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A STOCHASTICALLY QUASI-OPTIMAL SEARCH ALGORITHM FOR THE MAXIMUM OF THE SIMPLE RANDOM WALK

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Odlyzko [*Random Structures Algorithms* **6** (1995) 275–295] exhibited an asymptotically optimal algorithm, with respect to the average cost, among algorithms that find the maximum of a random walk by using only probes and comparisons. We extend Odlyzko's techniques to prove that his algorithm is indeed asymptotically optimal *in distribution* (with respect to the stochastic order). We also characterize the limit law of its cost. Computing its moments in two ways allows us to recover a surprising identity concerning Euler sums.

1. Introduction.

1.1. *The model.* In a remarkable paper, Odlyzko [31] introduces a new model for the study of various searching strategies in an unknown environment. The model is as follows: consider a simple symmetric random walk $\omega = (S_k(\omega))_{k=0,\dots,n}$ [i.e., $S_0(\omega) = 0$, $S_{k+1}(\omega) = S_k(\omega) \pm 1$ for $0 \le k \le n-1$ and the 2^n such Bernoulli paths are equiprobable]. Set

$$M_n(\omega) = \max\{S_k(\omega) \mid 0 \le k \le n\}.$$

The searcher incurs a cost each time he probes a position, at a given time l, of the random walker. Therefore, the cost of an algorithm is identified by the number of probes it uses. The simplest algorithm consists of probing each position l, for $1 \le l \le n$; its cost is n, but Figure 1 suggests that n probes is a crude upper bound for most of the Bernoulli paths. In the example of Figure 1, the searcher knows, before the first probe, that $0 \le M_{10} \le 10$. The first probe shows that $S_{10} = 0$ and, accordingly, $0 \le M_{10} \le 5$ [Figure 1(β)]. The second probe indicates $S_5 = 1$, leading to $1 \le M_{10} \le 3$, as one can see in Figure 1(γ). Finally, the third probe [Figure 1(δ)] gives $M_{10} = 3$, for a final cost of three probes.

1.2. *Optimality with respect to the stochastic order.* Recall that, given two random variables X and Y, X is said to be *stochastically smaller* than Y ($X \le_S Y$) if and only if, for any x,

$$\Pr(X \ge x) \le \Pr(Y \ge x),$$

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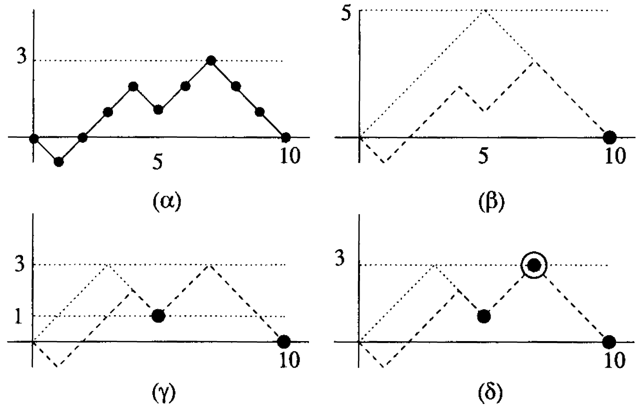

FIG. 1. *A search algorithm with $n = 10$ and $M_n = 3$.*

or, equivalently, if, for any bounded increasing function f,

$$E[f(X)] \le E[f(Y)].$$

Let A_n denote a set of algorithms that solve a given problem for any input ω of size n. For $a \in A_n$, let $C_a(\omega)$ denote the cost incurred by a to process ω. Finally, assume that the set of inputs of size n is endowed with a probability distribution, so that C_a can be seen as a random variable, and set

$$\Phi_n(x) = \min_{a \in A_n} \Pr\left(\frac{C_a}{c_n} \ge x\right).$$

A sequence of algorithms $(a(n))_{n \ge 1}$, $a(n) \in A_n$, is asymptotically optimal in distribution (with respect to the stochastic order) if there exists a normalizing coefficient c_n (in the specific problem we are interested in, $c_n = \sqrt{n}$) such that $\lim_n \Phi_n(x)$ is the tail function $F(x)$ of a proper distribution or of the point mass at some positive constant, and if, for any x,

$$\lim_n \Phi_n(x) = \lim_n \Pr\left(\frac{C_{a(n)}}{c_n} \ge x\right).$$

For the sake of brevity, $(a(n))_{n \ge 1}$ will be called *stochastically quasi-optimal* in A_n.

In this paper, A_n will denote the set of adaptive algorithms, that is, algorithms that decide the next probe taking into account the results of all previous probes. The information produced by a sequence of k probes is described by a sequence $s = (s_0, s_1, s_2, \dots, s_k)$ of $k+1$ points of the lattice, including $s_0 = (0, 0)$, $k = |s|$ being the number of probes (the cost already incurred) in state s. Let $s_i = (x_i, y_i) : y_i$ is the position of the random walker at time x_i. The chronology of probes brings no useful information to the searcher, so it will be convenient to assume that the sequence x_i is strictly increasing and does not reflect this chronology. We only consider sequences s that can be embedded in a Bernoulli path, to be precise, sequences such that $0 \le x_i \le n$, x_i is a strictly increasing sequence of integers, $|y_{i+1} - y_i| \le x_{i+1} - x_i$ and $y_{i+1} - y_i + x_{i+1} - x_i$ is even. Let P_n be the set of all such sequences and, for $s \in P_n$, let $\max(s)$ denote $\max_{0 \le i \le |s|} y_i$. If all the Bernoulli paths (i.e., all the sequences σ of P_n such that $|\sigma| = n$) that contain s as a subsequence have the same maximum, then s is called a *final state*.

Then an algorithm a can be described as a *decision tree*, more precisely, a rooted labeled tree where the labels of leaves are final states. The label of the root is $((s_0), x)$, $1 \le x \le n$, in which x is the first probe decided by the searcher, and, more generally, the label of an internal node is a couple $(s, x) \in P_n \times \{1, 2, \dots, n\}$: s is the information collected by the searcher after the $|s|$th probe, and $x = a(s) \notin \{x_i \mid 0 \le i \le |s|\}$ is the $(|s|+1)$st probe decided by the searcher, based on s. An internal node (s, x) has one child for each of the possible outcomes of the probe at time $x = a(s)$.

1.3. *Main results.* For $a \in A_n$, let $C_a(\omega)$ be the number of probes that a requires to find the maximum of the path ω. Odlyzko [31] proves that

$$\min_{a \in A_n} E(C_a) = c_0\sqrt{n} + o(\sqrt{n}),$$

where c_0 is given by

$$c_0 = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \frac{dy}{y} \int_0^1 \frac{1}{\sqrt{w}} \exp\left(-\frac{y^2}{2w}\right) \operatorname{erf}\left(\frac{y}{\sqrt{2(1-w)}}\right) dw,$$

and he describes a sequence of algorithms, $(\mathrm{Od}(n))_{n \ge 1}$, such that

$$E\left(C_{\mathrm{Od}(n)}\right) = c_0\sqrt{n} + o(\sqrt{n});$$

$\mathrm{Od}(n)$ is called the *average-case asymptotic optimality*. A description of $\mathrm{Od}(n)$ is given in Section 1.4. Chassaing [11] and Hwang [21] independently computed

$$c_0 = \sqrt{\frac{8}{\pi}} \log 2.$$

In this paper, we prove that $(\mathrm{Od}(n))_{n \ge 1}$ is stochastically quasi-optimal. Set

$$I = \frac{1}{2} \int_0^1 \frac{dt}{M_1^{\mathcal{B}} - \mathcal{B}_t},$$

where $(\mathcal{B}_t)_{t \ge 0}$ is a linear Brownian motion and $M_1^{\mathcal{B}} = \max\{\mathcal{B}_t \mid 0 \le t \le 1\}$; recall that

$$\Phi_n(x) = \min_{a \in A_n} \Pr\left(\frac{C_a}{\sqrt{n}} \ge x\right).$$

THEOREM 1.1. *For any x,*

$$\lim_n \Phi_n(x) = \Pr(I \ge x).$$

THEOREM 1.2. *For any* x,

$$\lim_n \Pr\left(\frac{C_{\mathrm{Od}(n)}}{\sqrt{n}} \geq x\right) = \Pr(I \geq x).$$

A simple description of I's distribution (previously unknown, to our knowledge) is given by the Laplace transform of $1/I^2$.

THEOREM 1.3. *For* $t \geq 0$,

$$E\left[\exp\left(-\frac{t}{I^2}\right)\right] = \frac{2\log(\cosh\sqrt{2t}\,)}{\sinh^2\sqrt{2t}}.$$

Consequences of Theorem 1.3 are detailed in Section 4, including expressions of the density of probability of I in terms of Jacobi's theta function and of the moments of I in terms of the Riemann zeta function.

REMARKS. (i) For comparison sorting, one can find average-case optimality results in textbooks ([1], page 92, [26], pages 193–195, or [36], Chapter 1); for the estimation of (essentially) linear operators in the presence of noisy information, in [34] and [41]. Some references are also given in [31], for example, [3, 20, 22, 45]. More recent results for combinatorial algorithms are given in [2, 10, 14, 30].

(ii) Stochastic quasi-optimality completes average-case asymptotic optimality but does not entail it. Actually, the relationship between stochastic and average-case asymptotic optimality is very much like the relationship between the weak convergence of a sequence of random variables and the convergence of their expectations: the weak convergence is more informative than the convergence of first moments. In this special example, $\mathrm{Od}(n)$ satisfies some kind of asymptotic $\mathcal{L}_1$-optimality (see Lemmas 2.1 and 2.2 and Theorem 2.3), entailing the average-case asymptotic optimality of $\mathrm{Od}(n)$. Furthermore, we have the following striking property:

PROPOSITION 1.4. *If* $b(n)$ *is an average-case asymptotically optimal sequence of algorithms, then* $C_{b(n)}/\sqrt{n}$ *converges weakly to* I *and is therefore stochastically quasi-optimal.*

(iii) Stochastic optimality is not merely a theoretical refinement of average-case optimality: for instance, in queueing systems, a decrease in the service time *with respect to the stochastic order* warrants a decrease in the *average* sojourn time, while a decrease in the *average* service time does not (cf. [4], Chapter 4, or [39], Section 5.2).

1.4. *Description of the algorithm* Od(n). Let d be a positive number. Basic to the algorithm is the existence of a set $\Omega_d(n)$ of sample paths with slow variation, such that $\Pr(\overline{\Omega}_d(n)) = o(n^{-1})$: $\Omega_d(n)$ is the set of sample paths ω such that, for any positive k and m,

$$m + k \le n \quad \Longrightarrow \quad |S_{m+k} - S_m| \le d\sqrt{k \log n}.$$

The bound on $\Pr(\overline{\Omega}_d(n))$ is proven in Section 2.2. If at any step Od(n) detects that ω belongs to $\overline{\Omega}_d(n)$, then Od(n) probes the n positions, with almost no contribution to $E(C_{\mathrm{Od}(n)})$. In a first stage, Od(n) searches a tight estimate M^* of M_n, that is, an estimate that satisfies

$$M_n - M^* \le cn^{1/8}(\log^{1/2} n) \tag{1.1}$$

for every $\omega \in \Omega_d(n)$ (and $M_n \ge M^*$ for any ω), at a low cost. To reach this goal, Od(n) probes the sample path ω at positions $(jl)_{j=1,\dots,\lfloor n/l \rfloor}$, where $l = \lfloor n^{1/2} \log n \rfloor$. Then Od$(n)$ scans the vicinity of the "large" S_{jl}. At this moment, M^* is the maximum of the probed positions.

In the second stage, Od(n) covers the sample path from left to right (regardless of the probes of the first stage), as follows: assume that, at step t, Od(n) just probed m. At step $t + 1$:

- If S_m is close to M^* (i.e., $M^* - S_m \le n^{1/6}$), then Od(n) probes S_{m+1}.
- If S_m is far away from M^* (i.e., $M^* - S_m \ge n^{1/6}$), then Od(n) probes S_{m+k}, where

$$k = 2(M^* - S_m) - 10c(M^* - S_m)^{1/2}(\log n)^{1/2}. \tag{1.2}$$

According to [31], pages 285 and 286, this value of k ensures that the random walk does not exceed M^* on $[m, m + k]$.

The paper is organized as follows: Section 2.1 contains the main steps of the proofs of Theorems 1.1 and 1.2. These steps are detailed in Sections 2.2 and 2.3. Section 3 contains the proof of Theorem 1.3, and some consequences are listed in Section 4.

2. Proofs of Theorems 1.1 and 1.2.

2.1. *Main steps.* Set

$$V_n = \frac{1}{2} \sum_{k=1}^{n} \frac{1}{M_n - S_k + 1}.$$

As in [31], we prove that, on one hand, $V_n(1 + o(1))$ is a lower bound for the cost C_a of any algorithm a in A_n (Lemma 2.1), while, on the other hand, $C_{\mathrm{Od}(n)}$ is well approximated by V_n (Lemma 2.2). Set

$$W_n(\omega) = \inf\{C_a(\omega) \mid a \in A_n\}.$$

LEMMA 2.1.

$$\frac{(V_n - W_n)_+}{\sqrt{n}} \xrightarrow{\mathcal{L}_1} 0.$$

LEMMA 2.2.

$$\frac{C_{\mathrm{Od}(n)} - V_n}{\sqrt{n}} \xrightarrow{\mathcal{L}_1} 0.$$

Compared with [31] or [11], note that, at a small additional cost, we replace convergence of expectations by $\mathcal{L}_1$-convergence. Both lemmas are consequences of a counting principle explained briefly at the end of this section. The full proof is given in Section 2.2. We also need the following result.

THEOREM 2.3.

$$\frac{V_n}{\sqrt{n}} \xrightarrow{law} I = \frac{1}{2}\int_0^1 \frac{dt}{M_1^{\mathcal{B}} - \mathcal{B}_t}.$$

Theorem 2.3 is not a consequence of Donsker's invariance principle, as I is not a continuous functional of the Brownian path. The proof of Theorem 2.3 is delayed to Section 2.3, where we exhibit a coupling of $V_n/\sqrt{n}$ and I through a Skorohod embedding. Theorem 1.2 is an immediate consequence of Lemma 2.2 and Theorem 2.3.

PROOF OF THEOREM 1.1. By the definitions of Φ_n and W_n, we have

$$\Pr\big(C_{\mathrm{Od}(n)} \geq x\sqrt{n}\big) \geq \Phi_n(x) \geq \Pr\big(W_n \geq x\sqrt{n}\big). \tag{2.1}$$

On the other hand, as $C_{\mathrm{Od}(n)} \geq W_n$, we obtain

$$\frac{(C_{\mathrm{Od}(n)} - V_n)_+}{\sqrt{n}} \geq \frac{(V_n - W_n)_-}{\sqrt{n}},$$

so that Lemmas 2.1 and 2.2 yield

$$\frac{W_n - V_n}{\sqrt{n}} \xrightarrow{\mathcal{L}_1} 0.$$

Finally, according to Theorem 2.3, both sides of (2.1) have the same limit, and

$$\lim_n \Phi_n(x) = \Pr(I \geq x). \qquad \square$$

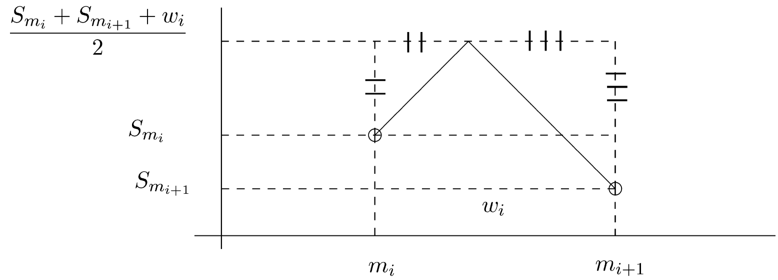

FIG. 2.

REMARK. Let us explain briefly the (somewhat unexpected) appearance of V_n. If the speed $v(y)$ of a traveler at point y of the line satisfies $v(y) \le z(y)$, then the duration t of the journey from point 0 to point x satisfies

$$t \ge \int_0^x \frac{dy}{z(y)}. \tag{2.2}$$

But every algorithm a in A_n defines an increasing sequence $\tilde{a}$ of positive integer-valued random variables $\{m_1, \dots, m_{C_a}\}$, the sequence $\tilde{a}$ of positions probed by a during its run time, and two consecutive positions m_i and m_{i+1} of $\tilde{a}$ satisfy, as will be explained below,

$$m_{i+1} - m_i \le 2M_n - S_{m_i} - S_{m_{i+1}}, \tag{2.3}$$

$$\le 2(M_n - S_{m_i} + 1)\big(1 + o(n)\big). \tag{2.4}$$

Thus, $m_{i+1} - m_i$ can be seen as the speed $v(m_i)$ of the algorithm at point m_i and

$$Z_k^{(n)} = 2(M_n - S_k + 1)$$

as the speed limit at point k, yielding, by analogy to (2.2), a lower bound V_n for the running time of the algorithm.

Note that we *do not* assume the m_i's to be numbered chronologically: the lower bound V_n holds also for algorithms that *do not* probe the sample path from left to right. Inequality (2.4) follows from the slow variation of paths of the set Ω_n.

Now we see in Figure 2 that the upper envelope of paths that agree with the result of probes at positions m_i and m_{i+1} has a maximum, on the interval $[m_i, m_{i+1}]$, given by

$$\frac{S_{m_i} + S_{m_{i+1}} + w_i}{2};$$

inequality (2.3) just says that this maximum is not greater than M_n, so a new probe inside $]m_i, m_{i+1}[$ would be pointless.

2.2. *A counting principle.* Lemmas 2.1 and 2.2 rely on a stochastic analog of inequality (2.2), Theorem 2.4, which we prove in this section. Theorem 2.5 states the corresponding equality, a counting principle giving approximately the length of a sequence subject to some conditions. We learned this principle from [31], where Odlyzko uses a variant to prove the average-case asymptotic optimality of $(\mathrm{Od}(n))_{n\geq 1}$. We believe Odlyzko's principle can be interesting in its own right: it allows the proof of asymptotic optimality of other algorithms (cf. [11, 30], and it also leads to the limit law of the couple (height, width) of simple trees (cf. [12]), giving incidentally a combinatorial explanation of a result of Jeulin on the local time of the Brownian excursion through the study of simple trees. We try to give Odlyzko's principle in general form, in order to spare the reader many references to very similar arguments in this paper, and also in [12].

Let $(Z_k^{(n)})_{0\leq k\leq n}$ denote an array of positive integer-valued random variables. Let α be a positive constant, less than 1, and let $G(n)$ and $H(n)$ be two sequences of positive numbers. We set

$$U_n = \sum_{k=1}^{n} \frac{1}{Z_k^{(n)}}$$

and

$$\tilde{U}_n = \sum_{Z_k^{(n)} \leq H(n)} \frac{1}{Z_k^{(n)}}.$$

Note that $U_n \leq n$ so that $u_n = E(U_n)$ is well defined. We assume that there exists a set Ω_n of sample paths such that

$$\Pr(\overline{\Omega}_n) = o\left(\frac{u_n}{n}\right), \tag{2.5}$$

and that, for any $\omega \in \Omega_n$,

$$\forall k, l, \qquad \left|Z_{k+l}^{(n)} - Z_k^{(n)}\right| \leq l^{\alpha} G(n).$$

Furthermore, we assume that

$$G(n) = o\big(H(n)^{1-\alpha}\big)$$

and that

$$E(\tilde{U}_n) = o(u_n). \tag{2.6}$$

We consider a finite or denumerable set $\tilde{A}_n$ of strictly increasing sequences a of positive integer-valued *random variables* $\{m_1, \ldots, m_{\ell(a)}\}$, whose length $\ell(a)$ is *random*, such that $m_0 = 0$, $m_{\ell(a)} = n$ and the increments

$$w_i = m_{i+1} - m_i$$

satisfy

$$w_i \leq Z_{m_i}^{(n)} \vee H(n), \tag{2.7}$$

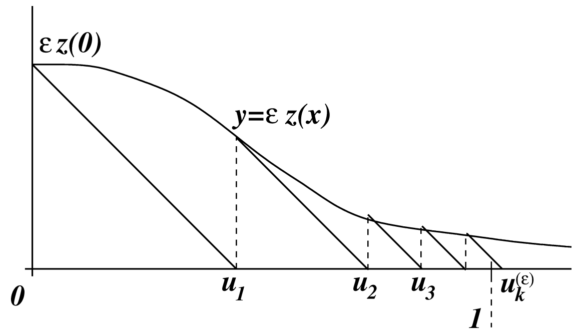

FIG. 3.

at least when $\omega \in \Omega_n$ [at the end of this section, we shall choose $Z_k^{(n)} \simeq 2(M_n - S_k + 1)$ in order to prove Lemmas 2.1 and 2.2]. Let ℓ_n denote the infimum of lengths of sequences in $\tilde{A}_n$.

Assumptions (2.5) and (2.6) will seem less arbitrary if one considers a simple problem of calculus, where a strictly increasing sequence $u_k^{(\varepsilon)}$ satisfies $u_0^{(\varepsilon)} = 0$ and $u_{k+1}^{(\varepsilon)} - u_k^{(\varepsilon)} = \varepsilon z(u_k^{(\varepsilon)})$. The first index k_ε such that $u_k^{(\varepsilon)} \geq 1$ (see Figure 3) is easily seen to satisfy

$$\begin{aligned} k_\varepsilon &\sim \int_0^1 \sum_{k \geq 0} \frac{1}{\varepsilon z(u_k^{(\varepsilon)})} \mathbb{1}_{[u_k^{(\varepsilon)}, u_{k+1}^{(\varepsilon)}]}(x)\, dx \\ &\sim \frac{1}{\varepsilon} \int_0^1 \frac{dy}{z(y)}, \end{aligned}$$

under the assumption of continuity of z, if z is positive. One can view assumption (2.5) as a stochastic analog of the latter. As regards assumption (2.6), it is the stochastic analog of a—strong—assumption of integrability for $1/z$, if z has a pole.

THEOREM 2.4.

$$\frac{(U_n - \ell_n)_+}{u_n} \xrightarrow{L_1} 0.$$

THEOREM 2.5. *Assume that, for a sequence* $a_n = (m_i^{(n)})_{0 \leq i \leq \ell(a_n)} \in \tilde{A}_n$,

$$E\big(\#\{i \,|\, Z_{m_i}^{(n)} \leq H(n)\} I_{\Omega_n}\big) = o(u_n)$$

and that there exist a positive number δ *and a sequence* ε_n *of positive numbers decreasing to* 0 *such that, for any integer* n *and any* $\omega \in \Omega_n$, *the sequence* a_n *satisfies*

$$w_i^{(n)} \geq (1 - \varepsilon_n) Z_{m_i}^{(n)} \qquad \textit{if } (1+\delta) Z_{m_i}^{(n)} \geq H(n).$$

Then

$$\frac{U_n - \ell(a_n)}{u_n} \xrightarrow{L_1} 0.$$

PROOF OF THEOREM 2.4. We shall prove that, for n large enough, independently of the sequence $a \in \tilde{A}_n$, there exists a constant K such that

$$(2.8)\qquad \big(U_n - \ell(a)\big)_+ \le n\ I_{\overline{\Omega}_n} + \big(\tilde{U}_n + KU_nG(n)H(n)^{\alpha-1}\big)\ I_{\Omega_n},$$

so that

$$(U_n - \ell_n)_+ \le nI_{\overline{\Omega}_n} + \tilde{U}_n + KU_nG(n)H(n)^{\alpha-1}.$$

Then assumptions (2.5) and (2.6) yield Theorem 2.4 at once.

The obvious formula

$$\ell(a) = \sum_{i=0}^{\ell(a)-1} \sum_{k=m_i+1}^{m_{i+1}} \frac{1}{w_i}$$

reflects the fact that $1/w_i$ is somehow the density of the presence of points of the sequence a over the interval $[m_i, m_{i+1}]$. Inequality (2.7) entails that

$$\ell(a) \ge \sum_i \sum_{k=m_i+1}^{m_{i+1}} \frac{1}{Z_{m_i}^{(n)}}.$$

Theorem 2.4 follows from the fact that we can replace $1/Z_{m_i}^{(n)}$ by $1/Z_k^{(n)}$ in the last sum, due to the assumption (2.5) of slow variation of $Z_{\cdot}^{(n)}$, obtaining thus a uniform lower bound U_n. This approximation is tight only if $Z_{m_i}^{(n)}$ and $Z_k^{(n)}$ are far enough from 0, but relation (2.6) takes care of the terms close to 0. So, in order to obtain (2.8), we bound $(U_n - C_a)_+$ by n for ω in $\overline{\Omega}_n$, and we proceed as follows for ω in Ω_n:

$$\begin{aligned}\big(U_n - \ell(a)\big)_+ &= \left(\sum_i \sum_{k=m_i}^{m_{i+1}} \frac{w_i - Z_k^{(n)}}{Z_k^{(n)} w_i}\right)_+ \\ &\le \sum_i \sum_{k=m_i}^{m_{i+1}} \frac{1}{Z_k^{(n)}} \left(\frac{w_i - Z_k^{(n)}}{w_i}\right)_+.\end{aligned}$$

- Assume that $w_i \ge H(n)/3$. For $k \in [m_i, m_{i+1}]$, we have

$$\left(\frac{w_i - Z_k^{(n)}}{w_i}\right)_+ \le \left(\frac{Z_{m_i}^{(n)} - Z_k^{(n)}}{w_i}\right)_+ \le \frac{G(n) w_i^{\alpha}}{w_i} \le KG(n)H(n)^{\alpha-1}.$$

- Assume that $Z^{(n)}_{m_i} - w_i \ge H(n)/3$. For $k \in [m_i, m_{i+1}]$, we have

$$w_i - Z^{(n)}_k \le \frac{-H(n) + Z^{(n)}_{m_i} - Z^{(n)}_k}{3} \le \frac{-H(n) + H(n)^\alpha G(n)}{3},$$

  the last term being negative if n is large enough, independently of a and ω, so that $(w_i - Z^{(n)}_k)_+ = 0$.
- Finally, let Δ be the set of indices i such that $w_i \le H(n)/3$ and $Z^{(n)}_{m_i} - w_i \le H(n)/3$. For $k \in [m_i, m_{i+1}]$, we have

$$Z^{(n)}_k = Z^{(n)}_k - Z^{(n)}_{m_i} + \bigl(Z^{(n)}_{m_i} - w_i\bigr) + w_i \le \frac{H(n)^\alpha G(n)}{3} + \frac{2H(n)}{3} \le H(n)$$

  for n large enough so that

$$\begin{aligned}\left(\sum_{i\in\Delta}\sum_{k=m_i+1}^{m_{i+1}} \frac{1}{Z^{(n)}_k} - \sum_{k=m_i+1}^{m_{i+1}} \frac{1}{w_i}\right)_+ &\le \sum_{i\in\Delta}\sum_{k=m_i+1}^{m_{i+1}} \frac{1}{Z^{(n)}_k} \\ &\le \tilde{U}_n.\end{aligned}$$

$\square$

PROOF OF THEOREM 2.5. Similarly, let Δ' be the set of indices i such that $(1+\delta)Z^{(n)}_{m_i} \le H(n)$. For $\omega \in \Omega_n$, we have

$$\left|\sum_{i\in\Delta'}\sum_{k=m_i+1}^{m_{i+1}} \frac{1}{Z^{(n)}_k} - \sum_{k=m_i+1}^{m_{i+1}} \frac{1}{w_i}\right| \le \#\Delta' + \sum_{i\in\Delta'}\sum_{k=m_i+1}^{m_{i+1}} \frac{1}{Z^{(n)}_k}.$$

But, for $i \in \Delta'$ and $k \in [m_i, m_{i+1}]$, we have

$$\begin{aligned} Z^{(n)}_k &\le Z_{m_i} + G(n) w_i^\alpha \\ &\le \frac{H(n)}{1+\delta} + G(n) H(n)^\alpha \\ &\le H(n), \end{aligned}$$

provided that n is large enough, so that

$$\sum_{i\in\Delta'}\sum_{k=m_i+1}^{m_{i+1}} \frac{1}{Z^{(n)}_k} \le \tilde{U}_n$$

and, finally,

$$\left|\sum_{i\in\Delta'}\sum_{k=m_i+1}^{m_{i+1}} \frac{1}{Z^{(n)}_k} - \sum_{k=m_i+1}^{m_{i+1}} \frac{1}{w_i}\right| \le \#\{i \,|\, Z^{(n)}_{m_i} \le H(n)\} + \tilde{U}_n.$$

On the other hand, for $i \notin \Delta'$, we have $(1+\delta)Z_{m_i}^{(n)} \geq H(n)$, so that, for $k \in [m_i, m_{i+1}]$,

$$\left|\frac{w_i - Z_k^{(n)}}{w_i}\right| \leq \frac{|w_i - Z_{m_i}^{(n)}|}{w_i} + \frac{|Z_{m_i}^{(n)} - Z_k^{(n)}|}{w_i}$$
$$\leq \frac{\varepsilon_n}{1-\varepsilon_n} + G(n) w_i^{\alpha-1}$$
$$\leq \frac{\varepsilon_n}{1-\varepsilon_n} + \left(\frac{1-\varepsilon_n}{1+\delta}\right)^{\alpha-1} G(n) H_n^{\alpha-1}$$

and

$$\left|\sum_{i\notin\Delta'}\sum_{k=m_i+1}^{m_{i+1}} \frac{1}{Z_k^{(n)}} - \frac{1}{w_i}\right| \leq \sum_{i\notin\Delta'}\sum_{k=m_i+1}^{m_{i+1}} \frac{1}{Z_k^{(n)}}\left|\frac{w_i - Z_k^{(n)}}{w_i}\right|$$
$$\leq U_n o(n).$$

Finally,

$$\frac{|U_n - \ell(a_n)|}{u_n} \leq \frac{U_n o(n) + \tilde{U}_n + \#\{i | Z_{m_i}^{(n)} \leq H(n)\} I_{\Omega_n} + n I_{\overline{\Omega}_n}}{u_n}. \qquad \square$$

PROOF OF LEMMA 2.1. We set $\alpha = 1/2$, $G(n) = c\sqrt{\log n}$, $H(n) = \log^3 n$ and, for a in A_n, we consider the sequence $\tilde{a} = \{m_1, \ldots, m_{\ell(\tilde{a})}\}$ obtained once we sort positions probed by a in increasing order, $\tilde{a}$ being eventually supplemented by position n. So the finite set $\tilde{A}_n$ of such sequences satisfies

$$\ell_n - 1 \leq W_n \leq \ell_n. \tag{2.9}$$

For $\omega \in \Omega_n$, (2.3) yields

$$w_i \leq 2(M_n - S_{m_i}) + c\sqrt{w_i \log n}$$

and, when $w_i \geq H(n)$,

$$w_i \leq 2\frac{M_n - S_{m_i} + 1}{1 - c/\log n}.$$

Thus, choosing

$$Z_k^{(n)} = 2\frac{M_n - S_k + 1}{1 - c/\log n},$$

on one hand we see that

$$\frac{U_n - V_n}{u_n} \xrightarrow{L_1} 0$$

and, on the other hand, owing to (2.9) and to Theorem 2.4,

$$\frac{(U_n + 1 - W_n)_+}{u_n} \xrightarrow{L_1} 0.$$

To complete the proof, we recall that, according to [11, 21, 31],

$$E(V_n) \simeq \sqrt{\frac{8}{\pi}} \log 2\sqrt{n},$$

so that

$$u_n = \Theta(\sqrt{n}).$$

Also, we have to check assumptions (2.5) and (2.6). First, for n large enough (otherwise, let $\overline{\Omega}_n$ be void), let Ω_n be the set $\Omega_d(n)$, defined in Section 1.4, of sample paths ω such that, for any positive k and m,

$$m + k \le n \quad \Longrightarrow \quad |S_{m+k} - S_m| \le d\sqrt{k \log n},$$

and let us choose d smaller than $c/2$. Due to Chernoff's bounds (see, e.g., [8], page 12):

$$\forall x \ge 0, \forall k, \qquad \Pr(|S_k| > x) \le 2 \exp\left(-\frac{x^2}{2k}\right), \tag{2.10}$$

we deduce as in [31]

$$\Pr(\Omega_d(n)) \ge 1 - 2n^{2-d^2/2}, \tag{2.11}$$

so, finally, $c = 5$ does the job. To check assumption (2.6), we introduce the local time $D_q(n)$ of $M_n - S_k$ at height q, that is,

$$D_q(\omega) = \#\{k | 0 \le k \le n \text{ and } M_n - S_k = q\}.$$

According to [31], we have the following result.

LEMMA 2.6. *There exists a constant C such that, for every $n \in N$,*

$$E(D_q(n)) \le C(q + 1).$$

Therefore,

$$\begin{aligned} \tilde{U}_n &\le \frac{1 - c/\log n}{2} \sum_{M_n - S_k \le \log^3 n} \frac{1}{M_n - S_k + 1} \\ &\le \sum_{q=0}^{\log^3 n} \frac{D_q(n)}{q + 1} \end{aligned}$$

and

$$E(\tilde{U}_n) = \mathcal{O}(\log^3 n). \qquad \square$$

PROOF OF LEMMA 2.2. Let $a(n) = (m_1, \dots, m_{R(n)})$ be the sequence of positions probed during the second stage of $\mathrm{Od}(n)$ (see Section 1.4). According to [31], the first stage needs $\mathcal{O}(\sqrt{n}/\log n)$ probes, on average, so that the proof of Lemma 2.2 reduces to

$$\frac{R(n) - V_n}{\sqrt{n}} \xrightarrow{L_1} 0;$$

that is, it reduces to proving that $a(n)$ satisfies the assumptions of Theorem 2.5. For α, $G(n)$ and Ω_n, we keep to the same definitions as in the preceding proof, but, due to the definition of $\mathrm{Od}(n)$, we take $H(n) = 4n^{1/6}$. Assumption (2.7) is satisfied for

$$Z_k^{(n)} = 2(M_n - S_k + 1),$$

so that this time U_n is equal to V_n; checking (2.6), we get $E(\tilde{U}_n) = \mathcal{O}(n^{1/6})$. We have

$$\begin{aligned}\#\{i \mid Z_{m_i}^{(n)} \le H(n)\} &\le \#\{k \mid Z_k^{(n)} \le H(n)\} \\ &\le \sum_{q=0}^{H(n)} D_q;\end{aligned}$$

thus, due to Lemma 2.6,

$$E\big(\#\{i \mid Z_{m_i}^{(n)} \le H(n)\} I_{\Omega_c(n)}\big) = O\big(H(n)^2\big).$$

Using (1.1) and (1.2), one can see that, for any $\omega \in \Omega_n$, and provided that $Z_{m_i}^{(n)} \ge 3n^{1/6}$ (i.e., we choose $\delta = 1/3$ in Theorem 2.5),

$$\begin{aligned}\frac{w_i}{Z_{m_i}^{(n)}} &\ge \frac{Z_{m_i}^{(n)} - 2 - 2cn^{1/8}\log^{1/2} n - 10c\sqrt{Z_{m_i}^{(n)} \log n}}{Z_{m_i}^{(n)}} \\ &\ge 1 - 10cn^{-1/12}\sqrt{\log n}\end{aligned}$$

for n large enough (otherwise, let $\varepsilon_n = 1$), independent of $\omega \in \Omega_n$. $\square$

2.3. *Proof of Theorem* 2.3. Set

$$I_n = \frac{V_n}{\sqrt{n}} = \frac{1}{2\sqrt{n}} \sum_{k=1}^{n} \frac{1}{M_n - S_k + 1};$$

we are to prove that I_n converges in distribution to I.

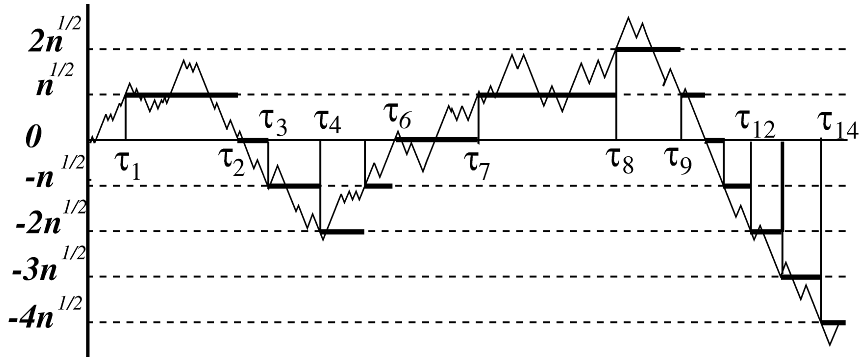

FIG. 4. *The Skorohod embedding.*

The operator T defined by

$$T(f) = \int_0^1 \frac{dx}{\max f - f(x)}$$

does not satisfy the continuity assumption for Donsker's invariance principle, so we exhibit a coupling of I_n and I through a Skorohod embedding and prove that $I - I_n$ converges to 0 in probability. Let $(\mathcal{B}_t)_{t\geq 0}$ denote a standard Brownian motion and let $(S_k^{(n)})_{0\leq k\leq n}$ be the embedded random walk associated with $\mathcal{B}$ through the standard construction due to Knight [24] (see Figures 4 and 5): consider the successive times $(\tau_k^{(n)})_{0\leq k\leq n}$, defined by $\tau_0^{(n)} = 0$ and for $k \geq 1$, $\tau_k^{(n)} = \inf\{t > \tau_{k-1}^{(n)}, |\mathcal{B}_t - \mathcal{B}_{\tau_{k-1}^{(n)}}| = n^{-1/2}\}$, where the Brownian motion has moved by $n^{-1/2}$, and set

$$S_k^{(n)} = \sqrt{n}\mathcal{B}(\tau_k^{(n)}).$$

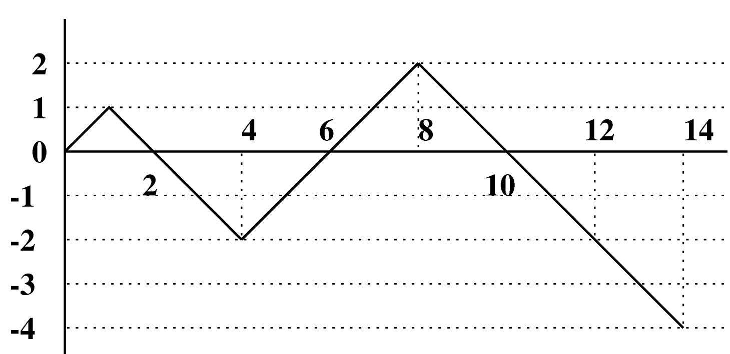

FIG. 5. *The embedded random walk.*

The sample path $(S_k^{(n)})_{k=0,\dots,n}$ has the same law as $\omega = (S_k(\omega))_{k=0,\dots,n}$ introduced in the beginning of the paper. Note that, for every $k > 0$, $n(\tau_{k+1}^{(n)} - \tau_k^{(n)})$ has the same law as α_1, the hitting time of 1 for a reflected Brownian motion, already defined in Section 3.

Consider the piecewise constant process $\beta_t^{(n)}$ equal to $\mathcal{B}(\tau_{k-1}^n) = n^{-1/2} S_{k-1}^{(n)}$ for t in the interval $]\,\tau_{k-1}^{(n)}, \tau_k^{(n)}]$; hence, by construction,

$$\left|\mathcal{B}(t) - \beta_t^{(n)}\right| \le \frac{1}{\sqrt{n}}.$$

Let $M_n^{(\beta)}$ denote the maximum of $(\beta_t^{(n)})_{0 \le t \le n}$ and set, for $0 < \varepsilon < 1/2$,

$$\begin{aligned}
\tilde{\tau}_n &= \tau_n^{(n)},\\
\tilde{I}_n &= \frac{1}{2\sqrt{n}} \sum_{k=1}^{n} \frac{1}{M_n^{(n)} - S_k + n^{-\varepsilon+1/2}},\\
W_1 &= \int_0^{\tilde{\tau}_n} \frac{dt}{M_n^{(\beta)} - \beta_t^{(n)} + n^{-\varepsilon}} - 2\tilde{I}_n\\
&= \sqrt{n} \sum_{k=1}^{n} \frac{\tau_k^{(n)} - \tau_{k-1}^{(n)}}{M_n^{(n)} - S_k^{(n)} + n^{-\varepsilon+1/2}} - 2\tilde{I}_n\\
&= \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{n(\tau_k^{(n)} - \tau_{k-1}^{(n)}) - 1}{M_n^{(n)} - S_k^{(n)} + n^{-\varepsilon+1/2}},\\
W_2 &= 2I - \int_0^{\tilde{\tau}_n} \frac{dt}{M_n^{(\beta)} - \beta_t^{(n)} + n^{-\varepsilon}}\\
&= \int_0^1 \frac{dt}{M_1^{\mathcal{B}} - \mathcal{B}_t} - \int_0^{\tilde{\tau}_n} \frac{dt}{M_n^{(\beta)} - \beta_t^{(n)} + n^{-\varepsilon}}.
\end{aligned}$$

We are to prove that $\tilde{I}_n - I_n$, W_1 and W_2 converge to 0 in probability. First,

$$\begin{aligned}
E(I_n - \tilde{I}_n) &\le n^{-\varepsilon} E\left(\sum_{k=1}^{n} \frac{1}{(M_n^{(n)} - S_k^{(n)} + 1)^2} \right)\\
&= n^{-\varepsilon} E\left(\sum_{q=0}^{n} \frac{D_q(n)}{(q+1)^2} \right)\\
&= O(n^{-\varepsilon} \log n),
\end{aligned}$$

the last line due to Lemma 2.6. Since $(\tau_k^{(n)})_{0 \le k \le n}$ and $\mathcal{B}(\tau_k^{(n)})_{0 \le k \le n}$ are independent, and the r.v.'s $n(\tau_k^{(n)} - \tau_{k-1}^{(n)}) - 1$ are independent, centered, with variance 1, we obtain

$$\begin{aligned} E(W_1^2) &= \frac{1}{n}E\Bigg(\sum_{k=1}^{n}\frac{1}{(M_n^{(n)}-S_k^{(n)}+n^{-\varepsilon+1/2})^2}\Bigg)\\ &\le \frac{1}{n}E\Bigg(\sum_{q=0}^{n}\frac{D_q(n)}{(q+1)^2}\Bigg)\\ &= O\bigg(\frac{\log n}{n}\bigg), \end{aligned}$$

so that $W_1 \overset{\mathcal{L}_2}{\to} 0$.

Next, we consider W_2. Noting that $\tilde{\tau}_n$ has the same law as the average of n independent copies of α_1 and that α_1 belongs to any $\mathcal{L}_p$ (see, e.g., [28]), we deduce that

$$\|\tilde{\tau}_n - 1\|_{\mathcal{L}_p} = O\bigg(\frac{1}{\sqrt{n}}\bigg). \tag{2.12}$$

Now

$$|W_2| \le A + B + C,$$

where

$$\begin{aligned} A &= \int_0^{1\wedge\tilde{\tau}_n}\frac{|\beta_t^{(n)}-\mathcal{B}_t| + |M_n^{(\beta)}-M_1^{\mathcal{B}}| + n^{-\varepsilon}}{(M_n^{(\beta)}-\beta_t^{(n)}+n^{-\varepsilon})(M_1^{\mathcal{B}}-\mathcal{B}_t)}\,dt\\ &\le \big(2n^{-\varepsilon}+|M_n^{(\beta)}-M_1^{\mathcal{B}}|\big)\int_0^{1\wedge\tilde{\tau}_n}\frac{1}{(M_n^{(\beta)}-\beta_t^{(n)}+n^{-\varepsilon})(M_1^{\mathcal{B}}-\mathcal{B}_t)}\,dt,\\ B &= \int_{\tilde{\tau}_n}^{1}\frac{dt}{M_1^{\mathcal{B}}-\mathcal{B}_t}, \qquad I_{\tilde{\tau}_n\le 1},\\ C &= \int_1^{\tilde{\tau}_n}\frac{dt}{M_n^{(\beta)}-\beta_t^{(n)}+n^{-\varepsilon}}, \qquad I_{\tilde{\tau}_n\ge 1}. \end{aligned}$$

By Hölder's inequality, we have successively

$$B \le (1-\tau_n)^{1/p}\bigg(\int_{\tau_n}^{1}\frac{dt}{(M_1^{\mathcal{B}}-\mathcal{B}_t)^q}\bigg)^{1/q}, \qquad I_{\tilde{\tau}_n\le 1},$$

and

$$E[B] \le E\big[|1-\tau_n|\big]^{1/p}E\bigg[\int_0^1\frac{dt}{(M_1^{\mathcal{B}}-\mathcal{B}_t)^q}\bigg]^{1/q},$$

the last term in both equations being defined provided that $q < 2$, so, due to (2.12), for any positive η, we have

$$n^{-\eta+1/4} B \xrightarrow{L_1} 0. \tag{2.13}$$

We also have

$$C \le n^{\varepsilon} |\tilde{\tau}_n - 1|,$$

which converges to 0 in any L_p due to (2.12). Finally,

$$A \le n^{-\varepsilon} \big(2 + n^{\varepsilon} |M_n^{(\beta)} - M_1^{\mathcal{B}}|\big)(A_1 + A_2), \tag{2.14}$$

where

$$A_1 = \int_0^1 \frac{dt}{(M_n^{(\beta)} - \beta_t^{(n)} + n^{-\varepsilon})(M_1^{\mathcal{B}} - \mathcal{B}_t)}, \qquad I_{\tilde{\tau}_n \ge 1},$$

$$A_2 = \int_0^{\tilde{\tau}_n} \frac{dt}{(M_n^{(\beta)} - \beta_t^{(n)} + n^{-\varepsilon})(M_1^{\mathcal{B}} - \mathcal{B}_t)}, \qquad I_{\tilde{\tau}_n \le 1}.$$

At the end of this section we shall prove the following result.

LEMMA 2.7. *For any positive* η,

$$n^{-\eta+1/4} |M_n^{(\beta)} - M_1^{\mathcal{B}}| \xrightarrow{prob.} 0.$$

It follows that, on the right-hand side of (2.14), $2 + n^{\varepsilon}|M_n^{(\beta)} - M_1^{\mathcal{B}}|$ converges in probability to 2, provided that $\varepsilon = 1/5$, for instance. We have, for $\alpha < 1/2$,

$$\begin{aligned} A_1 &\le n^{\alpha} \int_0^1 \frac{I_{\tilde{\tau}_n \ge 1 \text{ and } M_1^{\mathcal{B}} - \mathcal{B}_t \ge n^{-\alpha}}}{M_n^{(\beta)} - \beta_t^{(n)} + n^{-\varepsilon}} + n^{\varepsilon} \int_0^1 \frac{I_{M_1^{\mathcal{B}} - \mathcal{B}_t \le n^{-\alpha}}}{M_1^{\mathcal{B}} - \mathcal{B}_t}\, dt \\ &\le \frac{n^{2\alpha}}{2} + n^{\varepsilon} \int_0^1 \frac{I_{M_1^{\mathcal{B}} - \mathcal{B}_t \le n^{-\alpha}}}{M_1^{\mathcal{B}} - \mathcal{B}_t}\, dt. \end{aligned}$$

The second inequality follows from

$$M_n^{(\beta)} - \beta_t^{(n)} \ge M_1^{\mathcal{B}} - \mathcal{B}_t - \frac{2}{\sqrt{n}},$$

when $\tilde{\tau}_n \ge 1$. We shall prove the following result.

LEMMA 2.8. *For* $0 < \alpha \le 1/2$,

$$E\left(\int_0^{\tilde{\tau}_n} \frac{I_{\{M_n^{(\beta)} - \beta_t \le n^{-\alpha}\}}}{M_n^{(\beta)} - \beta_t^{(n)} + n^{-\varepsilon}}\, dt\right) = \mathcal{O}(n^{-\alpha}).$$

This entails

$$E(A_1) \le \frac{n^{2\alpha}}{2} + \mathcal{O}(n^{\varepsilon-\alpha}).$$

Finally, for $\alpha = 1/20$, for instance, we obtain

$$n^{-\varepsilon}\big(2 + n^{\varepsilon}|M_n^{(\beta)} - M_1^{\mathcal{B}}|\big)A_1 \overset{\text{prob.}}{\to} 0.$$

Similarly,

$$\begin{aligned} A_2 &\le n^{\alpha}\int_0^{\tilde{\tau}_n} \frac{I_{M_n^{(\beta)}-\beta_t^{(n)}\ge n^{-\alpha}}}{M_1^{\mathcal{B}} - \mathcal{B}_t}\,dt + \int_0^{\tilde{\tau}_n} \frac{I_{M_n^{(\beta)}-\beta_t^{(n)}\le n^{-\alpha}}}{(M_1^{\mathcal{B}} - \mathcal{B}_t)(M_n^{(\beta)} - \beta_t^{(n)} + n^{-\varepsilon})}\,dt \\ &\le \frac{n^{2\alpha}}{2} + \int_0^{\tilde{\tau}_n} \frac{I_{M_n^{(\beta)}-\beta_t^{(n)}\le n^{-\alpha}}}{(M_1^{\mathcal{B}} - \mathcal{B}_t)(M_n^{(\beta)} - \beta_t^{(n)} + n^{-\varepsilon})}\,dt \\ &\le \frac{n^{2\alpha}}{2} + n^{\varepsilon}\int_0^{\tilde{\tau}_n} \frac{I_{M_1^{\mathcal{B}}-\mathcal{B}_t\le n^{-\varepsilon}}}{M_1^{\mathcal{B}} - \mathcal{B}_t}\,dt + n^{\varepsilon}\int_0^{\tilde{\tau}_n} \frac{I_{M_n^{(\beta)}-\beta_t^{(n)}\le n^{-\alpha},M_1^{\mathcal{B}}-\mathcal{B}_t\ge n^{-\varepsilon}}}{M_n^{(\beta)} - \beta_t^{(n)} + n^{-\varepsilon}}\,dt. \end{aligned}$$

The second inequality is a consequence of

$$M_1^{\mathcal{B}} - \mathcal{B}_t \ge M_n^{(\beta)} - \beta_t^{(n)} - n^{-1/2},$$

when $\tilde{\tau}_n \le 1$. We finally obtain

$$E(A_2) \le \frac{n^{2\alpha}}{2} + \mathcal{O}(1) + \mathcal{O}(n^{\varepsilon-\alpha})$$

and

$$n^{-\varepsilon}\big(2 + n^{\varepsilon}|M_n^{(\beta)} - M_1^{\mathcal{B}}|\big)A_2 \overset{\text{prob.}}{\to} 0$$

as consequences of Lemma 2.8 and of the next lemma.

LEMMA 2.9. *For* $\alpha > 0$,

$$E\left(\int_0^1 \frac{I_{\{M_1^{\mathcal{B}}-\mathcal{B}_t\le n^{-\alpha}\}}}{M_1^{\mathcal{B}} - \mathcal{B}_t}\,dt\right) = \mathcal{O}(n^{-\alpha}).$$

We end the section with the proofs of Lemmas 2.7–2.9.

PROOF OF LEMMA 2.7. Let η and a be positive. Assuming first that $\tilde{\tau}_n > 1$, we obtain

$$|M_1^{\mathcal{B}} - M_n^{(\beta)}| \le \frac{1}{\sqrt{n}} + M_{\tilde{\tau}_n}^{\mathcal{B}} - M_1^{\mathcal{B}}.$$

So, for any positive $\phi(n)$,

$$\Pr\big(n^{-\eta+1/4}|M_n^{(\beta)} - M_1^{\mathcal{B}}|I_{\tilde{\tau}_n>1} > a\big) \le A_n + B_n,$$

where

$$
\begin{aligned}
A_n &= \Pr\big(\tilde{\tau}_n - 1 > \phi(n)\big),\\
B_n &= \Pr\biggl(n^{-\eta+1/4}\biggl(\frac{1}{\sqrt{n}} + M^{\mathcal{B}}_{1+\phi(n)} - M^{\mathcal{B}}_1\biggr) > a\biggr)\\
&\le \Pr\biggl(n^{-\eta+1/4}\biggl(\frac{1}{\sqrt{n}} + M^{\mathcal{B}}_{\phi(n)}\biggr) > a\biggr)\\
&= \Pr\biggl(n^{-\eta+1/4}\biggl(\frac{1}{\sqrt{n}} + \sqrt{\phi(n)}|N|\biggr) > a\biggr)
\end{aligned}
$$

for N a standard Gaussian r.v., so choosing $\phi(n) = n^{\eta-1/2}$ gives the desired result. The case $\tilde{\tau}_n > 1$ can be proven with the same method. □

PROOF OF LEMMA 2.8. Lemma 2.6 yields

$$
\begin{aligned}
E\biggl(\int_0^{\tilde{\tau}_n} \frac{I_{\{M_n^{(\beta)} - \beta_t^{(n)} \le n^{-\alpha}\}}}{M_n^{(\beta)} - \beta_t^{(n)} + 1/\sqrt{n}}\,dt\biggr) &= \frac{1}{\sqrt{n}} E\biggl(\sum_{k=0}^{n} \frac{I_{\{M_n - S_k \le n^{-\alpha+1/2}\}}}{M_n - S_k + 1}\biggr)\\
&= \frac{1}{\sqrt{n}} E\biggl(\sum_{j=0}^{n^{-\alpha+1/2}} \frac{D_q(n)}{q+1}\biggr)\\
&\le \frac{C}{\sqrt{n}} n^{-\alpha+1/2}.
\end{aligned}
$$

□

PROOF OF LEMMA 2.9. Using the Denisov decomposition of the Brownian path around the place θ where it reaches its maximum, through two independent Brownian meanders m_s^+ and m_s^-, the decomposition that is described in the first pages of the next section [see (3.2)], we obtain

$$
\int_0^1 \frac{I_{\{M_1^{\mathcal{B}} - \mathcal{B}_t \le n^{\alpha}\}}}{M_1^{\mathcal{B}} - \mathcal{B}_t}\,dt = \sqrt{\theta}\int_0^1 \frac{I_{\{m_s^- \sqrt{\theta} \le n^{-\alpha}\}}}{m_s^-}\,ds + \sqrt{1-\theta}\int_0^1 \frac{I_{\{m_s^+ \sqrt{1-\theta} \le n^{-\alpha}\}}}{m_s^+}\,ds,
$$

the two last terms, say A and B, being identically distributed. We have

$$
A = \sqrt{\theta}\int_{0 \le x \le n^{-\alpha}\theta^{-1/2}} \frac{L_x}{x}\,dx,
$$

where L_x denotes the local time of the Brownian meander. On the other hand, according to [16],

$$
\begin{aligned}
E(L_x) &= 2\int_0^x \exp\biggl(-\frac{(x+y)^2}{2}\biggr)\,dy\\
&= \sqrt{2\pi}\ \Pr(x \le |N| \le 2x),
\end{aligned}
$$

where N is standard Gaussian. Finally,

$$\begin{aligned}
E(A) &= K_1 \int_0^1 \int_{0\le x\sqrt{y}\le n^{-\alpha}} \Pr(x \le |N| \le 2x)\frac{dx}{x}\frac{dy}{\sqrt{1-y}} \\
&\le K_2 \int_0^1 \int_{0\le x\sqrt{y}\le n^{-\alpha}} \exp\Big(-\frac{x^2}{2}\Big)\,dx\frac{dy}{\sqrt{1-y}} \\
&= K_2\Big(2\int_0^{n^{-\alpha}} \exp\Big(-\frac{x^2}{2}\Big)\,dx + \int_{n^{-\alpha}}^{+\infty}\int_0^{n^{-2\alpha}x^{-2}} \frac{dy}{\sqrt{1-y}}\exp\Big(-\frac{x^2}{2}\Big)\,dx\Big) \\
&\le 2K_2\Big(n^{-\alpha} + \int_{n^{-\alpha}}^{+\infty}\Big(1-\sqrt{1-n^{-2\alpha}x^{-2}}\Big)\exp\Big(-\frac{x^2}{2}\Big)\,dx\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{n^{-\alpha}}^{+\infty}\Big(1-\sqrt{1-n^{-2\alpha}x^{-2}}\Big)\exp\Big(-\frac{x^2}{2}\Big)\,dx \\
&\qquad = n^{-\alpha}\int_1^{+\infty}\Big(1-\sqrt{1-u^{-2}}\Big)\exp(-n^{-2\alpha}u^2)\,du \\
&\qquad \le n^{-\alpha}\int_1^{+\infty}\Big(1-\sqrt{1-u^{-2}}\Big)\,du. \qquad \square
\end{aligned}$$

3. Proof of Theorem 1.3. Theorem 1.3 is a consequence of the following remarkable property of I.

THEOREM 3.1.

$$\sqrt{S}I \overset{law}{=} \operatorname{arctanh} U + \operatorname{arctanh} V, \tag{3.1}$$

where U and V are independent uniform variables on $[0,1]$ *and S is an exponential variable, with mean value* 2, *independent of I. The function* arctanh *is defined as usual by*

$$\operatorname{arctanh}(u) = \frac{1}{2}\log\Big(\frac{1+u}{1-u}\Big).$$

PROOF. On one hand,

$$\Pr(\sqrt{S}I \ge x) = \Pr\Big(S \ge \frac{x^2}{I^2}\Big) = E\Big(\exp\Big(-\frac{x^2}{2I^2}\Big)\Big)$$

and, on the other hand,

$$\begin{aligned}\Pr(\operatorname{arctanh} U + \operatorname{arctanh} V \geq x) &= \Pr\left(\frac{U+V}{1+UV} \geq \tanh x\right)\\ &= \frac{1}{\sinh^2 x}\log(\cosh^2 x).\end{aligned} \qquad \square$$

PROOF OF THEOREM 3.1. We shall produce two random variables X and Y with the same law as $\operatorname{arctanh} U$ and such that

$$\sqrt{S} I = X + Y.$$

Let $g = \sup\{s \leq 1 : \mathcal{B}_s = 0\}$ be the last 0 in $[0, 1]$ of the Brownian motion. The Brownian meander m is the process

$$m_t = \left|\frac{1}{\sqrt{1-g}}\mathcal{B}_{g+(1-g)t}\right| \qquad \text{for } 0 \leq t \leq 1.$$

Let θ be the unique time t in $]0,1[$ such that

$$M_1^{\mathcal{B}} = B_\theta.$$

Using the decomposition due to Denisov [15] (see also [6]) of the sample path $(\mathcal{B}_s)_{0\leq s\leq 1}$,

$$\begin{aligned}\mathcal{B}_s = {} & \sqrt{\theta}(m_1^- - m_{1-s/\theta}^-) I_{0\leq s\leq\theta}\\ & + \left(\sqrt{\theta} m_1^- - \sqrt{1-\theta} m_{(s-\theta)/(1-\theta)}^+\right) I_{\theta\leq s\leq 1},\end{aligned} \tag{3.2}$$

where $(m_s^-, 0 \leq s \leq 1)$ and $(m_s^+, 0 \leq s \leq 1)$ are two independent Brownian meanders also independent from θ, we get

$$I = \int_0^1 \frac{du}{M_1^{\mathcal{B}} - \mathcal{B}_u} = \sqrt{\theta}\left(\int_0^1 \frac{du}{m_u^-}\right) + \sqrt{1-\theta}\left(\int_0^1 \frac{du}{m_u^+}\right).$$

Set

$$X = \sqrt{\theta S}\left(\int_0^1 \frac{du}{2m_u^-}\right)$$

and

$$Y = \sqrt{(1-\theta)S}\left(\int_0^1 \frac{du}{2m_u^+}\right).$$

Before we continue, we have to state a few facts:

(i) θ, $1-\theta$ and g are arcsine distributed;

(ii) $S\theta$ and $S(1-\theta)$ are independent and satisfy

$$\sqrt{S\theta} \overset{\text{law}}{=} \sqrt{S(1-\theta)} \overset{\text{law}}{=} |N|, \tag{3.3}$$

where N denotes a Gaussian random variable with mean 0 and variance 1;

(iii) $(1/2)\int_0^1 du/m_u =^{\text{law}} \sup_{0\le t\le 1} |b_t|$, where $b_t, 0 \le t \le 1$, is a standard Brownian bridge;

(iv) Z, defined by $Z =^{\text{def}} |N| \sup_{t\le 1} |b_t|$, where N is independent of b, satisfies $\Pr(Z \le x) = \tanh(x)$.

Points (i) and (ii) are well known, and (iii) is just [5], formula 5)c)i), page 69 (see also [7]); (iv) may be obtained using excursion theory (see [35], Chapter 12, Exercise (4.24), question 3), but below we shall give a nice proof which is a special instance of the methods developed in [9]. Owing to (i) and (ii), X and Y are independent and identically distributed, and (3.1) reduces to proving that $\tanh(X)$ is uniform. This is just (iv), since (3.3) and (iii) entail that X and Z have the same law. □

PROOF OF (iv). Let $A_t = \sup_{s\le t}\{\mathcal{B}_s^2\}$ and, for $0 \le t \le 1$, let $b(t) = (1/\sqrt{g})\mathcal{B}_{tg}$, so that

$$A_g = g \sup_{0\le t\le 1} \{b_t^2\}.$$

Since $(b(t))_{0\le t\le 1}$ is a Brownian bridge independent of g, we get

$$SA_g \overset{\text{law}}{=} Z^2.$$

Set $\alpha_t = \inf\{s > 0 \mid A_s > t\}$ and $d_t = \inf\{u \ge t \mid \mathcal{B}_u = 0\}$. For $a \ge 0$, we have

$$\{A_g < a\} = \{g < \alpha_a\} = \{1 < d_{\alpha_a}\}.$$

Using the scaling property of Brownian motion, we obtain

$$d_{\alpha_a} \overset{\text{law}}{=} a d_{\alpha_1}$$

and, thus,

$$A_g \overset{\text{law}}{=} \frac{1}{d_{\alpha_1}}.$$

Finally,

$$\begin{aligned}\Pr(Z \ge x) &= \Pr(SA_g \ge x^2)\\ &= \Pr(S \ge x^2 d_{\alpha_1}) = E\left[\exp\left(-\frac{x^2}{2} d_{\alpha_1}\right)\right].\end{aligned}$$

But α_1 and $d_{\alpha_1} - \alpha_1$ are independent due to the Markov property, and

$$d_{\alpha_1} - \alpha_1 \overset{\text{law}}{=} \inf\{t > 0 \mid \mathcal{B}_t = 1\},$$

so, using the well-known Laplace transforms of α_1 and $d_{\alpha_1} - \alpha_1$, we get

$$\Pr(Z \geq x) = \frac{1}{\cosh(x)} \exp(-x) = 1 - \tanh(x). \qquad \square$$

4. Further results.

4.1. *Joint law of $M_1^{\mathcal{B}}$ and I.* Through Skorohod embedding, it is straightforward to extend Theorem 2.3 to the following:

$$\frac{1}{\sqrt{n}}(V_n, M_n, M_n - S_n) \overset{\text{law}}{\to} (I, M_1^{\mathcal{B}}, M_1^{\mathcal{B}} - \mathcal{B}_1).$$

Furthermore, arguments similar to those of Theorem 3.1 (see the computations in [32], page 268) yield that

$$|N|\left(\int_0^1 \frac{du}{m_u}, m_1\right) \overset{\text{law}}{=} (2X, \tanh(X)\Gamma),$$

where m_1 is the value of the meander $(m_u)_{0\leq u\leq 1}$ at time 1. As a consequence,

$$\sqrt{S}(I, M_1^{\mathcal{B}}, M_1^{\mathcal{B}} - \mathcal{B}_1) \overset{\text{law}}{=} (X + Y, (\tanh X)\Gamma, (\tanh Y)\Gamma'), \tag{4.1}$$

where X, Y, Γ and Γ' are independent, and Γ and Γ' are $\gamma_{2,1}$-distributed random variables. From (4.1), one finds

$$E\left(\frac{1}{M^2} \exp\left(-\frac{h^2}{2M^2}\right) I_{I/M \leq a/h}\right) = \int_0^y \frac{du}{u^2} \exp\left(-\frac{h}{u}\right) \frac{y-u}{1-yu},$$

where $y = \tanh(a)$. This shows, in some sense, how M_n and $C_{\mathrm{Od}(n)}$ are related.

4.2. *Density of I.* Writing Theorem 1.3 in the form

$$E\left[\exp\left(-\frac{a^2}{2I^2}\right)\right] = \frac{\log(\cosh^2 a)}{a^2} \frac{a^2}{\sinh^2 a}, \tag{4.2}$$

we observe the familiar Laplace transform $(a/\sinh(a))^2$ in $a^2/2$: indeed, if Σ_1 denotes the hitting time of 1 for a Bessel process of dimension 3, starting from 0, then (see [13], pages 168 and 169)

$$E\left(\exp\left(-\frac{\lambda^2}{2}\Sigma_1\right)\right) = \frac{\lambda}{\sinh \lambda}.$$

So $(a/\sinh(a))^2$ appears as the Laplace transform of the sum Σ_2 of two independent copies of Σ_1. As noted by Chung [13], Σ_1 and Σ_2 are surprisingly related to the supremum $M_1^{|\mathrm{b}|}$ of the absolute value of the Brownian bridge, that is, the asymptotic distribution of the Kolmogorov–Smirnov statistic, respectively to the maximum M_1^{e} of the normalized Brownian excursion, by

$$\Sigma_1 \overset{\text{law}}{=} \frac{4}{\pi^2}(M_1^{|\mathrm{b}|})^2, \qquad \Sigma_2 \overset{\text{law}}{=} \frac{4}{\pi^2}(M_1^{\mathrm{e}})^2;$$

on this topic, see also the nice developments in [29]. The distributions of these variables are discussed in detail in [7], and we have kept the notation of that paper for ease of reference. The distribution of M_1^{e} is also the distribution or limit distribution of:

- the maximum of a Bernoulli excursion (cf. [23, 38, 40],
- the normalized height for several classes of trees of size n (cf. [19, 18]),
- the range of the Brownian bridge (cf. [27, 33, 35, 42]).

Denote by U a uniform r.v. and let

$$\Sigma_1^{\sharp} = \inf\{t : |\mathcal{B}_t| = 1\} \quad \text{and} \quad G = \sup\{t < \Sigma_1^{\sharp} : \mathcal{B}_t = 0\}.$$

Then Williams' decomposition (cf. [43, 44]) tells us that G and $(\Sigma_1^{\sharp} - G)$ are independent and that $\sup_{u \le G} |\mathcal{B}_u| =^{\text{law}} V$, where V is uniform on [0, 1], whereas Knight [25] showed that

$$E\left(\exp\left(-\frac{\lambda^2}{2}G\right)\right) = \frac{\tanh(\lambda)}{\lambda}, \qquad E\left(\exp\left(-\frac{\lambda^2}{2}(\Sigma_1^{\sharp} - G)\right)\right) = \frac{\lambda}{\sinh(\lambda)}.$$

THEOREM 4.1. *We have*

$$\frac{1}{I^2} \stackrel{law}{=} UG + \Sigma_2 \stackrel{law}{=} H\Sigma_2^{\sharp} + \Sigma_2,$$

where, on the right-hand side, U, G and Σ_2 are independent, H, $\Sigma_2^{\sharp}$ and Σ_2 are independent, $\Sigma_2^{\sharp}$ is the sum of two independent copies of $\Sigma_1^{\sharp}$ and H has density

$$\Pr(H \in dh) = \left(\frac{1}{\sqrt{h}} - 1\right) I_{0<h<1}\, dh.$$

PROOF. Owing to (4.2), the first identity of Theorem 4.1 reduces to

$$\begin{aligned} E[\exp(-\lambda UG)] &= \frac{1}{\lambda} E\left[\int_0^{\lambda} \exp(-xG)\, dx\right] \\ &= \frac{1}{\lambda}\int_0^{\lambda} \frac{\tanh\sqrt{2x}}{\sqrt{2x}}\, dx \\ &= \frac{\log(\cosh\sqrt{2\lambda})}{\lambda}. \end{aligned}$$

Williams' decomposition entails that $G =^{\text{law}} V^2\Sigma_2^{\sharp}$. Finally, UV^2 is easily shown to be distributed as H, yielding the second identity in law. □

According to [5], page 74, the density of G is

$$\sum_{n=1}^{+\infty} \exp\left(-\frac{\tilde{n}^2 x}{2}\right), \qquad x > 0,\ \tilde{n} = \pi\left(n - \frac{1}{2}\right).$$

Thus, UG has density

$$\theta_1(x) = \sum_{n=1}^{+\infty} \int_x^{+\infty} \exp\left(-\frac{\tilde{n}^2 u}{2}\right) \frac{du}{u}.$$

According to [13, 23], we have

$$\begin{aligned} \Pr(H \le x) &= \sum_{-\infty \le n \le +\infty} (1 - 4n^2x^2) \exp(-2x^2n^2) \\ &= \frac{\pi\sqrt{\pi}}{x^3\sqrt{2}} \theta'\left(\frac{\pi}{2x^2}\right), \end{aligned}$$

where θ denotes the usual Jacobi (theta) function

$$\theta(x) = \sum_{-\infty \le n \le +\infty} e^{-\pi n^2 x}, \qquad x > 0,$$

and the second equality makes use of the Poisson identity

$$\theta(x) = \frac{1}{\sqrt{x}} \theta\left(\frac{1}{x}\right)$$

(for a probabilistic insight on the Poisson identity and other related identities, see [5], pages 72–74, or [38]).

Thus, the density of Σ_2 is given by

$$\theta_2(x) = \frac{\partial}{\partial x} \sum_{-\infty \le n \le +\infty} (1 - n^2\pi^2 x) e^{-n^2\pi^2 x/2},$$

and the density Θ of $1/I^2$ is given by

$$\Theta(x) = \int_0^x \theta_1(y)\theta_2(x-y)\, dy, \qquad x \ge 0. \tag{4.3}$$

We finally obtain the following result.

THEOREM 4.2. *If we denote by γ the density of I, then*

$$\gamma(u) = \frac{2}{u^3} \Theta\left(\frac{1}{u^2}\right), \qquad u > 0.$$

Using the power series expansion for $x \log(1+x)/(1-x)^2$, one can write the Laplace transform, taken at $a^2/2$, of $1/I^2$,

$$\frac{\log(\cosh^2 a)}{\sinh^2 a} = \frac{2a}{\sinh^2 a} + 8 \sum_{n=1}^{\infty} b_n e^{-2na},$$

where

$$b_n = n\overline{H}_n - n\log 2 - 1_{\{n \text{ odd}\}},$$
$$\overline{H}_n = \sum_{j=1}^{n} \frac{(-1)^{j-1}}{j},$$

and we obtain the following formula, to be compared with (4.3):

$$\Theta(x) = \frac{2}{\sqrt{2\pi}} \int_0^x \frac{\theta_2(x-t)}{\sqrt{t}}\,dt + 8\sqrt{\frac{2}{\pi x^3}} \sum_{n=1}^{\infty} n b_n \exp\left(-\frac{2n^2}{x}\right).$$

4.3. *Moments of I.* Let Γ (resp. ζ) denote the Euler gamma (resp. Riemann zeta) function. According to the notation of Flajolet and Salvy [17], set

$$S_{1,p}^{-+} = \sum_{n=1}^{+\infty} \frac{\overline{H}_n}{n^p},$$

$\tilde{\zeta}(0) = 1/2$, $\tilde{\zeta}(1) = \log 2$ and, for $k \geq 2$,

$$\tilde{\zeta}(k) = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^k} = (1 - 2^{1-k})\zeta(k).$$

PROPOSITION 4.3. $E(I) = \sqrt{8/\pi}\log 2$ *and, for* $n \geq 2$,

$$E(I^n) = \frac{4\Gamma((n+1)/2)}{\sqrt{2^n\pi}} \sum_{k=0}^{n} \tilde{\zeta}(k)\tilde{\zeta}(n-k), \tag{4.4}$$

which is also equal to

$$E(I^n) = \frac{4\Gamma((n+1)/2)}{\sqrt{\pi 2^n}} \big(2S_{1,n-1}^{-+} - 2\log 2\zeta(n-1) + (n-1)\zeta(n) - \tilde{\zeta}(n)\big). \tag{4.5}$$

As a consequence, $E(I^2) = \log^2 2 + \pi^2/12$, *and* $\operatorname{Var}(I) = (1 - 8/\pi)\log^2 2 + \pi^2/12 \simeq 0.0794565\ldots.$

PROOF.

- Using (3.1), we obtain

$$E(I^n) = \frac{E((X+Y)^n)}{E(S^{n/2})}.$$

Straightforward computations yield

$$E(S^{n/2}) = \Gamma\left(\frac{n}{2}+1\right)2^{n/2}$$

and

$$E(X^k) = \frac{k!}{2^{k-1}}\tilde{\zeta}(k),$$

yielding (4.4).

- Owing to Theorem 1.3, we have, for every positive function f,

$$\int_0^{+\infty} f(a)\frac{2\log(\cosh a)}{(\sinh a)^2}\,da = E\left[\int_0^{+\infty} f(a)\exp\left(-\frac{a^2}{2I^2}\right)da\right]$$
$$= E\left[I\int_0^{+\infty} f(xI)\exp\left(-\frac{x^2}{2}\right)dx\right].$$

Hence, the choice $f(x) \equiv x^{n-1}$ yields

$$E(I^n) = \frac{2^{1-n/2}}{\Gamma(n/2)}\int_0^{+\infty} a^{n-1}\frac{2\log(\cosh a)}{\sinh^2 a}\,da.$$

Using the duplication formula for the gamma function and the same power series expansions as in the preceding section, we obtain (4.5). □

Note that we recover in this way a surprising identity concerning Euler sums, due to Sitaramachandra Rao [37] and rederived by residue computations in [17]:

$$2S_{1,n-1}^{-+} = 2\log 2\,\zeta(n-1) - (n-1)\zeta(n) + 2\tilde{\zeta}(n) + \sum_{k=1}^{n-1}\tilde{\zeta}(k)\tilde{\zeta}(n-k).$$

4.4. *Laplace transform of I.*

COROLLARY 4.4. *The Laplace transform of I^2 is given by*

$$\Psi_{I^2}\left(\frac{\lambda^2}{2}\right) = E\left[\exp\left(-\frac{\lambda^2 I^2}{2}\right)\right]$$
$$= E\left[J_o\big(2\lambda(X+Y)\big)\right] \tag{4.6}$$
$$= \frac{1}{\pi}\int_{-1}^{1}\big(E(\exp(2i\lambda r X))\big)^2\frac{dr}{\sqrt{1-r^2}},$$

where X and Y have the same meaning as in the proof of Theorem 3.1 *and, moreover,*

$$J_o(\xi) \overset{def}{=} \frac{1}{\pi}\int_{-1}^{1}\exp(i\xi r)\frac{dr}{\sqrt{1-r^2}}.$$

PROOF. We multiply both sides of the identity in law (3.1) by $\varepsilon\sqrt{g}$, where ε is a symmetric Bernoulli variable [i.e., $\Pr(\varepsilon = \pm 1) = \frac{1}{2}$], g is arcsine distributed, independent of ε, and the pair (ε, g) is independent of either side of (3.1). Consequently, we obtain, on the left-hand side of (3.1), NI, where N is a standard Gaussian variable, independent of I. Now, we take the characteristic functions, with respect to the argument λ, of both sides. On the left-hand side we obtain

$$E\left[\exp\left(-\frac{\lambda^2 I^2}{2}\right)\right],$$

whereas on the right-hand side we obtain

$$E\left[J_o(\lambda(X+Y))\right],$$

since the c.f. of $\varepsilon\sqrt{g}$ is

$$J_o(\xi) \equiv \frac{1}{\pi}\int_{-1}^{1} \exp(i\xi r)\frac{dr}{\sqrt{1-r^2}}.$$

It may be of some interest to make formula (4.6) in Corollary 4.4 more explicit. For this purpose, we introduce the incomplete beta function:

$$B_x(a,b) = \int_0^x u^{a-1}(1-u)^{b-1}\,du.$$

We are now able to express the characteristic function of X in terms of $B_{1/2}$. $\square$

PROPOSITION 4.5. *The c.f. of X is identical to*

$$E[\exp(i\alpha X)] = 2B_{1/2}\left(1-\frac{i\alpha}{2},\ 1+\frac{i\alpha}{2}\right), \tag{4.7}$$

so that

$$E[\cos(\alpha X)] = \frac{\pi\alpha/2}{\sinh(\pi\alpha/2)}, \qquad E[\sin(\alpha X)] = 2\int_0^{1/2} \sin\left(\frac{\alpha}{2}\log\left(-1+\frac{1}{u}\right)\right)du.$$

PROOF. (i) Recall that the distribution of X has density $\mathbb{1}_{\{x\ge 0\}}/(\cosh(x))^2$. We shall show the general identity

$$\int_0^{+\infty} \exp(\alpha x)\frac{1}{(\cosh x)^\beta}\,dx = 2^{\beta-1}B_{1/2}\left(\frac{\beta-\alpha}{2}, \frac{\beta+\alpha}{2}\right) \tag{4.8}$$

for $\beta > 0$ and $\operatorname{Re}\alpha < \beta$, $\alpha \in C$. Assuming for a moment that formula (4.8) holds, we deduce formula (4.7) by taking $\beta = 2$ and replacing α by $i\alpha$.

(ii) We now prove formula (4.8). Making the change of variable $t = \exp(-2x)$ on the left-hand side of (4.8), which we now denote $l(\alpha,\beta)$, we obtain

$$l(\alpha,\beta) = 2^{\beta-1}\int_0^1 \frac{1}{(1+t)^\beta} t^{(\beta-\alpha)/2-1}\,dt,$$

and making a further change of variable $u = t/(1+t)$, we have

$$l(\alpha,\beta) = 2^{\beta-1}B_{1/2}\left(\frac{\beta-\alpha}{2}, \frac{\beta+\alpha}{2}\right)$$

in (4.8), formula as:

(iii) Denoting by $C(\alpha)$ the left-hand side of (4.7), we deduce from (4.8) that

$$\operatorname{Re} C(\alpha) = 2 \operatorname{Re} B_{1/2}(1 - i\alpha/2, 1 + i\alpha/2) = B(1 - i\alpha/2, 1 + i\alpha/2),$$

so that with the help of the classical formula of complements for the gamma function, we obtain

$$\operatorname{Re} C(\alpha) = \frac{\pi\alpha/2}{\sinh(\pi\alpha/2)}.$$

The formula for $\operatorname{Im} C(\alpha)$ can be deduced directly from (4.7). □

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