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AVOIDING THE ORIGIN: A FINITE-FUEL STOCHASTIC CONTROL PROBLEM

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We consider a model for the control of a satellite—fuel is expended in a linear fashion to move a satellite following a diffusion—the aim being to keep the satellite above a critical level. Under suitable assumptions on the drift and diffusion coefficients, it is shown that the probability of the satellite falling below the critical level is minimized by a policy that moves the satellite a certain distance above the critical level and then imposes a reflecting boundary at this higher level until the fuel is exhausted.

1. Introduction. In Jacka (1999) we considered a problem that can loosely be described as that of controlling a satellite using a finite amount of fuel.

A controller can expend fuel to change the satellite's speed. The controller's aim is, in a general sense, to keep the satellite from crashing or breaking up for as long as possible. We assume that this happens (or at least is irreversible) as soon as the satellite's speed falls below some critical value v_0. Shifting the origin, the problem becomes one of expending fuel to keep v from falling below 0.

We assumed that an expenditure of fuel Δy will produce a change proportional to Δy in v. We also assumed the possibility of rescue (at a speed-dependent rate α). The results were that under certain assumptions on the drift and diffusion coefficients we showed that the optimal control was to reflect the diffusion upward from 0 until the fuel was exhausted.

In this paper we are able to substantially generalize some of the results given in Jacka (1999) and to give a global bound on the optimal payoff.

Denoting our controlled diffusion by X, we assume that X is given by

$$X_t = x + \int_0^t \sigma(X_s)\,dB_s + \int_0^t \mu(X_s)\,ds + \xi_t, \tag{1.1}$$

where ξ is the (cumulative) fuel control, and that X is killed at rate $\alpha(X)$. Our results are given below; where X^0 denotes the uncontrolled diffusion [i.e., the (killed) solution to (1.1) with $\xi = 0$].

THEOREM 1.1. *Define G by*

$$G(x) = \mathbb{P}_x(X^0 \textit{ hits } 0 \textit{ before dying}).$$

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(a) *Suppose that* $\bar{g} = \inf_{x \ge 0} G'(x)/G(x)$. *Then the function* V^l, *given by*

$$V^l(x, y) = G(x) \exp(\bar{g} y),$$

is a lower bound for V, *the probability of the optimally controlled process falling below* 0.

(b) *Suppose that* $\bar{x} = \operatorname{argmin}(g) = \inf\{x : g(x) = \bar{g}\}$. *Then, for any* $x \ge \bar{x}$, V *is equal to* V^l.

THEOREM 1.2. *Suppose that, for any* $0 \le x \le z \le \bar{x}$,

$$\tfrac{1}{2}\sigma^2(x)G''(z) + \mu(x)G'(z) - \alpha(x)G(z) \ge 0. \tag{1.2}$$

Then V *is given by*

$$V(x, y) = \begin{cases} G(x)\exp\bigl(g(\bar{x})y\bigr), & x \ge \bar{x}, \\ G(\bar{x})\exp\bigl(g(\bar{x})\bigl(y - (\bar{x} - x)\bigr)\bigr), & x \le \bar{x} \le x + y, \\ G(x + y), & x + y \le \bar{x}. \end{cases} \tag{1.3}$$

THEOREM 1.3. *Suppose that* σ, μ *and* α *satisfy Assumption* 1. *Then* G *has a unique minimum, located in* $[0, M]$, G *satisfies* (1.2) *and* V *is given by* (1.3).

ASSUMPTION 1. σ^2 *is bounded away from* 0, *and* $\rho \stackrel{\text{def}}{=} \mu/\sigma^2$, $\tilde{\alpha} \stackrel{\text{def}}{=} \alpha/\sigma^2$ *are both* C^1, *and there exists* $0 \le M \le \infty$ *such that* ρ *and* $\tilde{\alpha}$ *are increasing on* $[0, M)$ *and decreasing on* (M, ∞).

Finite-fuel control problems were introduced in Bather and Chernoff (1967), but see also Harrison and Taylor (1977), Beneš, Shepp and Witsenhausen (1980) and Harrison and Taksar (1983). Further work has been done by Karatzas and Shreve (1986), connecting finite-fuel problems to related optimal stopping problems, and by Karatzas, Ocone, Wang and Zervos (2000). For a problem related to the one considered here, see Weerasinghe (1991).

2. Some preliminaries and a verification lemma.

2.1. We take a suitably rich, filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t; t \ge 0), \mathbb{P})$. We assume now that σ, μ and α are given functions:

$$\sigma : \mathbb{R} \to (0, \infty),$$

$$\mu : \mathbb{R} \to \mathbb{R},$$

$$\alpha : \mathbb{R} \to [0, \infty)$$

satisfying:

ASSUMPTION 0. σ^2, μ *and* α *are all locally Hölder continuous and, for some suitable* λ, $\sigma^2 \ge \lambda > 0$.

2.2. We define, for each $y \geq 0$, the control set

$$\mathcal{C}_y = \left\{\text{previsible processes } \xi \text{ of bounded variation: } \int_0^\infty |d\xi_t| \leq y\right\},$$

and $\mathcal{C} = \bigcup \mathcal{C}_y$ and, for each $\xi \in C$, each $x \geq 0$,

$$\tilde{X}_t^\xi(x) = x + \int_0^t \sigma(\tilde{X}_{s-}^\xi)\, dB_s + \int_0^t \mu(\tilde{X}_{s-}^\xi)\, ds + \xi_t, \tag{2.1}$$

where B is an $(\mathcal{F}_t)$ Brownian motion.

Then our "controlled diffusion," X^ξ, is the process given by (2.1) killed at a random time ζ, where ζ is a nonnegative random variable with (conditional) hazard rate $\alpha(\tilde{X}_{t-}^\xi)$ [i.e., $\mathbb{P}(\zeta \geq t | \tilde{X}_s^\xi : s < t) = \exp(-\int_0^t \alpha(\tilde{X}_{s-}^\xi)\, ds)$].

When X is killed it is sent to a coffin state ∂ (which can be thought of as $+\infty$).

2.3. Let us give some additional notation. We define, for any right-continuous, adapted processes X and Y (taking values in $\mathbb{R} \cup \{\partial\} \times \mathbb{R}$), the stopping times

$$\tau(X) = \inf\{t \geq 0 : X_t < 0\},$$
$$T(X) = \inf\{t \geq 0 : X_t = \partial\}.$$

The problem which we wish to solve in the following discussion is as follows.

PROBLEM 1. Find V, where

$$V(x, y) = \inf_{\xi \in \mathcal{C}_y} \mathbb{P}_x\big(\tau(X^\xi) < T(X^\xi)\big).$$

We shall show that, under Assumption 1, the optimal control strategy is to "immediately jump the diffusion to $\bar{x}$ (or as far up as we can) if the diffusion starts below $\bar{x}$ and then reflect the diffusion at $\bar{x}$ until we run out of fuel." Therefore, let us first calculate the candidate payoff under this policy.

2.4. The key concept in the calculations we want to make is that of the uncontrolled diffusion X^0, which is the killed version of the diffusion given by (2.1) when $\xi \equiv 0$.

Notice that the infinitesimal generator for this (killed) diffusion is L, given (for C^2 functions) by

$$L : f \mapsto \tfrac{1}{2}\sigma^2 f'' + \sigma^2 \rho f' - \alpha f$$

(at least for functions defined as 0 at ∂).

Define

$$G(x) = \mathbb{P}_x\big(\tau(X^0) < T(X^0)\big).$$

Thus, G is the probability that X^0 diffuses below 0 before being killed.

LEMMA 2.1. *G satisfies*

$$LG = 0, \tag{2.2}$$

and, defining $\tau_z(X) = \inf\{t \geq 0 : X_t \leq z\}$, *for any* $x \geq z$,

$$\mathbb{P}_x\bigl(\tau_z(X^0) < T(X^0)\bigr) = G(x)/G(z).$$

PROOF. The proof that (2.2) holds is a standard martingale argument, which runs as follows: under Assumption 0, L is uniformly elliptic on the interval $[0, a]$, for any a. Therefore [see Friedman (1975) or Theorem 3.6.6 of Pinsky (1995)], there is a unique solution, h, to the Dirichlet problem:

$$Lh = 0 \qquad \text{in } (0, a), \quad \text{with } h(0) = 1,\ h(a) = G(a).$$

Moreover, denoting by $\tilde{X}^0$ the uncontrolled and unkilled diffusion [i.e., the unkilled solution to (1.1) with $\xi = 0$], it follows from Ito's lemma that

$$h(x) = \mathbb{E}_x \exp\biggl(-\int_0^{S_a} \alpha(\tilde{X}^0_s)\,ds\biggr) h(\tilde{X}^0_{S_a}),$$

where $S_a = \inf\{t : \tilde{X}^0_t \notin (0, a)\}$. Thus, setting $h(\partial) = 0$,

$$h(x) = \mathbb{E}_x h(X^0_{S_a}).$$

Now, by Assumption 0, $\tilde{X}^0$ is regular on $(0, a)$ [see Rogers and Williams (1987), V. 45, and Theorem 2.2.1 of Pinsky (1995)], so $X^0_{S_a}$ either hits 0 or a or dies so that

$$h(x) = \mathbb{P}_x(X^0 \text{ hits } 0 \text{ before } a) + \mathbb{P}_x(X^0 \text{ hits } a \text{ before } 0)\mathbb{P}_a(X^0 \text{ hits } 0) = G(x).$$

Thus, $G = h$, and so satisfies (2.2), on $[0, a]$. Since a is arbitrary, (2.2) follows.

The second claim follows from the fact that X^0 is skip-free downward, so that, in order to go below 0, X^0 must first pass below z. Thus,

$$G(x) = \mathbb{P}_x(X^0 \text{ hits } z \text{ before dying})\mathbb{P}_z(X^0 \text{ hits } 0 \text{ before dying}). \qquad \square$$

REMARK. Notice also that G is decreasing in x and, of course, bounded between 0 and 1.

2.5. To calculate the payoff to Problem 1, notice that, for $X^{\hat{\xi}}$ (the killed diffusion with control as specified) to go below 0 before dying, it needs first to hit the interval $[0, \bar{x}]$ before dying and then to use up the "fuel" y before dying. Thus, if $\hat{V}$ denotes the payoff under control strategy $\hat{\xi}$, we must have

$$\hat{V}(x, y) = \begin{cases} G(x)\hat{V}(\bar{x}, y)/G(\bar{x}), & x \geq \bar{x}, \\ \hat{V}\bigl(\bar{x}, y - (\bar{x} - x)\bigr), & x \leq \bar{x} \leq x + y, \\ G(x + y), & x + y \leq \bar{x}. \end{cases} \tag{2.3}$$

So, to find $\hat{V}$, we need only find $\hat{V}(\bar{x}, y)$. We could do this formally, using a martingale argument and a version of Itô's formula suitable for processes such as $X^{\hat{\xi}}$ (which have both jumps and singular but continuous drift components), but we prefer a heuristic argument since our control lemma will deal with the formal arguments for us. We can think of the reflecting component of our candidate optimal control $\hat{\xi}$ as consisting of a series of infinitesimal jumps of size dy (one occurs each time the diffusion returns to $\bar{x}$). If we think of the control in this manner, we see that

$$\begin{aligned}\hat{V}(\bar{x}, y) &= \hat{V}(\bar{x} + dy, y - dy)\\ &= G(\bar{x} + dy)\hat{V}(\bar{x}, y - dy)/G(\bar{x}),\end{aligned}$$

so that, defining

$$g(x) = G'(x)/G(x),$$

$$\frac{(d/dy)\hat{V}(\bar{x}, y)}{\hat{V}(\bar{x}, y)} = g(\bar{x}). \tag{2.4}$$

Finally, using the boundary condition $\hat{V}(\bar{x}, 0) = G(\bar{x})$, we obtain from (2.3) and (2.4) the candidate payoff

$$\hat{V}(x, y) = \begin{cases} G(x)\exp\bigl(g(\bar{x})y\bigr), & x \ge \bar{x},\\ G(\bar{x})\exp\bigl(g(\bar{x})\bigl(y - (\bar{x} - x)\bigr)\bigr) & x \le \bar{x} \le x + y,\\ G(x + y), & x + y \le \bar{x}\end{cases} \tag{2.5}$$

(at least for $0 \le x$ and $y \ge 0$), with the extension

$$\hat{V}(x, y) = 1 \qquad \text{for } x < 0.$$

Note that $\hat{V}$ is right-continuous in x and $0 \le \hat{V} \le 1$.

2.6. Let us now state our verification lemma.

LEMMA 2.2. (a) *Suppose that* $f : \mathbb{R}_+^2 \to \mathbb{R}$ *satisfies*:

(i) $0 \le f \le 1$ *and* $f \in C^{2,1}(\mathbb{R}_+^2)$,
(ii) $-|f_x| - f_y \ge 0$ *on* $\mathbb{R}_+^2$,
(iii) $Lf \ge 0$ *on* $\mathbb{R}_+^2$

and

(iv) *for each* y, $f(x, y) \to 0$ *as* $x \to \infty$.

Then $f \le V$, *where* V *is the optimal payoff to Problem* 1.

(b) *Suppose*, *in addition*, *that* f *satisfies*:

(v) $f(0, 0) = 1$,

(vi) $(f_x - f_y)Lf = 0$

and

(vii) $f_x|_{x=0} - f_y|_{x=0} = 0$ *for* $y > 0$.

Then

$$f = V.$$

PROOF. (a) Given (x, y) and a control $\xi \in \mathcal{C}_y$, define Y^ξ (the fuel process) by

$$Y_t^\xi = y - \int_0^t |d\xi|,$$

so that Y_t^ξ denotes the fuel remaining at time t when following policy ξ. Now define τ^* to be the first time that the controlled process goes below 0 or is killed, so that

$$\tau^*(X^\xi) = \tau(X^\xi) \wedge T(X^\xi),$$

and then define the process S^ξ by

$$S_t^\xi = 1_{(t<\tau^*)} f(X_t^\xi, Y_t^\xi) + 1_{(t \geq \tau^* = \tau)}.$$

Then the generalized version of Itô's lemma (and the Feynman–Kac formula) tells us that

$$\begin{aligned} dS_t^\xi = 1_{(t<\tau)} \times \{&Lf(X_{t-}^\xi, Y_{t-}^\xi)\,dt + (f_x d\xi_t^c - f_y|d\xi_t^c|) \\ &+ \big(f(X_{t-}^\xi + \Delta\xi_t, Y_{t-}^\xi - |\Delta\xi_t|) - f(X_{t-}^\xi, Y_{t-}^\xi)\big)\} \\ &+ 1_{(t=\tau^*=\tau)}\big(1 - f(X_{t-}^\xi, Y_{t-}^\xi)\big) + dN_t^\xi, \end{aligned} \tag{2.6}$$

where N_t^ξ is a local martingale.

Now let us consider the first three terms within the braces on the right-hand side of (2.6). The first is nonnegative by virtue of condition (iii); the second is nonnegative by virtue of condition (ii), while condition (ii) also implies that the third term is nonnegative [since

$$f(x+\eta, y-\eta) - f(x, y) = \int_0^\eta \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right)(x+u, y-u)\,du,$$

while

$$f(x-\eta, y-\eta) - f(x, y) = \int_0^\eta \left(-\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right)(x-u, y-u)\,du].$$

The fourth term on the right-hand side is nonnegative by virtue of condition (i). Thus,

$$dS_t^\xi = dA_t^\xi + d\tilde{N}_t^\xi,$$

where A^{ξ} is a suitable increasing process and $\tilde{N}^{\xi}$ is a local martingale. It follows that M^{ξ} is a local submartingale, and since it is bounded (between 0 and 1), by condition (i), it is a uniformly integrable submartingale.

Thus,

$$f(x, y) \le \mathbb{E}S_0^{\xi} \le \mathbb{E}S_\infty^{\xi} = \mathbb{E}\big(1_{(\tau^*=\tau(X^{\xi})<\infty)} + 1_{(\tau^*=\infty)} \lim_t f(X_t^{\xi}, Y_t^{\xi})\big)$$
$$= \mathbb{P}_x\big(\tau(X^{\xi}) < T(X^{\xi}, Y^{\xi})\big),$$

the last equality following from the fact that, under our assumptions, $\mathbb{P}(\tau^* = \infty) = 0$ unless $\tilde{X}^{\xi}$ is transient, in which case it follows from condition (iv) that $\lim_t f(X_t^{\xi}, Y_t^{\xi}) = 0$.

Since ξ is an arbitrary element of $\mathcal{C}_y$, we have established that

$$f \le V.$$

(b) To establish the converse, define, for each $y > 0$,

$$I_y = \{x \ge 0 : f_x = f_y\},$$

and, for each $\varepsilon > 0$, define a policy ξ^{ε} by

$$d\xi_t^{\varepsilon} = \begin{cases} 0, & X_{t-}^{\xi^{\varepsilon}} \notin I_{Y_{t-}^{\xi^{\varepsilon}}}, \\ \delta^{\varepsilon}\big(X_{t-}^{\xi^{\varepsilon}}, Y_{t-}^{\xi^{\varepsilon}}\big), & X_{t-}^{\xi^{\varepsilon}} \in I_{Y_{t-}^{\xi^{\varepsilon}}}, \end{cases}$$

where

$$\delta^{\varepsilon}(x, y) = \inf\big\{\delta > 0 : |f_x(x+\delta, y-\delta) - f_y(x+\delta, y-\delta)| \ge \varepsilon\big\} \wedge y.$$

Now consider $S^{\xi^{\varepsilon}}$. If we return to (2.6), it follows from condition (vi) and the form of ξ^{ε} that the first term within the braces on the right-hand side of (2.6) is 0 (Lebesgue a.e.), the second is 0 (since ξ^{ε} increases only by jumps) and the third is bounded above by $\varepsilon d\xi_t^{\varepsilon}$ [from the definition of $\delta^{\varepsilon}(x, y)$]. The fourth term on the right-hand side of (2.6) is 0, owing to condition (v) and the fact that $\tau(X^{\xi^{\varepsilon}}) \ge \inf\{t : Y^{\xi^{\varepsilon}} = 0\}$ [which follows from condition (vi), which implies that $0 \in I_y$ for each $y > 0$]. Thus, $A^{\xi^{\varepsilon}}$, the increasing process in the decomposition of the uniformly integrable submartingale $S^{\xi^{\varepsilon}}$, satisfies

$$A_\infty^{\xi^{\varepsilon}} \le \varepsilon \xi_\infty^{\varepsilon} \le \varepsilon y,$$

and so

$$\mathbb{P}_x\big(\tau(X^{\xi^{\varepsilon}}) < T(X^{\xi^{\varepsilon}}, Y^{\xi^{\varepsilon}})\big) \le f(x, y) + \varepsilon y,$$

and the result follows since ε is arbitrary. $\square$

3. Proofs of Theorems 1.1–1.3.

PROOF OF THEOREM 1.1. Set

$$f(x,y) = G(x)\exp(\bar{g}y).$$

Now we know that $G(x) \xrightarrow{x\to\infty} 0$, so condition (iv) of Lemma 2.2 is satisfied. Moreover, $LG = 0$, so condition (iii) is satisfied. G is decreasing, so $g \le 0$ while $0 \le G \le 1$ and f is clearly $C^{2,1}$ so condition (i) is satisfied. Thus, to prove part (a), we need only establish that f satisfies condition (ii). Now

$$-|f_x| - f_y = G(x)\big(g(x) - \bar{g}\big),$$

which is nonnegative by assumption.

To establish part (b), observe that if $x \ge \bar{x}$, then, defining ξ^ε as in the proof of part (b) of Lemma 2.2 (but with I_y set to $\{\bar{x}\}$), we see that $V(x,y) \le f(x,y) + \varepsilon y$. $\square$

REMARK. If we are slightly more precise, we may actually prove that the optimal policy (for initial $x \ge \bar{x}$) is to use the fuel to reflect the controlled diffusion upward from $\bar{x}$ until we run out of fuel [see Jacka (1999) for details of the argument].

Armed with our verification lemma and with our candidate solution $\hat{V}$, all we have to do to prove Theorem 1.2 is to establish that $\hat{V}$ satisfies conditions (iii)–(vii) of Lemma 2.2.

PROOF OF THEOREM 1.2. We have already dealt with the case where $\bar{x} = 0$ in Theorem 1.1 so assume that $\infty > \bar{x} > 0$. Conditions (i), (iv), (v) and (vii) follow easily. Now it is fairly straightforward to show from (2.5) that

$$L\hat{V} = \begin{cases} 0, & x \ge \bar{x}, \\ \hat{V}(x,y)\big(\frac{1}{2}\sigma^2(x)\bar{g}^2 + \mu(x)\bar{g} - \alpha(x)\big), & x \le \bar{x} \le x+y, \\ \frac{1}{2}\sigma^2(x)G''(x+y) + \mu(x)G'(x+y) - \alpha(x)G(x+y), & \\ & x+y \le \bar{x}, \end{cases}$$

while

$$-|\hat{V}_x| - \hat{V}_y = \begin{cases} G(x)\exp(\bar{g}y)\big(g(x) - \bar{g}\big), & x \ge \bar{x}, \\ 0, & x \le \bar{x}. \end{cases}$$

Thus, (vi) follows immediately and (ii) follows from the fact that $\bar{g}$ is the global minimum of g. So all that remains to prove is condition (iii). Now if we recall that $\bar{x} > 0$ it follows that

$$0 = g'(\bar{x}) = G''(\bar{x})/G(\bar{x}) - g(\bar{x})^2,$$

so that

$$\bar{g}^2 = G''(\bar{x})/G(\bar{x})$$

and

$$L\hat{V} = \begin{cases} 0, & x \geq \bar{x}, \\ e^{\bar{g}(y-(\bar{x}-x))}\big(\frac{1}{2}\sigma^2(x)G''(\bar{x}) + \mu(x)G'(\bar{x}) - \alpha(x)G(\bar{x})\big), & \\ & x \leq \bar{x} \leq x + y, \\ \frac{1}{2}\sigma^2(x)G''(x+y) + \mu(x)G'(x+y) - \alpha(x)G(x+y), & \\ & x + y \leq \bar{x}, \end{cases}$$

and the required inequality follows from (1.2).

Now suppose that $\bar{x} = \infty$. Then $\hat{V}(x, y) = G(x + y)$, $\hat{V}_x \leq 0$ and $\hat{V}_x - \hat{V}_y \equiv 0$, while $L\hat{V} \geq 0$ by (1.2), so conditions (i) to (vii) of Lemma 2.2 follow easily. □

PROOF OF THEOREM 1.3. First we prove that, under Assumption 1, $\bar{x} \in [0, M]$.

Define the operator $\tilde{L} : C^2 \to C$ by

$$\tilde{L} : f \mapsto \tfrac{1}{2} f'' + \rho f' - \tilde{\alpha} f$$

and observe that (since $\tilde{L}G = 0$)

$$\tilde{L}G' = \tilde{\alpha}' G - \rho' G'.$$

Thus, since G is positive and decreasing, it follows from Assumption 1 that

$$\tilde{L}G' \geq 0 \qquad \text{on } [0, M] \tag{3.1}$$

and

$$\tilde{L}G' \leq 0 \qquad \text{on } [M, \infty). \tag{3.2}$$

Now, if $M < \infty$, define

$$k(x) = G'(x)G(M) - G(x)G'(M) = G(x)G(M)\big(g(x) - g(M)\big).$$

It is easy to see that

$$\tilde{L}k(x) = G(M)\big(\tilde{L}G'(x)\big),$$

so that, by (3.2),

$$\tilde{L}k \leq 0 \qquad \text{on } [M, \infty),$$

whereas it follows from the definition of G that $k(x) \xrightarrow{x\to\infty} 0 = k(M)$. So, applying the strong minimum principle to k [see, e.g., Friedman (1975)], we see that k has no negative minimum on $[M, \infty)$ and hence

$$g(x) \geq g(M) \qquad \text{on } [M, \infty).$$

It follows that the global minimum of g, $\bar{g}$, is attained on $[0, M]$.

Now assume that g has another local minimum, m_1 on $[0, M]$, and define

$$h(x) = G'(x)G(m_1) - G(x)G'(m_1) = G(x)G(m_1)\big(g(x) - g(m_1)\big).$$

It follows from (3.1) that

$$\tilde{L}h \geq 0 \qquad \text{on } [0, M],$$

and so it follows from the strong maximum principle that k has no positive maximum on $[\bar{x} \wedge m_1, \bar{x} \vee m_1]$. But $k(m_1) = 0$ and $k(\bar{x}) \leq 0$ and so $k \leq 0$ on $[\bar{x} \wedge m_1, \bar{x} \vee m_1]$, which contradicts the assumption that m_1 is a local minimum. Thus, g has only one local minimum on $[0, M]$ attained at $\bar{x}$.

Finally, to establish (1.2), observe that (since $\tilde{L}G = 0$), for all $x \leq z \leq \bar{x} \leq M$,

$$\begin{aligned} &\tfrac{1}{2}G''(z) + \rho(x)G'(z) - \tilde{\alpha}(x)G(z) \\ &\qquad = -\big(\rho(z) - \rho(x)\big)G'(z) + \big(\tilde{\alpha}(z) - \tilde{\alpha}(x)\big)G(z) \geq 0 \end{aligned} \tag{3.3}$$

by Assumption 1. Note that, in the case where $M = \infty$, (3.3) holds for all $0 \leq x \leq z < \infty$ and so, as in the proof of Theorem 1.2,

$$V(x, y) = G(x + y) \qquad \text{for all } (x, y) \in \mathbb{R}_+^2. \qquad \square$$

REMARK. We have recovered here, as a special case, some of the results of Jacka (1999): if $\tilde{\alpha}$ and ρ are decreasing (corresponding to $M = 0$ in Assumption 1), then the optimal control is a reflecting barrier at 0; if $\tilde{\alpha}$ and ρ are increasing (corresponding to $M = \infty$ in Assumption 1), then the optimal control is to immediately expend all the fuel in a single, upward, jump.

4. Concluding remarks.

4.1. *Problems of existence.* We have been somewhat cavalier about the existence of our controlled diffusions, and the corresponding optimal controls. In fact, under our assumptions, provided we stop the process at the first explosion time and interpret the state (after explosion) as ∂, our analysis goes through, and optimal controls and corresponding controlled diffusions exist. Refer to Section 6 of Jacka (1999) for details.

4.2. *Generalizations.* We remarked in Jacka (1999) that "a general solution, for fairly arbitrary diffusion characteristics, is much more complex, probably combining jumps and reflecting barriers (which may be abandoned when fuel runs low)." As we have seen, at least some of this is true. A general solution would be interesting.

As we observed in Jacka (1999), it is possible to obtain the solution to a discounted version of the original problem by changing the killing rate. Suppose that we want to find

$$\inf_{\xi \in \mathcal{C}_y} \mathbb{E} e^{-r\tau(X^\xi)}. \tag{4.1}$$

Define ${}^rX^\xi$ (for each $r \geq 0$) in a similar fashion to X^ξ: we still use $\tilde{X}^\xi$ given by (2.1), but we kill it at the random time ζ^r, defined in the same way as ζ except that the hazard function is α^r, given by

$$\alpha^r \equiv r + \alpha.$$

Then it is easy to see that

$$\mathbb{E} e^{-r\tau(X^\xi)} = \mathbb{P}\big(\tau({}^rX^\xi) < T({}^rX^\xi)\big),$$

so that the solution to (4.1) is the solution to Problem 1, with the hazard rate α^r.

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