The Annals of Applied Probability
1996, Vol. 6, No. 2, 561–571

A LARGE-DIMENSIONAL INDEPENDENT AND IDENTICALLY DISTRIBUTED PROPERTY FOR NEAREST NEIGHBOR COUNTS IN POISSON PROCESSES

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For an arbitrary point of a homogeneous Poisson point process in a d-dimensional Euclidean space, consider the number of Poisson points that have that given point as their rth nearest neighbor ($r = 1, 2, \dots$). It is shown that as d tends to infinity, these nearest neighbor counts ($r = 1, 2, \dots$) are iid asymptotically Poisson with mean 1. The proof relies on Rényi's characterization of Poisson processes and a representation in the limit of each nearest neighbor count as a sum of countably many dependent Bernoulli random variables.

1. Introduction. Let Π denote the countable random set of points of a Poisson process with constant intensity rate λ_d in $\mathbb{R}^d$. For an arbitrary point Q in Π, we are interested in the distribution of the number, $N_{d,r}$, of points in Π that have Q as their rth nearest neighbor (with respect to $\mathscr{L}_p$ distance, $1 \le p \le \infty$), $r = 1, 2, \dots$. The main objective of this paper is to show the following limit theorem:

THEOREM 1. *As $d \to \infty$, $N_{d,1}, N_{d,2}, \dots$ are iid asymptotically Poisson with mean* 1.

Here, the convergence should be understood as "convergence in distribution." Newman, Rinott and Tversky (1983) showed such convergence for the first component $N_{d,1}$ (nearest neighbors) under Euclidean distance ($p = 2$). Newman and Rinott (1985) extended this to include any $\mathscr{L}_p$ distance ($1 \le p \le \infty$) and simplified the argument. The key step in their approach is to establish

$$\lim_{d \to \infty} \int_{A_{d,k}} \exp(-V_{d,k}(u_1, \dots, u_k))\, du_1 \cdots du_k = 1,$$

where

$$A_{d,k} := \left\{(u_1, \dots, u_k) \in \mathbb{R}^{kd} : \|u_i\| \le \|u_i - u_j\|,\ 1 \le i \ne j \le k\right\}$$

and where $V_{d,k}(u_1, \dots, u_k)$ denotes the volume of the union of the $\mathscr{L}_p$ balls $S_i := B(u_i, \|u_i\|)$ (centered at u_i, of radius $\|u_i\|$) in $\mathbb{R}^d$, $1 \le i \le k$, $\|\cdot\|$ being the

Received September 1994.
AMS 1991 *subject classifications.* Primary 60G55; secondary 60B12, 60D05.
Key words and phrases. Nearest neighbor counts, Poisson point process.

$\mathscr{L}_p$ norm. In contrast, we shall show that, in the limit as $d \to \infty$, each $N_{d,r}$ can be expressed as a sum of countably many dependent Bernoulli random variables. The proof of the theorem follows as a consequence of the dependent structure of these Bernoulli random variables, along with an application of Rényi's characterization of Poisson processes.

Interest in nearest neighbor counts arises in ecological applications [see Roberts (1969), Cox (1981) and references therein] and in psychological studies [see Schwarz and Tversky (1980) and Tversky, Rinott and Newman (1983)]. Theoretical results concerning the probability that a random point is the rth nearest neighbor of its own sth nearest neighbor can be found in Roberts (1969), Schwarz and Tversky (1980), Cox (1981), Picard (1982), Schilling (1986) and Henze (1986, 1987). It should be remarked that, in addition to the Poisson process setting, some of these authors considered a set of n iid observations in $\mathbb{R}^d$ that have a continuous density and that behave locally like a Poisson process when n is large.

2. Preliminary results. Without loss of generality, let the Poisson process Π be "centered" so that the arbitrary point $Q = \mathbf{0} = (0, \dots, 0)$, the origin in $\mathbb{R}^d$. Order the remaining points by letting Q_k denote the kth nearest neighbor of Q in Π. Thus $\Pi = \{Q = \mathbf{0}, Q_1, Q_2, \dots\}$.

A standard argument shows that $\|Q_k\|^d$ has a gamma pdf of the form

$$f_k(v) = (\mathscr{V}_d \lambda_d)^k v^{k-1} \exp(-\mathscr{V}_d \lambda_d v)/(k-1)!, \qquad v > 0,$$

where $\mathscr{V}_d$ is the volume of the $\mathscr{L}_p$ unit ball $B(\mathbf{0}, 1)$. Since the distribution of $N_{d,r}$ does not depend on λ_d, we may and will take $\lambda_d = 1/\mathscr{V}_d$, so that

$$f_k(v) = \frac{v^{k-1}}{\Gamma(k)} e^{-v}, \qquad v > 0. \tag{2.1}$$

With this choice for λ_d, it can be readily shown that the joint pdf of $\|Q_1\|^d, \|Q_2\|^d, \dots, \|Q_k\|^d$ assumes the form

$$f_{1,\dots,k}(v_1, \dots, v_k) = \exp(-v_k), \qquad 0 < v_1 < v_2 < \cdots < v_k, \tag{2.2}$$

independent of d. For each $k = 1, 2, \dots,$ let $T_{d,k} = r$ if Q_k has $\mathbf{0}$ as its $(r+1)$st nearest neighbor $(r = 0, 1, \dots)$. Thus, $N_{d,r} = \sum_{k=1}^{\infty} I(T_{d,k} = r - 1)$ $(r = 1, 2, \dots)$ and $\sum_{r=0}^{\infty} I(T_{d,k} = r) = 1$ $(k = 1, 2, \dots)$, where $I(A)$ denotes the indicator of event A.

PROPOSITION 1. *For given nonnegative integers $r_1, \dots, r_k$, as $d \to \infty$,*

$$P(T_{d,i} = r_i, i = 1, \dots, k) \to p_k(r_1, \dots, r_k),$$

where

$$\begin{aligned} &p_k(r_1, \dots, r_k) \\ &\quad = \int_0^\infty \int_{v_1}^\infty \cdots \int_{v_{k-1}}^\infty \exp(-v_k) \prod_{i=1}^k \left\{ \frac{v_i^{r_i}}{r_i!} \exp(-v_i) \right\} dv_k \cdots dv_1. \end{aligned} \tag{2.3}$$

The proof of Proposition 1 borrows heavily from Newman and Rinott (1985) (hereafter referred to as NR), particularly the proof of Lemma 6. As a preliminary, we now state some lemmas.

LEMMA 1 (Lemma 4 of NR). *If the pdf of a random vector U in $\mathbb{R}^d$ can be expressed in the form $g(\|u\|)$ for some function g, then the distribution of $U/\|U\|$ does not depend on g.*

We denote the distribution of $U/\|U\|$ (which is concentrated on the $\mathscr{L}_p$ unit sphere centered at $\mathbf{0}$) by $\mathscr{P}_d$.

LEMMA 2. *If U_1 and U_2 are iid with distribution $\mathscr{P}_d$, then $\|U_1 - U_2\| \to c$ in probability as $d \to \infty$ for some constant $c > 1$ (depending on the value of p).*

LEMMA 3. *If U_1 and U_2 are iid with distribution $\mathscr{P}_d$, and $S_i = B(U_i, \|U_i\|)$, $i = 1, 2$, then $E \operatorname{vol}(S_1 \cap S_2)/\mathscr{V}_d \to 0$ as $d \to \infty$ [where $\operatorname{vol}(S)$ denotes the volume of set S].*

LEMMA 4. *If U has distribution $\mathscr{P}_d$, $S = B(U, \|U\|)$ and $S_0 = B(\mathbf{0}, 1)$, then as $d \to \infty$, $E \operatorname{vol}(S \cap S_0)/\mathscr{V}_d \to 0$.*

PROOF OF PROPOSITION 1. For $0 < v_1 < v_2 < \cdots < v_k$, let

$$p_k^d(v_1, \dots, v_k) := P\big(T_{d,i} = r_i,\, i = 1, \dots, k \mid \|Q_i\|^d = v_i,\, i = 1, \dots, k\big).$$

Note by (2.2) that

$$P(T_{d,i} = r_i, i = 1, \dots, k) = \int_0^\infty \int_{v_1}^\infty \cdots \int_{v_{k-1}}^\infty p_k^d(v_1, \dots, v_k) \exp(-v_k)\, dv_k \cdots dv_1.$$

We want to show that as $d \to \infty$,

$$p_k^d(v_1, \dots, v_k) \to \prod_{i=1}^k \left\{ \frac{v_i^{r_i}}{r_i!} \exp(-v_i) \right\}, \tag{2.4}$$

which together with the bounded convergence theorem yields Proposition 1. Observe that the ith factor on the right-hand side of (2.4) is just the probability that an $\mathscr{L}_p$ ball of radius $v_i^{1/d}$ in $\mathbb{R}^d$ contains exactly r_i points, that is, the probability that a point at distance $v_i^{1/d}$ from the origin (such as Q_i when $\|Q_i\|^d = v_i$) has $\mathbf{0}$ as its $(r_i + 1)$st nearest neighbor.

For the remainder of the proof, we will condition on the event

$$\mathscr{E}_d := \big[\|Q_i\|^d = v_i, i = 1, \dots, k\big],$$

with $0 < v_1 < v_2 < \cdots < v_k$ fixed, and let S_i denote the ball $B(Q_i, v_i^{1/d})$ for $i = 1, \dots, k$.

The proof of (2.4) depends upon showing, with (conditional) probability approaching 1 as $d \to \infty$, that the r_i points in S_i (besides Q_i) required to make $T_{d,i} = r_i$ do not reside in S_j (and hence do not include Q_j) for any $j \neq i$, nor in the ball $S_0 := B(\mathbf{0}, v_k^{1/d})$. Consequently, they reside in the set

$$\tilde{S}_i := S_i - \bigcup_{\substack{j=0\\ j\neq i}}^{k} S_j \qquad (i = 1, \dots, k).$$

It follows immediately that (2.4) is equivalent to

$$\text{(2.5)} \qquad \tilde{p}_k^d(v_1, \dots, v_k) \to \prod_{i=1}^{k} \left\{ \frac{v_i^{r_i}}{r_i!} \exp(-v_i) \right\} \quad \text{as } d \to \infty,$$

where $\tilde{p}_k^d(v_1, \dots, v_k)$ denotes the conditional probability, given $\mathscr{E}_d$, that there are exactly r_i points in $\tilde{S}_i$ for $i = 1, \dots, k$. The bases for these claims, and hence of this equivalence, are two facts, which do require justification:

(a) For $j \neq i$, $1 \leq i, j \leq k$, $P(Q_j \in S_i \mid \mathscr{E}_d) \to 0$ as $d \to \infty$.
(b) For $j \neq i$, $0 \leq i, j \leq k$, $E(\mathrm{vol}(S_i \cap S_j) \mid \mathscr{E}_d)/\mathscr{V}_d \to 0$ as $d \to \infty$.

To show (a), let $U_1 = Q_i/v_i^{1/d}$ and $U_2 = Q_j/v_j^{1/d}$ in Lemma 2, which are conditionally independent, given $\mathscr{E}_d$, with common distribution $\mathscr{P}_d$, and observe, on account of the triangle inequality, that

$$\|Q_i/v_i^{1/d} - Q_j/v_i^{1/d}\| \geq \|U_1 - U_2\| - |1 - \big(v_j/v_i\big)^{1/d}| \to c - 0 > 1$$

in probability as $d \to \infty$. This establishes (a).

Fact (b) requires two separate arguments: for $1 \leq i < j \leq k$ and for $0 = i < j \leq k$. For $1 \leq i < j \leq k$, defines $S_1' = S_i/v_i^{1/d}$ and $S_2' = S_j/v_j^{1/d}$, observe, on account of the triangle inequality, that

$$\mathrm{vol}\big(S_i \cap S_j\big)/v_j = \mathrm{vol}\Big(\Big(S_1'\big(v_i/v_j\big)^{1/d}\Big) \cap S_2'\Big) \leq \mathrm{vol}(S_1' \cap S_2')$$

and then apply Lemma 3 to establish (b). For $0 = i < j \leq k$, define $S' = S_j/v_j^{1/d}$, observe, on account of the triangle inequality, that

$$\mathrm{vol}\big(S_0 \cap S_j\big)/v_k = \mathrm{vol}\Big(B(\mathbf{0}, 1) \cap \Big(S'\big(v_j/v_k\big)^{1/d}\Big)\Big) \leq \mathrm{vol}(B(\mathbf{0}, 1) \cap S')$$

and then apply Lemma 4 to establish (b).

To get at the verification of (2.5), we refine $\mathscr{E}_d$ to events of the form

$$\hat{\mathscr{E}}_d = \big[(Q_1, \dots, Q_k) = (q_1, \dots, q_k)\big],$$

where $\|q_i\|^d = v_i$, $i = 1, \dots, k$. Further, let $\hat{p}_k^d(q_1, \dots, q_k)$ denote the conditional probability, given $\hat{\mathscr{E}}_d$, that there are exactly r_i points in $\tilde{S}_i$ for $i = 1, \dots, k$. With this level of conditioning, the sets $S_0, \tilde{S}_1, \dots, \tilde{S}_k$ are disjoint and fixed. Since $\hat{\mathscr{E}}_d$ depends only upon the behavior of the Poisson point process *within* S_0, and $\hat{p}_k^d(q_1, \dots, q_k)$ is concerned only with behavior *outside* S_0, it follows that

$$\hat{p}_k^d(q_1, \dots, q_k) = \prod_{i=1}^{k} \left\{ \frac{\tilde{v}_i^{r_i}}{r_i!} \exp(-\tilde{v}_i) \right\},$$

where $\tilde{v}_i = \tilde{v}_i(q_1, \dots, q_k) := \mathrm{vol}(\tilde{S}_i)/\mathscr{V}_d$, $i = 1, \dots, k$. Thus,

$$\begin{aligned}\tilde{p}_k^d(v_1, \dots, v_k) &= E\big(\hat{p}_k^d(Q_1, \dots, Q_k) \mid \mathscr{E}_d\big)\\ &= E\left(\prod_{i=1}^k \left\{\frac{\tilde{v}_i(Q_1, \dots, Q_k)^{r_i}}{r_i!}\exp(-\tilde{v}_i(Q_1, \dots, Q_k))\right\}\middle|\mathscr{E}_d\right).\end{aligned}$$

In turn, it follows, on account of (a) and (b), that as $d \to \infty$,

$$\tilde{v}_i(Q_1, \dots, Q_k) - v_i = \left\{\mathrm{vol}\big(\tilde{S}_i\big) - \mathrm{vol}(S_i)\right\}/\mathscr{V}_d \to 0$$

in probability, so that the difference

$$\begin{aligned}\tilde{p}_k^d(v_1, \dots, v_k) &- \prod_{i=1}^k \left\{\frac{v_i^{r_i}}{r_i!}\exp(-v_i)\right\}\\ &= E\left(\prod_{i=1}^k \left\{\frac{\tilde{v}_i(Q_1, \dots, Q_k)^{r_i}}{r_i!}\exp(-\tilde{v}_i(Q_1, \dots, Q_k))\right\}\right.\\ &\qquad\left. - \prod_{i=1}^k \left|\left\{\frac{v_i^{r_i}}{r_i!}\exp(-v_i)\right\}\middle|\mathscr{E}_d\right.\right) \to 0\end{aligned}$$

as $d \to \infty$, thereby establishing (2.5). □

PROOF OF LEMMA 2. The following arguments are very similar to those from line 20 on page 801 through line 2 of page 802 in NR. For $1 \le p < \infty$, let $W^{(1)}$ and $W^{(2)}$ be iid random vectors in $\mathbb{R}^d$ with iid components ($W_i^{(j)}$, $i = 1, \dots, d$) and pdf of the form

$$f(w_1, \dots, w_d) = (\mathrm{const.}) \prod_{i=1}^d \exp\big(-|w_i|^p\big).$$

For $p = \infty$, let $W^{(1)}$ and $W^{(2)}$ be iid random vectors in $\mathbb{R}^d$, each with independent components that are uniformly distributed on the interval $(-1, 1)$. By Lemma 1, (U_1, U_2) has the same distribution as $(W^{(1)}/\|W^{(1)}\|, W^{(2)}/\|W^{(2)}\|)$.

For $1 \le p < \infty$, by the law of large numbers, as $d \to \infty$,

$$\|W^{(j)}\|^p/d \to E|W_1^{(1)}|^p \quad \text{and} \quad \|W^{(1)} - W^{(2)}\|^p/d \to E|W_1^{(1)} - W_1^{(2)}|^p$$

in probability, so that, as $d \to \infty$,

$$\|U_1 - U_2\|^p = \left\|\frac{W^{(1)}}{\|W^{(1)}\|} - \frac{W^{(2)}}{\|W^{(2)}\|}\right\|^p \to \frac{E|W_1^{(1)} - W_1^{(2)}|^p}{E|W_1^{(1)}|^p}$$

in probability. The right-hand side is strictly greater than 1 since $E|W_1^{(1)} - x|^p$ is an even and convex function of x, and hence strictly increasing in $|x|$ for $1 \le p < \infty$.

For $p = \infty$, $\|W^{(j)}\| \to 1$ and $\|W^{(1)} - W^{(2)}\| \to 2$, so that

$$\|U_1 - U_2\| = \left\| \frac{W^{(1)}}{\|W^{(1)}\|} - \frac{W^{(2)}}{\|W^{(2)}\|} \right\| \to 2. \qquad \square$$

The proof of Lemma 3 is established as part of the proof of Lemma 6 of NR (cf. lines 13–14, page 801).

PROOF OF LEMMA 4. For an arbitrary but *fixed* n, let $U_1, \dots, U_n$ be iid with distribution $\mathscr{P}_d$ and let $S_i = B(U_i, 1)$. Then

$$\begin{aligned} 1 &\geq \mathscr{V}_d^{-1} E \operatorname{vol}\left(\left(\bigcup_{i=1}^{n} S_i \right) \cap S_0 \right) \\ &\geq \mathscr{V}_d^{-1} \sum_{i=1}^{n} E \operatorname{vol}(S_i \cap S_0) - \mathscr{V}_d^{-1} \sum_{1 \leq i \neq j \leq n} E \operatorname{vol}(S_i \cap S_j) \\ &= n \mathscr{V}_d^{-1} E \operatorname{vol}(S_1 \cap S_0) - n(n-1) \mathscr{V}_d^{-1} E \operatorname{vol}(S_1 \cap S_2). \end{aligned}$$

According to Lemma 3, the latter goes to zero as $d \to \infty$, and since n is arbitrary, the lemma follows. $\square$

3. Proof of Theorem 1. Let $T_1, T_2, \dots$ be nonnegative integer-valued random variables defined on a (rich) probability space such that

$$P(T_1 = r_1, \dots, T_k = r_k) = p_k(r_1, \dots, r_k) \tag{3.1}$$

given in (2.3). The existence of the probability space is guaranteed by the fact that the collection of distributions p_k, $k = 1, 2, \dots$, satisfies Kolmogorov's consistency condition. [Later, we need to assume that the probability space is sufficiently rich to admit uniform random variables independent of the T_k's. In truth, any nonatomic probability space is sufficiently rich to support all of our random variables; cf. Halmos (1950), page 173.]

Proposition 1 implies the weak convergence

$$(T_{d,1}, T_{d,2}, \dots) \Rightarrow (T_1, T_2, \dots) \quad \text{as } d \to \infty. \tag{3.2}$$

A simple, useful stochastic description of $(T_1, T_2, \dots)$ is possible, arising from functional relationships among the densities (2.2), (2.3) and (3.1). The components of $(T_1, T_2, \dots)$ can be viewed as conditionally independent Poisson distributed random variables with means $\lambda_1, \lambda_2, \dots,$ respectively, given the random vector $\lambda = (\lambda_1, \lambda_2, \dots)$, where the distribution of λ is, for any fixed dimension d, the same as that for $(\|Q_1\|^d, \|Q_2\|^d, \dots)$. Indeed, $\lambda_1, \lambda_2, \dots$ can be viewed as a Poisson point process with constant intensity rate 1 on the positive real line.

Since $\|Q_k\|^d$ has the density appearing in (2.1) it follows immediately that T_k has a negative binomial marginal distribution:

$$P(T_k = r) = \int_0^\infty \left\{ \frac{v^r}{r!} e^{-V} \right\} \frac{v^{k-1}}{\Gamma(k)} e^{-v}\, dv = 2^{-(k+r)} \binom{k+r-1}{r}, \tag{3.3}$$
$$r = 0, 1, \dots .$$

Thus the distribution of T_k is the k-fold convolution of the (geometric) distribution of T_1.

More generally, for any set B in $\{0, 1, \dots\}^k$,

$$\begin{aligned} &P((T_1, \dots, T_k) \in B) \\ &= \int_0^\infty \int_{v_1}^\infty \cdots \int_{v_{k-1}}^\infty \exp(-v_k) \sum_{(r_1, \dots, r_k) \in B} \prod_{i=1}^k \left\{ \frac{v_i^{r_i}}{r_i!} \exp(-v_i) \right\} dv_k \cdots dv_1. \end{aligned} \tag{3.4}$$

Let

$$N_r := \sum_{k=1}^\infty I(T_k = r - 1), \qquad r = 1, 2, \dots,$$

and observe from (3.3) that

$$EN_r = \sum_{k=1}^\infty 2^{-(k+r-1)} \binom{k+r-2}{r-1} = 1, \qquad r = 1, 2, \dots .$$

Likewise, $EN_{d,r} = 1$, $r = 1, 2, \dots$, because each point of Π has exactly one rth nearest neighbor and hence, by (3.2),

$$(N_{d,1}, N_{d,2}, \dots) \Rightarrow (N_1, N_2, \dots) \quad \text{as } d \to \infty.$$

Our task reduces to proving that $N_1, N_2, \dots$ are iid Poisson with mean 1.

To this end, let U_k, $k = 1, 2, \dots$, be iid uniform random variables on $(0, 1]$ that are independent of $(T_1, T_2, \dots)$ and consider the set of points $\Pi_0 = \{T_k + U_k : k = 1, 2, \dots\}$, a point process on $(0, \infty)$. Let $M(\mathscr{S})$ denote the number of points of Π_0 in the set $\mathscr{S}$. Clearly, $M((r-1, r]) = N_r$, and given $N_r = n$, these n points are uniformly distributed on $(r-1, r]$. The theorem follows if it can be shown that Π_0 is a Poisson process with constant intensity rate 1.

By Rényi's theorem [see, e.g., page 34 of Kingman (1993)], it suffices to show that

$$P(M(\mathscr{S}) = 0) = \exp(-L(\mathscr{S})) \tag{3.5}$$

for every finite union $\mathscr{S}$ of bounded intervals, when $L(\mathscr{S})$ denotes the Lebesgue measure of $\mathscr{S}$. Without loss of generality, we may consider sets of the form $\bigcup_{r=0}^n \mathscr{S}_r$, $n = 0, 1, \dots$, where $\mathscr{S}_r$ is a finite union of intervals in $(r, r+1]$. From the construction of Π_0, it is apparent that the distributions of $M(\bigcup_{r=0}^n \mathscr{S}_r)$ and $M(\bigcup_{r=0}^n (r, r + L(\mathscr{S}_r)])$ are the same. Thus, it suffices to consider sets $\mathscr{S}$ of the form $\bigcup_{r=0}^n (r, r + a_r]$ with $0 \le a_r \le 1$ and $n = 0, 1, \dots$.

To begin with, let $\mathscr{S}$ be a suitable set. Then

$$P(M(\mathscr{S}) = 0) = \lim_{k \to \infty} P(T_i + U_i \notin \mathscr{S}, i = 1, \dots, k) \tag{3.6}$$

and by the inclusion–exclusion principle,

$$(3.7)\qquad P(T_i + U_i \notin \mathscr{S},\ i = 1,\dots,k) = 1 - \mathscr{S}_{1,k} + \mathscr{S}_{2,k} - \cdots \pm \mathscr{S}_{k,k},$$

where

$$\mathscr{S}_{m,k} := \sum_{1 \le j_1 < j_2 < \cdots < j_m \le k} P\big(T_{j_i} + U_{j_i} \in \mathscr{S},\ i = 1,\dots,m\big), \qquad m = 1,\dots,k.$$

Clearly, $\mathscr{S}_{m,k}$ is nondecreasing in k and in the limit, as $k \to \infty$, attains the value

$$(3.8)\qquad \mathscr{S}_m := \sum_{1 \le j_1 < j_2 < \cdots < j_m < \infty} P\big(T_{j_i} + U_{j_i} \in \mathscr{S},\ i = 1,\dots,m\big).$$

We shall show that

$$(3.9)\qquad \mathscr{S}_m = \frac{L(\mathscr{S})^m}{m!}, \qquad m = 1,2,\dots,$$

which, in view of (3.6) and (3.7), establishes (3.5). (The required limiting operations follow from the dominated convergence theorem and

$$\sum_{m=0}^{\infty} \frac{L(\mathscr{S})^m}{m!} = e^{L(\mathscr{S})} < \infty.)$$

So the task is to establish (3.9).

It is instructive to start with the simple case $\mathscr{S} = (r, r+1]$, $r = 0,1,\dots,$ where we need to show that $\mathscr{S}_m = (m!)^{-1}$, $m = 1,2,\dots$. With (3.4) and $k \ge j_m$, a typical summand in (3.8) becomes

$$\begin{aligned}
&P\big(T_{j_i} + U_{j_i} \in \mathscr{S},\ i = 1,\dots,m\big)\\
&\quad = P\big(T_{j_i} = r,\ i = 1,\dots,m\big)\\
&\quad = \int_0^\infty \int_{v_1}^\infty \cdots \int_{v_{k-1}}^\infty \exp(-v_k) \prod_{i=1}^m \left\{ \frac{v_{j_i}^r}{r!} \exp(-v_{j_i}) \right\} dv_k \cdots dv_1\\
&\quad = \frac{(-1)^{rm}}{(r!)^m} \frac{\partial^{rm}}{\partial t_1^r \cdots \partial t_m^r}\\
&\qquad \times \int_0^\infty \int_{v_1}^\infty \cdots \int_{v_{k-1}}^\infty \exp\left(-v_k - \sum_{i=1}^m t_i v_{j_i}\right) dv_k \cdots dv_1 \bigg|_{t_1 = \cdots = t_m = 1},
\end{aligned}$$

with the integral assuming the value

$$\frac{1}{(1+t_m)^{k_m}(1+t_m+t_{m-1})^{k_{m-1}} \cdots (1+t_1+\cdots+t_m)^{k_1}},$$

where $k_1 = j_1$ and $k_i = j_i - j_{i-1}$ for $i > 1$, independently of k. Hereafter, it will be understood that the partial derivatives with respect to $t_1, \dots, t_m$ are evaluated at $t_1 = \cdots = t_m = 1$. Thus, due to symmetry,

$$\begin{aligned}\mathscr{S}_m &= \frac{(-1)^{rm}}{(r!)^m} \sum_{k_1,\dots,k_m \geq 1} \frac{\partial^{rm}}{\partial t_1^r \cdots \partial t_m^r} \\ &\qquad \times \left(\frac{1}{(1+t_m)^{k_m}(1+t_m+t_{m-1})^{k_{m-1}} \cdots (1+t_1+\cdots+t_m)^{k_1}} \right) \\ &= \frac{(-1)^{rm}}{(r!)^m} \sum_{k_1,\dots,k_m \geq 1} \frac{\partial^{rm}}{\partial t_1^r \cdots \partial t_m^r} \\ &\qquad \times \left(\frac{1}{(1+t_1)^{k_1}(1+t_1+t_2)^{k_2} \cdots (1+t_1+\cdots+t_m)^{k_m}} \right) \\ &= \frac{(-1)^{rm}}{(r!)^m} \frac{\partial^{rm}}{\partial t_1^r \cdots \partial t_m^r} \\ &\quad \times \sum_{k_1,\dots,k_m \geq 1} \frac{1}{(1+t_1)^{k_1}(1+t_1+t_2)^{k_2} \cdots (1+t_1+\cdots+t_m)^{k_m}} \\ &= \frac{(-1)^{rm}}{(r!)^m} \frac{\partial^{rm}}{\partial t_1^r \cdots \partial t_m^r} \left(\frac{1}{t_1(t_1+t_2)\cdots(t_1+\cdots+t_m)} \right).\end{aligned}$$

Since

$$\frac{\partial^{rm}}{\partial t_1^r \cdots \partial t_m^r} \left(\frac{1}{t_{\pi_1} \cdots (t_{\pi_1} + \cdots + t_{\pi_m})} \right)$$

is the same for all permutations $\pi = (\pi_1, \dots, \pi_m)$ on $\{1, \dots, m\}$ (when evaluated at $t_1 = \cdots = t_m = 1$) and since

$$\sum_{\pi} \frac{1}{t_{\pi_1} \cdots (t_{\pi_1} + \cdots + t_{\pi_m})} = \frac{1}{t_1 t_2 \cdots t_m}$$

(which can be readily shown by induction on m), we have

$$\begin{aligned}&\frac{\partial^{rm}}{\partial t_1^r \cdots \partial t_m^r} \left(\frac{1}{t_1 \cdots (t_1 + \cdots + t_m)} \right) \\ &\qquad = \frac{1}{m!} \frac{\partial^{rm}}{\partial t_1^r \cdots \partial t_m^r} \left(\frac{1}{t_1 t_2 \cdots t_m} \right) = \frac{(-1)^{rm}}{m!} (r!)^m.\end{aligned}$$

Consequently,

$$\mathscr{S}_m = \frac{(-1)^{rm}}{(r!)^m} \frac{(-1)^{rm}}{m!} (r!)^m = \frac{1}{m!}$$

as claimed.

Now for the general case with $\mathscr{S} = \bigcup_{r=0}^{n}(r, r+a_r]$, $0 \le a_0, a_1, \dots, a_n \le 1$, a typical summand in (3.8) becomes, for $k \ge j_m$ [cf. (3.4)],

$$\begin{aligned}
&P\big(T_{j_i} + U_{j_i} \in \mathscr{S},\ i = 1, \dots, m\big)\\
&\quad = P\big(T_{j_i} + U_{j_i} \in (r, r+a_r] \text{ for some } r = 0, \dots, n; \text{ for each } i = 1, \dots, m\big)\\
&\quad = \int_0^\infty \int_{v_1}^\infty \cdots \int_{v_{k-1}}^\infty \exp(-v_k) \prod_{i=1}^m \left\{ \sum_{r_i=0}^n a_{r_i} \frac{v_{j_i}^{r_i}}{r_i!} \exp(-v_{j_i}) \right\} dv_k \cdots dv_1\\
&\quad = \sum_{0 \le r_1, \dots, r_m \le n} \left(\frac{a_{r_1}}{r_1!}\right) \cdots \left(\frac{a_{r_m}}{r_m!}\right)\\
&\qquad\qquad \times \int_0^\infty \int_{v_1}^\infty \cdots \int_{v_{k-1}}^\infty \exp(-v_k) \prod_{i=1}^m \left\{ v_{j_i}^{r_i} \exp(-v_{j_i}) \right\} dv_k \cdots dv_1\\
&\quad = \sum_{0 \le r_1, \dots, r_m \le n} \prod_{i=1}^m \left(\frac{a_{r_i}}{r_i!}\right) (-1)^{r_1 + \cdots + r_m}\\
&\qquad\qquad \times \frac{\partial^{r_1 + \cdots + r_m}}{\partial t_1^{r_1} \cdots \partial t_m^{r_m}} \int_0^\infty \int_{v_1}^\infty \cdots \int_{v_{k-1}}^\infty \exp\left(-v_k - \sum_{i=1}^m t_i v_{j_i}\right) dv_k \cdots dv_1\\
&\quad = \sum_{0 \le r_1, \dots, r_m \le n} \prod_{i=1}^m \left(\frac{a_{r_i}}{r_i!}\right) (-1)^{r_1 + \cdots + r_m}\\
&\qquad \times \frac{\partial^{r_1 + \cdots + r_m}}{\partial t_1^{r_1} \cdots \partial t_m^{r_m}} \left(\frac{1}{(1+t_m)^{k_m} (1 + t_m + t_{m-1})^{k_{m-1}} \cdots (1 + t_1 + \cdots + t_m)^{k_1}} \right),
\end{aligned}$$

where $k_1 = j_1$ and $k_i = j_i - j_{i-1}$ for $i > 1$. Consequently,

$$\begin{aligned}
\mathscr{S}_m &= \sum_{0 \le r_1, \dots, r_m \le n} \prod_{i=1}^m \left(\frac{a_{r_i}}{r_i!}\right) (-1)^{r_1 + \cdots + r_m} \sum_{k_1, \dots, k_m \ge 1} \frac{\partial^{r_1 + \cdots + r_m}}{\partial t_1^{r_1} \cdots \partial t_m^{r_m}}\\
&\qquad\qquad \times \left(\frac{1}{(1+t_m)^{k_m} (1 + t_m + t_{m-1})^{k_{m-1}} \cdots (1 + t_1 + \cdots + t_m)^{k_1}} \right)\\
&= \sum_{0 \le r_1, \dots, r_m \le n} \prod_{i=1}^m \left(\frac{a_{r_i}}{r_i!}\right) (-1)^{r_1 + \cdots + r_m} \sum_{k_1, \dots, k_m \ge 1} \frac{\partial^{r_1 + \cdots + r_m}}{\partial t_1^{r_1} \cdots \partial t_m^{r_m}}\\
&\qquad\qquad \times \left(\frac{1}{(1+t_1)^{k_1} (1 + t_1 + t_2)^{k_2} \cdots (1 + t_1 + \cdots + t_m)^{k_m}} \right)\\
&= \sum_{0 \le r_1, \dots, r_m \le n} \prod_{i=1}^m \left(\frac{a_{r_i}}{r_i!}\right) (-1)^{r_1 + \cdots + r_m} \frac{\partial^{r_1 + \cdots + r_m}}{\partial t_1^{r_1} \cdots \partial t_m^{r_m}}\\
&\qquad\qquad \times \left(\frac{1}{t_1 (t_1 + t_2) \cdots (t_1 + \cdots + t_m)} \right)
\end{aligned}$$

$$= \frac{1}{m!} \sum_{\pi} \sum_{0 \le r_1, \dots, r_m \le n} \prod_{i=1}^{m} \left(\frac{a_{r_i}}{r_i!} \right) (-1)^{r_1 + \cdots + r_m} \frac{\partial^{r_1 + \cdots + r_m}}{\partial t_1^{r_1} \cdots \partial t_m^{r_m}}$$

$$\times \left(\frac{1}{t_{\pi_1} \cdots (t_{\pi_1} + \cdots + t_{\pi_m})} \right)$$

$$= \frac{1}{m!} \sum_{0 \le r_1, \dots, r_m \le n} \prod_{i=1}^{m} \left(\frac{a_{r_i}}{r_i!} \right) (-1)^{r_1 + \cdots + r_m} \frac{\partial^{r_1 + \cdots + r_m}}{\partial t_1^{r_1} \cdots \partial t_m^{r_m}} \left(\frac{1}{t_1 t_2 \cdots t_m} \right)$$

$$= \frac{1}{m!} \sum_{0 \le r_1, \dots, r_m \le n} \prod_{i=1}^{m} \left(\frac{a_{r_i}}{r_i!} \right) (r_1!) \cdots (r_m!) = \frac{1}{m!} (a_0 + \cdots + a_n)^m,$$

which establishes (3.9) and completes the proof. □

REFERENCES

COX, T. F. (1981). Reflexive nearest neighbors. *Biometrics* **37** 367–369.

HALMOS, P. R. (1950). *Measure Theory*. Van Nostrand, New York.

HENZE, N. (1986). On the probability that a random point is the jth nearest neighbor to its own kth nearest neighbor. *J. Appl. Probab.* **23** 221–226.

HENZE, N. (1987). On the fraction of random points with specified nearest-neighbor interrelations and degree of attraction. *Adv. in Appl. Probab.* **19** 873–895.

KINGMAN, J. F. C. (1993). *Poisson Processes*. Oxford Univ. Press.

NEWMAN, C. M. and RINOTT, Y. (1985). Nearest neighbors and Voronoi volumes in high-dimensional point processes with various distance functions. *Adv. in Appl. Probab.* **17** 794–809.

NEWMAN, C. M., RINOTT, Y. and TVERSKY, A. (1983). Nearest neighbors and Voronoi regions in certain point processes. *Adv. in Appl. Probab.* **15** 726–751.

PICARD, D. K. (1982). Isolated nearest neighbors. *J. Appl. Probab.* **19** 444–449.

ROBERTS, F. D. K. (1969). Nearest neighbors in a Poisson ensemble. *Biometrika* **56** 401–406.

SCHILLING, M. F. (1986). Mutual and shared neighbor probabilities: finite- and infinite-dimensional results. *Adv. in Appl. Probab.* **18** 388–405.

SCHWARZ, G. and TVERSKY, A. (1980). On the reciprocity of proximity relations. *J. Math. Psychol.* **22** 157–175.

TVERSKY, A., RINOTT, Y. and NEWMAN, C. M. (1983). Nearest neighbor analysis of point processes: applications to multidimensional scaling. *J. Math. Psychol.* **27** 235–250.

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