The Annals of Applied Probability
1996, Vol. 6, No. 3, 1041–1043

SAMPLE QUANTILES OF STOCHASTIC PROCESSES WITH STATIONARY AND INDEPENDENT INCREMENTS

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The purpose of this note is to obtain a representation of the distribution of the α-quantile of a process with stationary and independent increments as the sum of the supremum and the infimum of two rescaled independent copies of the process. This representation has already been proved for a Brownian motion. The proof is based on already known discrete time results.

1. Introduction. Let $(X(t),\ t \geq 0)$ be a process with stationary and independent increments with $X(0) = 0$ and consider the version with paths in $D[0, \infty)$ (see [2], page 306). For $0 < \alpha < 1$, define the α-quantile of $(X(s),\ 0 \leq s \leq t)$ by

$$M(\alpha, t) = \inf\left\{x\colon \int_0^t \mathbf{1}(X(s) \leq x)\, ds > \alpha t\right\}.$$

Our main result is the following theorem

THEOREM 1. *Let $X^{(1)}(t)$ and $X^{(2)}(t)$ be independent copies of $X(t)$. Then,*

$$\begin{pmatrix} M(\alpha, t) \\ X(t) \end{pmatrix} = \begin{pmatrix} \sup\limits_{0 \leq s \leq \alpha t} X^{(1)}(s) + \inf\limits_{0 \leq s \leq (1-\alpha)t} X^{(2)}(s) \\ X^{(1)}(\alpha t) + X^{(2)}((1-\alpha)t) \end{pmatrix} \quad \text{(in law)}.$$

This result was obtained for the special case when $X(t)$ is a Brownian motion by Dassios [4] and Embrechts, Rogers and Yor [5]. Using this result, one could calculate an expression for the joint probability density of $M(\alpha, t)$ and $X(t)$. The motivation for these calculations was a problem in mathematical finance, the pricing of the so-called α-quantile (α-percentile) options. For the pricing of these options, see [1, 4, 6].

To prove our theorem, we will use a similar discrete time result, obtained some time ago by Wendel [9] and Port [7].

2. Discrete time results. Consider the sequence $\mathbf{x} = (x_0, x_1, x_2, \dots)$. For integers $0 \leq j \leq n$, define the (j, n)th quantile of $\mathbf{x}$ for $j = 0, 1, 2, \dots, n$ by $M_{j,n}(\mathbf{x}) = \inf\{z\colon \sum_{i=0}^{n} \mathbf{1}(x_i \leq z) > j\}$.

Received May 1995; revised September 1995, April 1996.

AMS 1991 *subject classifications*. **PLEASE SUPPLY**

Key words and phrases. Sample quantiles, exchangeability, stationary and independent increments, look-back financial options.

In particular, $M_{0,n}(\mathbf{x}) = \min_{i=0,1,\dots,n}(x_i)$ and $M_{n,n}(\mathbf{x}) = \max_{i=0,1,\dots,n}\{x_i\}$. This definition coincides with the one in [7].

The following result is due to Wendel [9].

PROPOSITION 1. *Let $Y_1, Y_2, \dots$ be a sequence of i.i.d. random variables. Define $\mathbf{X} = (X_0, X_1, \dots)$ by $X_0 = 0$ and $X_n = \sum_{i=1}^n Y_i$ for $n = 1, 2, \dots$. Let $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ be two independent copies of $\mathbf{X}$. Then,*

$$\begin{pmatrix} M_{j,n}(\mathbf{X}) \\ X_n \end{pmatrix} = \begin{pmatrix} M_{j,j}(\mathbf{X}^{(1)}) + M_{0,n-j}(\mathbf{X}^{(2)}) \\ X_j^{(1)} + X_{n-j}^{(2)} \end{pmatrix} \quad (\textit{in law}). \tag{2.1}$$

Wendel actually states this result in characteristic function form. An extension of this result, involving the time the quantile is achieved, was obtained by Port [7]. A careful reading of Port's proof can persuade the reader that the result is actually true where $Y_1, Y_2, \dots, Y_n$ are exchangeable random variables, with $\mathbf{X}^{(1)}$ replaced by $(0, X_1, \dots, X_j)$ and $\mathbf{X}^{(2)}$ by $(0, X_{j+1} - X_j, \dots, X_n - X_j)$.

3. Proof of Theorem 1. Define the occupation time $L(x,t) = \int_0^t \mathbf{1}(X(s) \le x)\, ds$. We have that

$$\Pr(M(\alpha,t) \le x,\, X(t) \le a) = \Pr(L(x,t) > \alpha t,\, X(t) \le a). \tag{3.1}$$

Similarly for the discrete time process $\mathbf{X}$ define $L_n(x) = \sum_{i=0}^n \mathbf{1}(x_i \le x)$. We then have

$$\Pr\big(M_{j,n}(\mathbf{X}) \le x,\, X_n \le a\big) = \Pr\big(L_n(x) > j,\, X_n \le a\big). \tag{3.2}$$

Without loss of generality, we will prove Theorem 1 for $t = 1$. Let $(X(s), 0 \le s \le 1)$ be as in the Introduction, and for $r = 0, 1, \dots, n$ set $X_r = X(r/n)$. The process $(X_{[ns]}, 0 \le s \le 1)$ converges weakly to $(X(s), 0 \le s \le 1)$. Since $\int_0^1 \mathbf{1}(X(s) \le x)\, ds$ is a continuous functional of $(X(s), 0 \le s \le 1)$, we conclude that for all $0 < \alpha < 1$ such that $\Pr(L(x,t) > \alpha, X(t) \le a)$ is continuous,

$$\lim_{n\to\infty} \Pr(L_n(x) > [n\alpha],\, X_n \le a) = \Pr(L(x,1) > \alpha,\, X(1) \le a). \tag{3.3}$$

Take $X_r^{(1)} = X(r/n)$ for $r = 0, 1, \dots, [n\alpha]$, and $X_r^{(2)} = X(r + [n\alpha]/n) - X([n\alpha]/n)$ for $r = 0, 1, \dots, n - [n\alpha]$. Then $(M_{[n\alpha],[n\alpha]}(\mathbf{X}^{(1)}), X_{[n\alpha]})$ converges in distribution to $(\sup_{0\le s\le\alpha} X(s), X(\alpha))$ and $(M_{0,n-[n\alpha]}(\mathbf{X}^{(2)}), X_n - X_{[n\alpha]})$ converges in distribution to $(\inf_{0\le s\le 1-\alpha}(X(s+\alpha) - X(\alpha)), X(1) - X(\alpha))$. We note that $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ are independent and so from (2.1), (3.1), (3.2) and (3.3) we conclude

$$\begin{pmatrix} M(\alpha,1) \\ X(1) \end{pmatrix} = \begin{pmatrix} \sup_{o\le s\le\alpha} X(s) + \inf_{0\le s\le(1-\alpha)} (X(\alpha+s) - X(\alpha)) \\ X(\alpha) + X(1) - X(\alpha) \end{pmatrix} \quad (\text{in law}). \tag{3.4}$$

Taking $X^{(1)}(s) = X(s)$ and $X^{(2)}(s) = X(\alpha + s) - X(a)$ completes the proof of the theorem.

Equation (3.4) is also true if $(X(t),\ t \geq 0)$ is a process with exchangeable increments. If $(X(t),\ t \geq 0)$ is defined on a probability space $(\Omega, \mathscr{F}, \Pr)$, according to [8], Theorem 1, page 202, there exists a nontrivial σ-algebra $\mathscr{F}_1 \subset \mathscr{F}$ such that $(X(t),\ t \geq 0)$ has stationary and independent increments with respect to $\mathscr{F}_1$. This was first proved by Bühlman [3]. Conditioning on $\mathscr{F}_1$ and taking expectations, we obtain (3.4).

Acknowledgment. The author thanks the referee for pointing out important references.

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