The Annals of Applied Probability
1996, Vol. 6, No. 3, 992–1005

RANK INVERSIONS IN SCORING MULTIPART EXAMINATIONS

BY SIMEON M. BERMAN

Courant Institute of Mathematical Sciences

Let (X_i, Y_i), $i = 1, \ldots, n$, be independent random vectors with a standard bivariate normal distribution and let s_X and s_Y be the sample standard deviations. For arbitrary p, $0 < p < 1$, define $T_i = pX_i + (1-p)Y_i$ and $Z_i = pX_i/s_X + (1-p)Y_i/s_Y$, $i = 1, \ldots, n$. The couple of pairs (T_i, Z_i) and (T_j, Z_j) is said to be discordant if either $T_i < T_j$ and $Z_i > Z_j$ or $T_i > T_j$ and $Z_i < Z_j$. It is shown that the expected number of discordant couples of pairs is asymptotically equal to $n^{3/2}$ times an explicit constant depending on p and the correlation coefficient of X_i and Y_i. By an application of the Durbin–Stuart inequality, this implies an asymptotic lower bound on the expected value of the sum of $(\mathrm{rank}(Z_i) - \mathrm{rank}(T_i))^+$. The problem arose in a court challenge to a standard procedure for the scoring of multipart written civil service examinations. Here the sum of the positive rank differences represents a measure of the unfairness of the method of scoring.

1. Summary. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a sample of independent random vectors having a common bivariate normal distribution with standard normal marginals and correlation coefficient r. Let s_X^2 and s_Y^2 be the sample variances. For arbitrary fixed p, $0 < p < 1$, define the pairs (T_i, Z_i), $i = 1, \ldots, n$, as

$$
\begin{aligned}
T_i &= pX_i + (1-p)Y_i, \\
Z_i &= pX_i/s_X + (1-p)Y_i/s_Y, \quad i = 1, \ldots, n.
\end{aligned} \tag{1.1}
$$

For each pair (i, j) with $i \neq j$, the couple $((T_i, Z_i),\ (T_j, Z_j))$ is said to be concordant if either $T_i < T_j$ and $Z_i < Z_j$ or $T_i > T_j$ and $Z_i > Z_j$; otherwise the couple is said to be discordant. This concept has a central role in the work of Kendall on the coefficient of concordance [4]. Our main result is an explicit asymptotic formula for the expected number of discordant couples of pairs $((T_i, Z_i),\ (T_j, Z_j))$ for $n \to \infty$:

$$
\frac{(n/\pi)^{3/2}(1-r^2)p(1-p)}{p^2 + (1-p^2) + 2rp(1-p).} \tag{1.2}
$$

The event that the couple $((T_1, Z_1),\ (T_2, Z_2))$ is discordant is obviously the union of the events

$$
\left\{ p(X_1 - X_2) + (1-p)(Y_1 - Y_2) > 0,\ \frac{p(X_1 - X_2)}{s_X} + \frac{(1-p)(Y_1 - Y_2)}{s_Y} < 0 \right\},
$$

$$
\left\{ p(X_1 - X_2) + (1-p)(Y_1 - Y_2) < 0,\ \frac{p(X_1 - X_2)}{s_X} + \frac{(1-p)(Y_1 - Y_2)}{s_Y} > 0 \right\}.
$$

Received July 1995; revised February 1996.
AMS 1991 *subject classifications*. Primary 62F07, 62H20.
Key words and phrases. Bivariate normal distribution, concordance, ranks, sample variance.

Since there are $n(n-1)/2$ couples and the probability of discordance is the same for all of them, the expected number of discordant couples is

$$n(n-1)P\biggl(p(X_1-X_2)+(1-p)(Y_1-Y_2)>0,\quad \frac{p(X_1-X_2)}{s_X}+\frac{(1-p)(Y_1-Y_2)}{s_Y}<0\biggr). \tag{1.3}$$

Since $s_X \to 1$ and $s_Y \to 1$ almost surely, the probability in (1.3) converges to 0 as $n \to \infty$ and the main mathematical contribution here is the determination of the exact asymptotic form. The calculations presented in the derivation involve (1) the reduction of the probability in (1.3) to the case where the random vector (s_X, s_Y) is independent of the random vector $(X_1 - X_2, Y_1 - Y_2)$; (2) the estimation of the quadrant probabilities of the bivariate normal distribution in terms of the correlation, and (3) large deviation results for the gamma distribution for large values of the factorial parameter.

2. Background. The probability problem considered in this paper arose from the author's consultation with attorneys in legal proceedings in the Supreme Court of the State of New York against the City of New York. The plaintiffs were police officers who challenged the method used by the City Department of Personnel to mark a written examination for promotion to the position of sergeant in the New York City Housing Authority Police (Exam 0537, NYC Housing Authority Police Department. Test Date: November 17, 1990). The examination consisted of two parts, a "technical" part and an "ability" part. The examination announcement stated that the scores on the two parts would be combined in a weighted sum to obtain a final score, and that the weights 0.48 and 0.52 would be attached to the technical and ability parts, respectively. When the examination was marked, the graders first normalized the pairs of scores of the candidates by subtracting the mean technical and ability scores, respectively, and then dividing by the corresponding standard deviations, s_T and s_A. Thus the effective weights applied to the scores on the two parts were proportional to $0.48/s_T$ and $0.52/s_A$ rather than 0.48 and 0.52. The issues raised by the plaintiffs were as follows:

1. Any change in the weights attached to a pair of examination scores can itself change the relative ranks of the combined scores of individuals. The plaintiffs, several of those who failed the examination, argued that they would have passed if the announced weights 0.48 and 0.52 had been used without normalization because their combined score ranks would have been higher.
2. The law governing examinations states that the weights announced for scoring an examination cannot be altered after the examination has been marked. The plaintiffs argued that the use of weights $0.48/s_T$ and $0.52/s_A$ leads to such an alteration because the standard deviations are obtained from the examination scores themselves.

In response to the contention in issue 2, the city claimed that the normalization of scores by division by the standard deviations is an established statistical procedure used to account for the relative reliability of the two parts of the examination. As consultant for the plaintiffs, the author did not object to the use of the population standard deviations if they were known, but argued that the use of the sample standard deviation in weighting the scores introduced dependence among the scores. Then he presented a simple example demonstrating that the ranks of the weighted scores of two individuals can be reversed on the basis of changes in the scores of other individuals, a manifestly unfair and unreasonable consequence. In particular, the relative ranks of individuals who took the examination are likely to vary according to the performance of others who took the same examination. It is the purpose of this paper to furnish a precise measure of the change of relative ranks.

Deciding to reject the motion of the plaintiffs, the judge argued that sampling theory had no place in the analysis. He wrote [3], "Petitioner's assumption that a sampling error was present in the subject examination is incorrect. The candidates who took the examination constituted the entire population, and not a sample.... A population refers to the complete set of observations or measurements about which we would like to draw conclusions, and a sample consists of part of that population.... It is reasonable for respondents to treat the total number of test scores generated by candidates who took Examination 0537 as a population." The obvious response to this (which the author never had the opportunity to present) is that the use of the standard deviation as an inverse measure of reliability has no rational justification outside a probability context.

3. The probability model. Let the scores on the two parts of the examination for a randomly selected individual be represented as a random vector (X, Y), where X and Y are the scores on parts 1 and 2, respectively. Let p and $1-p$ be the prior weights attached to the two parts, where $0 < p < 1$. Let μ_X and μ_Y be the expected values of X and Y and let σ_X and σ_Y be the corresponding standard deviations. For a given pair (X, Y), the "true" weighted normalized score is defined as

$$T = \frac{(X-\mu_X)p}{\sigma_X} + \frac{(Y-\mu_Y)(1-p)}{\sigma_Y}.$$

In the ideal situation, these scores should be ranked and the candidates appointed in order of rank as positions become available. However, since the means and variances are generally unknown before sampling, the T-scores are replaced by the corresponding Z-scores, defined as

$$Z = \frac{(X-\overline{X})p}{s_X} + \frac{(Y-\overline{Y})(1-p)}{s_Y}, \tag{3.1}$$

where $\overline{X}$ and $\overline{Y}$ are the sample means and s_X and s_Y are the sample standard deviations. Under the assumption of a bivariate normal distribution of (X, Y) the joint distribution of (T, Z) is independent of the unknown means and variances; therefore, without loss of generality, we may consider just the special case $\mu_X = \mu_Y = 0$, $\sigma_X = \sigma_Y = 1$. Thus, we let (T_i, Z_i) be the score pair associated with (X_i, Y_i) by the relations

$$(3.2)\quad T_i = pX_i + (1-p)Y_i, \qquad Z_i = \frac{p(X_i - \overline{X})}{s_X} + \frac{(1-p)(Y_i - \overline{Y})}{s_Y}.$$

Since the ranks of (Z_i) in (1.1) are identical with the ranks of those in (3.2), the concordant pair couples of (1.1) are the same as those of (3.2). Hence the formula (1.2) is also valid for the discordances of the pairs (3.2).

We propose a measure of "unfairness" M_n for the results of an examination for which the Z-scores are used in the place of the T-scores to rank the candidates:

$$(3.3)\qquad M_n = \sum_{i=1}^{n} (\operatorname{rank} Z_i - \operatorname{rank} T_i)^+,$$

the sum of the positive rank differences. It is shown that

$$(3.4)\qquad \liminf_{n\to\infty} \frac{E(M_n)}{\sqrt{n}} \geq \frac{2(1-r^2)p(1-p)}{3\pi^{3/2}(p^2 + (1-p)^2 + 2rp(1-p))}.$$

In particular, this shows that the expected measure of unfairness increases asymptotically at least at a rate equal to the square root of the number of candidates.

4. Preliminary results on the normal and gamma distributions. Let (X, Y) have a standard bivariate normal distribution with correlation r, $|r| < 1$. For constants a, a', b and b', put $R = \text{correlation}(aX + bY, a'X + b'Y)$, so that

$$(4.1)\qquad R = \frac{aa' + bb' + ra'b + rab'}{(a^2 + b^2 + 2rab)^{1/2}(a'^2 + b'^2 + 2ra'b')^{1/2}}.$$

It follows that

$$(4.2)\qquad 1 - R^2 = \frac{(ab' - a'b)^2(1-r^2)}{(a^2 + b^2 + 2rab)(a'^2 + b'^2 + 2ra'b')}$$

and

$$(4.3)\qquad (1-R^2)^{1/2} \leq \frac{|b'|\,|a-a'| + |a'|\,|b-b'|}{|aa'|(1-r^2)^{1/2}}.$$

Next we note

$$(4.4)\quad (1-x^2)^{1/2} \leq \sin^{-1} 1 - \sin^{-1} x \leq 2(1-x^2)^{1/2}/(1+x) \quad \text{for } |x| < 1.$$

LEMMA 4.1. *Let U and V have a standard bivariate normal distribution with correlation R. Then*

$$(4.5)\qquad \frac{(1-R^2)^{1/2}}{2\pi} \leq P(U < 0,\ V > 0) \leq \frac{(1-R^2)^{1/2}}{\pi(1+R)}.$$

PROOF. Classical results of Sheppard and Stieltjes (see [1], page 290) imply $P(U < 0,\ V > 0) = (2\pi)^{-1}(\sin^{-1} 1 - \sin^{-1} R)$ and so (4.5) follows from (4.4). □

LEMMA 4.2. *Let $T = T_x$ be random variable with gamma density* $(\Gamma(x))^{-1} \cdot y^{x-1}e^{-y},\ y > 0$, *for arbitrary* $x > 1$. *Then*

$$\lim_{x\to\infty} xE((x/T)^{1/2} - 1)^2 = \tfrac{1}{4}. \tag{4.6}$$

PROOF. By direct computation we find that

$$xE((x/T)^{1/2} - 1)^2 = x\left[\frac{2x-1}{x-1} - \frac{2\sqrt{x}\Gamma(x-\frac{1}{2})}{\Gamma(x)}\right]. \tag{4.7}$$

By the precise form of Stirling's formula ([1], page 130),

$$\log \Gamma(x) = (x - \tfrac{1}{2})\log x - x + \tfrac{1}{2}\log 2\pi + (12x)^{-1} + O(x^{-3}),$$

it follows that

$$\log\left[\frac{\sqrt{x}\Gamma(x-\frac{1}{2})}{\Gamma(x)}\right] = (x-1)\log\left(1 - \frac{1}{2x}\right) + \frac{1}{2} + O(x^{-2}).$$

By the logarithmic expansion, the right-hand member above is equal to $3/(8x) + O(x^{-2})$. It follows that (4.7) is equal to

$$x\left[\frac{2x-1}{x-1} - 2\exp\left(\frac{3}{8x} + O(x^{-2})\right)\right],$$

which, by an elementary calculation, converges to $\frac{1}{4}$ for $x \to \infty$. □

LEMMA 4.3. *For every $\varepsilon > 0$ and $x > 1$*,

$$\int_{x(1+\varepsilon)}^{\infty} y^{x-1}e^{-y}\,dy \le \Gamma(x)(e^{-\varepsilon}(1+\varepsilon))^x\varepsilon^{-1}. \tag{4.8}$$

PROOF. By the change of variable $z = y - x(1+\varepsilon)$, followed by a simple algebraic operation, the left-hand member of (4.8) is equal to

$$\exp(-x(1+\varepsilon))(x(1+\varepsilon))^{x-1}\int_0^{\infty}\exp(-z)\left(1 + \frac{z}{x(1+\varepsilon)}\right)^{x-1} dz.$$

By $1+t < e^t$ for $t > 0$, the integral in the latter expression is at most equal to

$$\int_0^{\infty}\exp(-z\varepsilon/(1+\varepsilon))\,dz, \quad \text{or} \quad (1+\varepsilon)/\varepsilon,$$

so that the entire expression is at most equal to

$$e^{-x}x^{x-1}(e^{-\varepsilon}(1+\varepsilon))^x\varepsilon^{-1}. \tag{4.9}$$

The bound on the right-hand side of (4.8) now follows from (4.9) and

$$\Gamma(x) \ge \int_0^x y^{x-1}e^{-y}\,dy = e^{-x}\int_0^x y^{x-1}\,dy = e^{-x}x^{x-1}. \tag{4.10}$$

□

LEMMA 4.4. *For every* ε, $0 < \varepsilon < 1$ *and* $x > 1$

$$\int_0^{x(1-\varepsilon)} y^{x-1}e^{-y}\,dy \le \Gamma(x+1)(e^{\varepsilon}(1-\varepsilon))^{x-1}(e-e^{\varepsilon}). \tag{4.11}$$

PROOF. By the substitution $z = (y-x)/x$, the left-hand member of (4.11) becomes

$$x^x e^{-x}\int_{-1}^{-\varepsilon}(1+z)^{x-1}\exp(-z(x-1))\exp(-z)\,dz.$$

The latter is at most equal to

$$x^x e^{-x}((1-\varepsilon)e^{\varepsilon})^{x-1}\int_{-1}^{-\varepsilon} e^{-z}\,dz, \tag{4.12}$$

because $e^{-z}(1+z)$ is an increasing function on $(-1, -\varepsilon)$. The bound on the right-hand side of (4.11) now follows from (4.10). □

Let s_n^2 be the sample variance of sample of n observations from a standard normal distribution, where the sum of squares is divided by $n-1$. Since $(n-1)s_n^2$ has the gamma distribution with parameter $x = \frac{1}{2}(n-1)$, Lemmas 4.3 and 4.4 imply

$$\begin{aligned} P(|s_n^2 - 1| > \varepsilon) \le{}& (e^{-\varepsilon}(1+\varepsilon))^{(n-1)/2}\varepsilon^{-1} \\ &+ \tfrac{1}{2}(n-1)(e-e^{\varepsilon})(e^{\varepsilon}(1-\varepsilon))^{(n-3)/2} \end{aligned} \tag{4.13}$$

for every $n > 2$ and $0 < \varepsilon < 1$.

For arbitrary real numbers $x_1, x_2 \dots$, put

$$\overline{X}_n = n^{-1}\sum_{i=1}^n x_i, \qquad S_n = \sum_{i=1}^n (x_i - \overline{X}_n)^2.$$

Then

$$S_{n-1} \le S_n. \tag{4.14}$$

LEMMA 4.5. *Let* $X_1, \dots, X_n$ *be a sample from a standard normal distribution and let* s_n^2 *be the sample variance. Then for every* $b > 0$ *and* $n > 2 + 1/b$,

$$\begin{aligned} P(|s_n^2 - s_{n-1}^2| > 4b) \le{}& \frac{1}{b(n-2)-1}\left(\frac{b(n-2)}{e^{b(n-2)-1}}\right)^{(n-1)/2} \\ &+ 6[1-\Phi((b(n-2))^{1/2})], \end{aligned} \tag{4.15}$$

where Φ *is the standard normal distribution.*

PROOF. From the algebraic identity

$$(n-2)(s_n^2 - s_{n-1}^2) + s_n^2 = X_n^2 - n\overline{X}_n^2 + (n-1)\overline{X}_{n-1}^2,$$

it follows that

$$|s_n^2 - s_{n-1}^2| \le (n-2)^{-1}(s_n^2 + X_n^2 + n\overline{X}_n^2 + (n-1)\overline{X}_{n-1}^2).$$

From this and the fact that $\sqrt{n}\,\overline{X}_n$ has a standard normal distribution, it follows that

$$P(|s_n^2 - s_{n-1}^2| > 4b) \le P(s_n^2 > b(n-2)) + 3P(|X_1| > (b(n-2))^{1/2}),$$

and the latter is equivalent to

$$P(\tfrac{1}{2}(n-1)s_n^2 > \tfrac{1}{2}b(n-1)(n-2)) + 6(1 - \Phi((b(n-2))^{1/2})).$$

Since $\frac{1}{2}(n-1)s_n^2$ has the gamma distribution with parameter $x = \frac{1}{2}(n-1)$, Lemma 4.3 with $\varepsilon = b(n-2) - 1$ implies (4.15). □

As an application of (4.15), we have, for every δ, $0 < \delta < 1$ and $b = 1/4n^{\delta}$,

$$\lim_{n\to\infty} \sqrt{n}P(|s_n^2 - s_{n-1}^2| > n^{-\delta}) = 0. \tag{4.16}$$

It also follows from (4.16) by the triangle inequality that

$$\lim_{n\to\infty} \sqrt{n}P(|s_n^2 - s_{n-2}^2| > n^{-\delta}) = 0. \tag{4.17}$$

LEMMA 4.6. *Let s_X^2 and s_Y^2 be the sample variances for a sample of n pairs (X, Y) from a standard bivariate normal distribution with correlation r, $|r| < 1$. Then the pair*

$$\sqrt{n}(1 - 1/s_X), \qquad \sqrt{n}(1 - 1/s_Y) \tag{4.18}$$

has, for $n \to \infty$, a limiting standard bivariate normal distribution with correlation r^2.

PROOF. It follows from the distributional properties of the sample variances from a bivariate normal distribution ([1], page 366) that the pair $(\frac{1}{2}\sqrt{n}(s_X^2 - 1),\ \frac{1}{2}\sqrt{n}(s_Y^2 - 1))$ has a limiting standard bivariate normal distribution with correlation r^2. For real $u \to 1$, it is clear that $1 - 1/u \sim \frac{1}{2}(u^2 - 1)$. Therefore, the latter pair has the same limiting distribution as the pair (4.18). □

5. Reduction of the asymptotics to the case where X_1, X_2, Y_1 and Y_2 are omitted in the computation of s_X and s_Y. As noted in Section 1, we will reduce the analysis of (1.3) for $n \to \infty$ to the case where two X's and two Y's are omitted in the calculation of s_X and s_Y, respectively. The argument extends to any fixed finite number of X's and Y's. For the proof it suffices to consider just the case of the omission of one X and one Y. Let (X_i, Y_i) be independent observations from a standard normal population with correlation r, $|r| < 1$. For $n \ge 2$, put

$$\overline{X}_n = (1/n)\sum_{i=1}^{n} X_i, \qquad s^2(X, n) = (n-1)^{-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2,$$

$$\overline{Y}_n = (1/n)\sum_{i=1}^{n} Y_i, \qquad s^2(Y, n) = (n-1)^{-1}\sum_{i=1}^{n}(Y_i - \overline{Y}_n)^2.$$

THEOREM 5.1. *Put* $X = X_n$ *and* $Y = Y_n$. *Then*

$$\sqrt{n}P\left\{\frac{Xp}{s(X,n)} + \frac{Y(1-p)}{s(Y,n)} > 0,\ \frac{Xp}{s(X,n-1)} + \frac{Y(1-p)}{s(Y,n-1)} < 0\right\} \tag{5.1}$$

converges to 0 *for* $n \to \infty$.

PROOF. Write the probability in (5.1) as the sum of

$$\begin{aligned} P\biggl\{&\frac{Xp}{s(X,n)} + \frac{Y(1-p)}{s(Y,n)} > 0,\\ &\frac{Xp}{s(X,n-1)} + \frac{Y(1-p)}{s(Y,n-1)} < 0,\ X > 0\biggr\} \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} P\biggl\{&\frac{Xp}{s(X,n)} + \frac{Y(1-p)}{s(Y,n)} > 0,\\ &\frac{Xp}{s(X,n-1)} + \frac{Y(1-p)}{s(Y,n-1)} < 0,\ X < 0\biggr\}. \end{aligned} \tag{5.3}$$

We will show that the term (5.2) is $o(n^{-1/2})$ for $n \to \infty$; the same is true for (5.3), and the proof is analogous.

By (4.16), the term (5.2) is equal to

$$\begin{aligned} P\biggl\{&\frac{Xp}{s(X,n)} + \frac{Y(1-p)}{s(Y,n)} > 0,\ \frac{Xp}{s(X,n-1)} + \frac{Y(1-p)}{s(Y,n-1)} < 0,\\ &X > 0,\ |s^2(Y,n) - s^2(Y,n-1)| \le n^{-\delta}\biggr\} + o(n^{-1/2}). \end{aligned}$$

The latter is equal to

$$\begin{aligned} P\biggl\{&\frac{Xp}{s(X,n)} + \frac{Y(1-p)}{s(Y,n)} > 0,\ \frac{Xp}{s(X,n-1)} + \frac{Y(1-p)}{s(Y,n-1)} < 0,\\ &X > 0,\ Y < 0,\ |s^2(Y,n) - s^2(Y,n-1)| \le n^{-\delta}\biggr\} + o(n^{-1/2}) \end{aligned} \tag{5.4}$$

because the first two inequalities in the preceding event imply that X and Y are of opposite sign. By an application of (4.14) and (4.16), the term (5.4) is at most equal to

$$\begin{aligned} P\biggl\{&\frac{Xp}{s(X,n-1)}\left(\frac{n-1}{n-2}\right)^{1/2} + \frac{Y(1-p)}{(s^2(Y,n-1) + n^{-\delta})^{1/2}} > 0,\\ &\frac{Xp}{s(X,n-1)} + \frac{Y(1-p)}{s(Y,n-1)} < 0\biggr\} + o(n^{1/2}). \end{aligned} \tag{5.5}$$

Since $\frac{1}{2}(n-2)s^2(X,n-1)$ and $\frac{1}{2}(n-2)s^2(Y,n-1)$ have the common standard gamma distribution with parameter $\frac{1}{2}(n-2)$, (4.13) implies that for every ε,

$0 < \varepsilon < 1$, the inequalities $|s^2(X, n-1) - 1| < \varepsilon$ and $|s^2(Y, n-1) - 1| < \varepsilon$ hold everywhere except for an event of probability on the order

$$n(e^{\varepsilon}(1-\varepsilon))^{(1/2)n} + (e^{-\varepsilon}(1+\varepsilon))^{(1/2)n}.$$

Therefore the term (5.5) is equal to

$$\begin{aligned} P\biggl\{ &\frac{Xp}{s(X,n-1)}\left(\frac{n-1}{n-2}\right)^{1/2} + \frac{Y(1-p)}{(s^2(Y,n-1)+n^{-\delta})^{1/2}} > 0, \\ &\frac{Xp}{s(X,n-1)} + \frac{Y(1-p)}{s(Y,n-1)} < 0, \\ &|s^2(X,n-1)-1| < \varepsilon,\ |s^2(Y,n-1)-1| < \varepsilon \biggr\} + o(n^{-1/2}) \end{aligned} \tag{5.6}$$

for every ε, $0 < \varepsilon < 1$.

Note that (X, Y), that is, (X_n, Y_n), is independent of $(s(X, n-1),\ s(Y, n-1))$. To simplify the notation, put $s_X = s(X, n-1)$ and $s_Y = s(Y, n-1)$, and define

$$a = \frac{p}{s_X}, \qquad b = \frac{1-p}{s_Y}, \qquad a' = \left(\frac{n-1}{n-2}\right)^{1/2}\frac{p}{s_X}, \qquad b' = \frac{1-p}{(s_Y^2 + n^{-\delta})^{1/2}}. \tag{5.7}$$

Fix s_X and s_Y and write the probability in (5.6) as the expected value of the conditional probability, given s_X and s_Y:

$$E\bigl[P(aX + bY < 0,\ a'X + b'Y > 0 | s_X, s_Y)\ 1_{\{|s_X^2 - 1| < \varepsilon,\ |s_Y^2 - 1| < \varepsilon\}}\bigr]. \tag{5.8}$$

By (5.7), a, b, a' and b' are fixed by the values of s_X and s_Y.

Defining R as (4.1) and applying Lemma 4.1, we find that (5.8) is at most equal to

$$\frac{1}{\pi}E\left[\frac{(1-R^2)^{1/2}}{1+R} 1_{\{|s_X^2 - 1| < \varepsilon,\ |s_Y^2 - 1| < \varepsilon\}}\right]. \tag{5.9}$$

By (4.3) and (5.7),

$$\begin{aligned} &(1-R^2)^{1/2} 1_{\{|s_X^2 - 1| < \varepsilon,\ |s_Y^2 - 1| < \varepsilon\}} \\ &\quad \le \frac{(1-p)}{(1-\varepsilon+n^{-\delta})^{1/2}} \frac{p}{(1-\varepsilon)^{1/2}}\left[\left(\frac{n-1}{n-2}\right)^{1/2} - 1\right] \\ &\qquad + \frac{((n-1)/(n-2))^{1/2}(p(1-p))/\sqrt{1-\varepsilon}\,[(1-\varepsilon)^{-1/2} - (1-\varepsilon+n^{-\delta})^{-1/2}]}{(1+\varepsilon)^{-1}p^2[(n-1)/(n-2)]^{1/2}(1-r^2)^{1/2}}. \end{aligned}$$

The right-hand member is on the order $n^{-\delta}$ for δ such that $0 < \delta < 1$; hence, the expression (5.9) is at most

$$O(n^{-\delta})E\bigl[1_{\{|s_X^2 - 1| < \varepsilon,\ |s_Y^2 - 1| < \varepsilon\}}(1+R)^{-1/2}\bigr]. \tag{5.10}$$

Next we show that the expectation in (5.10) is bounded. If $r \geq 0$, then R is almost surely positive by virtue of the definition of R, so that $(1+R)^{-1/2} \leq 1$. The main part of the proof is for $-1 < r < 0$. We will show that for sufficiently small ε, $\varepsilon > 0$, depending on r, there exists $n(\varepsilon)$ such that

$$(1+R)^{-1/2} 1_{\{|s_X^2-1|<\varepsilon,\ |s_Y^2-1|<\varepsilon\}} \leq 1$$

almost surely, for all $n \geq n(\varepsilon)$. For this it suffices to show that $R \geq 0$ on the set $\{|s_X^2 - 1| < \varepsilon,\ |s_Y^2 - 1| < \varepsilon\}$ almost surely for all $n \geq n(\varepsilon)$.

For the proof, note that, by (4.1) and (5.7), R is equal to the ratio of

$$\begin{aligned} &\frac{p^2}{s_X^2}\left(\frac{n-1}{n-2}\right)^{1/2} + \frac{(1-p)^2}{s_Y(s_Y^2+n^{-\delta})^{1/2}} \\ &\qquad + \frac{rp(1-p)}{s_X}\left[\left(\frac{n-1}{n-2}\right)^{1/2}\frac{1}{s_Y} + \frac{1}{(s_Y^2+n^{-\delta})^{1/2}}\right] \end{aligned} \tag{5.11}$$

to the square root of

$$\begin{aligned} &\left(\frac{p^2}{s_X^2} + \frac{(1-p)^2}{s_Y^2} + \frac{2rp(1-p)}{s_X s_Y}\right) \\ &\qquad \times \left(\frac{p^2}{s_Y^2}\frac{n-1}{n-2}\frac{(1-p)^2}{s_Y^2+n^{-\delta}} + 2rp(1-p)\left(\frac{n-1}{n-2}\right)^{1/2}\frac{1}{s_X(s_Y^2+n^{-\delta})^{1/2}}\right) \end{aligned} \tag{5.12}$$

on the set $\{|s_X^2 - 1| < \varepsilon,\ |s_Y^2 - 1| < \varepsilon\}$. The term (5.11) is, for $r < 0$, at least equal to

$$\begin{aligned} H_n(\varepsilon) &= \frac{p^2}{1+\varepsilon} + \frac{(1-p)^2}{((1+\varepsilon)(1+\varepsilon+n^{-\delta}))^{1/2}} \\ &\qquad + \frac{rp(1-p)}{\sqrt{1-\varepsilon}}\left(\frac{n-1}{n-2}\right)^{1/2}[(1-\varepsilon)^{-1/2} + (1-\varepsilon+n^{-\delta})^{-1/2}]. \end{aligned}$$

For each n, $H_n(\varepsilon)$ is a decreasing function of ε, $0 \leq \varepsilon < 1$, and it converges, for $n \to \infty$, to the decreasing function

$$H(\varepsilon) = \frac{p^2 + (1-p)^2}{1+\varepsilon} + \frac{2rp(1-p)}{1-\varepsilon}, \qquad 0 \leq \varepsilon < 1.$$

Since $0 < H(0) < 1$, it follows that $0 < H(\varepsilon) < 1$ for all sufficiently small $\varepsilon > 0$. Therefore, by the uniform convergence of monotone functions to a continuous limit, $0 < H_n(\varepsilon) < 1$ for sufficiently large n, denoted $n \geq n(\varepsilon)$. Therefore, the numerator (5.11) is almost surely bounded on the indicated set, and the denominator, the square root of (5.12), is almost surely bounded away from 0. This completes the proof that the expected value in (5.10) is bounded in the case $-1 < r < 0$. It follows that (5.9) is $O(n^{-\delta})$, which, for $\delta > \frac{1}{2}$, is $o(n^{-1/2})$. Hence the expression (5.8) is $o(n^{-1/2})$ and, therefore, so is (5.2). □

6. Asymptotic probability of a single discordance.

THEOREM 6.1. *In a sample* $(X_1, Y_1), \ldots, (X_n, Y_n)$ *from a standard bivariate normal distribution with correlation* r, *put* $X = X_n$ *and* $Y = Y_n$ *and let* s_X *and* s_Y *be the sample standard deviations. Then*

$$\lim_{n\to\infty} \sqrt{n} P\left(\frac{Xp}{s_X} + \frac{Y(1-p)}{s_Y} > 0,\ Xp + Y(1-p) < 0\right) = \frac{(1-r^2)p(1-p)}{\pi^{3/2}(p^2+(1-p)^2+2rp(1-p))}. \tag{6.1}$$

PROOF. By Theorem 5.1, (6.1) holds if and only if the same relation holds when the sample standard deviations are calculated on the basis of observations excluding X_n and Y_n. Hence, in proving (6.1) we may assume that $(X, Y,)$ and (s_X, s_Y) are independent random vectors. For fixed (s_X, s_Y) put

$$R = \text{correlation}(Xp/s_X + Y(1-p)/s_Y,\ Xp + Y(1-p)). \tag{6.2}$$

Then, by the proof of Lemma 4.1, the expression following lim in (6.1) is

$$\frac{\sqrt{n}}{2\pi} E\left\{\int_R^1 \frac{dy}{(1-y^2)^{1/2}}\right\}. \tag{6.3}$$

For arbitrary ε, $0 < \varepsilon < 1$, the expression (6.3) is, by (4.13) and the elementary formula $\int_{-1}^{1}(1-y^2)^{-1/2}\,dy = \pi$, asymptotically equal, for $n \to \infty$, to

$$\frac{\sqrt{n}}{2\pi} E\left(\int_R^1 \frac{dy}{(1-y^2)^{1/2}} \quad 1_{\{|s_X^2-1|<\varepsilon,\ |s_Y^2-1|<\varepsilon\}}\right). \tag{6.4}$$

It follows from (4.1) and the particular form (6.2) of R that the latter is equal to

$$\begin{aligned} &p^2/s_X + (1-p)^2/s_Y + rp(1-p)(1/s_X + 1/s_Y) \\ &\quad \times [\{p^2/s_X^2 + (1-p)^2/s_Y^2 + (2rp(1-p))/(s_X s_Y)\}^{1/2} \\ &\quad \times [p^2 + (1-p)^2 + 2rp(1-p)]^{1/2}]^{-1}. \end{aligned} \tag{6.5}$$

Define

$$G(r) = p^2 + (1-p)^2 + 2rp(1-p), \qquad |r| < 1, \tag{6.6}$$

and choose $\varepsilon > 0$ so small that

$$|r| < \frac{1-\varepsilon}{1+\varepsilon}. \tag{6.7}$$

It follows from (6.5) that, on the set $\{|s_X^2 - 1| < \varepsilon,\ |s_Y^2 - 1| < \varepsilon\}$,

$$R \ge \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{1/2} \quad \text{for } r \ge 0 \tag{6.8}$$

and

$$R \geq \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{1/2} \frac{G(r(1+\varepsilon)/(1-\varepsilon)^{1/2})}{[G(r(1-\varepsilon)/(1+\varepsilon))G(r)]^{1/2}} \quad \text{for } r \leq 0. \tag{6.9}$$

Since the right-hand members of (6.8) and (6.9) converge to 1 for $\varepsilon \to 0$, it follows that for every $\delta > 0$, there exists $\varepsilon = \varepsilon(\delta)$ sufficiently small so that for $0 < \varepsilon' \leq \varepsilon$,

$$R\ 1_{\{|s_X^2-1|<\varepsilon',\ |s_Y^2-1|<\varepsilon'\}} \geq (1-\delta)\ 1_{\{|s_X^2-1|<\varepsilon',\ |s_Y^2-1|<\varepsilon'\}}. \tag{6.10}$$

By (4.4) and (6.10), the expression (6.4) has the lower and upper bounds

$$\frac{\sqrt{n}}{2\pi} E\{(1-R^2)^{1/2}\ 1_{\{|s_X^2-1|<\varepsilon,\ |s_Y^2-1|<\varepsilon\}}\} \tag{6.11}$$

and

$$\frac{\sqrt{n}}{(2-\delta)\pi} E[(1-R^2)^{1/2}\ 1_{\{|s_X^2-1|<\varepsilon,\ |s_Y^2-1|<\varepsilon\}}],$$

respectively. Since δ may be taken arbitrarily close to 0, the expression (6.4) has the same limit, if any, as (6.11).

By (4.2), with $a = p,\ b = 1-p,\ a' = p/s_X$ and $b' = (1-p)/s_X$, we have

$$\begin{aligned}&(n(1-R^2))^{1/2}\\ &\quad = \frac{\sqrt{n}\, p(1-p)(1-r^2)^{1/2}|1/s_X - 1/s_Y|}{[p^2/s_X^2 + (1-p)^2/s_Y^2 + (2rp(1-p))/(s_X s_Y)]^{1/2}(H(r))^{1/2}}.\end{aligned} \tag{6.12}$$

The right-hand member has, on the set $\{|s_X^2-1| < \varepsilon,\ |s_Y^2-1| < \varepsilon\}$, the lower bound

$$\begin{aligned}&(H(r))^{-1}(n(1-\varepsilon))^{1/2} p(1-p)(1-r^2)^{1/2}\left|\frac{1}{s_X} - \frac{1}{s_Y}\right|, \qquad r \geq 0,\\ &(n(1-\varepsilon))^{1/2} p(1-p)(1-r^2)^{1/2}\left|\frac{1}{s_X} - \frac{1}{s_Y}\right|\left(H\left(r\frac{1-\varepsilon}{1+\varepsilon}\right)H(r)\right)^{1/2},\\ &\qquad\qquad r \leq 0,\end{aligned} \tag{6.13}$$

and the upper bound

$$\begin{aligned}&\frac{(n(1+\varepsilon))^{1/2} p(1-p)(1-r^2)^{1/2}|1/s_X - 1/s_Y|}{H(r)}, \qquad r \geq 0,\\ &\frac{(n(1+\varepsilon))^{1/2} p(1-p)(1-r^2)^{1/2}|1/s_X - 1/s_Y|}{(H(r(1+\varepsilon)/(1-\varepsilon))H(r))^{1/2}}, \qquad r \leq 0.\end{aligned} \tag{6.14}$$

Next we prove that

$$\lim_{n\to\infty} \sqrt{n} E[|s_X^{-1} - s_Y^{-1}|\ 1_{\{|s_X^2-1|<\varepsilon,\ |s_Y^2-1|<\varepsilon\}}] = \frac{2(1-r^2)^{1/2}}{\pi} \tag{6.15}$$

for every $\varepsilon > 0$. For the proof, note that, by Lemma 4.6, $\sqrt{n}(1/s_X - 1/s_Y)$ has a limiting $N(0, 2(1-r^2))$ distribution. Furthermore, it has a bounded second moment

$$E(\sqrt{n}(1/s_X - 1/s_Y))^2 \le 2E(\sqrt{n}(1/s_X - 1))^2 + 2E(\sqrt{n}(1/s_Y - 1))^2 = 4E(\sqrt{n}(1/s_X - 1))^2, \tag{6.16}$$

which, by Lemma 4.2 and the fact that $\frac{1}{2}(n-2)s_X^2$ has a gamma distribution with parameter $x = \frac{1}{2}(n-2)$, has the limit 2. It follows from the convergence of the distribution of $\sqrt{n}(1/s_X - 1/s_Y)$ and the boundedness of the second moment that

$$\lim_{n\to\infty} E\left(\sqrt{n}\left|\frac{1}{s_Y} - \frac{1}{s_X}\right|\right) = \frac{2(1-r^2)^{1/2}}{\pi^{1/2}}.$$

The result (6.15) follows from this by noting that (Cauchy–Schwarz inequality)

$$\limsup_{n\to\infty} E\big[\sqrt{n}|s_X^{-1} - s_Y^{-1}|\ 1_{\{|s_X^2-1|>\varepsilon \text{ or } |s_Y^2-1|>\varepsilon\}}\big] \le 2\limsup_{n\to\infty} E^{1/2}\left(n\left(\frac{1}{s_X} - \frac{1}{s_Y}\right)\right)^2 [P(|s_X^2 - 1| > \varepsilon)]^{1/2}.$$

In accordance with (4.13) and (6.16), the right-hand member above is equal to 0. By substituting from (6.12) into (6.11) and then employing the bounds in (6.13) and (6.14) and the limit relation (6.15) and then invoking the continuity of $H(r)$ and the arbitrariness of $\varepsilon > 0$, we conclude that (6.11) has the limit $(1-r^2)p(1-p)/H(r)\pi^{3/2}$. This completes the proof of (6.1). □

7. Proof of (1.2). The asymptotic value of the probability in (1.3) is equal to that of the probability in (6.1). The formal difference between the two expressions is that the vector $(X_1 - X_2, Y_1 - Y_2)$ in (1.3) is replaced by (X, Y) in (6.1). This difference does not affect the limit relation (6.1). First of all, there is no change in the value of the probability in (1.3) if $(X_1 - X_2, Y_1 - Y_2)$ is replaced by $(1/\sqrt{2})(X_1 - X_2, Y_1 - Y_2)$, and the latter vector has the same distribution as (X_1, Y_1). Second, the argument in the proof of Theorem 6.1 about the reduction of (6.1) to the case where (X, Y) and (s_X, s_Y) are independent, based on Theorem 5.1, extends to the case where $(X_1 - X_2, Y_1 - Y_2)$ and (s_X, s_Y) are independent. Indeed, as noted before Theorem 5.1, the conclusion of the latter extends to the case where several observations are omitted from the computations of (s_X, s_Y). It then follows from (6.1) that (1.3) is asymptotically equal to (1.2). □

8. Proof of (3.4). We prove the following general result.

LEMMA 8.1. *For arbitrary real pairs (T_i, Z_i), put $t_i = \text{rank}(T_i)$ and $z_i = \text{rank}(Z_i)$, $i = 1, \dots, n$, and let Q_n be the number of discordant couples of* *pairs*. *Then*

$$\sum_{i=1}^{n}(z_i - t_i)^+ \geq (2/3n)Q_n. \tag{8.1}$$

PROOF. Since $\sum_{i=1}^{n} t_i = \sum_{i=1}^{n} z_i = \frac{1}{2}n(n+1)$, we have $\sum_{i=1}^{n}(z_i - t_i)^+ = \sum_{i=1}^{n}(z_i - t_i)^-$; hence,

$$\sum_{i=1}^{n} |z_i - t_i| = \sum_{i=1}^{n}(z_i - t_i)^+ + \sum_{i=1}^{n}(z_i - t_i)^- = 2\sum_{i=1}^{n}(z_i - t_i)^+. \tag{8.2}$$

Since $\max_{1\leq i\leq n} |z_i - t_i| \leq n$, it follows that $\sum_{i=1}^{n}(z_i - t_i)^2 \leq n\sum_{i=1}^{n} |z_i - t_i|$, and so (8.2) implies

$$\sum_{i=1}^{n}(z_i - t_i)^+ \geq (1/2n)\sum_{i=1}^{n}(z_i - t_i)^2. \tag{8.3}$$

The Durbin–Stuart inequality [2] implies $\frac{1}{2}\sum_{i=1}^{n}(z_i - t_i)^2 \geq (2/3)Q_n$, and so (8.1) follows from (8.3). □

The result (3.4) follows by taking expectations in (8.1) and applying (1.2) to get EQ_n. □

REFERENCES

[1] CRAMÉR, H. (1946). *Mathematical Methods of Statistics*. Princeton Univ. Press.

[2] DURBIN, J. and STUART, A. (1951). Inversions and rank correlation coefficients. *J. Roy. Statist. Soc. Ser. B* **13**.

[3] EVANS, M. (1993). Supreme Court of the State of New York, County of New York, IAS Part 36, Index No. 8623/92. Filed September 3, 1993, County Clerk's Office, New York.

[4] KENDALL, M. G. (1962). *Rank Correlation Methods*, 3rd ed. Charles Griffin, London.

COURANT INSTITUTE OF MATHEMATICAL SCIENCES
NEW YORK UNIVERSITY
251 MERCER STREET
NEW YORK, NEW YORK 10012
E-MAIL: berman@cims.nyu.edu