^\infty dx \\
&\qquad\qquad \times \int_x^\infty \bigl(1 - P\bigl[W_1^{+,(x,y-x)}(h_i) = 1,\ i = 0, 1, \dots, m\bigr]\bigr)F_{\text{on}}^{(\infty)}(dy)\biggr\} \\
&\quad = e^{-\mu_{\text{off}}/\mu_{\text{on}}} \\
&\qquad \times \exp\biggl\{-\frac{1}{\mu_{\text{on}}}\sum_{j=0}^{m-1}\int_{h_j}^{h_{j+1}} \bigl((1 - F_{\text{on}}^{(\infty)}(u)) \\
&\qquad\qquad\qquad \times q_{m-j}^{(\infty)}(h_{j+1} - u, \dots, h_m - u)\bigr)\,du\biggr\} \\
&\quad = P[Z^{(\infty)}(h_i) = 1,\ i = 0, 1, \dots, m]
\end{aligned}
$$

by (4.21). Together with stationarity, this proves that $Z^*$ and $Z^{(\infty)}$ have the same finite-dimensional distributions. $\square$

We next proceed to give a representation of the limiting process $Z^{(\infty)}$ in the case $0 \le q < 1$. Let $\bigl(W_j^{(q)}\bigr)$ be i.i.d. terminating on/off processes with (defective) *on* distribution $qF_{\text{on}}^{(\infty)}$ and *off* distribution $F_{\text{off}}$, each one of which starts with an *off* period, and independent of a time homogeneous Poisson random measure $M_q$ on $\mathbb{R}$ with intensity $p\mu_{\text{on}}^{-1}$. Let

$$
Z_q^*(t) = \prod_{\Gamma_j \in M_q, \Gamma_j \le t} W_j^{(q)}(t - \Gamma_j), \qquad t \in \mathbb{R}. \tag{5.14}
$$

THEOREM 5.4. *The stochastic process $Z_q^*$ defined by* (5.14) *is a version of the limiting process $Z^{(\infty)}$ in the case $q < 1$.*

PROOF. Since the random measure $M_q$ is time homogeneous, the process $Z_q^*$ in (5.14) is obviously stationary. Let $k = 0, 1, \dots$ and $0 = h_0 < h_1 < \cdots < h_k$. Observe that the number of terminating on/off processes that arrive before time 0 and run an *off* period at least one of the times $h_0 < h_1 < \cdots < h_k$ has a Poisson distribution with mean

$$
\tau_{-1} = p\mu_{\text{on}}^{-1}\int_0^\infty q_{k+1}^{(\infty)}(x, h_1 + x, \dots, h_k + x)\,dx,
$$

while the number of the terminating on/off processes that arrive between times $h_j$ and $h_{j+1}$ and run an *off* period at least one of the times $h_{j+1} < \cdots < h_k$ has a Poisson distribution with mean

$$\tau_j = p\mu_{\text{on}}^{-1} \int_{h_j}^{h_{j+1}} q_{k-j}^{(\infty)}(h_{j+1} - x, \dots, h_k - x)\, dx,$$

$j = 0, 1, \dots, k$. Moreover, these $k + 2$ Poisson random variables are independent. We conclude that

$$P[Z^*(h_i) = 1,\ i = 0, 1, \dots, k] = \exp\left(- \sum_{j=-1}^{k} \tau_j\right). \tag{5.15}$$

Note that

$$\begin{aligned}
\tau_{-1} = p\mu_{\text{on}}^{-1} \int_0^\infty \Bigg[ &\overline{F_{\text{off}}}(x) + q \int_0^x q_{k+1}^{(\infty)}(x - y, h_1 + x - y, \dots, \\
&\qquad h_k + x - y) F_{\text{on}}^{(\infty)} * F_{\text{off}}(dy) \\
&+ q \int_0^x \left( \sum_{j=0}^{k-1} \int_{h_j + x - y}^{h_{j+1} + x - y} q_{k-j}^{(\infty)}(h_{j+1} + x - y - z, \dots, \right. \\
&\qquad \left. h_k + x - y - z) F_{\text{on}}^{(\infty)}(dz) \right) \\
&\qquad \times F_{\text{off}}(dy) \Bigg] dx \\
= p\mu_{\text{on}}^{-1}\mu_{\text{off}} + q\tau_{-1} & \\
+ \sum_{j=0}^{k-1} \int_0^\infty &\left( \int_0^x \left( \int_{h_j + x - y}^{h_{j+1} + x - y} q_{k-j}^{(\infty)}(h_{j+1} + x - y - z, \dots, \right.\right. \\
&\qquad \left. h_k + x - y - z) F_{\text{on}}^{(\infty)}(dz) \right) \\
&\qquad \left. \times F_{\text{off}}(dy) \right) pq \frac{dx}{\mu_{\text{on}}}.
\end{aligned}$$

Now, for each $j = 0, 1, \dots, k - 1$, simple algebra gives

$$\begin{aligned}
&\int_0^\infty \left( \int_0^x \left( \int_{h_j + x - y}^{h_{j+1} + x - y} q_{k-j}^{(\infty)}(h_{j+1} + x - y - z, \dots, \right.\right. \\
&\qquad \left.\left. h_k + x - y - z) F_{\text{on}}^{(\infty)}(dz) \right) F_{\text{off}}(dy) \right) dx \\
&\quad = \int_0^\infty \left( \int_{h_j + x}^{h_{j+1} + x} q_{k-j}^{(\infty)}(h_{j+1} + x - z, \dots, h_k + x - z) F_{\text{on}}^{(\infty)}(dz) \right) dx \\
&\quad = \int_{h_j}^{h_{j+1}} \left(1 - F_{\text{on}}^{(\infty)}(x)\right) q_{k-j}^{(\infty)}(h_{j+1} - x, \dots, h_k - x)\, dx.
\end{aligned}$$

Since it is clear that $\tau_{-1} < \infty$, we conclude that

$$\tau_{-1} = \frac{\mu_{\text{off}}}{\mu_{\text{on}}} + \frac{q}{\mu_{\text{on}}} \sum_{j=0}^{k-1} \int_{h_j}^{h_{j+1}} \bigl(1 - F_{\text{on}}^{(\infty)}(x)\bigr) q_{k-j}^{(\infty)}(h_{j+1} - x, \dots, h_k - x)\, dx,$$

and so

$$\sum_{j=-1}^{k} \tau_j = \frac{\mu_{\text{off}}}{\mu_{\text{on}}} + \mu_{\text{on}}^{-1} \sum_{j=0}^{k-1} \int_{h_j}^{h_{j+1}} \bigl(1 - qF_{\text{on}}^{(\infty)}(x)\bigr) q_{k-j}^{(\infty)}(h_{j+1} - x, \dots, h_k - x)\, dx,$$

and so the statement of the theorem follows from (5.15) and (4.21). $\square$

REMARK 5.5. By constructing the on/off process involved in the representation of the limiting process $Z^{(\infty)}$ in Theorems 5.3 and 5.4 to have their sample paths in the the appropriate cadlag space, we immediately see that the above theorems give us a construction of a version of the limiting process $Z^{(\infty)}$ with sample paths in the cadlag space as well. This is a version we will work with in the sequel.

REMARK 5.6. We can relate Theorem 5.4 to Theorem 5.1 as follows. Recall that $\nu = \frac{p}{\mu_{\text{on}}}\mathbb{L} + \frac{q}{\mu_{\text{on}}} m^{(\infty)}$ so that PRM($\nu$) can be represented as an independent superposition of two Poisson processes, $M_{11}$ and $M_{12}$, where $M_{11}$ has mean measure $\frac{p}{\mu_{\text{on}}}\mathbb{L}$ so that it is homogeneous, and $M_{12}$ has mean measure $\frac{q}{\mu_{\text{on}}} m^{(\infty)}$. The leads to a representation which is a slight elaboration of (5.2), namely, for $t \geq 0$,

$$Z^*(t) = \prod_{\Gamma_j \in M_{11}} W_j^{(+,\Gamma_j)}(t) \prod_{\eta_j \in M_{12}} W_j^{(+,\eta_j)}(t) \prod_{\gamma_j \in M_2} W_j^{(-,\gamma_j)}(t). \tag{5.16}$$

Note that

$$\prod_{\Gamma_j \in M_{11}} W_j^{(+,\Gamma_j)}(t) = \prod_{\Gamma_j \in M_{11}, \Gamma_j \leq t} W_j^q(t - \Gamma_j)$$

in law. The representation (5.16) is for $t \in \mathbb{R}_+$. Write the analogous representation for a process on $[-T, \infty)$ and we will need ingredients,

$$\sum_j \varepsilon_{\Gamma_j} \sim \text{PRM}\Bigl(\frac{p}{\mu_{\text{on}}}\mathbb{L}\Bigr),$$

$$\sum_j \varepsilon_{\eta_j} \sim \text{PRM}\Bigl(\frac{q}{\mu_{\text{on}}} m_T^{(\infty)}(\cdot)\Bigr),$$

$$\sum_j \varepsilon_{\gamma_j} \sim \text{PRM}\Bigl(\frac{\mu_{\text{off}}}{\mu_{\text{on}}} F_{\text{off},T}^{(0)}(\cdot)\Bigr),$$

with the notation explained in the next line. For any $-\infty < s < t < \infty$,

$$\lim_{T\to\infty} m_T^{(\infty)}(s,t] = \lim_{T\to\infty} m^{(\infty)}(s+T,t+T] = \lim_{T\to\infty}\int_{s+T}^{t+T}\big(1-F_{\text{on}}^{(\infty)}(u)\big)\,du$$
$$\le \lim_{T\to\infty}(t-s)\big(1-F_{\text{on}}^{(\infty)}(T+s)\big) = 0,$$

and similarly,

$$\lim_{T\to\infty} F_{\text{off},T}^{(0)}(s,t] = \lim_{T\to\infty} F_{\text{off}}^{(0)}(s+T,t+T] = 0.$$

This suggests that in the reconstruction of (5.16) on $[-T,\infty)$, when $T\to\infty$, the second and third factors become negligible.

The clarity that is provided by representation (5.14) is only obtained by having a representation on all of $\mathbb{R}$.

An immediate conclusion of Theorem 5.4 is the following generalization of Proposition 3.2.

COROLLARY 5.7. (i) *If $q=0$ then $Z^{(\infty)}(\cdot)$ is the indicator process of a stationary* on/off *process where the* on *distribution is exponential with parameter $1/\mu_{\text{on}}$ and the* off *distribution is the busy period distribution of an $M/G/\infty$ queue whose input is a Poisson process with rate $1/\mu_{\text{on}}$ and whose service length distribution is $F_{\text{off}}$.*

(ii) *If $0\le q<1$ and $F_{\text{on}}^{(\infty)}=\delta_{\{0\}}$ then $Z^{(\infty)}(\cdot)$ is the indicator process of a stationary* on/off *process where the* on *distribution is exponential with parameter $p/\mu_{\text{on}}$ and the* off *distribution is the busy period distribution of an $M/G/\infty$ queue whose input is a Poisson process with rate $p/\mu_{\text{on}}$ and whose service length distribution $\sum_{n=1}^{\infty} pq^{n-1}F_{\text{off}}^{n*}$ is a geometric convolution of the original* off *distributions $F_{\text{off}}$.*

(iii) *If $0\le q<1$ and $F_{\text{on}}^{(\infty)}$ is the exponential distribution with parameter $\lambda$ then $Z^{(\infty)}(\cdot)$ is the indicator process of a stationary* on/off *process modulated by a state process, $N$, as follows. If $N=n\ge 0$ then i.i.d. interruptions arrive to the system according to a Poisson process with rate $n\lambda+p\mu_{\text{on}}^{-1}$. Each interruption has $F_{\text{off}}$ distribution. When an interruption arrives, the state process $N$ will stay equal to $n$ with probability $p\mu_{\text{on}}^{-1}/(n\lambda+p\mu_{\text{on}}^{-1})$ and will move to state $n-1$ with probability $n\lambda/(n\lambda+p\mu_{\text{on}}^{-1})$. When an interruption leaves the system, the state process $N$ stays the same with probability $p=1-q$ and goes up by one with probability $q$. In particular, if $N=n\ge 0$ at the beginning of an* on *period, this* on *period will have exponential distribution with parameter $n\lambda+p\mu_{\text{on}}^{-1}$.*

Note for (i) that when $q=0$, each $W_j^{(q)}(\cdot)$ indicates an *off* period distributed according to $F_{\text{off}}$ followed by an *on* period of infinite length. So *off* periods or interruptions arrive according to a homogeneous Poisson process and form a

Poisson clump. Similarly, for (ii), each *on* period is 0 with probability $q$ and infinite with probability $p := 1 - q$ so $W_j^{(q)}(\cdot)$ indicates an initial *off* period followed by a geometric number of additional *off* periods and then an infinite *on* period.

With a bit of extra work we can push the conclusions in the first two parts of the previous corollary further.

PROPOSITION 5.8. *Let* $T_{\mathrm{off}} = \inf\{t \geq 0 : Z^{(\infty)}(t) = 0\}$. *Then for every* $s > 0$,

$$
\begin{aligned}
P\big[Z^{(\infty)}(0) = 1,\ T_{\mathrm{off}} > s\big] & \\
&= \exp\biggl\{-\biggl(\frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}}} + \nu[0,s]\biggr)\biggr\} \qquad (5.17)\\
&= \exp\biggl\{-\biggl[\frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}}} + \frac{1}{\mu_{\mathrm{on}}}\int_0^s \big(1 - qF_{\mathrm{on}}^{(\infty)}(u)\big)\,du\biggr]\biggr\}.
\end{aligned}
$$

*In particular,* $Z^{(\infty)}(\cdot)$ *is the indicator process of a stationary on/off process with exponentially distributed* on *periods if and only if* $q = 0$ *or if* $0 \leq q < 1$ *and* $F_{\mathrm{on}}^{(\infty)} = \delta_{\{0\}}$.

PROOF. In the notation of representation (5.2) and Theorem 5.1, we have

$$
\begin{aligned}
P\big[Z^{(\infty)}(0) = 1,\ T_{\mathrm{off}} > s\big] &= P\big[M_2[0,\infty) = 0\big]P\big[M_1[0,s] = 0\big]\\
&= \exp\biggl\{-\biggl(\frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}}} + \nu[0,s]\biggr)\biggr\}.
\end{aligned}
$$

If $Z^{(\infty)}(\cdot)$ is the indicator process of a stationary on/off process with exponentially distributed *on* periods, then we must have, for some $a > 0$, $\nu[0,s] = as$, for all $s > 0$. This is possible only if $q = 0$ or if $0 \leq q < 1$ and $F_{\mathrm{on}}^{(\infty)} = \delta_{\{0\}}$. The converse follows from Corollary 5.7. □

**6. Rate of decay of covariances.** We need to understand the dependence structure of the limiting process $Z^{(\infty)}$ in order to assess the suitability of the hierarchical product models as explanations of measured traffic. In particular, it is of interest to understand in what cases the limiting process $Z^{(\infty)}$ will exhibit *long-range dependence*. In this section, we study the rate at which the covariance function

$$
R(t) = \mathrm{Cov}\big(Z^{(\infty)}(t), Z^{(\infty)}(0)\big) = P\big(Z(0) = Z(t) = 1\big) - \exp\biggl\{-2\frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}}}\biggr\} \qquad (6.1)
$$

decays as $t \to \infty$. Even though covariances provide only limited information on the length of memory in a stationary stochastic process, their rate of decay is, nonetheless, illuminating. This is a traditional way of studying dependence for stationary processes [see, e.g., Beran (1994) and references therein] and has been used in particular for on/off processes [see Heath, Resnick and Samorodnitsky (1998) or Willinger, Taqqu and Erramilli (1996) and references therein]. One of our goals is to compare the behavior of the covariances of the process $Z^{(\infty)}$ with those of on/off processes.

It turns out that the asymptotic behavior of the covariances is very different, depending on whether $q < 1$ or $q = 1$ and, in the latter case, whether $\mu_{\text{on}}^{(\infty)} < \infty$ or $\mu_{\text{on}}^{(\infty)} = \infty$. We consider, therefore, these three cases separately.

6.1. *The case $q = 1$ and $\mu_{\text{on}}^{(\infty)} < \infty$.* In this case it follows immediately from (4.13) or (5.5) that

$$\lim_{t\to\infty} R(t) = \exp\left\{-2\frac{\mu_{\text{off}}}{\mu_{\text{on}}}\right\}\left(\exp\left\{\frac{\mu_{\text{off}}^2}{\mu_{\text{on}}(\mu_{\text{on}}^{(\infty)} + \mu_{\text{off}})}\right\} - 1\right), \tag{6.2}$$

a nonzero limit because of lack of ergodicity; see Corollary 5.2 and Proposition 4.4.

6.2. *The case $q = 1$ and $\mu_{\text{on}}^{(\infty)} = \infty$.* In this case $\lim_{t\to\infty} R(t) = 0$. It follows from (4.8) and (4.14) that

$$\begin{aligned} R(t) &= \exp\left\{-2\frac{\mu_{\text{off}}}{\mu_{\text{on}}}\right\}\left\{\exp\left[\frac{\mu_{\text{off}}}{\mu_{\text{on}}}\left(1 - \int_0^t \frac{1 - F_{\text{off}}(t-s)}{\mu_{\text{off}}} p(s)\, ds\right)\right] - 1\right\} \\ &\sim \frac{\mu_{\text{off}}}{\mu_{\text{on}}} \exp\left\{-2\frac{\mu_{\text{off}}}{\mu_{\text{on}}}\right\}\left(1 - \int_0^t \frac{1 - F_{\text{off}}(t-s)}{\mu_{\text{off}}} p(s)\, ds\right) \qquad (6.3) \\ &= \frac{\mu_{\text{off}}}{\mu_{\text{on}}} \exp\left\{-2\frac{\mu_{\text{off}}}{\mu_{\text{on}}}\right\}\left(\overline{F_{\text{off}}^{(0)}}(t) + \int_0^t \frac{1 - F_{\text{off}}(t-s)}{\mu_{\text{off}}} (1 - p(s))\, ds\right) \end{aligned}$$

as $t \to \infty$. Recall that $p(s)$ is the probability of being in the *on* state at time $s$ in an on/off process with *on* distribution $F_{\text{on}}^{(\infty)}$ and *off* distribution $F_{\text{off}}$ and starting with an *on* period. Denoting

$$E(t) = \int_0^t \frac{1 - F_{\text{off}}(t-s)}{\mu_{\text{off}}} (1 - p(s))\, ds, \qquad t \ge 0, \tag{6.4}$$

we see that the rate of decay of the covariance is determined by the slower rate of decay, that of $\overline{F_{\text{off}}^{(0)}}(t)$ and that of $E(t)$. Note that

$$1 - p(t) = \int_0^t \psi(t-x) U^{(\infty)}(dx),$$

where

$$\psi(t) = \int_0^t \big(1 - F_{\text{off}}(t-x)\big) F_{\text{on}}^{(\infty)}(dx), \qquad t \ge 0.$$

Therefore,

$$E(t) = \frac{1}{\mu_{\text{off}}} \int_0^t g(t-x) U^{(\infty)}(dx), \tag{6.5}$$

where

$$g(t) = \int_0^t h(t-x) F_{\text{on}}^{(\infty)}(dx) \tag{6.6}$$

and

$$h(t) = \int_0^t \bigl(1 - F_{\text{off}}(x)\bigr)\bigl(1 - F_{\text{off}}(t-x)\bigr)\, dx, \qquad t \ge 0. \tag{6.7}$$

Observe that

$$h(t) \le \mu_{\text{off}}\left(1 - F_{\text{off}}\left(\frac{t}{2}\right)\right), \qquad t \ge 0, \tag{6.8}$$

and so the function $h$ is directly Riemann intergrable [Remark 3.10.5 in Resnick (1992)]. Therefore so is the function $g$ [Proposition 2.16(d) in Çinlar (1975)]. This fact and representation (6.5) of the function $E$ as a convolution of $g$ with a renewal function $U^{(\infty)}$ show that one can try using a key renewal theorem to find the rate at which the function $E$ converges to zero. Since the renewal function $U^{(\infty)}$ corresponds to an infinite mean distribution $F_{\text{on}}^{(\infty)} * F_{\text{off}}$, one has to use a "heavy tailed" key renewal theorem.

To use the available "heavy tailed" key renewal theorems we have to assume regular variation of the tail of the distribution $F_{\text{on}}^{(\infty)}$. Specifically, assume that

$$1 - F_{\text{on}}^{(\infty)}(x) = x^{-\alpha} L(x), \qquad x \to \infty, \tag{6.9}$$

for some $0 < \alpha \le 1$, and a slowly varying function $L$. No special assumptions will be imposed on the distribution $F_{\text{off}}$. We will see that the rate of decay of the function $E$ is determined by the tail of $F_{\text{on}}^{(\infty)}$. However, we already see from (6.3) that the rate of decay of the entire covariance function $R$ depends on the tails of both distributions, $F_{\text{on}}^{(\infty)}$ and $F_{\text{off}}$.

Recall from Section 4 that $m^{(\infty)}(t) = \int_0^t (1 - F_{\text{on}}^{(\infty)}(s))\, ds$, $t \ge 0$.

THEOREM 6.1. *Under the regular variation assumption* (6.9),

$$\begin{aligned} &\liminf_{t\to\infty} \left( \overline{F_{\text{off}}^{(0)}}(t) + (m^{(\infty)}(t))^{-1} \frac{\mu_{\text{off}}}{\Gamma(\alpha)\Gamma(2-\alpha)} \right)^{-1} R(t) \\ &\qquad = \frac{\mu_{\text{off}}}{\mu_{\text{on}}} \exp\left\{ -2 \frac{\mu_{\text{off}}}{\mu_{\text{on}}} \right\}. \end{aligned} \tag{6.10}$$

*Furthermore, if* $1/2 < \alpha \le 1$ *then*

$$\lim_{t\to\infty}\left(\overline{F_{\mathrm{off}}^{(0)}}(t) + (m^{(\infty)}(t))^{-1}\frac{\mu_{\mathrm{off}}}{\Gamma(\alpha)\Gamma(2-\alpha)}\right)^{-1} R(t) = \frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}}}\exp\left\{-2\frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}}}\right\}. \tag{6.11}$$

PROOF. Since $\mu_{\mathrm{off}} < \infty$, we see that $1 - F_{\mathrm{off}}(t) = o(1 - F_{\mathrm{on}}^{(\infty)}(t))$ as $t \to \infty$. It follows from the standard properties of distributions with subexponential and, in particular, regularly varying tails, that $\overline{F_{\mathrm{on}}^{(\infty)} * F_{\mathrm{off}}}(t) \sim 1 - F_{\mathrm{on}}^{(\infty)}(t)$ and, hence, $m_1^{(\infty)}(t) \sim m^{(\infty)}(t)$ as $t \to \infty$ as well, where $m_1^{(\infty)}(t) = \int_0^t (1 - F_{\mathrm{on}}^{(\infty)} * F_{\mathrm{off}}(s))\,ds$, $t \ge 0$. See, for example, Embrechts, Goldie and Veraverbeke (1979). It follows from Theorem 4 in Erickson (1970) that

$$\begin{aligned}\liminf_{t\to\infty} m^{(\infty)}(t)E(t) &= \liminf_{t\to\infty} m_1^{(\infty)}(t)E(t)\\ &= \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)\mu_{\mathrm{off}}}\int_0^\infty g(x)\,dx\\ &= \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)\mu_{\mathrm{off}}}\int_0^\infty h(x)\,dx\\ &= \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)\mu_{\mathrm{off}}}\mu_{\mathrm{off}}^2 = \frac{\mu_{\mathrm{off}}}{\Gamma(\alpha)\Gamma(2-\alpha)},\end{aligned} \tag{6.12}$$

which together with (6.3) establishes the first claim of the theorem.

To prove (6.11), note that we can write

$$E(t) = \frac{1}{\mu_{\mathrm{off}}}\int_0^t r(t-x)F_{\mathrm{on}}^{(\infty)}(dx), \tag{6.13}$$

where

$$r(t) = \int_0^t h(t-x)U^{(\infty)}(dx), \qquad t \ge 0.$$

Since the function $1 - F_{\mathrm{off}}(t)$ is nonincreasing and integrable, we conclude that by (6.8) that $h(t) = o(1/t)$ as $t \to \infty$. It follows by Theorem 3 in Erickson (1970) that in the case $1/2 < \alpha \le 1$,

$$\begin{aligned}\lim_{t\to\infty} m^{(\infty)}(t)r(t) &= \lim_{t\to\infty} m_1^{(\infty)}(t)r(t)\\ &= \left(\frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)}\int_0^\infty h(x)\,dx\right) = \frac{\mu_{\mathrm{off}}^2}{\Gamma(\alpha)\Gamma(2-\alpha)}.\end{aligned} \tag{6.14}$$

Now we use (6.13) to write for $0 < \varepsilon < 1$,

$$\mu_{\mathrm{off}}E(t) = \int_0^{\varepsilon t} r(t-x)\,F_{\mathrm{on}}^{(\infty)}(dx) + \int_{\varepsilon t}^t r(t-x)\,F_{\mathrm{on}}^{(\infty)}(dx) := E_1(t) + E_2(t).$$

It follows from (6.14) that

$$\limsup_{t\to\infty} m^{(\infty)}(t)E_1(t) \le (1-\varepsilon)^{-(1-\alpha)} \frac{\mu_{\mathrm{off}}^2}{\Gamma(\alpha)\Gamma(2-\alpha)}$$

because by Karamata's theorem $m^{(\infty)}(t)$ is regularly varying with exponent $1-\alpha$ [see, e.g., Theorem 0.6 in Resnick (1987)]. On the other hand,

$$E_2(t) \le 1 - F_{\mathrm{on}}^{(\infty)}(\varepsilon t) = o\big(\big(m^{(\infty)}(t)\big)^{-1}\big)$$

as $t \to \infty$ because $\alpha > 1/2$. We conclude that

$$\limsup_{t\to\infty} m^{(\infty)}(t)E(t) \le (1-\varepsilon)^{-(1-\alpha)} \frac{\mu_{\mathrm{off}}}{\Gamma(\alpha)\Gamma(2-\alpha)},$$

and since this is true for all $0 < \varepsilon < 1$, we conclude that

$$\limsup_{t\to\infty} m^{(\infty)}(t)E(t) \le \frac{\mu_{\mathrm{off}}}{\Gamma(\alpha)\Gamma(2-\alpha)},$$

which together with (6.12) shows that

$$\lim_{t\to\infty} m^{(\infty)}(t)E(t) = \frac{\mu_{\mathrm{off}}}{\Gamma(\alpha)\Gamma(2-\alpha)}. \tag{6.15}$$

In combination with (6.3), the statement (6.15) proves the remaining part of the theorem. □

REMARK 6.2. It follows immediately from Theorem 6.1 that

$$\int_0^\infty R(t)\,dt = \infty. \tag{6.16}$$

This is sometimes taken to be an indication of long-range dependence [see, e.g., Beran (1994)]. Note, furthermore, that (6.16) always holds in the case $q = 1$ and $\mu_{\mathrm{on}}^{(\infty)} = \infty$, whether or not the assumption of regular variation (6.9) is satisfied. Indeed, observe simply that

$$\int_0^\infty \left(\int_0^t \big(1 - F_{\mathrm{off}}(t-s)\big)\big(1-p(s)\big)\,ds\right)dt = \mu_{\mathrm{off}} \int_0^\infty \big(1-p(s)\big)\,ds = \infty$$

since the amount of time spent in either state of a nonterminating alternating renewal process is infinite with probability 1. Now (6.16) follows from (6.3).

REMARK 6.3. Karamata's theorem mentioned above actually shows that if $\alpha < 1$ then

$$m^{(\infty)}(t) \sim \frac{1 - F_{\mathrm{on}}^{(\infty)}(t)}{1-\alpha},$$

and so Theorem 6.1 can be, in the case $\alpha < 1$, reformulated accordingly.

REMARK 6.4. It follows from Theorem 6.1 that (at least, in the range $1/2 < \alpha \le 1$) the rate of decay of the covariance function $R(t)$ is faster when the tail of the distribution $F_{\mathrm{on}}^{(\infty)}$ is heavier. This should be contrasted with the case of on/off processes where heavier tails tend to cause the covariance function to decay slower [see, e.g., Heath, Resnick and Samorodnitsky (1998)].

REMARK 6.5. It is an open question to what extent (6.11) is true in the case $0 < \alpha \le 1/2$. The main ingredient in the proof of a key renewal theorem we used to obtain (6.12), the fact that

$$\lim_{t\to\infty} m_1^{(\infty)}(t)\big(U^{(\infty)}(t+h) - U^{(\infty)}(t)\big) = \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)} \tag{6.17}$$

is false, in general, in the case $0 < \alpha \le 1/2$, as counterexamples in Williamson (1968) demonstrate (in the arithmetic case). This, clearly, rules out the expected key renewal theorem, at least for some directly Riemann intergrable functions. On the other hand, at least in the arithmetic case, (6.17) fails only on a "small" set, so there is hope that a key renewal theorem may still hold for a reasonably rich class of functions.

At the very least one would expect both (6.17) and the corresponding key renewal theorem to hold under some smoothness assumptions on the distribution $F_{\mathrm{on}}^{(\infty)} * F_{\mathrm{off}}$ and, in fact, Erickson (1971) does state a theorem of this kind, that assumes existence of a sufficiently regular density of $F_{\mathrm{on}}^{(\infty)} * F_{\mathrm{off}}$. Unfortunately, no proof is given, and we have not been able to locate the promised future publication in which the proof was to appear. It is unfortunate that there does not seem to have been much progress on heavy tailed key renewal theorems since Erickson (1970) [but the interesting recent paper of Doney (1997) may be a sign of important additional future developments].

6.3. *The case $q < 1$.* The decomposition (6.3) still holds in this case, but now $p(s)$ is the probability of being in the *on* state at time $s$ in a terminating on/off process with (defective) *on* distribution $qF_{\mathrm{on}}^{(\infty)}$ and *off* distribution $F_{\mathrm{off}}$ and starting with an *on* period. The expressions (6.5) and (6.6) are still valid, but now we have to replace (6.7) with

$$h(t) = q\int_0^t \big(1 - F_{\mathrm{off}}(x)\big)\big(1 - F_{\mathrm{off}}(t-x)\big)\,dx, \qquad t \ge 0. \tag{6.18}$$

We study the decay rate of the covariance function $R(t)$ under several different scenarios.

We start with the case $q = 0$.

THEOREM 6.6. *If $q = 0$ then*

$$\lim_{t\to\infty} \frac{R(t)}{\overline{F_{\mathrm{off}}^{(0)}}(t)} = \frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}}} \exp\left\{-2\frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}}}\right\}. \tag{6.19}$$

The proof is an immediate conclusion from (6.3).

Unless specified otherwise, in the remainder of this section we assume that $0 < q < 1$.

Observe that

$$h(t) \ge 2q\bigl(1 - F_{\text{off}}(t)\bigr) \int_0^{t/2} \bigl(1 - F_{\text{off}}(x)\bigr)\, dx \\ \sim 2q\mu_{\text{off}}\bigl(1 - F_{\text{off}}(t)\bigr), \qquad t \to \infty. \tag{6.20}$$

It turns out that under mild heavy tails assumptions on $F_{\text{off}}$ the above holds as an asymptotic equivalence

$$\lim_{t\to\infty} \frac{h(t)}{1 - F_{\text{off}}(t)} = 2q\mu_{\text{off}}. \tag{6.21}$$

By Fatou's lemma, the minimal asumption under which (6.21) holds is the assumption $F_{\text{off}} \in \mathcal{L}$, the class of *long tailed* distributions: $G \in \mathcal{L}$ if $\bar{G}(t - x)/\bar{G}(t) \to 1$ as $t \to \infty$ for all $x > 0$. A sufficient condition for (6.21): There is a function $H$ integrable on $(0, \infty)$ such that

$$\frac{(1 - F_{\text{off}}(x))(1 - F_{\text{off}}(t - x))}{1 - F_{\text{off}}(t)} \le H(x) \qquad \text{for all } 0 < x < \frac{t}{2}. \tag{6.22}$$

Examples include $F_{\text{off}}$ with a regularly varying tail, in which case one can take $H(x) = C\overline{F_{\text{off}}}(x)$ for some $C > 0$, or, say, $\overline{F_{\text{off}}}(x) = \exp\{-ax^\beta\}$ for $0 < \beta < 1$, in which case one can take $H(x) = \exp\{-a(2 - 2^\beta)x^\beta\}$. We do not know if the relation (6.21) holds under the assumption that $F_{\text{off}}$ belongs to the class $\mathcal{S}$ of *subexponential* distributions: recall that $G \in \mathcal{S}$ is $\overline{G * G}(t) \sim 2\overline{G}(t)$ as $t \to \infty$. We refer the reader to Embrechts, Goldie and Veraverbeke (1979) for information on subexponential distributions and some of their properties used below.

THEOREM 6.7. *Let* $F_{\text{off}} \in \mathcal{S}$ *and let* $1 - F_{\text{on}}^{(\infty)}(t) = o(\overline{F_{\text{off}}}(t))$ *as* $t \to \infty$. *Assume that* (6.21) *holds* [*for which* (6.22) *is a sufficient condition*]. *Then* (6.19) *holds*.

PROOF. We claim that under the conditions of the theorem,

$$\lim_{t\to\infty} \frac{g(t)}{1 - F_{\text{off}}(t)} = 2q\mu_{\text{off}} \tag{6.23}$$

as well. Indeed, it follows from (6.21) and the assumption $1 - F_{\text{on}}^{(\infty)}(t) = o(\overline{F_{\text{off}}}(t))$ that

$$g(t) \sim 2q\mu_{\text{off}} \int_0^t \bigl(1 - F_{\text{off}}(t - x)\bigr) F_{\text{on}}^{(\infty)}(dx) \\ = 2q\mu_{\text{off}}\Bigl(\overline{F_{\text{off}} * F_{\text{on}}^{(\infty)}}(t) - \overline{F_{\text{on}}^{(\infty)}}(t)\Bigr)$$

as $t \to \infty$. Using the assumptions $F_{\mathrm{off}} \in \mathcal{S}$ and $1 - F_{\mathrm{on}}^{(\infty)}(t) = o(\overline{F_{\mathrm{off}}}(t))$, it is a standard property of subexponential distributions that $\overline{F_{\mathrm{off}} * F_{\mathrm{on}}^{(\infty)}}(t) \sim \overline{F_{\mathrm{off}}}(t)$ as $t \to \infty$ [see Embrechts, Goldie and Veraverbeke (1979)], and so (6.23) follows. The latter fact also implies that $pU^{(\infty)}$ which is a geometric convolution of the distributions $F_{\mathrm{off}} * F_{\mathrm{on}}^{(\infty)}$ has a tail of the same order,

$$\frac{1 - pU^{(\infty)}(t)}{1 - F_{\mathrm{off}}(t)} = \frac{1}{p}. \tag{6.24}$$

Therefore, using once again the properties of subexponential random variables, we see that

$$\begin{aligned} E(t) &\sim 2q\mu_{\mathrm{off}} \int_0^t \big(1 - F_{\mathrm{off}}(t-x)\big) U^{(\infty)}(dx) \\ &= \frac{2q\mu_{\mathrm{off}}}{p}\Big(\overline{F_{\mathrm{off}} * pU^{(\infty)}}(t) - \overline{pU^{(\infty)}}(t)\Big) \\ &\sim \frac{2q\mu_{\mathrm{off}}}{p}\overline{F_{\mathrm{off}}}(t) \end{aligned}$$

as $t \to \infty$.

On the other hand,

$$\overline{F_{\mathrm{off}}}(t) = o\Big(\overline{F_{\mathrm{off}}^{(0)}}(t)\Big) \tag{6.25}$$

as $t \to \infty$ if $F_{\mathrm{off}} \in \mathcal{L} \supset \mathcal{S}$. Now the statement of the theorem follows from (6.25) and (6.3). This completes the proof. $\square$

REMARK 6.8. It is easy to see by looking carefully at the above argument that the full force of (6.21) is not needed for its conclusion. For example, the same argument will establish (6.19) under the following conditions: There is a distribution $G \in \mathcal{S}$ such that

$$g(t) = O\big(1 - G(t)\big), \qquad \big(1 - G(t)\big) = o\Big(\overline{F_{\mathrm{off}}^{(0)}}(t)\Big) \qquad \text{as } t \to \infty. \tag{6.26}$$

Note that the first part of (6.26) already implies that

$$\max\big(1 - F_{\mathrm{on}}^{(\infty)}(t), \overline{F_{\mathrm{off}}}(t)\big) = O\big(1 - G(t)\big)$$

as $t \to \infty$.

The situation is a bit different in the case of exponential tails. Assume that there is a $\beta > 0$ such that

$$\widehat{F_{\mathrm{on}}^{(\infty)}}(-\beta)\widehat{F_{\mathrm{off}}}(-\beta) = \frac{1}{q} \tag{6.27}$$

and

$$\int_0^\infty te^{\beta t} F_{\mathrm{on}}^{(\infty)}(dt) < \infty, \qquad \int_0^\infty te^{\beta t} F_{\mathrm{off}}(dt) < \infty \tag{6.28}$$

[recall that $\widehat{F_{\mathrm{on}}^{(\infty)}}(\theta) = \int_0^\infty e^{-\theta t} F_{\mathrm{on}}^{(\infty)}(dt)$ is the Laplace transform of $F_{\mathrm{on}}^{(\infty)}$, and $\widehat{F_{\mathrm{off}}}(\theta)$ is the Laplace transform of $F_{\mathrm{off}}$].

THEOREM 6.9. *Assume* (6.27) *and* (6.28) *hold. Then*

$$\begin{aligned} &\lim_{t\to\infty} e^{\beta t} R(t) \\ &\quad = \frac{1}{\mu_{\mathrm{on}}} \exp\left\{-2\frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}}}\right\} \\ &\qquad \times \frac{\widehat{F_{\mathrm{on}}^{(\infty)}}(-\beta)(\widehat{F_{\mathrm{off}}}(-\beta)-1)^2}{\widehat{F_{\mathrm{on}}^{(\infty)}}(-\beta)\int_0^\infty te^{\beta t} F_{\mathrm{off}}(dt) + \widehat{F_{\mathrm{off}}}(-\beta)\int_0^\infty te^{\beta t} F_{\mathrm{on}}^{(\infty)}(dt)}. \end{aligned} \tag{6.29}$$

PROOF. It is immediate that the function $\overline{F_{\mathrm{off}}}(t)e^{\beta t}$ is directly Riemann integrable. Using Proposition 2.16(d) in Çinlar (1975) twice, we see that so is the function $h^\sharp(t) = e^{\beta t} h(t)$, with $h$ given in (6.18), and, hence, so is the function $g^\sharp(t) = e^{\beta t} g(t)$, with $g$ given in (6.6). Therefore, Proposition 3.11.1 in Resnick (1992) applies, and for $E(t)$ given in (6.5) we have

$$\lim_{t\to\infty} e^{\beta t} E(t) = \frac{1}{\mu_{\mathrm{off}}} \frac{\int_0^\infty e^{\beta x} g(x)\,dx}{q\int_0^\infty xe^{\beta x} F_{\mathrm{on}}^{(\infty)} * F_{\mathrm{off}}(dx)}.$$

Notice that under the asumptions of the theorem,

$$\overline{F_{\mathrm{off}}^{(0)}}(t) = o(e^{-\beta t}) \qquad \text{as } t \to \infty.$$

Since

$$\int_0^\infty e^{\beta x} g(x)\,dx = \frac{q}{\mu_{\mathrm{off}}} \widehat{F_{\mathrm{on}}^{(\infty)}}(-\beta)\big(\widehat{F_{\mathrm{off}}}(-\beta)-1\big)^2$$

and

$$\begin{aligned} &\int_0^\infty xe^{\beta x} F_{\mathrm{on}}^{(\infty)} * F_{\mathrm{off}}(dx) \\ &\qquad = \widehat{F_{\mathrm{on}}^{(\infty)}}(-\beta)\int_0^\infty te^{\beta t} F_{\mathrm{off}}(dt) + \widehat{F_{\mathrm{off}}}(-\beta)\int_0^\infty te^{\beta t} F_{\mathrm{on}}^{(\infty)}(dt), \end{aligned}$$

the statement of the theorem follows from (6.3). $\square$

REMARK 6.10. It is interesting to note that in the case $q < 1$ the statement (6.16) indicating that the covariance function is not integrable holds if and only if the second moment of $F_{\text{off}}$ is infinite. Indeed, this is a direct consequence of (6.3) and the fact that

$$\int_0^\infty \left( \int_0^t (1 - F_{\text{off}}(t-s))(1-p(s))\,ds \right) dt = \mu_{\text{off}} \int_0^\infty (1-p(s))\,ds < \infty$$

since the amount of time spent in the *off* state of a terminating alternating renewal process with a finite mean *off* distribution is finite, regardless of the *on* distribution. Compare this with Remark 6.2.

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SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
AND
DEPARTMENT OF STATISTICAL SCIENCE
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853
E-MAIL: sid@orie.cornell.edu
gennady@orie.cornell.edu