The Annals of Applied Probability
2003, Vol. 13, No. 2, 742–773

THE SUPER-REPLICATION PROBLEM VIA PROBABILISTIC METHODS

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We study the minimal investment that is needed in order to super-replicate (i.e., hedge with certainty) continuous-time options under transaction costs. We deal with both exotic and path-independent European and American options. In all our examples we prove that the optimal strategy is the cheapest possible buy and hold. Our method is to study the problem in a discrete-time *shadow market* that is free of transaction costs where the options are perpetual. We also produce useful and precise estimates of potential capital gains in a transaction cost environment. We believe that our method is robust and has both theoretical and practical implications. One advantage of our approach, in contrast with the existing literature, is that we do not impose any trading strategies restrictions related to the no bankruptcy condition. Namely we allow hedging with unlimited borrowing and still the best one can do is buy and hold. Another advantage is that we do not assume that share prices are diffusions.

1. Introduction. This paper deals with finding the minimal investment that is needed, in the presence of transaction costs, to super-replicate options using their underlying financial assets (stocks and money accounts). In other words, the goal is to select, from all the portfolios that have values that exceed the payoff associated with the option, the one with the smallest initial value. The first interesting paper related to this problem was that by Bensaid, Lense, Pages and Scheinkman (1992). It showed that in discrete-time models, with transaction costs, super-replication of options is sometimes cheaper than exact replication. Davis and Clark (1994) formally conjectured that in the context of continuous-time Black–Scholes markets, the cheapest way to super-replicate a European style call option is by a trivial portfolio, that is, *buy and hold*. Soner, Shreve and Cvitanic (1995) proved the conjecture by analytic methods. In Levental and Skorohod (1997) the same result was proved by probabilistic methods for more general models and options. Recently (as we learned after finishing the first version on this paper), two papers that deal with path-independent European options were written: Cvitanic, Pham and Touzi (1998) studied the *one-dimensional* case (one underlying stock) and Bouchard and Touzi (1999) studied *multidimensional* issues (there are several underlying stocks). Both papers used analytic methods. In this paper we use

Received January 2001; revised April 2002.
AMS 2000 subject classifications. Primary 90A09, 60H30.
Key words and phrases. Transaction costs, super-replication, hedging options.

exclusively probabilistic methods. Our main idea is that if an investor can super-replicate an option in a continuous-time model with transaction costs, then it is possible with *even* smaller initial investment to hedge a *perpetual* discrete-time related option in a market which is *free* of transaction costs where share prices are martingales. The *financial insight* here is crystal clear: hedging options in continuous-time with transaction costs is roughly equivalent to hedging perpetual options in discrete-time models without transaction costs.

Our approach is simple, yet powerful, and has both theoretical and practical implications. It has two main advantages over the above-mentioned papers that use analytic methods. The first one has, in our opinion, financial importance. We do not assume any restrictions on potential losses of the portfolios. So even if the investor has an unlimited credit line, s/he will still need to resort to trivial hedging in all our examples. *Doubling schemes* are not going to help! In the literature (with the exception of Levental and Skorohod, 1997) mentioned above, there is always a restriction on potential hedging losses (*no bankruptcy condition*) and the reader might suspect that if those restrictions were eliminated, new opportunities of super-replication would appear. We show that this clearly is not the case. The second advantage is that we do not need to assume, as others have done, that share prices are represented by diffusions. Rather, we use nondegenerate continuous processes as share prices.

Our method is robust and to demonstrate this point we give examples of all sorts of options and models. We deal with European and American options, with exotic and path-independent options, with two-sided and one-sided transaction costs, and with payoff function that is defined in terms of the portfolio value (stock account + money account), or a pair of values (stock account, money account).

In all the examples that we bring here, the cheapest way to super-replicate is buy-and-hold. In this paper we do not deal either with options whose optimal solution is not buy-and-hold or with multi-dimensional options. Our approach can contribute to the super-replication problem of those options as well, but due to lengthy considerations, we will deal with these questions in a subsequent paper. In this regard, see Levental and Ryznar (2000).

Finally we describe the organization of the paper. In Section 2 we describe the model and state the main results. In Section 3 we provide useful and precise estimates of the potential capital gain when the stock price does not change by much. Section 4 contains the reduction of the problem to a discrete-time, transaction costs free, model. Section 5 generalizes the result to transaction costs that are one-sided. Section 6 deals with one-dimensional path-independent options. The last section, Section 7, is devoted to examples.

2. Model and main results. We consider a financial market in which one stock is traded in the time interval $0 \leq t < \infty$. The price of this stock is represented by a stochastic process $Z = \{Z(t) : 0 \leq t < \infty\}$, which is defined on a complete probability space (Ω, F, P). We will take Z to be a positive continuous process with respect to a filtration $\{F_t : 0 \le t < \infty\}$ that is right continuous and such that F_t contains all P null sets, $0 \le t < \infty$, and F_0 is a trivial σ-algebra. For ease of presentation we will assume that the interest rate equals to 0. In the model we present below these assumption entails no loss of generality. We will assume that Z both fluctuates and does not fluctuate with positive probability. To define this precisely we need some notation. Let $0 \le \tau < \infty$ be a stopping time and $\delta > 0$. We define

$$\tau^\delta = \begin{cases} \inf\{t : \tau \le t < \infty : Z(t) = Z(\tau)e^{\pm\delta}\}, \\ \infty, \quad \text{if no such } t \text{ exists.} \end{cases}$$

The following basic assumption about the process $Z(u), u \ge 0$, will hold throughout the paper.

ASSUMPTION 1. There exists δ_0 such that for all stopping times $0 \le \tau < \infty$, $0 < \delta < \delta_0$ and $0 < a < b < \infty$, we have, on the event $\{\tau < a\}$, a.s.:

$$P\big(a < \tau^\delta < b,\ Z(\tau^\delta) = Z(\tau)e^{\delta} | \mathcal{F}_\tau\big) > 0$$

and

$$P\big(a < \tau^\delta < b,\ Z(\tau^\delta) = Z(\tau)e^{-\delta} | \mathcal{F}_\tau\big) > 0.$$

This assumption holds for most market models including nondegenerate diffusions. Let us point out that Assumption 1 implies that on the event $\{\tau < a\}$, almost surely:

$$P(\tau^\delta > a | \mathcal{F}_\tau) > 0.$$

In the sequel we will work with the following sequence of stopping times which is defined for every $\delta > 0$:

$$\begin{aligned} &\tau_0 = \tau_0(\delta) = 0, \\ &\tau_n = \tau_n(\delta) = \inf\{\tau_{n-1} \le t : Z(t) = Z(\tau_{n-1})e^{\pm\delta}\}, \quad n \ge 1, \end{aligned} \tag{1}$$

with an obvious convention that if $\tau_k = \infty$ then $\tau_m = \infty$, for $m \ge k$. From now on we use this notation consistently so $\{\tau_n\}$ will always mean stopping times defined by (1) allowing the context to determine δ.

The following definition will be used throughout.

DEFINITION 1. Let $\delta > 0$. A sequence of random variables $\{Z_k\}$ is called δ-predictable if $Z_k \in F_{\tau_{k-1}}, k \ge 1$, where the sequence $\{\tau_k\}$ is defined above.

Next we define admissible portfolios and their respective capital gains in our model.

DEFINITION 2. A portfolio is an adapted cad-lag stochastic process $M = \{M(t) : t \ge 0\}$, which satisfies

$$P\left(\int_0^T |dM|(t) < \infty\right) = 1, \qquad 0 < T < \infty.$$

We denote the class of all portfolios by FV.

Let $M \in FV$. $M(t)$ represents the number of shares at time t. We define two nondecreasing processes M^+ and M^-, which are associated with M:

$$M^+(t) = \frac{M(0) + M(t) + \int_0^t |dM|(s)}{2},$$

$$M^-(t) = \frac{M(0) + M(t) + \int_0^t |dM|(s)}{2}.$$

The process $M^+(t)(M^-(t))$ represents the accumulated number of shares bought (sold) by the portfolio M up to time t, respectively.

Let $0 \le \lambda$, $\mu \le 1$ represent the fractional transaction costs when buying-λ or selling-μ.

DEFINITION 3. The capital gain generated by a portfolio M is a stochastic process $\{S_M(t) : t \ge 0\}$ defined by

$$S_M(t) = \int_0^t M(u)\, dZ(u) - \lambda \int_0^t Z(u)\, dM^+(u) - \mu \int_0^t Z(u)\, dM^-(u).$$

REMARK 1. For simplicity it is assumed that no transaction costs are being paid due to holding of $M(0)$ shares at time $t = 0$. However see Remark 2 below on the possibility of modifying our results if one changes this assumption

To every option we associate a payoff function. Throughout we deal with options which expire at $T = 1$. Let $G(t)$, $0 \le t \le 1$, be an adapted stochastic process that represents that payoff. In the paper we consider either path-independent options or exotic options. By a path-independent option we mean an option with $G(t) = g(Z(t))$, where $g : (0, \infty) \to R$ is a measurable function. The classical example is a *call option*, where $g(z) = [z - k]^+$ and $k > 0$. By an exotic option we mean a situation when the function $G(t)$, $0 \le t \le 1$, depends not only on $Z(t)$ but on the whole path of the process Z up to time t. One of the most interesting examples is the so-called *Asian type option*, where $G(t) = g((1/t)\int_0^t Z(u)\, du)$. Next we define the initial investment that is needed to super-replicate the option $G(t)$ with a portfolio $M \in FV$, where (A) is for the American style option and (B) is for the European style option. Let ST[0, 1] denote all stopping times such that $0 \le \tau \le 1$. Then

(A) $\quad y_M = \inf\{x \in R : x + S_M(\tau) \ge G(\tau) \text{ a.s., } \tau \in \mathrm{ST}[0, 1]\},$

(B) $\quad x_M = \inf\{x \in R : x + S_M(1) \ge G(1) \text{ a.s.}\}.$

Both x_M and y_M will be taken to be ∞ if the sets are empty.

DEFINITION 4. We define the super-replication cost of the European option $G(t)$ to be

$$b_E = b_E(G) = \inf\{x_M : M \in FV\}.$$

In the case of the American option the super-replication cost will be

$$b_A = b_A(G) = \inf\{y_M : M \in FV\}.$$

Observe that $b_E \le b_A$ since $x_M \le y_M$ for every $M \in FV$.

Economic interpretation. The hedging portfolio is actually a pair $(B(t), M(t))$, where $M \in FV$ represents the number of shares as before while B represents the money in a money market account. The value of the portfolio is $V(t) = M(t)Z(t) + B(t)$. We assume that the portfolio is *self-financing* which means that $V(t) = x + S_M(t)$, where x is the initial investment. When the payoff of the option is specified as a pair $(b(t), m(t))$, where $b(t)$ is money in the money market account and $m(t)$ is the the number of shares in the stock account, then the payoff $G(t)$ satisfies $G(t) = m(t)Z(t) + b(t)$. In this setup, Definition 4 makes economic sense if the option holder is willing to accept, at exercise time t, any portfolio $(B(t), M(t))$ as long as $V(t) = M(t)Z(t) + B(t) \ge m(t)Z(t) + b(t)$. Although this may be looked upon as an unrealistic assumption, it turns out that in proving results about super-replication with transaction costs, this is an efficient way of looking at the problem.

We also consider more complicated options of the form of a pair of functions $(G(t), m(t))$, where $G(t)$ is as above and $m(t)$ is another adapted process that represents the required number of shares at the exercise time t. Here the super-replication means that the portfolio value at time t after the transition to $m(t)$ shares still dominates $G(t)$. If as before the option payoff is specified as a pair $(b(t), m(t))$, then our setup is equivalent to demand that the hedging portfolio $(B(t), M(t))$ can be converted, after paying transaction costs, to a portfolio $(B'(t), M'(t))$ so that $B'(t) \ge b(t)$ and $M'(t) \ge m(t)$.

Now we define the initial investment that is needed to super-replicate the option $(G(t), m(t))$ with a portfolio $M \in FV$, where (A*) is for the American style option and (B*) is for the European style option:

$$\text{(A}^*\text{)}\qquad y_M^* = \inf\{x \in R : x + S_M(\tau) \ge G(\tau) + \lambda Z(\tau)[M(\tau) - m(\tau)]^- \\ + \mu Z(\tau)[M(\tau) - m(\tau)]^+ \text{ a.s., } \tau \in \mathrm{ST}[0, 1]\};$$

$$\text{(B}^*\text{)}\qquad x_M^* = \inf\{x \in R : x + S_M(1) \ge G(1) + \lambda Z(1)[M(1) - m(1)]^- \\ + \mu Z(1)[M(1) - m(1)]^+ \text{ a.s.}\}.$$

DEFINITION 5. We define the super-replication cost of the European option $(G(t), m(t))$ to be

$$b_E^* = b_E^*(G, m) = \inf\{x_M^* : M \in FV\}.$$

In the case of the American option the super-replication cost will be

$$b_A^* = b_A^*(G, m) = \inf\{y_M^* : M \in FV\}.$$

We deal with the second type options only in the case of path-independent options, that is, $(G(t), m(t))$ depends on t only through the share price $Z(t)$. One may conclude from the results of Section 6 that both methods of super-replication are equivalent in some sense.

In the paper we provide lower bounds for option super-replication prices. Our method is based on a discretization which transforms our problem to that of pricing options in a shadow market. This market is a binomial market and is free of transaction costs.

The shadow market. This is a discrete-time market with one stock. Let $z_0 > 0$ and fix $\delta > 0$. Let

$$\mathcal{Z}_\delta(n) = \left\{ \mathbf{z} = (z_0, \dots, z_n) : z_k = z_0 \exp\left(\delta \sum_{i=1}^{k} \varepsilon_i \right);\ \varepsilon_i = \pm 1 \right\}$$

and

$$\mathcal{Z}_\delta = \left\{ \mathbf{z} = (z_0, \dots) : z_k = z_0 \exp\left(\delta \sum_{i=1}^{k} \varepsilon_i \right);\ \varepsilon_i = \pm 1 \right\}.$$

The share price process in the shadow market is represented by the process $\{z_k\} \in \mathcal{Z}_\delta$. We will consider z_k, $k \geq 0$, as a random process defined on a product probability space $(\{\pm 1\}^\infty, \hat{P}_\delta)$, where

$$\hat{P}_\delta(\varepsilon_i = 1) = \frac{1 - e^{-\delta}}{e^\delta - e^{-\delta}}, \qquad \hat{P}_\delta(\varepsilon_i = -1) = \frac{e^\delta - 1}{e^\delta - e^{-\delta}}.$$

Note that $\{\varepsilon_i\}$ are i.i.d. under $\hat{P}_\delta$. By $\hat{E}_\delta$ denote the expectation with respect to $\hat{P}_\delta$. Let $\mathcal{F}_i = \sigma\{z_0, \dots, z_i\}$ be the natural filtration. Note that the sequence $\{z_i, \mathcal{F}_i\}$ is a positive martingale which satisfies the conditions

$$\hat{E}_\delta z_k = z_0 \quad \text{and} \quad \lim_{n \to \infty} z_n = 0, \qquad \hat{P}_\delta \text{ a.s.}$$

The last statement follows from $\hat{P}_\delta(\varepsilon_i = 1) < \hat{P}_\delta(\varepsilon_i = -1)$. We define three families of $\{\mathcal{F}_i\}$ stopping times:

$$\mathcal{S}_n(\delta) = \{\eta : \eta \leq n,\ \hat{P}_\delta \text{ a.s.}\},$$
$$\mathcal{S}_B(\delta) = \bigcup_n \mathcal{S}_n(\delta),$$
$$\mathcal{S}(\delta) = \{\eta : \eta < \infty,\ \hat{P}_\delta \text{ a.s.}\}.$$

The following definition will be used in the formulation of Theorems 1 and 2.

DEFINITION 6. Let $\delta > 0$ be fixed and let $\{\tau_k\}$ be a sequence defined in (1). Let $\hat{G}_k : \mathbf{R}^{k+1} \to \mathbf{R}$, $k \geq 1$, be a sequence of functions.

A. We will say that $\{\hat{G}_k\}$ underestimates G in the American sense if

$$P\big(G(\tau_k) \geq \hat{G}_k(Z(0), \dots, Z(\tau_k)),\ Z(\tau_k) = Z_k,\ 0 \leq k \leq n,\ \tau_n \leq 1\big) > 0. \tag{2}$$

B. We will say that $\{\hat{G}_k\}$ underestimates G in the European sense if

$$\begin{aligned} &P\big(G(\tau_n) \geq \hat{G}_n(Z(0), \dots, Z(\tau_n)),\ Z(\tau_k) = Z_k,\ 0 \leq k \leq n,\ \eta < \tau_n \leq 1\big) > 0, \\ &\qquad 0 < \eta < 1. \end{aligned} \tag{3}$$

Both (2) and (3) should hold for every $n \geq 1$ and any δ-predictable sequence $(Z_0, Z_1, \dots, Z_n) \in \mathcal{Z}_\delta(n)$.

The conditions in Definition 6 may not look very easy to verify, but usually $\{\hat{G}_k\}$ emerges in a very natural way as a discrete surrogate of G. Lemma 7 will serve us as a useful tool in verifying whether a certain sequence of functions underestimates G. In particular in the case of a path-independent option $G(t) = g(Z(t))$, $t \geq 0$, we can simply take $\hat{G}_k(z_0, \dots, z_k) = g(z_k)$ as we do in Section 6 We will demonstrate other examples in Section 7.

To prove results about European options we will need the following technical condition imposed on the payoff function $G(t)$: For every $n \geq 1$ and any $\varepsilon > 0$ there is $0 < \xi < 1$ such that on the event $\{\xi < \tau_n \leq 1\}$ we have a.s.

$$P\big(G(1) + \varepsilon \geq G(\tau_n), (\tau_n)^\beta \geq 1 | F_{\tau_n}\big) > 0 \qquad \text{for every } \beta > 0, \tag{4}$$

where $\{\tau_k\}$ is the sequence defined in (1).

Now we can state our results, which are organized as four theorems. Theorem 1 deals with options in the case of two-sided transaction costs. It gives an effective lower bound for the super-replication price for both the American and European types. In Examples 4 and 5 in Section 7 we show how to use Theorem 1 effectively. It is also a tool in proving Theorem 3.

THEOREM 1. *Assume that* $\min\{\lambda, \mu\} > 0$. *Let* $\delta > 0$ *and* $e^{2\delta} - 1 < \min\{\lambda, \mu\}$. *Let* $\hat{G}_k : \mathbf{R}^{k+1} \to \mathbf{R}$, $k \geq 1$, *be a sequence of functions.*

A. *If* $\{\hat{G}_k\}$ *underestimates G in the American sense, then*

$$b_A \geq \sup_{\eta \in \mathcal{S}_B(\delta)} \hat{E}_\delta \hat{G}_\eta.$$

Furthermore, if $\hat{G}_k \geq c$ *for every* $k \geq 1$, *where c is some constant, then*

$$b_A \geq \sup_{\eta \in \mathcal{S}(\delta)} \hat{E}_\delta \hat{G}_\eta.$$

B. *If* $\{\hat{G}_k\}$ *underestimates G in the European sense and G satisfies* (4), *then*

$$b_E \geq \sup_n \hat{E}_\delta \hat{G}_n.$$

Theorem 2 deals with options in the case of one-sided transaction costs. It gives an effective lower bound for the super-replication price for the American case only. As for the European case, it follows from Example 2 in Levental and Skorohod [(1997), page 435] that in our setup, where there are no restrictions on potential capital losses of the hedging portfolios, we may have $b_E = -\infty$. We apply Theorem 2 to prove the second part of Theorem 3.

THEOREM 2. *Let $\delta_0 > 0$. Suppose that for any $0 < \delta \le \delta_0$ there is a sequence of functions $\hat{G}_k = \hat{G}_k^\delta : \mathbf{R}^{k+1} \to \mathbf{R}$, $k \ge 1$, that underestimates G in the American sense* [*see* (2)].

A. *Let $\mu = 0$ and $\lambda > 0$. Define a sequence of functions*

$$\hat{H}_k = \hat{H}_k^\delta(z_0, \dots, z_k) = \min\{\hat{G}_k^\delta(z_0, \dots, z_k), \hat{G}_{k+3}^\delta(z_0, \dots, z_k, e^\delta z_k, e^{2\delta} z_k, e^{3\delta} z_k)\}.$$

Let $\eta \in \mathcal{S}_B(\delta)$ and $a_\eta(\delta) = \hat{E}_\delta \hat{H}_\eta^\delta$. Then

$$b_A \ge \limsup_{\delta \to 0} \sup_{\eta \in \mathcal{S}_B(\delta)} a_\eta(\delta).$$

Furthermore, if $\hat{G}_k^\delta \ge c$ for every $k \ge 1$ and some constant c, then

$$b_A \ge \limsup_{\delta \to 0} \sup_{\eta \in \mathcal{S}(\delta)} a_\eta(\delta).$$

B. *Let $\mu > 0$ and $\lambda = 0$. Define a sequence of functions*

$$\begin{aligned}\hat{H}_k &= \hat{H}_k^\delta(z_0, \dots, z_k)\\ &= \min\{\hat{G}_k^\delta(z_0, \dots, z_k), \hat{G}_{k+3}^\delta(z_0, \dots, z_k, e^{-\delta} z_k, e^{-2\delta} z_k, e^{-3\delta} z_k)\}.\end{aligned}$$

Let $\eta \in \mathcal{S}_B(\delta)$ and $a_\eta(\delta) = \hat{E}_\delta \hat{H}_\eta^\delta$. Then

$$b_A \ge \limsup_{\delta \to 0} \sup_{\eta \in \mathcal{S}_B(\delta)} a_\eta(\delta).$$

Furthermore, if $\hat{G}_k^\delta \ge c$ for every $k \ge 1$ and some constant c, then

$$b_A \ge \limsup_{\delta \to 0} \sup_{\eta \in \mathcal{S}(\delta)} a_\eta(\delta).$$

Now we present results related to path-independent options. Let the payoff function be of the form $G(t) = g(Z(t))$. Define the function

$$\bar{g}(z) = \sup\{\alpha g(z_1) + (1-\alpha) g(z_2) : 0 \le \alpha \le 1,\ z = \alpha z_1 + (1-\alpha) z_2,\ z_1, z_2 > 0\}.$$

THEOREM 3. *Assume that* $\inf_{z>0} g(z) > -\infty$ *and $g(z)$ is lower-semicontinuous. Let $z_0 = Z(0)$ be the initial share price. If* $\min\{\lambda, \mu\} > 0$, *then*

$$b_A(g) = b_E(g) = \bar{g}(z_0)$$

and the optimal super-replication strategy is buy and hold. *For one-sided transaction costs, that is,* $\min\{\lambda, \mu\} = 0$ *and* $\max\{\lambda, \mu\} > 0$, *we have*

$$b_A(g) = \bar{g}(z_0).$$

Finally we state a result about an option of the type $(g(Z(t)), m(Z(t)))$. First we need to introduce three functions related to $g(z)$ and $m(z)$:

$$\begin{aligned}
g_1(z) &= g\left(\frac{z}{1+\lambda}\right) + \frac{\lambda z}{1+\lambda} m\left(\frac{z}{1+\lambda}\right),\\
g_2(z) &= g\left(\frac{z}{1-\mu}\right) - \frac{\mu z}{1-\mu} m\left(\frac{z}{1-\mu}\right),\\
g_m(z) &= \max\{g_1(z), g_2(z)\}.
\end{aligned}$$

The next theorem should be compared to Cvitanic, Pham and Touzi (1999). In this regard the reader should keep in mind that, as explained in the Introduction, in our solution there are no restrictions on potential hedging losses and the share price is not necessarily represented by a diffusion.

THEOREM 4. *Let* $\max\{\lambda, \mu\} > 0$. *Assume that* $\inf_{z>0} g_m(z) > -\infty$ *and* $m(z)$, $b(z) = g(z) - zm(z)$ *are both lower-semicontinuous functions. Let* $z_0 = Z(0)$ *be the initial share price. Then*

$$b_A^*(g, m) = b_E^*(g, m) = \overline{g_m}(z_0)$$

and the optimal super-replication strategy is buy and hold.

REMARK 2. If the model is modified and one has to pay transaction costs for buying shares at $t = 0$, with parameter $\lambda^* \le \lambda$, then Theorems 3 and 4 are still correct with one change: In the conclusions replace z_0 by $z_0(1 + \lambda^*)$.

3. Useful inequalities. In the sequel we will need some precise estimates of the capital gain process over intervals when the price of the stock does not change much. We will have two major applications of these estimates. In Section 4 we use them to reduce the problem of super-replication in continuous-time models to that of the perpetual discrete model as explained in the Introduction. Later in Section 6 we use them to establish equivalence between super-replication of European and American path-independent options. We provide only the proof of Lemma 1A in its entirety since all the other proofs are similar. At this point we mention that our estimates are in the spirit of Lemma 3.2 in Levental and Skorohod (1997) and may be treated as extensions of that result. We recall that

$$S_M(t) = \int_0^t M(u)\, dZ(u) - \lambda \int_0^t Z(u)\, dM^+(u) - \mu \int_0^t Z(u)\, dM^-(u).$$

LEMMA 1. *Let* $0 \le \theta_1 \le \theta_2 < \infty$ *be two stopping times and let* $M \in FV$. *Assume that* $\mu = 0$ *and* $\lambda > 0$. *Let* $0 \le \xi \le \lambda$, $0 \le \rho \le \frac{\lambda-\xi}{1+\xi}$ *and* $\lambda' = \frac{\lambda-\xi}{1+\xi} - \rho$.

A. *Then on the event* $\{\frac{Z(\theta_1)}{1+\xi} \le Z(t) \le Z(\theta_1)(1+\rho),\ \theta_1 \le t \le \theta_2\}$ *we have*, *almost surely*,

$$S_M(\theta_2) - S_M(\theta_1) \le M(\theta_2)\bigl(Z(\theta_2) - Z(\theta_1)\bigr) - \lambda' Z(\theta_1)\bigl(M^+(\theta_2) - M^+(\theta_1)\bigr) \\ - \rho Z(\theta_1)\bigl(M(\theta_2) - M(\theta_1)\bigr).$$

B. *Then on the event* $\{\frac{Z(\theta_2)}{1+\xi} \le Z(t) \le Z(\theta_2)(1+\rho),\ \theta_1 \le t \le \theta_2\}$ *we have*, *almost surely*,

$$S_M(\theta_2) - S_M(\theta_1) \le M(\theta_1)\bigl(Z(\theta_2) - Z(\theta_1)\bigr) - \lambda' Z(\theta_2)\bigl(M^+(\theta_2) - M^+(\theta_1)\bigr) \\ - \rho Z(\theta_2)\bigl(M(\theta_2) - M(\theta_1)\bigr).$$

LEMMA 2. *Let* $0 \le \theta_1 \le \theta_2 < \infty$ *be two stopping times and let* $M \in FV$. *Assume that* $\mu > 0$ *and* $\lambda = 0$. *Let* $0 \le \xi \le \mu$, $0 \le \rho \le \frac{\mu-\xi}{1+\mu-\xi}$ *and* $\mu' = (\mu-\xi)(1-\rho) - \rho$.

A. *Then on the event* $\{Z(\theta_1)(1-\rho) \le Z(t) \le \frac{Z(\theta_1)}{1-\xi},\ \theta_1 \le t \le \theta_2\}$ *we have*, *almost surely*,

$$S_M(\theta_2) - S_M(\theta_1) \le M(\theta_2)\bigl(Z(\theta_2) - Z(\theta_1)\bigr) - \mu' Z(\theta_1)\bigl(M^-(\theta_2) - M^-(\theta_1)\bigr) \\ + \rho Z(\theta_1)\bigl(M(\theta_2) - M(\theta_1)\bigr).$$

B. *Then on the event* $\{Z(\theta_2)(1-\rho) \le Z(t) \le \frac{Z(\theta_2)}{1-\xi},\ \theta_1 \le t \le \theta_2\}$ *we have*, *almost surely*,

$$S_M(\theta_2) - S_M(\theta_1) \le M(\theta_1)\bigl(Z(\theta_2) - Z(\theta_1)\bigr) - \mu' Z(\theta_2)\bigl(M^-(\theta_2) - M^-(\theta_1)\bigr) \\ + \rho Z(\theta_2)\bigl(M(\theta_2) - M(\theta_1)\bigr).$$

The lemmas very easily yield the following corollaries for which proofs are omitted.

COROLLARY 1. *Let* $0 \le \theta_1 \le \theta_2 < \infty$ *be two stopping times and let* $M \in FV$. *Let* $0 \le \psi \le \mu$ *and* $0 \le \xi \le \lambda$. *Then on the event* $\{\frac{Z(\theta_2)}{1+\xi} \le Z(t) \le \frac{Z(\theta_2)}{1-\psi},\ \theta_1 \le t \le \theta_2\}$ *we have*, *almost surely*,

$$S_M(\theta_2) - S_M(\theta_1) \le M(\theta_1)\bigl(Z(\theta_2) - Z(\theta_1)\bigr) - \lambda' Z(\theta_2)\bigl(M^+(\theta_2) - M^+(\theta_1)\bigr) \\ - \mu' Z(\theta_2)\bigl(M^-(\theta_2) - M^-(\theta_1)\bigr),$$

where $\lambda' = \frac{\lambda-\xi}{1+\xi}$ *and* $\mu' = \frac{\mu-\psi}{1+\xi}$.

COROLLARY 2. *Let* $0 \le \theta_1 \le \theta_2 < \infty$ *be two stopping times and let* $M \in FV$.

A. *Assume that* $\mu = 0$ *and* $\lambda > 0$. *Let* $0 \le \xi \le \lambda$ *and* $0 \le \rho \le \frac{\lambda - \xi}{1+\xi}$. *Then on the event* $\{\frac{Z(\theta_2)}{1+\xi} \le Z(t) \le Z(\theta_2)(1+\rho),\ \theta_1 \le t \le \theta_2\}$ *we have*, *almost surely*,

$$\begin{aligned} &S_M(\theta_2) - S_M(\theta_1) \\ &\qquad \le (1+\rho)M(\theta_1)\big(Z(\theta_2) - Z(\theta_1)\big) - \rho\big(Z(\theta_2)M(\theta_2) - Z(\theta_1)M(\theta_1)\big). \end{aligned}$$

B. *Assume that* $\mu > 0$ *and* $\lambda = 0$. *Let* $0 \le \xi \le \mu$ *and* $0 \le \rho \le \frac{\mu - \xi}{1+\mu-\xi}$. *Then on the event* $\{Z(\theta_2)(1-\rho) \le Z(t) \le \frac{Z(\theta_2)}{1-\xi},\ \theta_1 \le t \le \theta_2\}$ *we have*, *almost surely*,

$$\begin{aligned} &S_M(\theta_2) - S_M(\theta_1) \\ &\qquad \le (1-\rho)M(\theta_1)\big(Z(\theta_2) - Z(\theta_1)\big) + \rho\big(Z(\theta_2)M(\theta_2) - Z(\theta_1)M(\theta_1)\big). \end{aligned}$$

PROOF OF LEMMA 1. A. We will use the integration by parts formula,

$$M(b)Z(b) - M(a)Z(a) = \int_a^b Z(s)\,dM(s) + \int_a^b M(s)\,dZ(s) \qquad \text{a.s.}, \tag{5}$$

where $0 \le a \le b \le 1$. Formula (5) holds path by path because it follows from the general integration by parts formula for any process that $Z(t)$ is continuous and $M(t) \in FV$; see Section VI.6 of Fomin and Kolmogorov (1957). Applying (5) we obtain

$$\int_{\theta_1}^{\theta_2} M(s)\,dZ(s) = M(\theta_2)\big(Z(\theta_2) - Z(\theta_1)\big) + \int_{\theta_1}^{\theta_2} \big(Z(\theta_1) - Z(s)\big)\,dM(s). \tag{6}$$

Note that

$$Z(\theta_1) - Z(s) \le \xi Z(s), \qquad Z(s) - Z(\theta_1) \le \rho Z(\theta_1) \qquad \text{for } \theta_1 \le s \le \theta_2.$$

The above estimates yields

$$\begin{aligned} &\int_{\theta_1}^{\theta_2} \big(Z(\theta_1) - Z(s)\big)\,dM(s) - \lambda \int_{\theta_1}^{\theta_2} Z(s)\,dM^+(s) \\ &\quad = \int_{\theta_1}^{\theta_2} \big(Z(\theta_1) - Z(s)\big)\,dM^+(s) - \lambda \int_{\theta_1}^{\theta_2} Z(s)\,dM^+(s) \\ &\qquad + \int_{\theta_1}^{\theta_2} \big(Z(s) - Z(\theta_1)\big)\,dM^-(s) \\ &\quad \le -(\lambda - \xi) \int_{\theta_1}^{\theta_2} Z(s)\,dM^+(s) + \rho \int_{\theta_1}^{\theta_2} Z(\theta_1)\,dM^-(s) \\ &\quad \le -\frac{\lambda - \xi}{1+\xi} Z(\theta_1) \int_{\theta_1}^{\theta_2} dM^+(s) + \rho \int_{\theta_1}^{\theta_2} Z(\theta_1)\,dM^-(s) \\ &\quad = -\bigg(\frac{\lambda - \xi}{1+\xi} - \rho\bigg) Z(\theta_1)\big(M^+(\theta_2) - M^+(\theta_1)\big) - \rho Z(\theta_1)\big(M(\theta_2) - M(\theta_1)\big) \end{aligned}$$

and the result follows from (6).

Part B. We use (5) to get

$$\int_{\theta_1}^{\theta_2} M(s)\,dZ(s) = M(\theta_1)\big(Z(\theta_2) - Z(\theta_1)\big) + \int_{\theta_1}^{\theta_2} \big(Z(\theta_2) - Z(s)\big)\,dM(s). \tag{7}$$

Using similar arguments as above and (7) we obtain the conclusion. □

4. Reduction to discrete-time perpetual American option. Throughout this section we assume that $\min\{\lambda, \mu\} > 0$. We start with a remark which is an obvious consequence of Corollary 1.

REMARK 3. Let $\delta > 0$ and $e^{2\delta} - 1 < \min\{\lambda, \mu\}$. Let $0 \le \theta_1 \le \theta_2 < \infty$ be two stopping times. Assume that $M \in FV$ and that $e^{-\delta} \le Z(t)/Z(\theta_1) \le e^{\delta}$, $\theta_1 \le t \le \theta_2$, a.s. Then

$$S_M(\theta_2) - S_M(\theta_1) \le M(\theta_1)\big(Z(\theta_2) - Z(\theta_1)\big), \qquad \text{a.s.}$$

The lemma below is crucial in proving Theorems 1 and 2. It is based on binomial representation. This binomial representation greatly simplifies Levental and Skorohod (1997) and also allows us to deal with exotic options without difficulties.

LEMMA 3. *Let $n \ge 1$. Let $\delta > 0$ and $\hat{G}_k : \mathbf{R}^{k+1} \to \mathbf{R}$, $0 \le k \le n$, be a sequence of functions.*

A. *Let $\eta \in \mathcal{S}_n(\delta)$. There exists a sequence of measurable functions $Z_k : \mathbf{R}^k \to \mathbf{R}^+$, $0 \le k \le n$, $Z_k = Z_k(M_0, \dots, M_{k-1})$ so that for every $(M_0, \dots, M_{n-1}) \in \mathbf{R}^n$, we have*

$$(Z_0, \dots, Z_n) \in \mathcal{Z}_\delta(n) \tag{8}$$

and

$$\max_{0 \le m \le n} \left\{ \hat{G}_m(Z_0, \dots, Z_m) - \sum_{k=0}^{m-1} M_k (Z_{k+1} - Z_k) \right\} \ge a_\eta, \tag{9}$$

where $a_\eta = \hat{E}_\delta \hat{G}_\eta(z_0, \dots, z_\eta)$.

B. *There exists a sequence of measurable functions $Z_k : \mathbf{R}^k \to \mathbf{R}^+$, $0 \le k \le n$, $Z_k = Z_k(M_0, \dots, M_{k-1})$ so that for every $(M_0, \dots, M_{n-1}) \in \mathbf{R}^n$ both (8) is satisfied and*

$$\hat{G}_n(Z_0, \dots, Z_n) - \sum_{k=0}^{n-1} M_k (Z_{k+1} - Z_k) \ge a_n, \tag{10}$$

where $a_n = \hat{E}_\delta \hat{G}_n(z_0, \dots, z_n)$.

PROOF. Observe that the random process $\hat{G}_k(z_0, \dots, z_k)$, $0 \le k \le n$, which is defined on the shadow market corresponding to δ, is adapted with respect to $\{\mathcal{F}_k\}$; hence, $\hat{G}_\eta$ is a well-defined random variable measurable with respect to $\mathcal{F}_\eta \subset \mathcal{F}_n$.

By the binomial representation property [see Harrison and Pliska (1981)] there exist $a_\eta \in \mathbf{R}$ and $M_k^* = M_k^*(z_0, \dots, z_k)$, $0 \le k \le n-1$, such that for every $\mathbf{z} \in \mathcal{Z}_\delta(n)$,

$$\hat{G}_\eta = a_\eta + \sum_{k=0}^{n-1} M_k^*(z_{k+1} - z_k).$$

Observe that $\{M_k^*\}$ depends on η. Also, this representation of $\hat{G}_\eta$ is unique (it does depend on δ). Since $M_k^* \in \mathcal{F}_k$ and $\{z_k, \mathcal{F}_k\}$ is a martingale it follows that $a_\eta = \hat{E}_\delta \hat{G}_\eta$. Next observe that

$$\hat{E}_\delta[\mathbf{1}_{\{\eta \le k\}} M_k^*(z_{k+1} - z_k) | \mathcal{F}_\eta] = \hat{E}_\delta\big[\hat{E}_\delta[\mathbf{1}_{\{\eta \le k\}} M_k^*(z_{k+1} - z_k) | \mathcal{F}_k] \mathcal{F}_\eta\big] = 0.$$

Hence from the fact that $\hat{G}_\eta$ is $\mathcal{F}_\eta$-measurable it follows that

$$\hat{G}_\eta = \hat{E}_\delta[\hat{G}_\eta | \mathcal{F}_\eta] = a_\eta + \sum_{k=0}^{\eta-1} M_k^*(z_{k+1} - z_k),$$

which implies that for any $\mathbf{z} \in \mathcal{Z}_\delta(n)$,

$$\max_{0 \le m \le n} \left\{ \hat{G}_m(z_0, \dots, z_m) - \sum_{k=0}^{m-1} M_k^*(z_{k+1} - z_k) \right\} \ge a_\eta. \tag{11}$$

Now we are going to determine $\{Z_k\}$. For every $(M_0, \dots, M_{n-1}) \in \mathbf{R}^n$, we define Z_k as follows: $Z_0 = z_0$ and

$$Z_{k+1} = \begin{cases} Z_k e^{\delta}, & \text{if } M_k < M_k^*, \\ Z_k e^{-\delta}, & \text{if } M_k \ge M_k^*, \end{cases} \qquad 0 \le k \le n-1.$$

With this choice of Z_k we get

$$\begin{aligned} &\sum_{k=0}^{m} M_k^*(Z_{k+1} - Z_k) - \sum_{k=0}^{m} M_k(Z_{k+1} - Z_k) \\ &\qquad = \sum_{k=0}^{m} (M_k^* - M_k)(Z_{k+1} - Z_k) \ge 0 \qquad \text{for } 0 \le m \le n-1. \end{aligned} \tag{12}$$

This combined with (11) implies that

$$\max_{0 \le m \le n} \left\{ \hat{G}_m(Z_0, \dots, Z_m) - \sum_{k=0}^{m-1} M_k(Z_{k+1} - Z_k) \right\} \ge a_\eta.$$

To get (10), we repeat the proof with $\eta = n$. By the binomial representation property there exist $a_n \in \mathbf{R}$, $a_n = \hat{E}_\delta \hat{G}_n$ and $M_k^* = M_k^*(z_0, \dots, z_k)$, $0 \le k \le n-1$, such that for every $\mathbf{z} \in \mathcal{Z}_\delta(n)$,

$$\hat{G}_n = a_n + \sum_{k=0}^{n-1} M_k^*(z_{k+1} - z_k). \tag{13}$$

Then we will construct $\{Z_k\}$ as above. Observe that (12) will hold with $m = n-1$, so combined with (13) will yield

$$\hat{G}_n(Z_0, \dots, Z_n) - \sum_{k=0}^{n-1} M_k(Z_{k+1} - Z_k) \ge a_n. \qquad \square$$

PROOF OF THEOREM 1. A. Let $n \ge 1$, $M \in FV$ and $\eta \in \mathcal{S}_n(\delta)$. By Remark 3, on the event $\{\tau_n \le 1\}$, we have

$$G(\tau_k) - S_M(\tau_k) \ge G(\tau_k) - \sum_{i=0}^{k-1} M(\tau_i)\big(Z(\tau_{i+1}) - Z(\tau_i)\big), \qquad 0 \le k \le n \text{ a.s.} \tag{14}$$

By Lemma 3A there are measurable functions $Z_k : \mathbf{R}^k \to \mathbf{R}^+$, $0 \le k \le n$, so that

$$\max_{0 \le m \le n} \left\{ \hat{G}_m - \sum_{k=0}^{m-1} M(\tau_k)(Z_{k+1} - Z_k) \right\} \ge a_\eta \qquad \text{a.s.,} \tag{15}$$

where $a_\eta = \hat{E}_\delta \hat{G}_\eta$, $Z_k = Z_k(M(\tau_0), \dots, M(\tau_{k-1}))$, $\hat{G}_m$ are evaluated at $\{Z_k\}$ and (8) holds. Note that $\{Z_k\}$ is δ-predictable. Denote

$$B = \big\{ Z(\tau_k) = Z_k,\ 0 \le k \le n,\ \tau_n \le 1 \big\}.$$

Since $\{\hat{G}_k\}$ underestimates G in the American sense, then by (2) and (14) we get

$$\begin{aligned} P\Bigg(B, &\max_{0 \le m \le n} \{G(\tau_m) - S_M(\tau_m)\} \\ &\ge \max_{0 \le m \le n} \left\{ \hat{G}_m - \sum_{k=0}^{m-1} M(\tau_k)(Z_{k+1} - Z_k) \right\} \Bigg) > 0. \end{aligned} \tag{16}$$

Combining (15) and (16) we obtain

$$P\left(\max_{0 \le m \le n} \{G(\tau_m) - S_M(\tau_m)\} \ge a_\eta,\ \tau_n \le 1 \right) > 0.$$

This shows that the initial investment y_M [required to dominate $G(t)$] must satisfy $y_M \ge a_\eta$, which completes the proof of the first part of **A**. If $\hat{G}_k \ge c$, $k \ge 0$, then by Fatou's lemma $\liminf_{n \to \infty} \hat{E}_\delta \hat{G}_{n \wedge \eta} \ge \hat{E}_\delta \hat{G}_\eta$, $\eta \in \mathcal{S}(\delta)$, which completes the proof of the second part of part A.

B. Let $n \geq 1$ and $M \in FV$. As before, by Remark 3, on the event $\{\tau_n \leq 1\}$ we have

$$G(\tau_n) - S_M(\tau_n) \geq G(\tau_n) - \sum_{i=0}^{n-1} M(\tau_i)\big(Z(\tau_{i+1}) - Z(\tau_i)\big) \qquad \text{a.s.} \tag{17}$$

By Lemma 3B there are measurable functions $Z_k : \mathbf{R}^k \to \mathbf{R}^+$, $0 \leq k \leq n$, so that

$$\hat{G}_n - \sum_{k=0}^{n-1} M(\tau_k)(Z_{k+1} - Z_k) \geq a_n \qquad \text{a.s.,} \tag{18}$$

where $Z_k = Z_k(M(\tau_0), \dots, M(\tau_{k-1}))$, $\hat{G}_n$ is evaluated at $\{Z_k\}$ and (8) holds. Note that $\{Z_k\}$ is δ-predictable. For $0 < \xi < 1$ denote

$$B = B(\xi) = \big\{Z(\tau_k) = Z_k,\ 0 \leq k \leq n,\ \xi \leq \tau_n \leq 1\big\}.$$

Since $\{\hat{G}_k\}$ underestimates G in the European sense, then by (3) and (17) we get

$$P\left(B, G(\tau_n) - S_M(\tau_n) \geq \hat{G}_n - \sum_{k=0}^{n-1} M(\tau_k)(Z_{k+1} - Z_k)\right) > 0. \tag{19}$$

Combining (18) and (19) we obtain

$$P\big(G(\tau_n) \geq a_n + S_M(\tau_n),\ \xi \leq \tau_n \leq 1\big) > 0, \qquad 0 < \xi < 1.$$

Now let $\varepsilon > 0$ and pick $\xi = \xi(\varepsilon)$ such that (4) holds. Next choose M^* such that the event $C = \{G(\tau_n) \geq a_n + S_M(\tau_n),\ \xi \leq \tau_n \leq 1,\ |M(\tau_n)| \leq M^*\}$ still has positive probability. Note that $C \in F_{\tau_n}$ so for any $\beta > 0$, by (4), we infer that

$$\begin{aligned} &P\big(G(1) + \varepsilon \geq a_n + S_M(\tau_n),\ \xi \leq \tau_n \leq 1 \leq (\tau_n)^\beta,\ |M(\tau_n)| \leq M^*\big) > 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 < \xi < 1. \end{aligned} \tag{20}$$

Next, on the event $\{\tau_n \leq 1 \leq (\tau_n)^\beta,\ |M(\tau_n)| \leq M^*\}$, by Corollary 1, we get for sufficiently small $\beta > 0$,

$$\begin{aligned} S(1) - S_M(\tau_n) &\leq M(\tau_n)\big(Z(1) - Z(\tau_n)\big) \leq M^*|Z(1) - Z(\tau_n)| \\ &\leq M^* Z(\tau_n)(e^\beta - 1) \leq M^* z_0 e^{n\delta}(e^\beta - 1). \end{aligned}$$

Combining this with (20) we obtain

$$P\big(G(1) \geq a_n - \varepsilon - M^* z_0 e^{n\delta}(e^\beta - 1) + S_M(1)\big) > 0.$$

This shows that the initial investment x_M [required to dominate $G(t)$ at $t = 1$] must satisfy $x_M \geq a_n - \varepsilon - M^* z_0 e^{n\delta}(e^\beta - 1)$. Since β, ε can be arbitrarily close to 0, we conclude that $b_E \geq a_n$, which completes the proof. $\square$

5. One-sided transaction costs. In this section we assume that $\mu = 0$ and $\lambda > 0$ or $\mu > 0$ and $\lambda = 0$. Our aim is to generalize Theorem 1 to the case of one-sided transaction costs. Let us recall that

$$S_M(t) = \int_0^t M(u)\,dZ(u) - \lambda \int_0^t Z(u)\,dM^+(u) - \mu \int_0^t Z(u)\,dM^-(u).$$

We start with two lemmas which enable the discretization of the problem.

LEMMA 4. *Assume that* $\mu = 0$ *and* $\lambda > 0$. *Let* $\delta > 0$ *and* $\alpha = e^{2\delta} - 1 < 1$. *Assume that* $e^{3\delta} - 1 \geq \frac{\alpha}{1-\alpha}$ *and* $e^{4\delta} \leq 1 + \lambda$. *If* $\{\tau_i\}$ *is the sequence defined in* (1), *then on the event* $\{M(\tau_n) \geq 0\}$ *we have*

$$S_M(\tau_n) \leq \sum_{i=0}^{n-1} \frac{M(\tau_i)}{1-\alpha}\big(Z(\tau_{i+1}) - Z(\tau_i)\big) + \frac{\alpha}{1-\alpha} M(0)Z(0) \qquad a.s. \tag{21}$$

On the event $\{M(\tau_n) < 0,\ Z(\tau_{n+3}) = e^{3\delta} Z(\tau_n)\}$ *we obtain*

$$S_M(\tau_{n+3}) \leq \sum_{i=0}^{n-1} \frac{M(\tau_i)}{1-\alpha}\big(Z(\tau_{i+1}) - Z(\tau_i)\big) + \frac{\alpha}{1-\alpha} M(0)Z(0) \qquad a.s. \tag{22}$$

PROOF. From Corollary 2A with $\rho = \frac{\alpha}{1-\alpha}$ we infer that, almost surely,

$$\begin{aligned} S_M(\tau_n) \leq {} & \sum_{i=0}^{n-1} \frac{M(\tau_i)}{1-\alpha}\big(Z(\tau_{i+1}) - Z(\tau_i)\big) + \frac{\alpha}{1-\alpha} M(0)Z(0) \\ & - \frac{\alpha}{1-\alpha} M(\tau_n)Z(\tau_n), \end{aligned} \tag{23}$$

so inequality (21) is obvious if $M(\tau_n) \geq 0$.

Now suppose that $M(\tau_n) < 0$ and $Z(\tau_{n+3}) = e^{3\delta} Z(\tau_n)$. Note that in this case

$$\frac{Z(\tau_{n+3})}{1+\lambda} \leq Z(t) \leq Z(\tau_{n+3}), \qquad \tau_n \leq t \leq \tau_{n+3},$$

so we can apply Lemma 1B with $\rho = 0$, $\theta_1 = \tau_n$ and $\theta_2 = \tau_{n+3}$ to get

$$S_M(\tau_{n+3}) - S_M(\tau_n) \leq M(\tau_n)\big(Z(\tau_{n+3}) - Z(\tau_n)\big) = M(\tau_n)(e^{3\delta} - 1)Z(\tau_n).$$

Next, combine this with (23) to obtain

$$\begin{aligned} S_M(\tau_{n+3}) &= S_M(\tau_n) + S_M(\tau_{n+3}) - S_M(\tau_n) \\ &\leq \sum_{i=0}^{n-1} \frac{M(\tau_i)}{1-\alpha}\big(Z(\tau_{i+1}) - Z(\tau_i)\big) + \frac{\alpha}{1-\alpha} M(0)Z(0) \\ &\quad + \left(e^{3\delta} - 1 - \frac{\alpha}{1-\alpha}\right) M(\tau_n)Z(\tau_n). \end{aligned}$$

Hence (22) follows since $(e^{3\delta} - 1 - \frac{\alpha}{1-\alpha})M(\tau_n)Z(\tau_n) \leq 0$. $\square$

A similar lemma as above can be proved in the case $\mu > 0$ and $\lambda = 0$. Its proof goes as the proof of the previous lemma, but we have to use Corollary 2B and then Lemma 2B to repeat the arguments we used above. Hence we omit the proof.

LEMMA 5. *Assume that* $\mu > 0$ *and* $\lambda = 0$. *Let* $\delta > 0$ *and* $\alpha = e^{2\delta} - 1$. *Assume that* $e^{3\delta} - 1 \geq \frac{\alpha}{1-\alpha}$ *and* $e^{4\delta} \leq (1-\mu)^{-1}$. *If* $\{\tau_i\}$ *is the sequence defined in* (1), *then on the event* $\{M(\tau_n) \leq 0\}$ *we have*

$$S_M(\tau_n) \leq \sum_{i=0}^{n-1} \frac{M(\tau_i)}{1+\alpha}\big(Z(\tau_{i+1}) - Z(\tau_i)\big) - \frac{\alpha}{1+\alpha} M(0)Z(0) \qquad a.s. \tag{24}$$

On the event $\{M(\tau_n) \geq 0,\ Z(\tau_{n+3}) = e^{-3\delta} Z(\tau_n)\}$ *we obtain*

$$S_M(\tau_{n+3}) \leq \sum_{i=0}^{n-1} \frac{M(\tau_i)}{1+\alpha}\big(Z(\tau_{i+1}) - Z(\tau_i)\big) - \frac{\alpha}{1+\alpha} M(0)Z(0) \qquad a.s. \tag{25}$$

Now we are in a position to prove an extension of Theorem 1 to the case of one-sided transaction costs.

PROOF OF THEOREM 2. A. Pick $0 < \delta \leq \delta_0$ which satisfies the assumptions of Lemma 4. Let $n \geq 1$, $M \in FV$ and $\eta \in \mathcal{S}_n(\delta)$. Denote $a_\eta = \hat{E}_\delta \hat{H}_\eta$. Let $\varepsilon > 0$ and assume that $\alpha = e^{2\delta} - 1$ is so small that

$$\frac{\alpha}{1-\alpha} M(0)Z(0) \leq \varepsilon.$$

By Lemma 3 applied to $\frac{M}{1-\alpha}$ and $\hat{H}_k$ there is a sequence of δ-predictable random variables $\tilde{Z}_k = \tilde{Z}_k(M(\tau_0), \dots, M(\tau_{k-1}))$, $0 \leq k \leq n$, so that

$$\max_{0 \leq m \leq n} \left\{ \hat{H}_m - \sum_{k=0}^{m-1} M(\tau_k)(\tilde{Z}_{k+1} - \tilde{Z}_k) \right\} \geq a_\eta \qquad \text{a.s.,} \tag{26}$$

where $(\tilde{Z}_0, \dots, \tilde{Z}_n) \in \mathcal{Z}_\delta(n)$ and $\hat{H}_m$ are evaluated at $\{\tilde{Z}_k\}$. We need a slight modification of $\{\tilde{Z}_i\}$. Let

$$m^* = \inf\left\{ m : \hat{H}_m - \sum_{k=0}^{m-1} M(\tau_k)(\tilde{Z}_{k+1} - \tilde{Z}_k) \geq a_\eta \right\}.$$

Now we are ready to define a new δ-predictable sequence $\{Z_i,\ 0 \leq i \leq n+3\}$:

$$Z_k = \tilde{Z}_k, \qquad 0 \leq k \leq m^*, \quad \text{and} \quad Z_{m^*+1} = e^{\delta} \tilde{Z}_{m^*}, \dots, Z_{n+3} = e^{(n-m^*+3)\delta} \tilde{Z}_{m^*}.$$

The new sequence is the same as the old one up to m^* and then it increases by the factor e^δ. Note that $\{Z_k\} \in \mathcal{Z}_\delta(n+3)$ is δ-predictable. Now if $M(\tau_{m^*}) \geq 0$ and $Z_k = Z(\tau_k)$, $0 \leq k \leq n+3$, then Lemma 4 and (26) yield

$$\hat{G}_{m^*} - S_M(\tau_{m^*}) \geq \hat{H}_{m^*} - S_M(\tau_{m^*}) \geq a_\eta - \frac{\alpha}{1-\alpha} M(0)Z(0) \geq a_\eta - \varepsilon, \tag{27}$$

where $\hat{H}_{m^*}$ and $\hat{G}_{m^*}$ are evaluated at $\{Z_k\}$. Next if $M(\tau_{m^*}) < 0$ and $Z_k = Z(\tau_k)$, $0 \le k \le n+3$, we have $Z(\tau_{m^*+3}) = e^{3\delta} Z(\tau_{m^*})$, so the second part of Lemma 4 gives

$$\hat{H}_{m^*} - S_M(\tau_{m^*+3}) \ge a_\eta - \frac{\alpha}{1-\alpha} M(0)Z(0) \ge a_\eta - \varepsilon.$$

Observe that $Z(\tau_{m^*+3}) = e^{3\delta} Z(\tau_{m^*})$ implies $\hat{H}_{m^*} \le \hat{G}_{m^*+3}$. Hence in this case from $Z_k = Z(\tau_k)$, $0 \le k \le n+3$, we infer that

$$\hat{G}_{m^*+3} - S_M(\tau_{m^*+3}) \ge a_\eta - \varepsilon. \tag{28}$$

Let

$$A = \big\{G(\tau_k) \ge \hat{G}_k\big(Z(0), \dots, Z(\tau_k)\big),\ Z(\tau_k) = Z_k,\ 0 \le k \le n+3,\ \tau_{n+3} \le 1\big\}.$$

Inequalities (27) and (28) imply that

$$A \subset \left\{\max_{0 \le m \le n+3} \{G(\tau_m) - S_M(\tau_m)\} \ge a_\eta - \varepsilon,\ \tau_{n+3} \le 1\right\}.$$

By (2) we have $P(A) > 0$ so we can conclude that the minimal super-replication cost y_M for the portfolio M is at least $a_\eta - \varepsilon$. This proves the first part of A. If $\hat{G}_k \ge c$, $k \ge 0$, then by Fatou's lemma $\liminf_{n\to\infty} \hat{E}_\delta \hat{H}_{n\wedge\eta} \ge \hat{E}_\delta \hat{H}_\eta$, $\eta \in \mathcal{S}(\delta)$, which completes the proof of the second part of A.

B. The proof is almost the same as that of part A, but instead of Lemma 4 use Lemma 5. $\square$

6. Path-independent options. In this section we apply our earlier results to deal with path-independent options. By a path-independent option we mean an option with a payoff function $G(t) = g(Z(t))$, where $g : (0,\infty) \to R$ is a measurable function. Our first problem is to find a minimal investment such that there is a portfolio M so that the value of the portfolio at the exercise time will exceed the payoff function. We do not put any restrictions on the number of shares at the exercise time. We recall definitions of the super-replication cost for the option $g(Z(t))$,

$$b_A(g) = \inf\big\{x \in R : \exists M \in FV \text{ such that } x + S_M(\tau) \ge g(Z(\tau)) \text{ a.s.}, \tau \in \mathrm{ST}[0,1]\big\},$$

$$b_E(g) = \inf\big\{x \in R : \exists M \in FV \text{ such that } x + S_M(1) \ge g(Z(1)) \text{ a.s.}\big\},$$

where $b_A(g)$ and $b_E(g)$ correspond to American and European options, respectively.

Now suppose that at the time of exercise we demand that the hedging portfolio contain exactly $m(z)$ shares, where z is the share price at the time of exercise and still the value of the portfolio is at least $g(z)$. In this case we say that the portfolio super-replicates the option $(g(Z(t)), m(Z(t)))$. Now the definitions of the super-replication cost for the option will be

$$b_A^*(g,m) = \inf\{x \in R : \exists M \in FV \text{ such that } x + S_M(\tau) \geq g(Z(\tau)) + \lambda Z(\tau)[M(\tau) - m(Z(\tau))]^- + \mu Z(\tau)[M(\tau) - m(Z(\tau))]^+ \text{ a.s., } \tau \in \mathrm{ST}[0,1]\},$$

$$b_E^*(g,m) = \inf\{x \in R : \exists M \in FV \text{ such that } x + S_M(1) \geq g(Z(1)) + \lambda[M(1) - m(Z(1))]^- + \mu[M(1) - m(Z(1))]^+ \text{ a.s.}\}.$$

We recall that

$$\bar{g}(z) = \sup\{\alpha g(z_1) + (1-\alpha) g(z_2) : 0 \leq \alpha \leq 1,\ z = \alpha z_1 + (1-\alpha) z_2,\ z_1, z_2 > 0\}.$$

LEMMA 6. *The function* $\bar{g}(z)$ *has the following properties*:

(a) $\bar{g}(z) \geq g(z)$ *and* $z > 0$;
(b) *it is a concave function*.

We start with a result that shows that in the case of two-sided transaction costs, American and European super-replication costs are the same. Note that this is not true for one-sided transaction costs. Once again we can mention Example 2 from Levental and Skorohod (1997), where it was shown that $b_E(g) \leq 0$ for $g(z) = z$, while $b_A(g) = z_0$.

PROPOSITION 1. *Let* $\min\{\lambda, \mu\} > 0$. *If* g *is lower-semicontinuous*, *then* $b_A(g) = b_E(g)$.

PROOF. If $x < b_A(g)$, then for any portfolio $M(t)$ there exists a stopping time τ such that either

$$V(\tau) = x + S_M(\tau) < g(Z(\tau)), \qquad \tau < 1,\ M(\tau) \geq 0, \tag{29}$$

or

$$V(\tau) = x + S_M(\tau) < g(Z(\tau)), \qquad \tau < 1,\ M(\tau) < 0, \tag{30}$$

holds with positive probability. Assume that (30) is true. Then we can also find positive constants $w^* < z^*$ so that, with positive probability,

$$x + S_M(\tau) < g(Z(\tau)), \qquad \tau < 1, \quad w^* \leq Z(\tau) \leq z^*, \quad M(\tau) < 0. \tag{31}$$

We claim that there is a continuous function $\tilde{g}(z) \leq g(z)$ for $z \in [w^*/2, 2z^*]$ such that, with positive probability,

$$x + S_M(\tau) < \tilde{g}(Z(\tau)), \qquad \tau < 1, \quad w^* \leq Z(\tau) \leq z^*, \quad M(\tau) < 0. \tag{32}$$

This follows from (31) and the fact that any lower-semicontinuous function on a compact interval is a pointwise limit of an increasing sequence of continuous functions. We will show that for any $\varepsilon > 0$, with positive probability,

$$x + S_M(1) < \tilde{g}\big(Z(1)\big) + \varepsilon \le g\big(Z(1)\big) + \varepsilon. \tag{33}$$

This would imply that $x < b_E(g)$ and hence prove that $b_A(g) \le b_E(g)$. For any $\delta > 0$ denote

$$B(\delta) = \{Z(\tau)e^{-\delta} \le Z(t) \le Z(\tau)e^{\delta},\ Z(\tau) \le Z(1),\ \tau \le t \le 1,\ w^* \le Z(\tau) \le z^*\}.$$

Fix $\varepsilon > 0$. Since $\tilde{g}(z)$ is uniformly continuous on $[w^*/2, 2z^*]$, then on the event $B(\delta)$ we get

$$\tilde{g}\big(Z(\tau)\big) \le g\big(Z(1)\big) + \varepsilon \tag{34}$$

for sufficiently small $\delta > 0$. It follows from Corollary 1 that on the event $B(\delta)$, for sufficiently small δ, we have, almost surely,

$$S_M(1) - S_M(\tau) \le M(\tau)\big(Z(1) - Z(\tau)\big) \le 0. \tag{35}$$

Combining (32), (34) and (35) we obtain (33) on the event

$$\{x + S_M(\tau) < \tilde{g}\big(Z(\tau)\big),\ \tau < 1\} \cap B(\delta),$$

which has positive probability for sufficiently small $\delta > 0$ by the assumptions on the process. $\square$

To apply Theorem 1 or 2 we have to indicate (for $\delta > 0$) a sequence of functions which underestimate the option $g(Z(t))$ in the American sense. If we take $\hat{G}_k(z_0, \dots, z_k) = g(z_k)$, then the condition (2) is reduced to

$$P(Z(\tau_k) = Z_i,\ 1 \le k \le n,\ \tau_n < 1) > 0$$

for every $n \ge 1$ and a δ-predictable sequence Z_i, $1 \le i \le n$, such that $Z_{i+1} = e^{\pm\delta} Z_i$, which holds by Lemma 7 proven below.

LEMMA 7. *Fix $\delta > 0$. Let $n \ge 0$ and let $a_1 < b_1 < \cdots < a_n < b_n < a_{n+1}$. Let $(Z_0, \dots, Z_n) \in \mathcal{Z}_\delta(n)$ be δ-predictable. Then the following holds*:

$$P\big(a_k < \tau_k < b_k, Z(\tau_k) = Z_k,\ 1 \le k \le n, \tau_{n+1} \ge a_{n+1}\big) > 0. \tag{36}$$

PROOF. First we prove by induction that

$$P\big(a_k < \tau_k < b_k, Z(\tau_k) = Z_k,\ 1 \le k \le n\big) > 0. \tag{37}$$

Let $A_n = \{a_i < \tau_i < b_i, Z(\tau_k) = Z_i,\ 1 \le k \le n\}$. Assume that $P(A_{n-1}) > 0$. Since

$$P(A_{n-1}, Z_n = Z_{n-1}e^{\delta}) + P(A_{n-1}, Z_n = Z_{n-1}e^{-\delta}) = P(A_{n-1}),$$

we may assume that $P(A_{n-1}, Z_n = Z_{n-1}e^\delta) > 0$. Denote $B_{n-1} = \{A_{n-1}, Z_n = Z_{n-1}e^\delta\}$. Notice that Z_n is $F_{\tau_{n-1}}$-measurable; hence, we have $B_{n-1} \in F_{\tau_{n-1}}$ and this yields

$$\begin{aligned} &P(A_{n-1}, Z_n = Z_{n-1}e^\delta, a_n < \tau_n < b_n, Z(\tau_n) = Z_n) \\ &\quad = P(B_{n-1}, a_n < \tau_n < b_n, Z(\tau_n) = Z(\tau_{n-1})e^\delta) \\ &\quad = E[\mathbf{1}_{B_{n-1}} P(a_n < \tau_n < b_n, Z(\tau_n) = Z(\tau_{n-1}e^\delta) | F_{\tau_{n-1}})]. \end{aligned}$$

Since $B_{n-1} \subset \{\tau_{n-1} < b_{n-1} < a_n\}$, then by Assumption 1,

$$P(a_n < \tau_n < b_n, Z(\tau_n) = Z(\tau_{n-1})e^\delta | F_{\tau_{n-1}}) > 0 \qquad \text{a.s. on } B_{n-1},$$

which ends the proof of (37). To get (36) from (37) we can choose $Z_{n+1} = Z_n e^\delta$ so the sequence $(Z_1, \dots, Z_{n+1})$ is predictable and we can apply (37) with n replaced by $n + 1$. The proof is complete. □

Now we are ready to prove Theorem 3 which says that $b_A(g) = b_E(g) = \bar{g}(z_0)$ if we pay both transaction costs and $b_A(g) = \bar{g}(z_0)$ in the case of one-sided transaction costs, where the initial share price is z_0.

PROOF OF THEOREM 3. We begin with the case of two-sided transaction costs. According to Proposition 1, it is enough to consider the American case only. Assume that $0 < w_1 < z_0 < w_2$ and $z_0 = \alpha w_1 + (1 - \alpha) w_2$ for some $0 < \alpha < 1$. Let $y = \alpha g(w_1) + (1 - \alpha) g(w_2)$. Then there is M such that

$$y + M(w_i - z_0) = g(z_i), \qquad i = 1, 2.$$

Next assume that we can find $\delta > 0$ small enough and integers $m < 0$ and $n > 0$ such that $w_1 = z_0 e^{m\delta}$ and $w_2 = z_0 e^{n\delta}$. On a shadow market which is defined for δ and uses probability $\hat{P}_\delta$, we choose a stopping time $\eta = \inf\{k : z_k \in \{w_1, w_2\}\}$. It is clear that $\eta < \infty$, $\hat{P}_\delta$ a.s. and $\hat{E}_\delta z_\eta = \lim_{n\to\infty} \hat{E}_\delta z_{\eta\wedge n} = z_0$. This follows from the fact that $z_{\eta\wedge n} \le w_2$, $\hat{P}_\delta$ a.s. and the martingale property. Next, note that

$$g(z_\eta) = y + M(z_\eta - z_0),$$

which implies that

$$\hat{E}_\delta g(z_\eta) = y + M(\hat{E}_\delta z_\eta - z_0) = y = \alpha g(w_1) + (1 - \alpha) g(w_2).$$

Therefore, by Theorem 1 we obtain

$$b_A \ge \hat{E}_\delta g(z_\eta) \ge \alpha g(w_1) + (1 - \alpha) g(w_2).$$

In the general case we can easily find a sequence $\delta_k \downarrow 0$ and integers $m_k < 0$ and $n_k > 0$ such that $w_1 = z_0 e^{m_k \delta_k}$ and $w_2(k) = z_0 e^{n_k \delta_k} \to w_2$ as $k \to \infty$. Moreover, there is a sequence $0 < \alpha_k < 1$ such that

$$z_0 = \alpha_k w_1 + (1 - \alpha_k) w_2(k).$$

From the first part of the proof it follows that

$$b_A \geq \alpha_k g(w_1) + (1-\alpha_k) g\bigl(w_2(k)\bigr).$$

Using lower-semicontinuity of $g(z)$ and passing with k to ∞ we infer that

$$b_A \geq \alpha g(w_1) + (1-\alpha) g(w_2)$$

in the general case and this concludes the proof of the lower bound in the case of two-sided transaction costs.

In the case of one-sided transaction costs we will sketch the proof of the lower bound only for $\mu = 0$, $\lambda > 0$. Let

$$h_\delta(z) = \min\bigl\{g(z), g(e^{3\delta} z)\bigr\}.$$

Due to Theorem 2A we have

$$b_A \geq \limsup_{\delta \to 0} \sup_{\eta \in \mathcal{S}(\delta)} \hat{E}_\delta h_\delta(z_\eta).$$

Note that $\hat{E}_\delta h_\delta(z_\eta) \geq \min\{\hat{E}_\delta g(z_\eta), \hat{E}_\delta g(e^{3\delta} z_\eta)\}$. Next we proceed as above with obvious changes and show that for any $0 < w_1 < z_0 < w_2$ such that $z_0 = \alpha w_1 + (1-\alpha) w_2$ for some $0 < \alpha < 1$ we have

$$\limsup_{\delta \to 0} \sup_{\eta \in \mathcal{S}(\delta)} \hat{E}_\delta h_\delta(z_\eta) \geq \alpha g(w_1) + (1-\alpha) g(w_2),$$

which gives us the desired estimate in this case.

Now we deal with the upper bound regardless of the type of transaction costs. Since the function $\bar{g}(z)$ is concave, then

$$g(z) \leq \bar{g}(z) \leq \bar{g}(z_0) + M^*(z - z_0),$$

where $M^* = (\frac{d\bar{g}}{dz})^+(z_0)$. This shows that $b_A \leq \bar{g}(z_0)$ and the optimal super-replication strategy is *buy and hold*. The proof is complete. $\square$

Now we proceed to options of the type $(g(Z(t)), m(Z(t)))$. Recall that

$$g_1(z) = g\left(\frac{z}{1+\lambda}\right) + \frac{\lambda z}{1+\lambda} m\left(\frac{z}{1+\lambda}\right),$$
$$g_2(z) = g\left(\frac{z}{1-\mu}\right) - \frac{\mu z}{1-\mu} m\left(\frac{z}{1-\mu}\right)$$

and

$$g_m(z) = \max\bigl\{g_1(z), g_2(z)\bigr\}.$$

The following proposition shows that the requirement of having a specified number of shares at the end of the hedge forces the equality of American and European super-replication prices regardless of the type of transaction costs.

PROPOSITION 2. *Assume that* $\max\{\mu, \lambda\} > 0$. *Suppose that* $b(z) = g(z) - zm(z)$ *and* $m(z)$ *are lower-semicontinuous. Then*

$$b^*_A(g, m) = b^*_E(g, m).$$

PROOF. If $x < b^*_A(g, m)$, then for any portfolio $M(t)$ there exists a stopping time τ such that, with positive probability,

$$\begin{aligned} V(\tau) = x + S_M(\tau) < g\big(Z(\tau)\big) + \lambda Z(\tau)\big[m\big(Z(\tau)\big) - M(\tau)\big]^+ \\ + \mu Z(\tau)\big[M(\tau) - m\big(Z(\tau)\big)\big]^+, \qquad \tau < 1. \end{aligned}$$

Hence we can also find $0 < \lambda_1 < \lambda$ or $\lambda_1 = 0$ if $\lambda = 0$, $0 < \mu_1 < \mu$ or $\mu_1 = 0$ if $\mu = 0$ and positive constants $w^* < z^*$, m^*, so that, with positive probability,

$$\begin{aligned} V(\tau) < g\big(Z(\tau)\big) + \lambda_1 Z(\tau)\big[m\big(Z(\tau)\big) - M(\tau)\big]^+ \\ + \mu_1 Z(\tau)\big[M(\tau) - m\big(Z(\tau)\big)\big]^+ \end{aligned} \tag{38}$$

and

$$\tau < 1, \qquad w^* \le Z(\tau) \le z^*, \qquad |M(\tau)| \le m^*. \tag{39}$$

Our task is to show that for any $\varepsilon > 0$, with positive probability,

$$\begin{aligned} x + S_M(1) < g\big(Z(1)\big) + \lambda Z(1)\big[m\big(Z(1)\big) - M(1)\big]^+ \\ + \mu Z(1)\big[M(1) - m\big(Z(1)\big)\big]^+ + \varepsilon. \end{aligned} \tag{40}$$

This would show that $x < b^*_E(g, m)$ and hence prove that $b^*_A(g, m) \le b^*_E(g, m)$. We can also assume that $m(Z(1)) = M(1)$. Note that

$$\begin{aligned} & g\big(Z(\tau)\big) + \lambda_1 Z(\tau)\big[m\big(Z(\tau)\big) - M(\tau)\big]^+ + \mu_1 Z(\tau)\big[M(\tau) - m\big(Z(\tau)\big)\big]^+ \\ &\quad = g\big(Z(\tau)\big) - \mu_1 Z(\tau) m\big(Z(\tau)\big) + (\lambda_1 + \mu_1) Z(\tau)\big[m\big(Z(\tau)\big) - M(\tau)\big]^+ \\ &\qquad + \mu_1 Z(\tau) M(\tau) \\ &\quad = f\big(Z(\tau)\big) + (\lambda_1 + \mu_1) Z(\tau)\big[m\big(Z(\tau)\big) - M(\tau)\big]^+ + \mu_1 Z(\tau) M(\tau), \end{aligned}$$

where $f(z) = g(z) - \mu_1 z m(z)$. Note that the assumptions about $m(z)$ and $b(z)$ imply that $f(z)$ is a lower-semicontinuous function. Arguing as in the proof of Proposition 1 and using (38) and (39) we can find continuous functions $\tilde{f}(z) \le f(z)$ and $\tilde{m}(z) \le m(z)$ for $z \in [w^*/2, 2z^*]$ such that we have, with positive probability,

$$V(\tau) < \tilde{f}\big(Z(\tau)\big) + (\lambda_1 + \mu_1) Z(\tau)\big[\tilde{m}\big(Z(\tau)\big) - M(\tau)\big]^+ + \mu_1 Z(\tau) M(\tau) \tag{41}$$

and

$$\tau < 1, \qquad w^* \le Z(\tau) \le z^*, \qquad |M(\tau)| \le m^*. \tag{42}$$

Let $A \in F_\tau$ denote the event described by (41) and (42). For any $\delta > 0$, denote

$$B(\delta) = \{Z(\tau)e^{-\delta} \le Z(t) \le Z(\tau)e^{\delta},\ \tau \le t \le 1,\ w^* \le Z(\tau) \le z^*\}.$$

Fix $\varepsilon > 0$. By uniform continuity of $\tilde{f}(z)$ and $\tilde{m}(z)$ on $[w^*/2, 2z^*]$, for sufficiently small $\delta > 0$, on the event $A \cap B(\delta)$ we obtain

$$\begin{aligned}
V(\tau) &< \tilde{f}(Z(\tau)) + (\lambda_1 + \mu_1)Z(\tau)[\tilde{m}(Z(\tau)) - M(\tau)]^+ + \mu_1 Z(\tau)M(\tau) \\
&\le \tilde{f}(Z(1)) + (\lambda_1 + \mu_1)Z(1)[\tilde{m}(Z(1)) - M(\tau)]^+ + \mu_1 Z(1)M(\tau) + \varepsilon/2 \\
&\le f(Z(1)) + (\lambda_1 + \mu_1)Z(1)[m(Z(1)) - M(\tau)]^+ + \mu_1 Z(1)M(\tau) + \varepsilon/2 \\
&= g(Z(1)) + \lambda_1 Z(1)[m(Z(1)) - M(\tau)]^+ \\
&\quad + \mu_1 Z(1)[M(\tau) - m(Z(1))]^+ + \varepsilon/2,
\end{aligned} \tag{43}$$

where the first step follows from (41). It remains to estimate the capital gain on the interval $[\tau, 1)$. Since $m(z)$ is lower-semicontinuous, then it is bounded from below on bounded intervals away from zero; hence

$$\inf_{w^*/2 < z < 2z^*} m(z) \ge C > -\infty.$$

We need to consider three cases.

Case 1. $\mu = 0$ and $\lambda > 0$. Apply Lemma 1A with $\rho = \xi = e^{\delta} - 1$ and $\lambda' = (1 + \lambda)e^{-\delta} - e^{\delta}$ to show that on the event $B(\delta)$ we have, almost surely,

$$\begin{aligned}
S_M(1) - S_M(\tau) &\le m(Z(1))(Z(1) - Z(\tau)) - \lambda' Z(\tau)[m(Z(1)) - M(\tau)]^+ \\
&\quad - (e^{\delta} - 1)Z(\tau)(m(Z(1)) - M(\tau)) \\
&\le -\lambda' Z(\tau)[m(Z(1)) - M(\tau)]^+ \\
&\quad + (m(Z(1)) - C)(Z(1) - Z(\tau)e^{\delta}) + C(Z(1) - Z(\tau)e^{\delta}) \\
&\quad + (e^{\delta} - e^{-\delta})Z(\tau)M(\tau) \\
&\le -\lambda' Z(\tau)[m(Z(1)) - M(\tau)]^+ + (e^{\delta} - e^{-\delta})Z(\tau)(|M(\tau)| + |C|) \\
&\le -\lambda' Z(\tau)[m(Z(1)) - M(\tau)]^+ + (e^{\delta} - 1)z^*(m^* + |C|) \\
&\le -\lambda' Z(\tau)[m(Z(1)) - M(\tau)]^+ + \varepsilon/2 \\
&\le -\lambda' e^{-\delta} Z(1)[m(Z(1)) - M(\tau)]^+ + \varepsilon/2,
\end{aligned} \tag{44}$$

for δ small enough. Next choose δ small enough so that $\lambda' e^{-\delta} \ge \lambda_1$ and then from (43) and (44) it follows that (40) holds on the event $A \cap B(\delta)$. By the assumptions on the process, its probability is positive.

Case 2. $\mu > 0$ and $\lambda = 0$. Using Lemma 2A, with $\rho = \xi = 1 - e^{-\delta}$ and by calculations as above we can estimate that on the event $B(\delta)$ we have, almost surely,

$$S_M(1) - S_M(\tau) \le -\mu' e^{-\delta} Z(1)\big[M(\tau) - m\big(Z(1)\big)\big]^+ + \varepsilon/2, \tag{45}$$

where $\mu' = \mu e^{-\delta} + e^{-2\delta} - 1$ and δ is small enough. The rest of the arguments are the same as in Case 1.

Case 3. $\mu > 0$ and $\lambda > 0$. We now use Corollary 1 with $\psi = 1 - e^{-\delta}$ and $\xi = e^{\delta} - 1$ to show that on the event $B(\delta)$ we have, almost surely,

$$\begin{aligned}
&S_M(1) - S_M(\tau) \\
&\quad\le M(\tau)\big(Z(1) - Z(\tau)\big) - \lambda' Z(1)\big[m\big(Z(1)\big) - M(\tau)\big]^+ \\
&\qquad - \mu' Z(1)\big[M(\tau) - m\big(Z(1)\big)\big]^+ \\
&\quad\le m^*(e^{\delta} - 1)z^* - \lambda' Z(1)\big[m\big(Z(1)\big) - M(\tau)\big]^+ \\
&\qquad - \mu' Z(1)\big[M(\tau) - m\big(Z(1)\big)\big]^+ \\
&\quad\le -\lambda' Z(1)\big[m\big(Z(1)\big) - M(\tau)\big]^+ - \mu' Z(1)\big[M(\tau) - m\big(Z(1)\big)\big]^+ + \varepsilon/2,
\end{aligned} \tag{46}$$

where $\mu' = (\mu + e^{-\delta} - 1)e^{-\delta}$, $\lambda' = (\lambda - e^{\delta} + 1)e^{-\delta}$ and δ is small enough. Again the rest of the arguments are the same as in Case 1. $\square$

PROOF OF THEOREM 4. We prove first that $b_A^*(g, m) \ge \overline{g_m}(z_0)$. If not, then from Theorem 3 there is $x > b_A^*(g, m)$ such that for any portfolio $M(t)$ there exists a stopping time τ such that, with positive probability,

$$V(\tau) = x + S_M(\tau) < g_m\big(Z(\tau)\big), \qquad \tau < 1. \tag{47}$$

Our task is to show that there is a stopping time $\theta > \tau$ such that, with positive probability,

$$\begin{aligned}
V(\theta) < g_m\big(Z(\theta)\big) &+ \lambda Z(\theta)\big[M(\theta) - m\big(Z(\theta)\big)\big]^- \\
&+ \mu Z(\theta)\big[M(\theta) - m\big(Z(\theta)\big)\big]^+, \qquad \theta < 1.
\end{aligned} \tag{48}$$

This would contradict the definition of $b_A^*(g, m)$. We consider two cases:

Case 1. Assume that

$$P\big(g_m\big(Z(\tau)\big) = g_1\big(Z(\tau)\big),\ V(\tau) < g_1\big(Z(\tau)\big),\ \tau < 1\big) > 0. \tag{49}$$

We also assume that $\mu = 0$ and $\lambda > 0$. Since $g(z)$ and $m(z)$ are lower-semicontinuous, there is $0 < \xi < \lambda$ and $0 < \rho < \frac{\lambda - \xi}{1+\xi}$ for which, with positive probability, $\tau < 1$ and

$$\begin{aligned}
V(\tau) &= x + S_M(\tau) \\
&< g\left(\frac{Z(\tau)}{1+\xi}\right) + \frac{\xi Z(\tau)}{1+\xi} m\left(\frac{Z(\tau)}{1+\xi}\right) - \rho Z(\tau)\left(M(\tau) - m\left(\frac{Z(\tau)}{1+\xi}\right)\right).
\end{aligned}$$

Define $\theta = \inf\{t > \tau : Z(t) = \frac{Z(\tau)}{1+\xi}\}$ and $\theta^* = \inf\{t > \tau : Z(t) = Z(\tau)(1+\rho)\}$. The above inequality says that on the event $\{\theta < \theta^*\}$ we have

$$V(\tau) < g\big(Z(\theta)\big) + \xi Z(\theta) m\big(Z(\theta)\big) - \rho Z(\tau)\big(M(\tau) - m\big(Z(\theta)\big)\big). \tag{50}$$

By Lemma 1A on the event $\{\theta < \theta^*\}$ we obtain

$$\begin{aligned} S_M(\theta) - S_M(\tau) &\le M(\theta)\big(Z(\theta) - Z(\tau)\big) - \rho Z(\tau)\big(M(\theta) - M(\tau)\big) \\ &= -\xi M(\theta) Z(\theta) - \rho Z(\tau)\big(M(\theta) - M(\tau)\big). \end{aligned} \tag{51}$$

The estimates (50) and (51) imply that

$$\begin{aligned} V(\theta) &= V(\tau) + S_M(\theta) - S_M(\tau) \\ &< g\big(Z(\theta)\big) + \xi Z(\theta) m\big(Z(\theta)\big) - \rho Z(\tau)\big(M(\tau) - m\big(Z(\theta)\big)\big) \\ &\quad - \xi M(\theta) Z(\theta) - \rho Z(\tau)\big(M(\theta) - M(\tau)\big) \\ &= g\big(Z(\theta)\big) + \big(\xi Z(\theta) + \rho Z(\tau)\big)\big(m\big(Z(\theta)\big) - M(\theta)\big) \\ &= g\big(Z(\theta)\big) + \big(\xi + (\xi+1)\rho\big) Z(\theta)\big(m\big(Z(\theta)\big) - M(\theta)\big) \\ &\le g\big(Z(\theta)\big) + \lambda Z(\theta)\big[m\big(Z(\theta)\big) - M(\theta)\big]^+ \end{aligned}$$

and (48) follows, since by (49) and the assumption on the process

$$P\big(g_m\big(Z(\tau)\big) = g_1\big(Z(\tau)\big),\ V(\tau) < g_1\big(Z(\tau)\big),\ \tau < \theta < 1,\ \theta < \theta^*\big) > 0.$$

Case 2. Assume that

$$P\big(g_m\big(Z(\tau)\big) = g_2\big(Z(\tau)\big),\ V(\tau) < g_2\big(Z(\tau)\big),\ \tau < 1\big) > 0.$$

We also assume that $\mu > 0$ and $\lambda = 0$. Since $g(z) - \mu z m(z)$ and $m(z)$ are lower-semicontinuous, which follows from the lower-semicontinuity of $m(z)$ and $b(z)$, there are $0 < \xi < \mu$ and $0 < \rho < \frac{\mu-\xi}{1+\mu-\xi}$ for which, with positive probability, $\tau < 1$ and

$$\begin{aligned} V(\tau) &= x + S_M(\tau) \\ &< g\left(\frac{Z(\tau)}{1-\xi}\right) - \mu \frac{Z(\tau)}{1-\xi} m\left(\frac{Z(\tau)}{1-\xi}\right) + \left(\frac{\mu-\xi}{1-\xi} - \rho\right) Z(\tau) m\left(\frac{Z(\tau)}{1-\xi}\right) \\ &\quad + \rho Z(\tau) M(\tau) \\ &= g\left(\frac{Z(\tau)}{1-\xi}\right) - \frac{\xi Z(\tau)}{1-\xi} m\left(\frac{Z(\tau)}{1-\xi}\right) + \rho Z(\tau)\left(M(\tau) - m\left(\frac{Z(\tau)}{1-\xi}\right)\right). \end{aligned}$$

Define $\theta = \inf\{t > \tau : Z(t) = \frac{Z(\tau)}{1-\xi}\}$ and $\theta^* = \inf\{t > \tau : Z(t) = Z(\tau)(1-\rho)\}$. Now, a similar argument as in Case 1 can be used, based on Lemma 2A. We omit the details.

We now prove the inequality $b_A^*(g,m) \le \overline{g_m}(z_0)$. By concavity of $\overline{g_m}(z)$ it follows that

$$g_m(z) \le \overline{g_m}(z) \le \overline{g_m}(z_0) + M^*(z - z_0), \tag{52}$$

where $M^* = (\frac{d\overline{g_m}}{dz})^+(z_0)$. We need to show that

$$\overline{g_m}(z_0) + M^*(z - z_0) \ge g(z) + \lambda z[M^* - m(z)]^- + \mu z[M^* - m(z)]^+ \quad \text{for every } z > 0. \tag{53}$$

Now fix z. There are two possible cases:

Case a. $M^* \le m(z)$. Use the definition of $g_m(z)$ with $z(1+\lambda)$ instead of z to get, from (52),

$$\overline{g_m}(z_0) + M^*\bigl(z(\lambda+1) - z_0\bigr) \ge g(z) + \lambda z m(z),$$

which implies (53).

Case b. $M^* > m(z)$. Use the definition of $g_m(z)$ with $z(1-\mu)$ instead of z to get, from (52),

$$\overline{g_m}(z_0) + M^*\bigl(z(1-\mu) - z_0\bigr) \ge g(z) - \mu z m(z),$$

which again implies (53). □

Now we provide an argument that justifies Remark 2, regardless of whether $\mu = 0$ or $\mu > 0$, in the setup of Theorem 4. Let $0 < \lambda^* < \lambda$ be a transaction cost parameter paid when one buys shares at $t = 0$. Define $\tau = \inf\{t > 0 : Z(t) = z_1 \text{ or } z_2\}$, where $z_1 = z_0(1+\lambda^*)$ and $z_2 = \frac{z_0(1+\lambda^*)}{1+\xi}$ with $0 < \lambda^* < \xi < \lambda$. Assume that we buy $M(0) \ge 0$ shares at $t = 0$ [since $\overline{g_m}(z)$ is increasing, this is the only sensible choice]. From Corollary 1 (with $\mu = 0$ and ξ chosen above) it follows that on the event $\{Z_\tau = z_1\}$, we get, for every portfolio $M(t)$,

$$S_M(\tau) \le M(0)(z_1 - z_0) = \lambda^* M(0) z_0.$$

On the other hand, by Theorem 4, for every $x > b_A^*(g,m)$ there is $M(t)$, so that on $\{Z_\tau = z_1\}$ we have

$$x + S_M(\tau) - \lambda^* M(0) z_0 \ge \overline{g_m}(z_1).$$

Hence by the last two inequalities we obtain $b_A^*(g,m) \ge \overline{g_m}(z_1)$. In fact, $b_A^*(g,m) = \overline{g_m}(z_1)$ because of the buy and hold strategy with $M(t) = M^* = (\frac{d\overline{g_m}}{dz})^+(z_1)$. To handle the case $\lambda = \lambda^*$, we let λ^* converge to λ and we use the continuity of $\overline{g_m}(z)$. The same argument works as well in the setup of Theorem 3.

7. Examples. In this section we demonstrate how to apply our results in some concrete situations. The first example deals with arbitrage issues in a market with transaction costs. Then we present two examples of *barrier options* and finally examples of an *Asian option* and a *lookback option*.

EXAMPLE 1 [see Levental and Ryznar (2000)]. *Assume that* $\min\{\mu, \lambda\} > 0$. *Then there are* no arbitrage opportunities. *That is*, *for every* $M \in FV$ *either* $S_M(1) = 0$, *a.s. or* $P(S_M(1) < 0) > 0$.

REMARK 4. We *do not assume any restrictions on the portfolios* (i.e., number of shares, capital losses, etc.). This is in contrast with the no transaction costs markets, where doubling schemes will allow arbitrage (even when Z is a geometric Brownian motion) unless some portfolio restrictions are assumed. Also, Assumption 1 and finite variation strategies do not prevent arbitrage in no transaction costs markets.

PROOF. Let $M \in FV$. Define $\tau = \inf\{t : M(t) \neq 0\}$. If $\tau \geq 1$ a.s., then obviously $S_M(1) \leq 0$ a.s. So we assume that $\{\tau < 1\}$ is not a null event. Let $\delta > 0$ be appropriately small. By Corollary 1 we can find $\lambda' > 0$ and $\mu' > 0$ such that on the event $A = \{\tau^\delta < 1,\ Z(\tau^\delta) = Z(\tau)e^{-\operatorname{sign}(M(\tau))\delta}\}$, sign(0) = 1, which has a positive probability given F_τ, we have

$$S_M(\tau^\delta) - S_M(\tau) \leq M(\tau)\big(Z(\tau^\delta) - Z(\tau)\big) - \lambda' Z(\tau^\delta)\big(M^+(\tau^\delta) - M^+(\tau)\big) \\ - \mu' Z(\tau^\delta)\big(M^-(\tau^\delta) - M^-(\tau)\big).$$

Observe that either $M(\tau) \neq 0$ or $\max\{M^+(\tau^\delta) - M^+(\tau),\ M^-(\tau^\delta) - M^-(\tau)\} > 0$, which implies that

$$S_M(\tau^\delta) - S_M(\tau) < 0$$

on the event A. Next we take $\alpha > 0$ small enough, so that on the event $\{(\tau^\delta)^\alpha \geq 1\}$, which has a positive probability given F_{τ^δ}, again by Corollary 1, we have

$$S_M(1) - S_M(\tau^\delta) \leq M(\tau^\delta)\big(Z(1) - Z(\tau^\delta)\big) \\ \leq |M(\tau^\delta)| Z(\tau^\delta)(e^\alpha - 1).$$

Hence choosing appropriately small α and adding the last two inequalities gives, with positive probability,

$$S_M(1) = S_M(1) - S_M(\tau) < 0. \qquad \square$$

Next we present a result related to down and out options, that is, options of the form

$$G(t) = g\big(Z(t)\big)\mathbf{1}_{\{\min_{0\leq u\leq t} Z(u) > c\}},$$

where $c > 0$ is a given constant. On the interval (c, ∞) define the function

$$\bar{g}_c(z) = \sup\{\alpha g(z_1) + (1-\alpha)g(z_2) : 0 \le \alpha \le 1,\ z = \alpha z_1 + (1-\alpha)z_2,\ z_1, z_2 > c\},$$

which is a concave majorant of $g(z)$ over that interval.

The following proposition can be proved along the same lines as the lower bound in Theorem 3 with the following change: The numbers $w_1 < z_0 < w_2$ that appear in the proof of Theorem 3, should satisfy $c < w_1 < z_0 < w_2$ here.

PROPOSITION 3. *Assume that* $\inf_{z>0} g(z) > -\infty$ *and* $g(z)$ *is lower-semicontinuous. Let* $z_0 = Z(0) \ge c$ *be the initial share price. If* $\max\{\lambda, \mu\} > 0$, *then*

$$b_A(G) \ge \bar{g}_c(z_0).$$

If $\min\{\lambda, \mu\} > 0$, *then*

$$b_A(G) = b_E(G) \ge \bar{g}_c(z_0).$$

Next we have an application of Proposition 3 to a down and out put option.

EXAMPLE 2. *Assume that* $\max\{\mu, \lambda\} > 0$. *Let* $z_0 = Z(0) \ge c$ *be the initial share price. For a down and out put option, that is,* $G(t) = [q - Z(t)]^+ \mathbf{1}_{\{\min_{0 \le u \le t} Z(u) > c\}}$, *where* $0 \le c < \min\{z_0, q\}$ *we have*

$$b_A(G) = q - c.$$

If $\min\{\mu, \lambda\} > 0$, *then*

$$b_A(G) = b_E(G) = q - c.$$

PROOF. The lower bound is an immediate application of Proposition 3. Since the function $g(z) = [q - z]^+$ is nonincreasing, we have $\bar{g}_c(z) = g(c)$ for $z > c$ and hence $b_A(G) \ge \bar{g}_c(z_0) = g(c) = q - c$ if we have one-sided transaction costs and $b_E(G) \ge \bar{g}_c(z_0) = q - c$ for two-sided transaction costs. The upper bound is trivial since the payoff function of the option is bounded by $g(c)$. $\square$

Now we proceed with a down and out call option:

$$G(t) = [Z(t) - q]^+ \mathbf{1}_{\{\min_{0 \le u \le t} Z(u) > c\}}.$$

EXAMPLE 3. *Assume that* $\max\{\mu, \lambda\} > 0$. *Let* $z_0 = Z(0) \ge c$ *be the initial share price and let* $0 \le c < \min\{z_0, q\}$. *Then for the down and out call option*,

$$z_0 - c \le b_A(G) \le z_0 - (1-\mu)c.$$

If $\min\{\mu, \lambda\} > 0$, *then*

$$z_0 - c \le b_A(G) = b_E(G) \le z_0 - (1-\mu)c.$$

PROOF. The lower bound is an immediate application of Proposition 3 since, for $g(z) = [z - q]^+$ we have $\bar{g}_c(z) = z - c$ for $z > c$. The upper bound is obtained by observing that a strategy of "hold one share and sell it when the price drops to c" super-replicates the option. $\square$

REMARK 5. In fact, $b_A(G) = z_0 - (1 - \mu)c$. A proof can be based on the lower bound achieved in Example 3 and on some calculations involving the binomial shadow market. Due to length considerations, we will not provide the details.

Finally we deal with an example of an Asian option.

EXAMPLE 4. *Assume that* $\min\{\lambda, \mu\} > 0$. *Let* $q > 0$ *and* $G(t) = [q - (1/t)\int_0^t Z(u)\,du]^+$. *Then*

$$b_A(G) = b_E(G) = q.$$

PROOF. We begin by constructing a sequence $\{\hat{G}_k\}$ which underestimates G in the European sense. Let $n \geq 1$, $0 < \eta < 1$ and let $\delta > 0$ be small. We choose

$$a_i = \frac{i}{n} - \frac{\delta}{n^2}, \quad b_i = \frac{i}{n}, \qquad 1 \leq i \leq n - 1,$$

$$a_n = \max\left\{\eta, 1 - \frac{\delta}{n^2}\right\}, \qquad b_n = 1.$$

Suppose now that $a_i < \tau_i < b_i$ for $1 \leq i \leq n$ and $\tau_{n+1} \geq 1$. Then

$$\frac{\tau_i - \tau_{i-1}}{\tau_n} \leq \frac{1 + \delta}{n(1 - \delta)}$$

and

$$\frac{1}{\tau_n}\int_0^{\tau_n} Z(u)\,du \leq \frac{1}{n}\sum_{i=0}^{n-1} Z(\tau_i)\frac{1 + \delta}{1 - \delta}e^{\delta}.$$

If we denote $c = c(\delta) = \frac{1+\delta}{1-\delta}e^{\delta}$, then the above inequality yields

$$G(\tau_n) \geq \left[q - \frac{c}{n}\sum_{i=0}^{n} Z(\tau_i)\right]^+$$

on the event $\{a_i < \tau_i < b_i,\ 1 \leq i \leq n, \tau_{n+1} \geq 1\}$. With the choice

$$\hat{G}_k = \left[q - \frac{c}{k}\sum_{i=0}^{k} z_i\right]^+$$

the condition (3) holds by Lemma 7. Before we apply Theorem 1, we need to verify the condition (4). First observe that

$$|G(t) - G(1)| \le 2|1 - t| \sup_{0\le u\le 1} |Z(u)|, \qquad 0 \le t \le 1. \tag{54}$$

Now take $0 < \beta$ and $\xi < 1$, and observe that on the event $\{\xi \le \tau_n \le 1 \le (\tau_n)^\beta\}$ we have that $\sup_{0\le u\le 1} |Z(u)| \le e^{n\delta+1}$, so by (54) we obtain

$$G(\tau_n) \le G(1) + 2|1 - \xi|e^{n\delta+1}.$$

By the assumptions on the process, we have $P(\xi \le \tau_n \le 1 \le (\tau_n)^\beta | F_{\tau_n}) > 0$ a.s. on the event $\{\xi \le \tau_n \le 1\}$, so the condition (4) is satisfied if ξ is chosen close enough to 1. Now we can apply Theorem 1 to get

$$b_A \ge b_E \ge \lim_{n\to\infty} \hat{E}_\delta \hat{G}_n.$$

Observe that $(1/n)\sum_{i=0}^n z_i \to 0$, $\hat{P}_\delta$ a.s., since $z_n \to 0$, $\hat{P}_\delta$ a.s.; hence $\hat{G}_n \to q$, $\hat{P}_\delta$ a.s., and

$$b_A \ge b_E \ge \lim_{n\to\infty} \hat{E}_\delta \hat{G}_n = q$$

by the bounded convergence theorem. The upper bound is again trivial.

Let us mention that if $G(t) = g((1/t)\int_0^t Z(u)\,du)$, where g is a decreasing continuous function on $[0,\infty)$, then similar reasoning as above leads to the conclusion

$$b_A = b_E = g(0). \qquad \square$$

EXAMPLE 5. *Assume that* $\min\{\lambda, \mu\} > 0$. *Let* $z_0 = Z(0)$ *be the initial share price. Let* $G(t) = \max_{0\le u\le t} Z(u) - Z(t)$. *Then*

$$b_A(G) = b_E(G) = \infty.$$

PROOF. Since $G(\tau_n) \ge e^{-\delta} \max_{0\le i\le n} Z(\tau_i) - Z(\tau_n)$, we take

$$\hat{G}_k(z_0, \dots, z_k) = e^{-\delta} \max_{0\le i\le k} z_i - z_k.$$

To apply Theorem 1 we have to verify the conditions (3) and (4), which can be done in a similar fashion as in the previous example. We omit the details. By Theorem 1 we get

$$b_A \ge b_E \ge \lim_{n\to\infty} \hat{E}_\delta \hat{G}_n = \lim_{n\to\infty} \hat{E}_\delta\left(\sup_{0\le i\le n} e^{-\delta} z_i - z_n\right) = e^{-\delta}\hat{E}_\delta \sup_{0\le i<\infty} z_i - z_0 = \infty.$$

To verify the last statement, suppose that $\hat{E}_\delta \sup_{0\le i<\infty} z_i < \infty$. Then by the dominated convergence theorem,

$$\lim_{n\to\infty} \hat{E}_\delta z_n = \hat{E}_\delta \lim_{n\to\infty} z_n,$$

but the right-hand side is equal to z_0 due to the martingale property and the left hand side is zero since z_k is a positive martingale which converges to 0, $\hat{P}_\delta$ a.s. □

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