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LIMIT THEOREM FOR LELAND'S STRATEGY[1]

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The Leland strategy for an approximate hedging of the call option under transactions costs is studied. The rate of convergence in the Kabanov–Safarian theorem for the Leland strategy is found. The limit theorem for the hedging portfolio is proved.

1. Introduction. In this paper we shall study the problem of option pricing in the presence of transactions costs. Leland [8] suggested an approach based on the idea of a periodic revision of a hedging portfolio using modified Black–Scholes strategy.

It turns out that (see [8, 9]) if the transaction cost vanishes when the number of revisions n tends to infinity, then the terminal value of the portfolio converges in probability to the pay-off function. For this setting [1] gives the diffusion limit for the hedging strategy. In [7] the properties of efficient strategies have been studied. The situation becomes crucial when the transaction cost does not vanish but is a positive constant. In this case Kabanov and Safarian [5] proved that Leland's strategy does not approximate the pay-off function and they computed the limiting hedging error. Grandits and Schachinger [4] have investigated the Leland strategy. They have found the estimates for the rate of convergence in the case when at the termination time the share price is in a small vicinity of the strike price.

In this paper we show that the rate of convergence in the Kabanov–Safarian theorem is of order $n^{1/4}$. The limiting distribution for the terminal value of the portfolio for Leland's strategy is a mixture of Gaussian distributions.

The paper is organized as follows. In Section 2 we fix assumptions and formulate the main results. In Section 3 we show how one needs to modify Leland's strategy to obtain the solution for the problem of option hedging. In Section 4 an asymptotic representation for the transactions costs is obtained. Asymptotic properties of the terminal value of the portfolio for Leland's strategy are gathered in Section 5. The proof of the main result is given in Section 6. Proofs of technical lemmas are given in the Appendix.

2. Basic results. Let us consider the continuous time model of financial market with two assets. The first asset is riskless (bond) and the second asset is

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risky (stock). The bond price is constant over time and equal to 1. The stock price dynamics is given by the stochastic differential equation

$$dS_t = \sigma S_t \, dw_t, \qquad S_0 > 0, \tag{2.1}$$

where σ is a positive constant, w is the Wiener process, that is,

$$S_t = S_0 \exp\{\sigma w_t - \sigma^2 t/2\}, \qquad 0 \le t \le 1. \tag{2.2}$$

We recall that a pair (β_t, γ_t) is an admissible self-financing strategy if γ_t is $\mathcal{F}_t$-measurable ($\mathcal{F}_t$ is σ algebra generated by $\{w_s,\ s \le t\}$) such that

$$\int_0^1 \gamma_t^2 \, dt < \infty \qquad \text{a.s.}$$

and

$$\beta_t = V_0 + \int_0^t \gamma_u \, dS_u - \gamma_t S_t,$$

where V_0 is the initial capital. Here β_t and γ_t are fractions of the wealth invested into bonds and stocks correspondingly. Then the value of the portfolio at time t is

$$V_t = \beta_t + \gamma_t S_t = V_0 + \int_0^t \gamma_u \, dS_u.$$

The European call option hedging problem with maturing at $T = 1$ and striking price K, that is, with the terminal payoff $H = H(S_1) = (S_1 - K)^+$, is to choose the admissible self-financing strategy $\{\beta_t,\ \gamma_t\}$ such that

$$V_1 = V_0 + \int_0^1 \gamma_u \, dS_u \ge H \qquad \text{a.s.}$$

The portfolio value of the hedging strategy is given by the Black–Scholes formula [2] $V_t = C(t, S_t, \sigma)$, where

$$C(t, s, \sigma) = s\Phi(d) - K\Phi\big(d - \sigma\sqrt{1-t}\big),$$

Φ is the standard normal distribution function with the density

$$\varphi(x) = e^{-x^2/2}/\sqrt{2\pi}$$

and

$$d = d(t, s, \sigma) = \frac{\eta(s)}{\sigma\sqrt{1-t}} + \frac{\sigma}{2}\sqrt{1-t}, \qquad \eta(s) = \ln(s/K). \tag{2.3}$$

The strategy γ_t at time t is defined by the equality

$$\gamma_t = \Phi(d_t), \qquad d_t = d(t, S_t, \sigma). \tag{2.4}$$

For this strategy

$$V_1 = H(S_1) \qquad \text{a.s.}$$

Assume that in the stock market the cost of a single transaction is a fixed fraction κ of its trading volume. The trading strategy suggested by Leland is the following. Denote by γ_t^n the number of shares of the stock in the portfolio at time t. Then γ_t^n is given by

$$\gamma_t^n = \sum_{i=1}^{n} \Phi(\nu_{t_{i-1}}) \chi_{(t_{i-1}, t_i]}(t), \tag{2.5}$$

where n denotes the numbers of revision intervals,

$$t_i = i/n, \qquad \nu_t = d(t, S_t, \theta), \qquad \theta = \theta_n = \sqrt{\sigma^2 + \sigma_0^2 n^{1/2}} \tag{2.6}$$

and σ_0 is a positive constant.

Then the portfolio value at t with the initial endowment $C(0, S_0, \theta)$ has the form

$$V_t^n = V_t^n(S) = C(0, S_0, \theta) + \int_0^t \gamma_u^n \, dS_u - \kappa J_t^n, \tag{2.7}$$

where

$$J_t^n = J_t^n(S) = \sum_{t_i \le t} S_{t_i} \left| \gamma_{t_i}^n - \gamma_{t_{i-1}}^n \right|. \tag{2.8}$$

Kabanov and Safarian [5] have proved that for $\sigma_0^2 = \sigma_{0,L}^2 = 2\sqrt{2}\kappa\sigma/\sqrt{\pi}$,

$$\mathbf{P} - \lim_{n\to\infty} J_1^n(S) = J(S_1, \sigma_0) \tag{2.9}$$

and

$$\mathbf{P} - \lim_{n\to\infty} V_1^n(S) = \hat{H}(S_1, \sigma_0). \tag{2.10}$$

Here

$$\hat{H}(s, \sigma_0) = H(s) + \min(s, K) - \kappa J(s, \sigma_0).$$

The function $J(s, \sigma_0)$ is defined by

$$J(s, \sigma_0) = s \int_0^\infty G\big(\lambda, \eta(s), \sigma_0\big) \varphi_1\big(\lambda, \eta(s)\big) \, d\lambda, \tag{2.11}$$

where

$$G(\lambda, \eta, \sigma_0) = \int_{-\infty}^{+\infty} g(\lambda, \eta, y/\sigma_0^2) \varphi(y) \, dy,$$

$$g(\lambda, \eta, y) = \left| f(\lambda, \eta) - \sigma y/\sqrt{\lambda} \right|, \qquad f(\lambda, \eta) = \frac{1}{4\sqrt{\lambda}} - \frac{\eta}{2\lambda\sqrt{\lambda}}, \tag{2.12}$$

$$\varphi_1 = \varphi_1(\lambda, \eta) = \varphi\big(\mu(\lambda, \eta)\big), \qquad \mu(\lambda, \eta) = \frac{\sqrt{\lambda}}{2} + \frac{\eta}{\sqrt{\lambda}}.$$

In this paper we will show that the limiting relationships (2.9) and (2.10) hold true for any $\sigma_0 > 0$ and we will get the following result.

THEOREM 2.1. *For any $\sigma_0 > 0$,*

$$n^{1/4}\big(J_1^n(S) - J(S_1, \sigma_0)\big) \Rightarrow \vartheta_1, \qquad n \to \infty \tag{2.13}$$

and

$$n^{1/4}\big(V_1^n - \hat{H}(S_1, \sigma_0)\big) \Rightarrow \vartheta_2, \qquad n \to \infty. \tag{2.14}$$

Here ϑ_i is a random variable with a mixed Gaussian distribution, that is,

$$\mathbf{P}\{\vartheta_i \le x\} = \mathbf{E}\Phi(x/\tilde{\vartheta}_i)\,,$$

where $\tilde{\vartheta}_i = \Lambda_i(S_1)$ and $\Lambda_i(\cdot)$ is a positive measurable function, $i = 1, 2$.

REMARK 2.2. The proof of Theorem 2.1 is based on the following idea. The portfolio value is conditioned on the terminal value of the underlying asset price S_1. As a result we obtain a function of the Brownian bridge and the central limit theorem for martingales can be applied to analyze it.

3. Application to the option hedging problem. First, we notice that Kabanov and Safarian [5] have shown that

$$\hat{H}(S_1, \sigma_{0,L}) \le H(S_1) \qquad \text{a.s.}$$

This means that Leland's strategy, that is, the strategy (2.5) with $\sigma_0 = \sigma_{0,L}$ does not solve the option hedging problem. Therefore, the natural question arises: How should one modify strategy (2.5) to obtain the solution for this problem?

In this section we show how one needs to choose the value of the parameter σ_0 in the portfolio (2.5) to achieve this goal.

Indeed, Theorem 2.1 implies that the limiting relationship (2.9) holds true for any $\sigma_0 > 0$. Furthermore, it is easy to see that

$$\lim_{\sigma_0 \to \infty} J(s, \sigma_0) = J^*(s),$$

where

$$J^*(s) = s\int_0^\infty |f(\lambda, s)|\varphi_1(\lambda, s)\,d\lambda = \begin{cases} s, & \text{if } s < K, \\ s/2, & \text{if } s = K, \\ J_0^*(s), & \text{if } s > K, \end{cases}$$

and

$$J_0^*(s) = 2s\int_{v^*}^\infty \varphi(v)\,dv, \qquad v^* = \sqrt{2\eta(s)}.$$

One can show that

$$J^*(s) \le \min(s, K)\,.$$

Then taking into account that

$$\int_0^\infty e^{-(v^2 + a^2/v^2)}\,dv = \frac{\sqrt{\pi}}{2}e^{-2|a|}, \tag{3.1}$$

we obtain

$$|J(s,\sigma_0) - J^*(s)| \le 2\sigma\sqrt{\frac{2}{\pi}}\,\frac{\min(s,K)}{\sigma_0^2}.$$

Therefore, for $\kappa \in (0,1)$,

$$\begin{aligned}\hat{H}(S_1,\sigma_0) - H(S_1) &\ge \min(S_1,K) - \kappa J^*(S_1) - \kappa|J(S_1,\sigma_0) - J^*(S_1)|\\ &\ge \min(S_1,K)\left(1-\kappa-2\sqrt{\frac{2}{\pi}}\frac{\kappa\sigma}{\sigma_0^2}\right) \qquad (3.2)\\ &= \rho\min(S_1,K) > 0 \qquad \text{a.s.}\end{aligned}$$

for all $\sigma_0^2 > \sigma_*$, where

$$\sigma_* = 2\sigma\sqrt{\frac{2}{\pi}}\,\frac{\kappa}{1-\kappa}.$$

It will be observed that inequality (3.2) is due to the high option price. This inequality holds because $C(0,S_0,\theta) \to S_0$. It means that it makes no sense to buy the option at this price in the situation where one can buy a share and keep it up to the termination moment. Such a strategy is called "buy and hold." There arises a natural question: Is it possible to lower the option price for the Leland strategy? To answer this question we propose to use, instead of the hedging with probability 1 (cf. [2, 8]), the hedging with probability $1-\varepsilon$, where ε is a small positive number (cf. [3]).

We propose to sell the option at the price αS_0, (where $0<\alpha<1$ is properly chosen) and the remaining part $(1-\alpha)S_0$ is to be paid by the option seller himself. In order to compensate his expenses and to pay the owner of option, the seller is assumed to include his expenses in the hedging problem; that is, the asymptotic terminal portfolio value must exceed the quantity $H(S_1)+(1-\alpha)S_0$. The inequality (3.2) shows that for some values of the parameter $\alpha \in (0,1)$ this probability is positive. To determine the portfolio it remains to choose α. We propose to define it as

$$\alpha_\varepsilon = \inf\{a>0 : F(a) \ge 1-\varepsilon\},$$

where $F(a) = \mathbf{P}(\hat{H}(S_1,\sigma_0) > H(S_1) + (1-a)S_0)$. From this it follows that $\mathbf{P}(\hat{H}(S_1,\sigma_0) \ge H(S_1) + (1-\alpha_\varepsilon)S_0) \ge 1-\varepsilon$.

Notice that α_ε belong to $[0,1)$ for $0<\varepsilon<\mathbf{P}(S_1 \le S_0)$. Indeed, by (3.2), $F(1)=1$ and by the definition of $\hat{H}(S_1,\sigma_0)$ in (2.10) for such $\varepsilon>0$,

$$F(0) \le \mathbf{P}\big(\min(S_1,\ K) > S_0\big) \le \mathbf{P}(S_1 > S_0) < 1-\varepsilon.$$

Furthermore, for all $x>0$,

$$\mathbf{P}\big(\hat{H}(S_1,\sigma_0) > H(S_1)+x\big) \le \mathbf{P}\big(\min(S_1,\ K) > x\big) \le \mathbf{P}(S_1 > x) < 1.$$

Therefore $\alpha_\varepsilon \to 1$ as $\varepsilon \to 0$. Taking this into account and the inequality (3.2), we obtain for sufficiently small $\varepsilon > 0$ the following inequality:

$$\begin{aligned} 1-\varepsilon &> \mathbf{P}\big(\hat{H}(S_1,\sigma_0) > H(S_1) + (1-a)S_0\big) \\ &\geq \mathbf{P}\big(\rho \min(S_1, K) > (1-a)S_0\big) \\ &= 1 - \mathbf{P}\big(w_1 \geq N(a)\big), \end{aligned}$$

where $1 - \rho K/S_0 < a < \alpha_\varepsilon$ and $N(a) = \frac{1}{\sigma} \ln \frac{\rho}{1-a} - \sigma/2$.

This implies that for sufficiently small $\varepsilon > 0$,

$$\varepsilon \leq (2\pi)^{-1/2} \int_{N(\alpha_\varepsilon)}^{\infty} e^{-t^2/2}\, dt \leq (2\pi)^{-1/2}\, e^{-N(\alpha_\varepsilon)^2/2}.$$

From here it follows that in this case α_ε satisfies the following asymptotic inequality:

$$1 - \alpha_\varepsilon \geq \rho \exp\Big\{-\sigma\sqrt{2|\ln(\sqrt{2\pi}\,\varepsilon)|} - \sigma^2/2\Big\}, \qquad \varepsilon \to 0.$$

This inequality implies that

$$\lim_{\varepsilon\to 0} \frac{1-\alpha_\varepsilon}{\varepsilon^\gamma} = +\infty$$

for all $\gamma > 0$. It means that the price of Leland's strategy can be substantially lowered as compared with the power function of parameter ε with an arbitrary small power.

4. Approximation for J_1^n. First we shall deal with the term which accounts for transaction costs. Keeping in mind the definition in (2.8), we can rewrite this term in the form

$$J_1^n(S) = \sum_{m=0}^{n-1} S_{u_m} |\Phi(q_{m+1}) - \Phi(q_{m+2})|, \tag{4.1}$$

where $u_m = 1 - t_m$, $q_m = \eta(S_{u_m})/\sqrt{\lambda_m} + \sqrt{\lambda_m}/2$ and $\lambda_m = \theta^2 t_m$.

If t_m is bounded away from zero then λ_m diverges to infinity and so $\Phi(q_m) - \Phi(q_{m-1}) \to 0$. Therefore it is enough to study only those components in the sum (4.1) for which $t_m \to 0$; that is, $S_{u_m} \to S_1$. It means that components in the sum (4.1) corresponding to such u_m are strongly dependent on S_1, that is, on the future. Therefore we cannot apply to this sum any classic limiting theorem. To overcome this difficulty it is proposed to pass to the conditional distribution by fixing the terminal value S_1. In order to do that we notice that (see [6], page 359) for any Borel sets $\Gamma_1, \ldots, \Gamma_k$, $0 < t_1 < \cdots < t_k < 1$ and $-\infty < b < \infty$.

$$P\big(w_{t_1} \in \Gamma_1, \ldots, w_{t_k} \in \Gamma_k | w_1 = b\big) = P\big(x_{t_1}^b \in \Gamma_1, \ldots, x_{t_k}^b \in \Gamma_k\big) \qquad \text{a.s.,} \tag{4.2}$$

where $x_t^b = bt + w_t - tw_1$ is a Brownian bridge. Therefore the conditional distribution of $J_1^n(S)$ under the condition $\{w_1 = b\}$ coincides with the nonconditional distribution of the random variable,

$$J_1^n(\hat{S}^b) = \sum_{m=0}^{n-1} \hat{S}_{u_m}^b \left|\Phi(\hat{q}_{m+1}^b) - \Phi(\hat{q}_{m+2}^b)\right|, \tag{4.3}$$

where

$$\hat{S}_t^b = S_0 e^{-\sigma^2 t/2 + \sigma x_t^b}, \qquad \hat{q}_m^b = \frac{\eta(\hat{S}_{u_m}^b)}{\sqrt{\lambda_m}} + \frac{\sqrt{\lambda_m}}{2}.$$

It follows that it is sufficient to find a limiting distribution for $J_1^n(\hat{S}^b)$ for almost all b, $-\infty < b < \infty$. Then integration over b will give the limiting distribution for $J_1^n(S)$. For this, we need to find a principal term for the sum (4.3). There arises a natural question: What components in the sum (4.3) are principal? That is, what is the exact rate of the convergence (2.9)? To answer this question we will apply Grandits and Schachinger's result [4] for the rate of this convergence. They proved that this rate is smaller than $n^{-1/4+\varepsilon}$ for any $\varepsilon > 0$. Therefore, we can neglect in the sum (4.3) all components which tend to zero faster than $n^{-1/4}$. For this, let us represent $\hat{q}_m^b$ in the following form:

$$\hat{q}_m^b = \mu_m^b - \frac{\sigma}{\sqrt{\lambda_m}} \tilde{w}_{t_m} + \frac{\tau^b}{\theta^2}\sqrt{\lambda_m}, \qquad \mu_m^b = \mu^b(\lambda_m) = \mu(\lambda_m, \eta^b), \tag{4.4}$$

where $\tau^b = \sigma^2/2 + \sigma(w_1 - b)$, $\eta^b = \eta(\hat{S}_1^b)$, $\tilde{w}_t = w_1 - w_{1-t}$ and the function μ is defined by (2.12). It is well known that $\{\tilde{w}_t, t \ge 0\}$ is Brownian motion. Notice that, if $\eta^b \neq 0$, then there exist positive constants C and α such that

$$\left|\Phi(\hat{q}_{m+1}^b) - \Phi(\hat{q}_m^b)\right| \le \left|1 - \Phi(\hat{q}_{m+1}^b)\right| + \left|1 - \Phi(\hat{q}_m^b)\right| \le C e^{-\alpha l_n}$$

for $\lambda_m \le l_n$ $(m \le n\, l_n/\theta^2)$ or $\lambda_m \ge l_n$ $(m \ge n\, l_n/\theta^2)$ for some l_n. By choosing $l_n = (\ln n)^3$ we get that all this components in the sum (4.3) are asymptotically smaller than n^{-p} for any $p > 0$. It follows that

$$J_1^n(\hat{S}^b) = \sum_{m=m_0}^{m_1} \hat{S}_{u_m}^b \left|\Phi(\hat{q}_{m+2}^b) - \Phi(\hat{q}_{m+1}^b)\right| + o_p(n^{-1/4}), \tag{4.5}$$

where $m_0 = m_0(n) = [n/(\theta^2 l_n)]$, $m_1 = m_1(n) = [l_n\, n/\theta^2]$, $[a]$ denotes the integer part of a and $o_p(n^{-\alpha})$ means a sequence of a random variables $\{\xi_n\}$ for which

$$\mathbf{P} - \lim_{n\to\infty} n^\alpha \xi_n = 0.$$

Then one needs to study the asymptotic behavior of the difference $\Phi(\hat{q}_{m+1}^b) - \Phi(\hat{q}_{m+2}^b)$. By the Taylor formula,

$$\Phi(\hat{q}_{m+1}^b) - \Phi(\hat{q}_m^b) = \varphi(\hat{q}_m^b)\Delta\hat{q}_{m+1}^b + \tfrac{1}{2}\dot{\varphi}(\tilde{q}_m)(\Delta\hat{q}_{m+1}^b)^2,$$

where $\varphi(\cdot)$ is the Gaussian(0, 1) density and $\min(\hat{q}^b_m, \hat{q}^b_{m+1}) \le \tilde{q}_m \le \max(\hat{q}^b_m, \hat{q}^b_{m+1})$. By (4.4) we can write the increment $\Delta\hat{q}^b_{m+1} = \hat{q}^b_{m+1} - \hat{q}^b_m$ in the following form:

$$\begin{aligned}\Delta\hat{q}^b_{m+1} &= \big(f^b_m + f_0(\lambda_m)\tilde{w}_{t_m} - f_1(\lambda_m)\, y_{m+1}\big)\Delta\lambda_{m+1} + r_{m+1}(\Delta\lambda_{m+1})^2 \\ &= \rho_{m+1}\Delta\lambda_{m+1} + r_{m+1}(\Delta\lambda_{m+1})^2,\end{aligned} \tag{4.6}$$

where $f^b_m = f(\lambda_m, \eta^b) = \frac{\partial}{\partial\lambda}\mu(\lambda_m, \eta^b)$, $f_0(\lambda) = \sigma/2\lambda^{3/2}$, $f_1(\lambda) = \sigma/\sigma_0^2\sqrt{\lambda}$, $y_m = (\tilde{w}_{t_m} - \tilde{w}_{t_{m-1}})/\Delta t_m$ and the residual term r_m satisfies the following inequality:

$$\mathbf{E}|r_m| \le C(\lambda_m^\gamma + \lambda_m^{-\gamma})$$

for some constants $C > 0$ and $\gamma > 0$. Taking into account that, for any $\gamma > 0$,

$$\sum_{m=m_0}^{m_1} (\lambda_m^\gamma + \lambda_m^{-\gamma})\,(\Delta\lambda_{m+1})^2 = \frac{\theta^2}{n}\sum_{m=m_0}^{m_1}(\lambda_m^\gamma + \lambda_m^{-\gamma})\Delta\lambda_{m+1} \le \frac{\theta^2}{n} l_n^{1+\gamma},$$

we obtain that

$$J_1^n(\hat{S}^b) = \sum_{m=m_0}^{m_1} \hat{S}^b_{u_m}\big|\varphi(\hat{q}^b_{m+1})\rho_{m+1}\big|\Delta\lambda_{m+2} + o_p(n^{-1/4}), \tag{4.7}$$

where ρ_m is the principal term in (4.6). By expanding the function $\varphi(\hat{q}^b_m)$ at the point μ^b_m by the Taylor formula and taking into account that $\dot{\varphi}(x) = -x\varphi(x)$, we can replace $\varphi(\hat{q}^b_m)$ in the sum (4.7) by

$$\varphi(\mu^b_m) + \dot{\varphi}(\mu^b_m)\,(\hat{q}^b_m - \mu^b_m) = \varphi(\mu^b_m)\big(1 - \mu^b_m\,(\hat{q}^b_m - \mu^b_m)\big),$$

because the expectation of the residual term in this equality is smaller than $C(\lambda_m + \lambda_m^{-1})/\theta^2$; therefore the expectation of the residual term in the sum (4.7) tends to zero faster than $\sum_{m=m_0}^{m_1}(\lambda_m + \lambda_m^{-1})\,\Delta\lambda_{m+1}/\theta^2 = o(n^{-1/4})$. According to (4.4) the principal term in the expression for $\hat{q}^b_m$ is equal to $\mu^b_m - \sigma\,\tilde{w}_{t_m}/\sqrt{\lambda_m}$. Therefore, one can replace $\varphi(\hat{q}^b_m)$ in (4.7) by $\varphi(\mu^b_m)\rho^1_m$ with $\rho^1_m = 1 + \omega^b_m\,\tilde{w}_{t_m}$ and $\omega^b_m = \omega(\lambda_m, \eta^b) = \mu(\lambda_m, \eta^b)\,\sigma/\sqrt{\lambda_m}$.

Furthermore, taking into account that $\varphi(\mu^b_m) = \varphi^b_1(\lambda_m) = \varphi_1(\lambda_m, \eta^b)$, we can write that

$$J_1^n(\hat{S}^b) = \sum_{m=m_0}^{m_1} \hat{S}^b_{u_m}\varphi^b_1(\lambda_{m+1})|\rho^1_{m+1}\,\rho_{m+1}|\Delta\lambda_{m+2} + o_p(n^{-1/4}).$$

To study the principal term in this equality, notice that

$$\rho^1_m\,\rho_m = L^b(\lambda_m, \tilde{w}_{t_m}, y_{m+1}) = A^b_1(\lambda_m, \tilde{w}_{t_m}) - A^b_2(\lambda_m, \tilde{w}_{t_m})y_{m+1} + \omega^b_m f_0(\lambda_m)\tilde{w}^2_{t_m},$$

where

$$\begin{aligned}A^b_1(\lambda, z) &= A_1(\lambda, z, \eta^b) = f(\lambda, \eta^b) + \big(f_0(\lambda) + \omega(\lambda, \eta^b)\, f(\lambda, \eta^b)\big)z, \\ A^b_2(\lambda, z) &= A_2(\lambda, z, \eta^b) = f_1(\lambda)\big(1 + \omega(\lambda, \eta^b)z\big)\end{aligned} \tag{4.8}$$

and that $\hat{S}^b_{u_m} = \hat{S}^b_1(1 - \sigma\,\tilde{w}_{t_m} + \tau^b\, t_m) = \hat{S}^b_1(1 - \sigma\,\tilde{w}_{t_m} + \theta^{-2}\tau^b\,\lambda_m)$.

From the above it is easy to deduce that $J^n_1(\hat{S}^b) = \hat{S}^b_1\,\overline{J}^b_n + o_p(n^{-1/4})$ with

$$\overline{J}^b_n = \sum_{m=m_0}^{m_1} (1 - \sigma\,\tilde{w}_{t_{m-1}})\varphi^b_1(\lambda_{m-1})\big|L^b(\lambda_{m-1}, \tilde{w}_{t_{m-1}}, y_m)\big|\Delta\lambda_m. \tag{4.9}$$

Notice that the last term is a sum of a sequence of random variables $(\zeta_m\,\Delta\lambda_m)$ which are $\mathcal{F}_m$ measurable ($\mathcal{F}_m = \sigma\{\tilde{w}_t,\, 0 \le t \le t_m\}$), that is, independent of the future. Now to study this term we can apply a limiting theorem method. For this, one has to find its martingale part, that is, to construct the Doob–Meyer decomposition. By making use of the standard method, we can represent $\overline{J}^b_n$ as the sum of the two following components:

$$\overline{J}^b_n = \sum_{m=m_0}^{m_1} \mathbf{E}(\zeta_m|\mathcal{F}_{m-1})\Delta\lambda_m + \sum_{m=m_0}^{m_1} \big(\zeta_m - \mathbf{E}(\zeta_m|\mathcal{F}_{m-1})\big)\Delta\lambda_m\,. \tag{4.10}$$

By putting $h^b(\lambda, z) = h(\lambda, \eta^b, z) = \int |L(\lambda, \eta^b, z, y)|\varphi(y)\,dy$ and taking into account that for sufficiently small z this function is twice continuously differentiable with respect to z, we get

$$\begin{aligned}\mathbf{E}(\zeta_m|\mathcal{F}_{m-1}) &= \big(1 - \sigma\,\tilde{w}_{t_{m-1}}\big)\varphi^b_1(\lambda_{m-1})h^b\big(\lambda_{m-1}, \tilde{w}_{t_{m-1}}\big)\\ &= \varphi^b_1(\lambda_{m-1})\big(h^b(\lambda_{m-1}, 0) + \varphi^b_1(\lambda_{m-1})\tilde{w}_{t_{m-1}}\,\psi^b(\lambda_{m-1}) + \varrho_m\,\tilde{w}^2_{t_{m-1}}\big),\end{aligned}$$

where $\psi^b(\lambda) = \psi^b(\lambda, \eta^b)$, $\psi^b(\lambda, \eta) = h_z(\lambda, 0, \eta) - \sigma\, h(\lambda, 0, \eta)$. Furthermore, since $\mathbf{E}\sup_{m_0\le m\le m_1} |\tilde{w}_{t_m}| = O(\sqrt{l}_n/n^{1/4})$ then for some $\gamma > 0$ the sequence $\{\varrho_m = (\lambda^\gamma + \lambda^{-\gamma})^{-1}\varrho^m\}$ is bounded in probability, that is,

$$\lim_{L\to\infty}\lim_{n\to\infty}\mathbf{P}\bigg(\sup_{m_0\le m\le m_1} |\tilde{\varrho}_m| > L\bigg) = 0.$$

It means that the first term in (4.10) equals

$$\begin{aligned}&\sum_{m=m_0}^{m_1} \varphi^b_1(\lambda_{m-1})h^b(\lambda_{m-1}, 0)\,\Delta\lambda_m\\ &\qquad + \sum_{m=m_0}^{m_1} \varphi^b_1(\lambda_{m-1})\psi^b(\lambda_{m-1})\tilde{w}_{t_{m-1}}\,\Delta\lambda_m + o_p(n^{-1/4}).\end{aligned}$$

Since $h^b(\lambda, 0) = G(\lambda, \eta^b, \sigma_0)$, the first term is the integral sum for $\int_0^\infty G(\lambda, \eta^b, \sigma_0)\varphi_1(\lambda, s)\,d\lambda$ and by Lemma A.1 and (2.11) one can replace this sum by $J(\hat{S}^b_1, \sigma_0)/\hat{S}^b_1$. Now notice that we can rewrite the second term in the last equality as

$$\begin{aligned}&\tilde{w}_{t_{m_0}}\sum_{m=m_0}^{m_1} \varphi^b_1(\lambda_{m-1})\psi^b(\lambda_{m-1})\,\Delta\lambda_m\\ &\qquad + \sum_{j=m_0}^{m_1-1} y_j\sqrt{\Delta t_j}\sum_{m=j+1}^{m_1} \varphi^b_1(\lambda_{m-1})\psi^b(\lambda_{m-1})\,\Delta\lambda_m.\end{aligned} \tag{4.11}$$

Since $\mathbf{E}|\tilde{w}_{t_{m_0-1}}| = O(\frac{1}{\sqrt{l_n}n^{1/4}})$, and for $\eta^b \neq 0$,

$$\lim_{n\to\infty} \sum_{m=m_0}^{m_1} \varphi_1^b(\lambda_{m-1})|\psi^b(\lambda_{m-1})|\,\Delta\lambda_m = \int_0^\infty \varphi_1^b(\lambda)|\psi^b(\lambda)|\,d\lambda < \infty,$$

we obtain that the first component in (4.11) is $o_p(n^{-1/4})$ and therefore by Lemma A.1, we get the following asymptotic representation:

$$\begin{aligned}\hat{S}_1^b\,\overline{J}_n^b &= J(\hat{S}_1^b,\sigma_0) + \hat{S}_1^b \frac{1}{\sigma_0^2}\sum_{j=m_0}^{m_1} y_j \Theta^b(\lambda_j)\Delta\lambda_j \\ &\quad + \hat{S}_1^b \sum_{m=m_0}^{m_1} (1-\sigma\tilde{w}_{t_{m-1}})\varphi_1^b(\lambda_{m-1})\,\xi_m\,\Delta\lambda_m + o_p(n^{-1/4}),\end{aligned} \tag{4.12}$$

where $\Theta^b(\lambda) = \Theta(\lambda,\eta^b) = \int_\lambda^\infty \varphi_1(z,\eta)\psi(z,\eta^b)\,dz$, $\xi_m = L^b(\lambda_{m-1},\tilde{w}_{t_{m-1}},y_m) - h^b(\lambda_{m-1},\tilde{w}_{t_{m-1}})$.

Finally, taking into account (see the Appendix, Section A2) that

$$\sum_{m=m_0}^{m_1} \tilde{w}_{t_{m-1}}\varphi_1^b(\lambda_{m-1})\,\xi_m\,\Delta\lambda_m = o_p(n^{-1/4}), \tag{4.13}$$

we can write the following asymptotic equality for J_1^n:

$$J_1^n(\hat{S}^b) = J(\hat{S}_1^b,\sigma_0) + \hat{S}_1^b M_n + o_p(n^{-1/4}), \tag{4.14}$$

where $J(s,\sigma_0)$ is defined by (2.11) and

$$M^n = \sum_{j=m_0}^{m_1} \big(\varphi_1^b(\lambda_{j-1})\xi_j + \sigma_0^{-2} y_j \Theta^b(\lambda_{j-1})\big)\Delta\lambda_j, \tag{4.15}$$

where $\varphi_1^b(\lambda) = \varphi_1^b(\lambda,\eta^b)$, $\Theta^b(\lambda) = \Theta(\lambda,\eta^b)$ with $\eta^b \neq 0$.

Equality (4.14) is the Doob–Meyer decomposition for $J_1^n(\hat{S}^b)$, where $\hat{S}_1^b M^n$ is the martingale term.

5. Representation for V_1^n. In this section we will study the terminal portfolio value V_1^n. We shall explain the appearance of the additional component $\min(s,K)$ in the limit $\hat{H}$ of (2.10). According to [5] we can represent V_1^n in the following form:

$$V_1^n(S) = H(S_1) + I^n(S) + \int_0^1 \big(\gamma_t^n - \Phi(\nu_t)\big)\,dS_t - \kappa J_1^n, \tag{5.1}$$

where

$$I^n(S) = \frac{\theta^2-\sigma^2}{2}\int_0^1 C_{ss}(t,S_t,\theta)S_t^2\,dt.$$

Here (see the Appendix, Section A3)

$$\mathbf{P}-\lim_{n\to\infty} n^{1/4}\int_0^1 \big(\gamma_t^n-\Phi(\nu_t)\big)\,dS_t=0. \tag{5.2}$$

Now let us consider the second term in the right-hand side of (5.1). Putting

$$U_t(s)=\frac{\varphi(d(t,s,\theta))s}{\sqrt{1-t}}, \tag{5.3}$$

we obtain that

$$\begin{aligned} I^n(S)&=\frac{\theta^2-\sigma^2}{2\theta}\int_0^1\frac{\varphi(\nu_t)S_t}{\sqrt{1-t}}\,dt=\frac{\theta^2-\sigma^2}{2\theta}\int_0^1 U_t(S_t)\,dt\\ &=\frac{\theta^2-\sigma^2}{2\theta}\int_0^1 U_t(S_1)\,dt+\frac{\theta^2-\sigma^2}{2\theta}\int_0^1\big(U_t(S_t)-U_t(S_1)\big)\,dt. \end{aligned}$$

Furthermore,

$$\begin{aligned} \frac{\theta^2-\sigma^2}{2\theta}\int_0^1 U_t(s)\,dt&=\frac{s(\theta^2-\sigma^2)}{2\sqrt{2\pi}\theta}\int_0^1\frac{\exp\{-\mu^2(\theta^2(1-t),\eta(s))/2\}}{\sqrt{1-t}}\,dt\\ &=\frac{s(\theta^2-\sigma^2)}{\sqrt{2\pi}\theta^2}\int_0^\theta \exp\big\{-\mu^2\big(v^2,\eta(s)\big)/2\big\}\,dv. \end{aligned}$$

Taking into account equality (3.1) we obtain a representation for V_1^n,

$$V_1^n(S)=H(S_1)+\min(S_1,K)+I_1^n-\kappa J_1^n(S)+o_p(n^{-1/4}),$$

where

$$I_1^n=I_1^n(S)=\frac{\theta^2-\sigma^2}{2\theta}\int_0^1\big(U_t(S_t)-U_t(S_1)\big)\,dt. \tag{5.4}$$

Then, one can show (see the Appendix, Section A3) that

$$I_1^n=-\frac{\sigma}{2}S_1F^n(w)+o_p(n^{-1/4}), \tag{5.5}$$

where

$$F^n(w)=\theta\sum_{j=m_0}^{m_1}\int_{t_{j-1}}^{t_j}\dot U_{1-v}(S_1)\,dv\,\big(w_1-w_{1-t_j}\big).$$

Therefore, we can represent V_1^n in the following form:

$$V_1^n(S)=\hat H(S_1)-\Delta^n(w)+o_p(n^{-1/4}), \tag{5.6}$$

where $\Delta^n(w)=\frac{\sigma}{2}S_1F^n(w)+\kappa\,(J_1^n(S)-J(S_1,\sigma_0))$.

Now one needs to consider the function $F^n(\cdot)$ on the trajectory of the Brownian bridge x_t^b, that is,

$$F^n(x^b) = \theta \sum_{j=m_0}^{m_1} \int_{t_{j-1}}^{t_j} \dot{U}_{1-v}(\hat{S}_1^b)\, dv\, (x_1^b - x_{1-t_j}^b).$$

By the definition of the function U_t in (5.3), we have that

$$\dot{U}_t(\hat{S}_1^b) = \theta\phi^b(\theta^2(1-t)) = \phi(\theta^2(1-t), \eta^b)$$

with

$$\phi(v,\eta) = \varphi_1(v,\eta)\left(\frac{1}{\sqrt{v}} - \mu(v,\eta)\frac{1}{v}\right). \tag{5.7}$$

Thus

$$\begin{aligned} F^n(x^b) &= \theta^2 \sum_{j=m_0}^{m_1} \int_{t_{j-1}}^{t_j} \phi^b(\theta^2 v)\, dv\, (\tilde{w}_{t_j} + t_j(b - w_1)) \\ &= \sum_{j=m_0}^{m_1} \int_{\lambda_{j-1}}^{\lambda_j} \phi^b(\lambda)\, d\lambda\, (\tilde{w}_{t_j} + t_j(b - w_1)). \end{aligned}$$

By the same method which we used to obtain the form (4.12) we can show that for $\eta^b \neq 0$,

$$F^n(x^b) = \frac{1}{\sigma_0^2} \sum_{m=m_0}^{m_1} y_m \phi_1^b(\lambda_{m-1})\Delta\lambda_m + o_p(n^{-1/4}), \tag{5.8}$$

where $\phi_1^b(\lambda) = \phi_1(\lambda, \eta^b) = \int_\lambda^{+\infty} \phi(u, \eta^b)\, du$.

6. Proof of Theorem 2.1. First we prove the convergence (2.14). From (5.6) it follows that to prove this it suffices to show that

$$n^{1/4}\, \Delta^n(w) \Rightarrow \vartheta_2 \qquad \text{as } n \to \infty. \tag{6.1}$$

To obtain this convergence it suffices to prove that for almost all $-\infty < b < \infty$,

$$n^{1/4}\, \Delta^n(x^b) \Rightarrow \mathcal{N}(0, \Lambda_2(\hat{S}_1^b)) \qquad \text{as } n \to \infty, \tag{6.2}$$

where $\Lambda_2(\cdot)$ is some positive mesurable function. We will show this convergence for all $-\infty < b < \infty$ for which $\eta^b \neq 0$, that is, $b \neq \frac{\sigma}{2} - \frac{1}{\sigma} \ln S_0/K$.

Now, by (4.14), (4.15) and (5.8) we can represent $\Delta^n(x^b)$ in the following asymptotic form:

$$\Delta^n(x^b) = \hat{S}_1^b \sum_{m=m_0}^{m_1} (\Theta_1^b(\lambda_{m-1}) y_m + \kappa\varphi_1^b(\lambda_{m-1})\xi_m)\Delta\lambda_m + o_p(n^{-1/4}),$$

where

$$\Theta_1^b(\lambda) = \Theta_1(\lambda, \eta^b) = \frac{\sigma}{2\sigma_0^2}\phi_1(\lambda, \eta^b) + \frac{1}{\sigma_0^2}\Theta(\lambda, \eta^b).$$

Then we can write that

$$\Delta^n(x^b) = \frac{\hat{S}_1^b}{n^{1/4}}\big(\sigma_0 + o(1)\big)\sum_{m=m_0}^{m_1} \varsigma_m + o_p(n^{-1/4}),$$

where

$$\varsigma_m = \big(\Theta_1^b(\lambda_{m-1})y_m + \kappa\varphi_1^b(\lambda_{m-1})\xi_m\big)\sqrt{\Delta\lambda_m}.$$

By applying Lemma A.2 (see the Appendix, Section A5) to $\{\varsigma_m\}$, we get the convergence (6.1). The proof of (2.14) implies the convergence (2.13).

APPENDIX

A1. Integral approximations.

LEMMA A.1. *If $\eta \neq 0$ then for any $-\infty < \alpha < \infty$,*

$$\lim_{n\to\infty} n^{1/4}\left(\int_0^{+\infty} \varphi_1(\lambda, \eta)\lambda^\alpha\, d\lambda - \sum_{m=m_0}^{m_1} \varphi_1(\lambda_{m-1}, \eta)\lambda_{m-1}^\alpha \Delta\lambda_m\right) = 0. \tag{A.1}$$

PROOF. Denoting $\tilde{\varphi}(\lambda) = \varphi_1(\lambda, \eta)\lambda^\alpha$ we get that

$$\begin{aligned} &\left|\int_0^{+\infty} \tilde{\varphi}(\lambda)\, d\lambda - \sum_{m=m_0}^{m_1} \tilde{\varphi}(\lambda_{m-1})\Delta\lambda_m\right| \\ &\quad\le \int_{\lambda_{m_1}}^{\infty} \tilde{\varphi}(\lambda)\, d\lambda + \int_0^{\lambda_{m_0}} \tilde{\varphi}(\lambda)\, d\lambda \\ &\qquad + \sum_{m=m_0}^{m_1} \int_{\lambda_{m-1}}^{\lambda_m} |\tilde{\varphi}(\lambda) - \tilde{\varphi}(\lambda_{m-1})|\, d\lambda. \end{aligned} \tag{A.2}$$

By the definitions of m_1 and m_0, we have that $\lambda_{m_1} \approx l_n$ and $\lambda_{m_0} \approx l_n^{-1}$ ($l_n = (\ln n)^3$); therefore,

$$\int_{\lambda_{m_1}}^{\infty} \tilde{\varphi}(\lambda)\, d\lambda + \int_0^{\lambda_{m_0}} \tilde{\varphi}(\lambda)\, d\lambda = o(n^{-p}), \qquad n \to \infty$$

for any $p > 0$.

The last term in (A.2) can be estimated as

$$\sum_{m=m_0}^{m_1} \int_{\lambda_{m-1}}^{\lambda_m} |\tilde{\varphi}(\lambda) - \tilde{\varphi}(\lambda_{m-1})| \, d\lambda$$
$$\le L \sum_{m=m_0}^{m_1} \int_{\lambda_{m-1}}^{\lambda_m} (\lambda_{m-1}^{-\gamma} + \lambda_{m-1}^{\gamma})(\lambda - \lambda_{m-1}) \, d\lambda$$

for some positive constants γ, L. From here the limiting relationship (A.1) follows. □

A2. Proof of (4.13). First notice that, by the definition of ξ_m, we can get an upper bound for its conditional variance

$$\mathbf{E}\big((\xi_m)^2 | \mathcal{F}_{m-1}\big) \le C\big((\lambda_{m-1})^{\gamma} + (\lambda_{m-1})^{-\gamma}\big)\big(1 + |\tilde{w}_{t_{m-1}}|^2\big) \tag{A.3}$$

for some positive constants $C > 0$ and $\gamma > 0$. Taking this into account, one can deduce that

$$\mathbf{E}\left(\sum_{m=m_0}^{m_1} \tilde{w}_{t_{m-1}} \varphi_1^b(\lambda_{m-1}) \, \xi_m \, \Delta\lambda_m\right)^2$$
$$= \sum_{m=m_0}^{m_1} \mathbf{E}\tilde{w}_{t_{m-1}}^2 \big(\varphi_1^b(\lambda_{m-1})\big)^2 \mathbf{E}\big((\xi_m)^2 | \mathcal{F}_{m-1}\big) \, (\Delta\lambda_m)^2$$
$$\le C \frac{1}{n} \sum_{m=m_0}^{m_1} \big((\lambda_{m-1})^{\gamma} + (\lambda_{m-1})^{-\gamma}\big) \, \Delta\lambda_m = \frac{1}{n} O(l_n^{1+\gamma})$$

and we get (4.13).

A3. Proof of (5.2). By the Lenglart inequality, for any $\delta > 0$, we get that

$$\mathbf{P}\left(\left|\int_0^1 (\Phi(\nu_t) - \gamma_t^n) S_t \, dw_t\right| > \delta n^{-1/4}\right)$$
$$\le (a/\delta)^2 + \mathbf{P}\left(\int_0^1 (\Phi(\nu_t) - \gamma_t^n)^2 S_t^2 \, dt > a^2/\sqrt{n}\right)$$
$$\le (a/\delta)^2 + \mathbf{P}\left(\sup_{0 \le t \le 1} S_t > L\right)$$
$$+ \mathbf{P}\left(\int_0^1 (\Phi(\nu_t) - \gamma_t^n)^2 \, dt > \mu^2/L^2\sqrt{n}\right)$$
$$\le (a/\delta)^2 + \mathbf{P}\left(\sup_{0 \le t \le 1} S_t > L\right)$$

$$+\mathbf{P}\left(\int_{1-\varepsilon_n}^{1}(\Phi(\nu_t)-\gamma_t^n)^2\,dt>\mu^2/2L^2\sqrt{n}\right)$$
$$+\mathbf{P}\left(\int_0^{1-\varepsilon_n}(\Phi(\nu_t)-\gamma_t^n)^2\,dt>a^2/2L^2\sqrt{n}\right),$$

where $a>0$ and $\varepsilon_n=1/(\theta^2 l_n)$. Since the process γ_t^n is bounded, the third term in the right-hand side of this inequality is equal to 0 for sufficiently large n. Therefore, to prove (5.2) it suffices to show that

$$\lim_{n\to\infty}\sqrt{n}\int_0^{1-\varepsilon_n}\mathbf{E}(\Phi(\nu_t)-\gamma_t^n)^2\,dt=0. \tag{A.4}$$

Indeed,

$$\int_0^{1-\varepsilon_n}(\Phi(\nu_t)-\gamma_t^n)^2\,dt=\sum_{k=1}^{n}\int_{\hat{t}_{k-1}}^{\hat{t}_k}(\Phi(\nu_t)-\Phi(\nu_{t_{k-1}}))^2\,dt,$$

where $\hat{t}_k=\min(t_k,(1-\varepsilon_n))$. By the Itô formula, we get

$$\nu_t-\nu_{t_{k-1}}=\int_{t_{k-1}}^{t}\left(\frac{\eta(S_u)}{2\theta(1-u)^{3/2}}-\frac{\theta}{4(1-u)^{1/2}}-\frac{\sigma^2}{2\theta(1-u)^{1/2}}\right)du$$
$$+\frac{\sigma}{\theta}\int_{t_{k-1}}^{t}\frac{1}{(1-u)^{1/2}}\,dw_u.$$

Then, for $t_{k-1}\le t\le t_k\le 1-\varepsilon_n$,

$$\begin{aligned}\mathbf{E}(\nu_t-\nu_{t_{k-1}})^2&\le\frac{2\sigma^2}{\theta^2}\int_{t_{k-1}}^{t}\frac{1}{1-u}\,du\\&\quad+2\mathbf{E}\left(\int_{t_{k-1}}^{t}\left(\frac{\eta(S_u)}{2\theta(1-u)^{3/2}}-\frac{\theta}{4(1-u)^{1/2}}-\frac{\sigma^2}{2\theta(1-u)^{1/2}}\right)du\right)^2\\&\le c\left(\frac{1}{\theta^2\varepsilon_n}+\int_{t_{k-1}}^{t}\left(\frac{1}{\theta^2(1-u)^3}+\frac{\theta^2}{(1-u)}\right)du\right)\Delta t_k\le c\frac{l_n^3}{n}\end{aligned}$$

for some constant $c>0$. The limiting relationship (A.3) follows from the Lipschitz condition for the function Φ and the last inequality.

A4. Proof of (5.5). First, let us show that

$$I_1^n=-\frac{\theta\sigma S_1}{2}\int_{1-\delta_n}^{1-\varepsilon_n}\dot{U}_t(S_1)(w_1-w_t)\,dt+o_p(n^{-1/4}), \tag{A.5}$$

where $\delta_n=l_n/\theta^2$ and $\varepsilon_n=1/(l_n\theta^2)$. For this we introduce the set

$$\Gamma_L=\left\{\inf_{0\le t\le 1}S_t\ge L^{-1},\ \sup_{0\le t\le 1}S_t\le L\right\}. \tag{A.6}$$

Notice that for $L^{-1} < s < L$ and for $0 \le t \le 1-\delta_n$,

$$d(t,s,\theta) = \frac{\eta(s)}{\theta(1-t)^{1/2}} + \frac{\theta}{2}(1-t)^{1/2} \ge \frac{\sqrt{l_n}}{2} - \frac{\ln LK}{\sqrt{l_n}} \ge \frac{\sqrt{l_n}}{4}$$

for sufficiently large n. Therefore, according to (5.3), on the set Γ_L, for $0 \le t \le 1-\delta_n$,

$$|U_t(S_t)| + |U_t(S_1)| \le L\frac{\varphi(\sqrt{l_n}/4)}{\sqrt{\delta_n}}.$$

Thus taking into account that $\lim_{L\to\infty} \mathbf{P}(\Gamma_L^c) = 0$, we get that

$$\mathbf{P} - \lim_{n\to\infty} n^{1/2} \int_0^{1-\delta_n} \big(U_t(S_1) - U_t(S_t)\big)\,dt = 0. \tag{A.7}$$

Furthermore, since

$$\dot{U}_t(s) = \frac{\varphi(d(t,s,\theta))}{\sqrt{1-t}} + \frac{\dot{\varphi}(d(t,s,\theta))}{\theta(1-t)},$$

we get

$$\begin{aligned}
&\mathbf{E}\int_{1-\delta_n}^1 |U_t(S_1) - U_t(S_t)|\,dt \\
&\qquad \le c\int_{1-\delta_n}^1 \bigg(\frac{1}{\sqrt{1-t}} + \frac{1}{\theta(1-t)}\bigg)\mathbf{E}|S_1 - S_t|\,dt \le c\big(\delta_n + \sqrt{\delta_n}/\theta\big)
\end{aligned}$$

for some constant $c > 0$. From here, taking the limiting relationship (A.7) into account, we can represent $I_1^n(S)$ in the following asymptotic form:

$$I_1^n(S) = -\frac{\theta}{2}\int_{1-\delta_n}^1 \big(U_t(S_1) - U_t(S_t)\big)\,dt + o_p(n^{-1/4}).$$

By the Itô formula, we have

$$U_t(S_1) - U_t(S_t) = \frac{\sigma^2}{2}\int_t^1 S_u^2 \ddot{U}_t(S_u)\,du + \sigma\int_t^1 S_u \dot{U}_t(S_u)\,dw_u,$$

where

$$\ddot{U}_t(s) = \frac{\dot{\varphi}(d(t,s,\theta))}{s\theta(1-t)} + \frac{\ddot{\varphi}(d(t,s,\theta))}{s\theta^2(1-t)^{3/2}}.$$

Notice furthermore that on the Γ_L,

$$\begin{aligned}
&\int_{1-\delta_n}^1 \int_t^1 S_u^2 \big|\ddot{U}_t(S_u)\big|\,du\,dt \\
&\qquad \le c\,L\int_{1-\delta_n}^1 \int_t^1 \bigg(\frac{1}{\theta(1-t)} + \frac{1}{\theta^2(1-t)^{3/2}}\bigg)\,du\,dt \le \frac{cLl_n}{n^{3/4}}
\end{aligned}$$

for some constant $c > 0$. It means that

$$\mathbf{P} - \lim_{n\to\infty} n^{1/2} \int_{1-\delta_n}^{1} \int_{t}^{1} S_u^2 \ddot{U}_t(S_u)\,du\,dt = 0\,.$$

Hence

$$I_1^n(S) = -\frac{\sigma\theta}{2} \int_{1-\delta_n}^{1} \int_{t}^{1} S_u \dot{U}_t(S_u)\,dw_u\,dt + o_p(n^{-1/4}).$$

By the same method we can obtain that

$$\begin{aligned}\int_{1-\delta_n}^{1} \int_{t}^{1} S_u \dot{U}_t(S_u)\,dw_u\,dt &= \int_{1-\delta_n}^{1} S_t \dot{U}_t(S_t)(w_1 - w_t)\,dt + o_p(n^{-1/2})\\ &= S_1 \int_{1-\delta_n}^{1} \dot{U}_t(S_1)\,(w_1 - w_t)\,dt + o_p(n^{-1/2})\\ &= S_1 \int_{1-\delta_n}^{1-\varepsilon_n} \dot{U}_t(S_1)\,(w_1 - w_t)\,dt + o_p(n^{-1/2}).\end{aligned}$$

From here (A.5) follows.

Let us rewrite the integral in the right-hand side of (A.5) as

$$\begin{aligned}\int_{1-\delta_n}^{1-\varepsilon_n} \dot{U}_t(S_1)(w_1 - w_t)\,dt &= \int_{\varepsilon_n}^{\delta_n} \dot{U}_{1-v}(S_1)(w_1 - w_{1-v})\,dv\\ &= \sum_{j=j_0}^{j_1} \int_{t_{j-1}}^{t_j} \dot{U}_{1-v}(S_1)(w_1 - w_{1-v})\,dv\\ &\quad - \int_{t_{j_0-1}}^{\varepsilon_n} \dot{U}_{1-v}(S_1)(w_1 - w_{1-v})\,dv\\ &\quad - \int_{\delta_n}^{t_{j_1}} \dot{U}_{1-v}(S_1)(w_1 - w_{1-v})\,dv,\end{aligned}$$

where

$$j_0 = \inf\{j \ge 1 : t_j \ge \varepsilon_n\} = [n\varepsilon_n] + 1 = m_0 + 1,$$
$$j_1 = \inf\{j \ge 1 : t_j \ge \delta_n\} = [n\delta_n] + 1 = m_1 + 1\,.$$

Taking into account that every component in this sum is $o_p(n^{-1/4})$ we obtain (5.5). □

A5. Limiting theorem for some integral sums. Let W be the Wiener process on [0, 1]. Let us consider the following random variables:

$$\xi_m = \left|A_1(\lambda_{m-1}, w_{m-1}) - A_2(\lambda_{m-1}, w_{m-1})y_m\right| - h(\lambda_{m-1}, w_{m-1}),$$

where $w_m = W_{t_m}$ and $y_m = (W_{t_m} - W_{t_{m-1}})\sqrt{n}$, $t_m = m/n$, $\lambda_m = \theta^2 t_m$ and

$$h(\lambda, z) = \int_{-\infty}^{+\infty} |A_1(\lambda, z) - A_2(\lambda, z)y|\varphi(y)\,dy,$$

$\varphi(\cdot)$ is the Gaussian(0, 1) density. We will suppose that the functions A_i satisfy the following conditions. The functions $A_i(\lambda, z)$ are differentiable with respect to z for $\lambda > 0$ and $-\infty < z < +\infty$ and there exist positive constants c and γ such that

$$\max\left(|A_i(\lambda, z)|, \left|\frac{\partial}{\partial z} A_i(\lambda, z)\right|\right) \leq c\,(\lambda^{\gamma} + \lambda^{-\gamma})(1 + |z|) \tag{A.8}$$

for all $\lambda > 0$ and $-\infty < z < +\infty$. We denote $a_i(\lambda) = A_i(\lambda, 0)$ $(i = 1, 2)$ and $h_0(\lambda) = h(\lambda, 0)$.

Let $q_i(\cdot)$ be bounded functions such that for any $-\infty < \alpha < +\infty$,

$$\int_0^{+\infty} q_1(\lambda)\lambda^{\alpha}\, d\lambda < \infty \quad \text{and} \quad \int_0^{+\infty} q_2^2(\lambda)\, d\lambda < \infty. \tag{A.9}$$

We will study the following sum:

$$N_n = \sum_{m=m_0}^{m_1} \varsigma_m,$$

where $\varsigma_m = (q_1(\lambda_{m-1})\xi_m + q_2(\lambda_{m-1}) y_m)\sqrt{\Delta\lambda_m}$.

LEMMA A.2. *Assume that conditions* (A.8) *and* (A.9) *are satisfied. Then*

$$N_n \Rightarrow \mathcal{N}(0, D) \qquad \text{as } n \to \infty,$$

where

$$D = \int_0^{+\infty} \big(q_1^2(\lambda) v_1^2(\lambda) + 2\, q_1(\lambda)\, q_2(\lambda) v_2(\lambda) + q_2^2(\lambda)\big)\, d\lambda$$

with

$$v_1^2(\lambda) = a_1^2(\lambda) + a_2^2(\lambda) - h_0^2(\lambda), \qquad v_2(\lambda) = \int_{-\infty}^{+\infty} |a_1(\lambda) - a_2(\lambda) y|\, y\, \varphi(y)\, dy.$$

PROOF. First notice that condition (A.7) enables that $|D| < \infty$. Furthermore, let us show that $D > 0$. Indeed, we can represent D as

$$D = \int_0^{+\infty} \big(\big(q_1(\lambda) v_1(\lambda) + q_2(\lambda)\rho(\lambda)\big)^2 + q_2^2(\lambda)\big(1 - \rho^2(\lambda)\big)\big)\, d\lambda,$$

where $\rho(\lambda) = v_2(\lambda)/v_1(\lambda)$. Notice that $\rho(\lambda) = \mathbf{E}(\tau(\lambda)\tau_0)/\sqrt{\mathbf{E}\tau^2(\lambda)}$, where $\tau_0 \sim \mathcal{N}(0, 1)$ and $\tau(\lambda) = |a_1(\lambda) - a_2(\lambda)\tau_0| - h_0(\lambda)$. By the Cauchy–Bunyakovskii–Schwarz inequality we get that $|\rho(\lambda)| \leq 1$. Thus $D \geq 0$. If $D = 0$ then there exists a constant $c \neq 0$ such that $\tau(\lambda) = c\tau_0$ a.s. It means that $|a_1(\lambda) - a_2(\lambda)\tau_0| = c\tau_0 + h_0(\lambda)$, but this is impossible, because τ_0 is a Gaussian random variable, thus $D > 0$.

Now, let us check conditions for the central limit theorem for martingales ([10], page 511) for the sequence $\{\varsigma_m, m_0 \leq m \leq m_1\}$. We need to check the following two conditions:

(C1) For all $\varepsilon > 0$,

$$\lim_{n\to 0} \sum_{m=m_0}^{m_1} \mathbf{E}\varsigma_m^2 \chi_{\{|\varsigma_m|\ge\varepsilon\}} = 0,$$

(C2)

$$\mathbf{P} - \lim_{n\to 0} \sum_{m=m_0}^{m_1} \mathbf{E}(\varsigma_m^2 | \mathcal{F}_{m-1}) = D,$$

where $\mathcal{F}_m = \sigma\{W_t,\, t \le t_m\}$.

Let us check now the first condition. Indeed, by Chebyshev's inequality we have that

$$\mathbf{E}\varsigma_m^2 \chi_{\{|\varsigma_m|\ge\varepsilon\}} \le \varepsilon^{-2}\mathbf{E}\varsigma_m^4 \le \frac{c}{\sqrt{n}}\big(\mathbf{E}\xi_m^4\, q_1^4(\lambda_{m-1}) + q_2^4(\lambda_{m-1})\big)\Delta\lambda_m.$$

Therefore, by condition (A.8) there exists $\gamma > 0$ such that for sufficiently large n,

$$\begin{aligned}\sum_{m=m_0}^{m_1} \mathbf{E}\varsigma_m^2 \chi_{\{|\varsigma_m|\ge\varepsilon\}} &\le \frac{c}{\sqrt{n}} \sum_{m_0}^{m_1} \big((\lambda_{m-1}^{\gamma} + \lambda_{m-1}^{-\gamma})q_1^4(\lambda_{m-1}) + q_2^4(\lambda_{m-1})\big)\Delta\lambda_m \\ &\le \frac{c}{\sqrt{n}} \int_0^{+\infty} \big((\lambda^{\gamma} + \lambda^{-\gamma})q_1^4(\lambda) + q_2^4(\lambda)\big)\, d\lambda < \infty\end{aligned}$$

and we get the needed convergence.

Finally, by (A.9) and taking into account that the functions A_i are differentiable, we obtain that

$$\begin{aligned}\mathbf{E}(\xi_m^2 | \mathcal{F}_{m-1}) &= v_1^2(\lambda_{m-1}) + \big(1 + o(1)\big), \\ \mathbf{E}(\xi_m y_m | \mathcal{F}_{m-1}) &= v_2(\lambda_{m-1}) + \big(1 + o(1)\big),\end{aligned}$$

because $\mathbf{E}W_{t_m}^2 = t_m = \theta^{-2}\lambda_m \le \lambda_{m_1}/\theta^2 \le l_n/\theta^2 \to 0$. Then

$$\begin{aligned}&\mathbf{E}(\varsigma_m^2 | \mathcal{F}_{m-1}) \\ &\quad = \Big(q_1^2(\lambda_{m-1})v_1^2(\lambda_{m-1})\big(1 + o(1)\big) \\ &\qquad + 2q_1(\lambda_{m-1})q_2(\lambda_{m-1})v_2(\lambda_{m-1})\big(1 + o(1)\big) + q_2^2(\lambda_{m-1}) + o(1)\Big)\Delta\lambda_m.\end{aligned}$$

From here we obtain the second condition. This proves the lemma. □

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