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A GAMBLING SYSTEM AND A MARKOV CHAIN[1]

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"Oscar's system" is a gambling system in which the aim is to win one betting unit, at least with high probability, and then start over again. The system can be modeled by an irreducible Markov chain in a subset of the two-dimensional integer lattice. We show that the Markov chain, which depends on a parameter p representing the single-trial win probability, is transient if $p < \frac{1}{2}$ and positive recurrent if $p \geq \frac{1}{2}$.

1. Introduction. "Oscar's system" was described by Allan N. Wilson in his book *The Casino Gambler's Guide* [(1965), page 247] as follows:

> This system is designed so that in each sequence of plays, the player gains a net profit of one unit, and then starts over again.... The first bet is one unit. Whenever a bet is lost, the next bet is the same size as the one just lost. Whenever a bet is won, the next bet is one unit larger—*unless* winning the next bet would produce for the sequence a profit exceeding one unit. If it would, the bet size is reduced to an amount that is just sufficient to produce a profit of one unit.

The system can be modeled by a Markov chain. We assume that at each trial (in a sequence of independent Bernoulli trials) the gambler wins or loses the amount of his bet, winning with probability p, where $0 < p < 1$. Let X_n denote the number of units the gambler needs to reach his goal following the nth trial and let Y_n denote the gambler's bet size at trial $n+1$. Then $\{(X_n, Y_n),\ n = 0, 1, 2, \ldots\}$ is a Markov chain in the state space

$$S = \{(0,0)\} \cup \{(i, j) \in \mathbf{N} \times \mathbf{N}\colon i \geq j\}, \tag{1.1}$$

where $\mathbf{N} = \{1, 2, 3, \ldots\}$, with initial state $(X_0, Y_0) = (1, 1)$ and transition probabilities

$$P((i,j),(k,l)) = \begin{cases} p, & \text{if } (k,l) = (i-j, (j+1) \wedge (i-j)), \\ q, & \text{if } (k,l) = (i+j, j), \\ 0, & \text{otherwise}, \end{cases} \tag{1.2}$$

for all $i \geq j \geq 1$, where $q = 1 - p$ and $a \wedge b = \min\{a, b\}$. If we assume that

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upon completion of a sequence a new one is begun, then

$$P((0,0),(k,l)) = \begin{cases} 1, & \text{if } (k,l) = (1,1), \\ 0, & \text{otherwise}, \end{cases} \tag{1.3}$$

and the chain is irreducible and aperiodic.

In a previous paper [Ethier (1996)] the author investigated the problem of finding the probability that the gambler achieves his one-unit goal before the system calls for a bet that exceeds a house limit of M units. The problem had been posed by Wilson (1962) in *The American Mathematical Monthly*. Let $\{(X_n, Y_n),\ n = 0, 1, 2, \ldots\}$ denote the Markov chain in S with transition probabilities (1.2) and (1.3) and define

$$T = \min\{n \geq 0\colon (X_n, Y_n) = (0,0) \text{ or } Y_n \geq M+1\}, \tag{1.4}$$

where the positive integer M is fixed. The problem was to find $Q(1,1)$, where in general

$$Q(i,j) = \mathbf{P}\{(X_T, Y_T) = (0,0) \mid (X_0, Y_0) = (i,j)\} \tag{1.5}$$

for all $(i,j) \in S$. These probabilities satisfy the linear system

$$\begin{aligned} Q(i,j) = pQ(i-j, (j+1) \wedge (i-j)) + qQ(i+j, j),& \\ j \leq i \leq I(j),\ 1 \leq j \leq M,& \end{aligned} \tag{1.6}$$

where $Q(0,0) = 1$ and $Q(i,j) = 0$ if $j \geq M+1$ or if both $1 \leq j \leq M$ and $i \geq I(j) + 1$. Here

$$I(j) = \left(\sum_{k=j}^{M+1} k \right) - 1 \tag{1.7}$$

and so the number of equations in the system is

$$\sum_{j=1}^{M} (I(j) - j + 1) = \frac{M(M+1)(M+2)}{3}. \tag{1.8}$$

In the author's earlier paper, a recursive algorithm was provided that yields the exact value of $Q(1,1)$ in a finite number of steps (depending on M).

Some numerical values for $Q(1,1)$ are given in Table 1. Results are included for $M = 50, 100, 150, \ldots, 350$ and three choices of p (motivated by roulette, craps and coin-tossing). They were obtained on a high-speed Hewlett–Packard computer with a 128-megabyte memory.

In the present paper we address the following questions. If Table 1 could be extended ad infinitum (and with arbitrary precision), would the sequence of probabilities in the second column converge to 1? What about the third column? More generally, is the Markov chain in S with transition probabilities (1.2) and (1.3) recurrent or transient? The following theorem provides the answers and a bit more.

THEOREM 1. *The Markov chain in S with transition probabilities* (1.2) *and* (1.3) *is transient if $p < \frac{1}{2}$ and positive recurrent if $p \geq \frac{1}{2}$.*

TABLE 1

$Q(1, 1)$, *the probability using Oscar's system of achieving a net profit of one unit before being forced to abort the system, as a function of the house limit M and the single-trial win probability p* [a]

M	$p = 9/19$	$p = 244/495$	$p = 1/2$
50	0.99078112	0.99620367	0.99734697
100	0.99464246	0.99841219	0.99904342
150	0.99587158	0.99902785	0.99947577
200	0.99646866	0.99930553	0.99965833
250	0.99681764	0.99946077	0.99975501
300	0.99704404	0.99955895	0.99981337
350	0.99720107	0.99962624	0.99985174

[a] All figures are rounded to eight decimal places. [From Ethier (1996).]

Thus, when $p < \frac{1}{2}$ as is typically the case in a gambling house, there is positive probability of not achieving the one-unit goal, even in the absence of a house limit. This is proved in Section 2 using a probabilistic argument based on the strong law of large numbers.

The case $p = \frac{1}{2}$ is the most interesting from the mathematical point of view. If the rules were simplified so that the gambler's bet size remained at one unit at each trial, the system could be modeled by a reflecting random walk in the set of nonnegative integers, which of course is null recurrent when $p = \frac{1}{2}$. Evidently, the increasing bet sizes in Oscar's system are responsible for the switch to positive recurrence in the critical case. In Section 3 we give an analytic proof based on Foster's (1953) criterion.

It seems that for most Markov chains depending on a parameter, null recurrence occurs at the boundary between transience and positive recurrence, but the present example is an exception.

2. Transience when $p < \frac{1}{2}$. Let $\xi_1, \xi_2, \ldots$ be a sequence of i.i.d. random variables on some probability space $(\Omega, \mathscr{F}, \mathbf{P})$ with $\mathbf{P}\{\xi_1 = 1\} = p = 1 - \mathbf{P}\{\xi_1 = 0\}$, where $0 < p < \frac{1}{2}$, and let i be a positive integer (to be specified later). We can construct the Markov chain $\{(X_m, Y_m),\ m = 0, 1, \ldots\}$ in S with initial state $(i, 1)$ and transition probabilities (1.2) and (1.3) in terms of the sequence $\xi_1, \xi_2, \ldots,$ at least up to, but not including, time

$$T_D = \min\{m \geq 1\colon X_m - Y_m \leq 0\}, \tag{2.1}$$

where $\min \varnothing = \infty$. Specifically, we define $(X_0, Y_0) = (i, 1)$ and

$$X_{m+1} = X_m - (2\xi_{m+1} - 1)Y_m, \qquad Y_{m+1} = Y_m + \xi_{m+1}, \tag{2.2}$$

for $m = 0, 1, \ldots,$ and we define T_D in terms of these sequences by (2.1).

Let $0 < \varepsilon < \frac{1}{4} \wedge p$ be such that $p - \varepsilon$ is rational and

$$(1 - 2(p - \varepsilon))(p - \varepsilon)\frac{(1 - 4\varepsilon)^2}{2} - 4\varepsilon(p + \varepsilon) > 0. \tag{2.3}$$

Since $Y_m - 1 = \xi_1 + \cdots + \xi_m$, the strong law of large numbers implies that there exists a positive-integer-valued random variable N such that

$$\mathbf{P}\{|m^{-1}(Y_m - 1) - p| \le \varepsilon \text{ for all } m \ge N\} = 1. \tag{2.4}$$

Choose a positive integer N_0 such that $\mathbf{P}\{N \le N_0\} \ge 1 - \varepsilon$. Then the event

$$A = \{m(p - \varepsilon) \le Y_m - 1 \le m(p + \varepsilon) \text{ for all } m \ge N_0\} \tag{2.5}$$

satisfies $\mathbf{P}(A) \ge 1 - \varepsilon$.

Given a (nonrandom) $\{0, 1\}$-valued sequence $\eta_1, \eta_2, \ldots$, we define, by analogy with (2.2), $(x_0, y_0) = (i, 1)$ and

$$x_{m+1} = x_m - (2\eta_{m+1} - 1)y_m, \qquad y_{m+1} = y_m + \eta_{m+1}, \tag{2.6}$$

for $m = 0, 1, \ldots$. Let

$$E = \{(\eta_m) \in \{0, 1\}^{\mathbf{N}}\colon m(p - \varepsilon) \le y_m - 1 \le m(p + \varepsilon) \text{ for all } m \ge N_0\} \tag{2.7}$$

and, for each $n \ge N_0$, choose $(\eta_m^{(n)}) \in E$ so that the corresponding sequence $(x_m^{(n)}, y_m^{(n)})$ [as in (2.6)] satisfies $y_n^{(n)} - 1 = [n(p + \varepsilon)]$ and

$$y_l^{(n)} = \min\{y_l\colon (\eta_m) \in E \text{ and } y_n - 1 = [n(p + \varepsilon)]\}, \qquad 1 \le l \le n - 1. \tag{2.8}$$

We claim that

$$X_n \ge x_n^{(n)} \text{on } A, \qquad n \ge N_0. \tag{2.9}$$

Before proving this, we indicate the underlying idea. The set E can be interpreted as the set of all admissible sequences of wins and losses [admissible in the sense that $\omega \in A$ implies $(\xi_m(\omega)) \in E$]. Given $n \ge N_0$, the sequence $(\eta_m^{(n)}) \in E$ is chosen to maximize the number of wins in the first n trials, and yet postpone them as long as possible [cf. (2.8)]. Inequality (2.9) says that $(\eta_m^{(n)})$ is the gambler's best-case scenario in E, insofar as his fortune following the nth trial is concerned. This is intuitively clear, since, under the sequence $(\eta_m^{(n)})$ of wins and losses, winning bets occur later (closer to trial n) when the bet size is larger.

To make this precise, note that if $\omega \in A$, the vector $(\eta_1^{(n)}, \ldots, \eta_n^{(n)})$ can be obtained from $(\xi_1(\omega), \ldots, \xi_n(\omega))$ by means of a finite number of "transpositions" of the form $(\eta_1, \ldots, \eta_n) \mapsto (\eta'_1, \ldots, \eta'_n)$, where, for some $k \in \{1, \ldots, n - 1\}$, $(\eta_k, \eta_{k+1}) = (1, 0)$, $(\eta'_k, \eta'_{k+1}) = (0, 1)$ and $\eta'_l = \eta_l$ for all $l \in \{1, \ldots, n\} - \{k, k+1\}$, followed by a finite number of "substitutions" of the form $(\eta_1, \ldots, \eta_n) \mapsto (\eta''_1, \ldots, \eta''_n)$, where, for some $k \in \{1, \ldots, n\}$, $\eta_k = 0$, $\eta_l = 1$ for $l = k+1, \ldots, n$, $\eta''_k = 1$ and $\eta''_l = \eta_l$ for all $l \in \{1, \ldots, n\} - \{k\}$. If (x_m, y_m), (x'_m, y'_m) and (x''_m, y''_m) are the corresponding sequences as in (2.6), then $x'_n = x_n - 1$ and $x''_n = x_n - 2y_{k-1} - (n - k)$. We conclude that (2.9) holds.

It follows from (2.9) that

$$X_n - Y_n \geq x_n^{(n)} - 1 - [n(p+\varepsilon)] \text{ on } A, \qquad n \geq N_0. \tag{2.10}$$

Notice that if we denote $x_n^{(n)}$ by $x_n^{(n)}(i)$ to indicate its dependence on i, then $x_n^{(n)}(i) = x_n^{(n)}(1) + i - 1$. Thus, if we could show that

$$x_n^{(n)}(1) - [n(p+\varepsilon)] \to \infty \quad \text{as } n \to \infty, \tag{2.11}$$

it would follow that, for i sufficiently large, $X_n - Y_n \geq 1$ on A for all $n \geq 1$, that is, $T_D = \infty$ on A, and so

$$\mathbf{P}\{T_D = \infty\} \geq 1 - \varepsilon > 0. \tag{2.12}$$

This would then imply that the Markov chain in S with transition probabilities (1.2) and (1.3) is transient when $p < \frac{1}{2}$.

It remains to establish (2.11), which is simply a property of a deterministic sequence. Note that the last l_n terms of $\eta_1^{(n)}, \ldots, \eta_n^{(n)}$ are all 1's, where l_n is the largest integer satisfying

$$\frac{[n(p+\varepsilon)] - l_n}{n - l_n} \geq p - \varepsilon, \tag{2.13}$$

hence

$$l_n = \left[\frac{[n(p+\varepsilon)] - n(p-\varepsilon)}{1 - p + \varepsilon}\right] \leq \frac{2n\varepsilon}{1 - p + \varepsilon} < 4n\varepsilon. \tag{2.14}$$

At each m for which $N_0 + 1 \leq m \leq n - l_n - 1$, if $\eta_m^{(n)} = 1$, then $\eta_{m+1}^{(n)} = 0$. [This follows from the fact that $b/a \geq p - \varepsilon$ implies $(b+1)/(a+2) \geq p - \varepsilon$.] So each win is followed immediately by a loss, and additional losses are interspersed at various times. To make things easier to analyze, we apply some of the transpositions described earlier to $(\eta_m^{(n)})$ to obtain $(\tilde{\eta}_m^{(n)})$, which will belong to $\{0, 1\}^{\mathbf{N}}$ but not to E. Recall that $p - \varepsilon = r/s$ for positive integers r and s with $2r < s$. For each segment $\{Ms+1, \ldots, (M+1)s\}$ ($M = 1, 2, \ldots$) in $\{N_0 + 1, \ldots, n - l_n\}$, the s terms of the sequence $(\eta_m^{(n)})$ do not depend on M. There are $s - 2r$ 0's and r (1,0)'s in some specific order. We define $(\tilde{\eta}_m^{(n)})$ by requiring that on each segment $\{Ms+1, \ldots, (M+1)s\}$ in $\{N_0+1, \ldots, n - l_n\}$ it consist of $s - 2r$ 0's followed by r (1,0)'s (in that order). Elsewhere it is the same as $(\eta_m^{(n)})$. Since $\tilde{x}_n^{(n)}(1) \leq x_n^{(n)}(1)$ by the argument in the second paragraph below (2.9), it will suffice to show that

$$\tilde{x}_n^{(n)}(1) - [n(p+\varepsilon)] \to \infty \quad \text{as } n \to \infty. \tag{2.15}$$

On each such segment $\{Ms+1, \ldots, (M+1)s\}$, the behavior of $(\tilde{x}_m^{(n)}(1))$ is very simple. The $s - 2r$ initial losses increase $\tilde{x}_\bullet^{(n)}(1)$ by $(s-2r)(Mr+1)$ since

$\tilde{y}^{(n)}_{Ms} - 1 = (r/s)Ms$, and the remaining r win–loss pairs increase it further by r. Consequently, we have

$$\begin{aligned}
\tilde{x}^{(n)}_n(1) &\geq \tilde{x}^{(n)}_{N_0}(1) + \sum_{M=[N_0/s]+1}^{[(n-l_n)/s]-1} \{(s-2r)(Mr+1)+r\} \\
&\qquad - \sum_{j=1}^{l_n} \{[n(p+\varepsilon)] - l_n + j\} \\
&= (s-2r)r\binom{[(n-l_n)/s]}{2} + (s-r)\left[\frac{n-l_n}{s}\right] \\
&\qquad - l_n[n(p+\varepsilon)] + \binom{l_n}{2} + O(1) \\
&\geq \left\{(1-2(p-\varepsilon))(p-\varepsilon)\frac{(1-4\varepsilon)^2}{2} - 4\varepsilon(p+\varepsilon)\right\}n^2 + O(n)
\end{aligned} \tag{2.16}$$

and hence (2.3) implies that (2.15) holds, as required.

3. Positive recurrence when $p \geq \frac{1}{2}$. Here we rely on Foster's (1953) criterion, which for our purposes can be stated as follows. [See Meyn and Tweedie (1993) for more-general formulations. See also Fayolle, Malyshev and Menshikov (1995).]

FOSTER'S CRITERION. *An irreducible Markov chain in a countable state space S with transition probabilities $\{P(i,j),\ i,j \in S\}$ is positive recurrent if and only if there exist a function $V\colon S \mapsto [0,\infty)$, a constant $\varepsilon > 0$ and an element i_0 of S such that*

$$\sum_{j\in S} P(i,j)V(j) - V(i) \leq -\varepsilon, \qquad i \in S - \{i_0\}, \tag{3.1}$$

and

$$\sum_{j\in S} P(i,j)V(j) < \infty, \qquad i = i_0. \tag{3.2}$$

Let us begin with the simplest (and least interesting) case of (1.1)–(1.3), namely, the case in which $\frac{1}{2} < p < 1$. Taking $V(i,j) \equiv i$ and $\varepsilon = p - q$, we find that

$$\begin{aligned}
&\sum_{(k,l)\in S} P((i,j),(k,l))V(k,l) - V(i,j) \\
&\qquad = pV(i-j,(j+1)\wedge(i-j)) + qV(i+j,j) - V(i,j) \\
&\qquad = p(i-j) + q(i+j) - i \\
&\qquad = -j\varepsilon \\
&\qquad \leq -\varepsilon, \qquad (i,j) \in S - \{(0,0)\}.
\end{aligned} \tag{3.3}$$

Thus, Foster's criterion [with $i_0 = (0,0)$] implies the positive recurrence when $p > \frac{1}{2}$.

For the remainder of this section, we assume (1.1)–(1.3) *with* $p = \frac{1}{2}$. Define the operator Δ on real-valued functions V on S by

$$\Delta V(i,j) = \sum_{(k,l)\in S} P((i,j),(k,l))V(k,l) - V(i,j)$$

$$= \begin{cases} \frac{1}{2}V(i-j,(j+1)\wedge(i-j)) \\ \qquad + \frac{1}{2}V(i+j,j) - V(i,j), & (i,j) \neq (0,0), \\ V(1,1) - V(0,0), & (i,j) = (0,0), \end{cases} \tag{3.4}$$

and observe that

$$\Delta V(i,j) = \Delta_1 V(i,j) + \Delta_2 V(i,j), \qquad (i,j) \in S - \{(0,0)\}, \tag{3.5}$$

where

$$\Delta_1 V(i,j) = \tfrac{1}{2}V(i-j, j\wedge(i-j)) + \tfrac{1}{2}V(i+j,j) - V(i,j) \tag{3.6}$$

and

$$\Delta_2 V(i,j) = \tfrac{1}{2}V(i-j,(j+1)\wedge(i-j)) - \tfrac{1}{2}V(i-j, j\wedge(i-j)). \tag{3.7}$$

This suggests that, to apply Foster's criterion, we should look for a nonnegative function V on S that is concave in the first variable and decreasing in the second variable.

We begin by considering the function V_1 on S given by

$$V_1(i,j) = i^\alpha, \tag{3.8}$$

where $0 < \alpha < 1$.

LEMMA 1. *Fix* $0 < \alpha < 1$ *and* $b \geq 0$. *Then*

$$(1+x)^\alpha \leq 1 + \alpha x - \frac{\alpha(1-\alpha)}{2(1+b)^2}x^2, \qquad -1 \leq x \leq b, \tag{3.9}$$

and

$$(1+x)^\alpha \geq 1 + \alpha x - \frac{\alpha(1-\alpha)}{2}x^2, \qquad 0 \leq x < \infty. \tag{3.10}$$

PROOF. Let $f(x)$ and $g(x)$ denote the left- and right-hand sides of (3.9). Then $f(0) = g(0)$, $f'(0) = g'(0)$ and $f'' \leq g''$ on $(-1, b]$, so (3.9) (except with $x = -1$) are consequences of the Taylor expansion

$$f(x) = f(0) + xf'(0) + x^2\int_0^1 (1-u)f''(ux)\,du, \qquad -1 < x \leq b, \tag{3.11}$$

and the analogous expansion for g. The case $x = -1$ then follows by continuity.

Inequality (3.10) is proved similarly. □

Using (3.9) with $b = 1$, we find that

$$\begin{aligned} \Delta V_1(i, j) &= \frac{1}{2}(i-j)^\alpha + \frac{1}{2}(i+j)^\alpha - i^\alpha \\ &= i^\alpha\left[\frac{1}{2}\left(1-\frac{j}{i}\right)^\alpha + \frac{1}{2}\left(1+\frac{j}{i}\right)^\alpha - 1\right] \\ &\le -\frac{\alpha(1-\alpha)}{8}\frac{j^2}{i^{2-\alpha}}, \qquad (i, j) \in S - \{(0,0)\}, \end{aligned} \tag{3.12}$$

and therefore that

$$\Delta V_1(i, j) < 0, \qquad (i, j) \in S - \{(0,0)\}, \tag{3.13}$$

and

$$\Delta V_1(i, j) \le -\frac{\alpha(1-\alpha)}{72}, \qquad (i, j) \in A, \tag{3.14}$$

where

$$A = \left\{(i, j) \in S - \{(0,0)\}\colon j \ge \tfrac{1}{3} i^{(2-\alpha)/2}\right\}. \tag{3.15}$$

Next, given $0 < \beta < 1$, we consider the function V_2 on S defined by

$$V_2(i, j) = \begin{cases} 0, & \text{if } (i/\beta)^\beta < j \le i, \\ (i - \beta j^{1/\beta})^2 j^{-2}, & \text{if } i^\beta < j \le (i/\beta)^\beta \wedge i, \\ i^{2(1-\beta)} - \beta(2-\beta) j^{2(1-\beta)/\beta}, & \text{if } 1 \le j \le i^\beta, \\ 0, & \text{if } (i, j) = (0,0). \end{cases} \tag{3.16}$$

Note that V_2, if extended via (3.16) to all of $\{(0,0)\} \cup \{(i, j) \in [1,\infty) \times [1,\infty)\colon i \ge j\}$, is continuous, as is its first partial derivative with respect to i. (The significance of this will become apparent later.)

LEMMA 2. *Assume* $\frac{1}{2} < \beta \le \frac{2}{3}$. *Then*

$$\Delta V_2(i, j) < \tfrac{9}{8}, \qquad (i, j) \in S - \{(0,0)\}, \tag{3.17}$$

and

$$\Delta V_2(i, j) < -(1-\beta)(2-\beta), \qquad (i, j) \in B, \tag{3.18}$$

where

$$B = \left\{(i, j) \in S - \{(0,0)\}\colon j \le \tfrac{1}{3} i^\beta\right\}. \tag{3.19}$$

REMARK. As we will see, the second term in the third line of (3.16) gives (3.18) (using $\beta \le \frac{2}{3}$). The first term in that line makes V_2 nonnegative and is harmless because of (3.13) (using $\beta > \frac{1}{2}$). Finally, the second line of (3.16) smooths out V_2 sufficiently to give (3.17).

PROOF OF LEMMA 2. With reference to (3.16), we define

$$
\begin{aligned}
E &= \{(i, j) \in S\colon (i/\beta)^\beta < j \le i\},\\
F &= \{(i, j) \in S\colon i^\beta < j \le (i/\beta)^\beta \wedge i\},\\
G &= \{(i, j) \in S\colon 1 \le j \le i^\beta\}.
\end{aligned}
\tag{3.20}
$$

Note that $E \cup F \cup G = S - \{(0,0)\}$ and that $(1,1) \in G$, $(2,2) \in F$ (using $\beta > \frac{1}{2}$) and $(i, i) \in E$ for all $i \ge 3$.

If $(i, j) \in B$, then $j \le \frac{1}{3} i^\beta < \frac{1}{3} i$, so $(i-j)^\beta > (\frac{2}{3} i)^\beta > \frac{2}{3} i^\beta \ge 2j \ge j+1$. Consequently, $(i, j) \in G$, $(i-j, j+1) \in G$ and $(i-j, j) \in G$. By (3.13), $\Delta_1 V_2(i, j) < 0$ and therefore

$$
\begin{aligned}
\Delta V_2(i, j) &= \Delta_1 V_2(i, j) + \Delta_2 V_2(i, j)\\
&< \Delta_2 V_2(i, j)\\
&= -\tfrac{1}{2}\beta(2-\beta)((j+1)^{2(1-\beta)/\beta} - j^{2(1-\beta)/\beta})\\
&\le -(1-\beta)(2-\beta),
\end{aligned}
\tag{3.21}
$$

where the second inequality uses the fact that $(j+1)^\gamma - j^\gamma = \int_j^{j+1} \gamma x^{\gamma-1}\,dx \ge \gamma$ if $\gamma \ge 1$, and this gives (3.18).

To establish (3.17), notice first that $V_2(i, \cdot)$ is nonincreasing for each $i \ge 1$. This implies that

$$
\Delta_2 V_2(i, j) \le 0, \qquad (i, j) \in S - \{(0,0)\}. \tag{3.22}
$$

Observe next that, for each $j \ge 1$, $V_2(\cdot, j)$ is 0 on $[j, \beta j^{1/\beta})$ (if $j \ge 3$), quadratic and convex on $[\beta j^{1/\beta} \vee j, j^{1/\beta})$ and concave on $[j^{1/\beta}, \infty)$. Since V_2, together with its first partial derivative with respect to i, is continuous,

$$
V_2(i, j) \le (i - \beta j^{1/\beta})^2 j^{-2}, \qquad (i, j) \in S - \{(0,0)\}, \tag{3.23}
$$

with equality holding if $(i, j) \in F$.

It remains to bound $\Delta_1 V_2$. There are several cases to consider:

Case 1. $(i, j) \in E$ and $(i+j, j) \in E$. Since $V_2(1,1) = (1-\beta)^2 < \frac{1}{4}$, $V_2(2,2) = \frac{1}{4}(2 - \beta 2^{1/\beta})^2 < \frac{1}{4}$ and $V_2(l, l) = 0$ for $l = 0$ and all $l \ge 3$, and since $j < i - j$ implies $(i-j, j) \in E$, we have $\Delta_1 V_2(i, j) = \frac{1}{2} V_2(i-j, j \wedge (i-j)) < 1/8$.

Case 2. $(i, j) \in E$ and $(i+j, j) \in F$. Here $i < \beta j^{1/\beta} \le i + j$, so $\Delta_1 V_2(i, j) = \frac{1}{2} V_2(i-j, j \wedge (i-j)) + \frac{1}{2}(i + j - \beta j^{1/\beta})^2 j^{-2} < \frac{1}{8} + \frac{1}{2}$.

Case 3. $(i, j) \in E$ and $(i+j, j) \in G$. [This can occur for only finitely many (i, j).] In view of (3.23), the argument used in Case 2 applies here, with the equality replaced by an inequality. (Note that $i < \beta j^{1/\beta} \le i + j$ continues to hold in this case.)

Case 4. $(i, j) \in F$. By (3.23),

$$
\begin{aligned}
\Delta_1 V_2(i, j) &\le \tfrac{1}{2}(i - j - \beta j^{1/\beta})^2 j^{-2} + \tfrac{1}{2} V_2(i-j, i-j)\\
&\qquad + \tfrac{1}{2}(i + j - \beta j^{1/\beta})^2 j^{-2} - (i - \beta j^{1/\beta})^2 j^{-2}\\
&< 1 + \tfrac{1}{8}.
\end{aligned}
\tag{3.24}
$$

Here we have bounded $\frac{1}{2}V_2(i-j, j\wedge(i-j))$ by the sum of the first two terms after the first inequality in (3.24).

Case 5. $(i, j) \in G$ and $(i-j, j\wedge(i-j)) \in G - \{(1,1)\}$. Since $j\wedge(i-j) = j$ and $(i+j, j) \in G$, we have $\Delta_1 V_2(i, j) < 0$ by (3.13).

Case 6. $(i, j) \in G$ and $(i-j, j\wedge(i-j)) \in F - \{(2,2)\}$. Again, $j\wedge(i-j) = j$ and $(i+j, j) \in G$. Also, $i = j^{1/\beta} + \theta j$ for some $\theta \in [0, 1)$, so

$$
\begin{aligned}
&\Delta_1 V_2(i, j)\\
&\quad = \tfrac{1}{2}(j^{1/\beta} + (\theta-1)j - \beta j^{1/\beta})^2 j^{-2}\\
&\qquad + \tfrac{1}{2}((j^{1/\beta} + (\theta+1)j)^{2(1-\beta)} - \beta(2-\beta)j^{2(1-\beta)/\beta})\\
&\qquad - ((j^{1/\beta} + \theta j)^{2(1-\beta)} - \beta(2-\beta)j^{2(1-\beta)/\beta})\\
&\quad = \tfrac{1}{2}j^{2(1-\beta)/\beta}(1-\beta + (\theta-1)j^{-(1-\beta)/\beta})^2\\
&\qquad + \tfrac{1}{2}j^{2(1-\beta)/\beta}((1+(\theta+1)j^{-(1-\beta)/\beta})^{2(1-\beta)} - \beta(2-\beta))\\
&\qquad - j^{2(1-\beta)/\beta}((1+\theta j^{-(1-\beta)/\beta})^{2(1-\beta)} - \beta(2-\beta))\\
&\quad \le \tfrac{1}{2}j^{2(1-\beta)/\beta}((1-\beta)^2 + 2(1-\beta)(\theta-1)j^{-(1-\beta)/\beta} + (\theta-1)^2 j^{-2(1-\beta)/\beta})\\
&\qquad + \tfrac{1}{2}j^{2(1-\beta)/\beta}\big(1 + 2(1-\beta)(\theta+1)j^{-(1-\beta)/\beta}\\
&\qquad\qquad - \tfrac{1}{9}(1-\beta)(1-2(1-\beta))(\theta+1)^2 j^{-2(1-\beta)/\beta} - \beta(2-\beta)\big)\\
&\qquad - j^{2(1-\beta)/\beta}\big(1 + 2(1-\beta)\theta j^{-(1-\beta)/\beta}\\
&\qquad\qquad - (1-\beta)(1-2(1-\beta))\theta^2 j^{-2(1-\beta)/\beta} - \beta(2-\beta)\big)\\
&\quad = \tfrac{1}{2}(\theta-1)^2 - \tfrac{1}{18}(1-\beta)(2\beta-1)(\theta+1)^2 + (1-\beta)(2\beta-1)\theta^2\\
&\quad < \tfrac{1}{2} + \tfrac{1}{9},
\end{aligned}
\tag{3.25}
$$

where the first inequality uses (3.9) with $b = 2$ and (3.10).

Case 7. $(i, j) \in G$ and $(i-j, j\wedge(i-j)) \in E \cup \{(0,0),(1,1),(2,2)\}$. [This can occur for only finitely many (i, j).] The term following the first equality in (3.25) is replaced by $\frac{1}{2}V_2(i-j, j\wedge(i-j)) < 1/8$, so from Case 6, $\Delta_1 V_2(i, j) < \frac{11}{18} + \frac{1}{8}$. [To see that this argument applies here, note that if $j < i-j$, then $(i-j, j) \in E$, so $i-j < \beta j^{1/\beta} < j^{1/\beta}$, while if $j \ge i-j$, then $i-j \le j < j^{1/\beta}$; in either case, $i = j^{1/\beta} + \theta j$ for some $\theta \in [0, 1)$.]

The resulting bound on $\Delta_1 V_2$, together with (3.22), gives (3.17) and proves the lemma. □

We now complete the argument. In addition to $\frac{1}{2} < \beta \le \frac{2}{3}$, assume that $2(1-\beta) \le \alpha < 1$ and define the nonnegative function V on S by

$$V = V_1 + \eta V_2, \tag{3.26}$$

where $\eta > 0$ remains to be determined. Then, by (3.14) and (3.17),

$$\Delta V(i, j) < -\frac{\alpha(1-\alpha)}{72} + \frac{9}{8}\eta, \qquad (i, j) \in A, \tag{3.27}$$

and by (3.13) and (3.18),

$$\Delta V(i, j) < -(1-\beta)(2-\beta)\eta, \qquad (i, j) \in B. \tag{3.28}$$

We can equalize the right-hand sides of (3.27) and (3.28) by taking

$$\eta = \frac{\alpha(1-\alpha)}{72(1-\beta)(2-\beta)+81}. \tag{3.29}$$

Since $(2-\alpha)/2 \le \beta$, we have $A \cup B = S - \{(0,0)\}$ and we can apply Foster's criterion with V as in (3.26), $\varepsilon = (1-\beta)(2-\beta)\eta$ and $i_0 = (0,0)$. This establishes the positive recurrence when $p = \frac{1}{2}$ and completes the proof of the theorem.

4. Additional remarks. (1) Back in the early 1960s, Julian Braun ran a computer simulation of Oscar's system, as reported by Wilson [(1965), pages 250–252]. He took the house limit to be $M = 500$ units and the single-trial win probability to be $p = 244/495$ (from the game of craps). There were 66 disasters in 280,000 sequences, which led to an estimate of $Q(1,1)$ of 0.99976429 in this case.

When $M = 500$ and $p = 244/495$, the linear system (1.6) has 41,917,000 equations [see (1.8)]. Using the algorithm referred to in Section 1 and a Silicon Graphics Power Challenge XL computer at the Utah Supercomputing Institute, we have found in this case that $Q(1,1) = 0.999740807832$, rounded to 12 decimal places. (This double-precision computation required about 200 megabytes of memory and about 7.5 hours of CPU time.) Although the simulation estimate of the complementary probability is off by about 9%, it is accurate to within 1 standard deviation for a simulation of its size.

(2) Let $p = \frac{1}{2}$ and let $\{(X_n, Y_n),\ n = 0, 1, \dots\}$ be the Markov chain in S with initial state $(X_0, Y_0) = (1,1)$ and transition probabilities (1.2) and (1.3). Let $\mathscr{F}_n = \sigma\{(X_m, Y_m)\colon m = 0, \dots, n\}$, let T_0 be the hitting time of the state $(0,0)$ and define $X_n^* = X_{n \wedge T_0}$. Then $\{X_n^*, \mathscr{F}_n,\ n = 0, 1, \dots\}$ is a martingale, but despite the fact that $\mathbf{E}[T_0] < \infty$, the optional stopping theorem does not apply. Indeed, $\mathbf{E}[X_{T_0}^*] = 0$ while $\mathbf{E}[X_0^*] = 1$. However, because the increments grow at most linearly ($|X_{n+1}^* - X_n^*| \le Y_n \le n+1$), a second-moment assumption on the stopping time will suffice for the applicability of the optional stopping theorem. [This follows from a minor modification of Theorem 4.7.5 of Durrett (1995).] In particular, $\mathbf{E}[T_0^2] = \infty$.

(3) Let $p = \frac{1}{2}$ and define T_0 as in the preceding remark. Recently, Borovkov (1996) has improved on our result that $\mathbf{E}[T_0] < \infty$ by showing that

$$\mathbf{P}\{T_0 > n\} = O(n^{-3/2} \log^{3/2} n) \quad \text{as } n \to \infty. \tag{4.1}$$

The proof involves combining embedding techniques with boundary problems for the Wiener process.

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