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BOUNDS ON MEASURES SATISFYING MOMENT CONDITIONS

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Given a semialgebraic set $S \subset \mathbb{R}^n$, we provide a numerical approximation procedure that provides upper and lower bounds on $\mu(S)$, for measures μ that satisfy some given moment conditions. The bounds are obtained as solutions of positive semidefinite programs that can be solved via standard software packages.

1. Introduction. Given a set Φ of measures μ on the Borel sets of $\mathbb{R}^n$, that satisfy some given moment conditions $\int x^\alpha \, d\mu = \gamma_\alpha$, $\alpha \in \Gamma$, and given a Borel set S, we investigate the problem $\mathbb{P} \to \sup_{\mu \in \Phi} \mu(S)$ of computing (or approximating) Tchebycheff-type upper (and lower) bounds on $\mu(S)$.

For basic theoretical results on Tchebycheff-type bounds, the interested reader is referred to the pioneering work of Isii [7], and later, among others, Anastassiou [1], Johnson and Taaffe [10], Bertsimas and Sethuraman [3], Bertsimas and Popescu [2], Smith [15], Whitt [16], in particular, for examples of important applications, notably in queuing, probability and finance.

Some problems of this type have elegant, and sometimes exact, solutions via the use of geometric probability techniques, provided the number of moment conditions is small, say two or three. For a nice account of such results, the interested reader is referred to [1]. However, as soon as the number of moment conditions is larger than, say three, one may not invoke these techniques any more since they involve the description of the convex hull of a "moment curve."

In a more recent work, Bertsimas and Sethuraman [3] (based on Bertsimas and Popescu [2]) have provided nice explicit solutions for the case where S is convex, Γ includes all first-order moment conditions, and μ is supported on $(\mathbb{R}^n)^+$, and the case where Γ includes all first- and second-order moment conditions and μ is supported on $\mathbb{R}^n$. In the latter case, solving $\mathbb{P}$ reduces to solving a convex optimization problem in $\mathbb{R}^n$, where one minimizes the "weighted distance" of the point γ_1 (with coordinates the first moments of γ) to the set S, the "weighted distance" being expressed via the matrix of second-order moments of γ. This upper bound strictly improves the well-known Chebyshev bound in the scalar case, which shows the potential of the approach. The methodology also applies if S is the union of a finite number of disjoint convex sets. They use semidefinite programming techniques applied to the dual problem $\mathbb{D} \to \inf\{\sum_{\alpha \in \Gamma} p_\alpha \gamma_\alpha \mid$

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$p(x) \geq \mathbb{1}_S(x)\}$, which under an interior point condition, has same value as the primal $\mathbb{P}$. On the negative side, they also show that, in general, solving $\mathbb{D}$ to obtain a tight upper bound is a NP-hard optimization problem that reduces to a separation problem of Grötschel, Lovász and Schrijver [6].

However, bounds obtained under few moment conditions are poor and not very informative (see for instance the discussion in Johnson and Taaffe [10] for Tchebycheff systems on the line). Therefore, there is a need for efficient practical computation of upper bounds (even not tight) in case of a larger number of moment conditions.

In this paper, we use the fact that in typical applications, the set S is a semialgebraic set (like a rectangle, an intersection of ellipsoids, ...) which permits us to derive an appropriate approximation procedure. Therefore, we restrict to the case of an arbitrary semialgebraic set S (neither necessarily convex, nor connected) and an arbitrary (finite) number of moment conditions. In contrast to the approach in [3], we directly consider the primal problem $\mathbb{P}$ instead of the dual $\mathbb{D}$. We decompose μ into the sum $\varphi + \psi$ of a measure ψ and a measure φ with support contained in S, and $\mathbb{P}$ becomes the problem of maximizing $\varphi(\mathbb{R}^n)$, the total mass of φ under the moment constraints. This permits immediately obtaining a family $\{\mathbb{Q}_r\}$ of relaxations of $\mathbb{P}$, each $\mathbb{Q}_r$ being a convex linear positive semidefinite (psd) program with as variables, the unknown moments up to $2r$ of φ and ψ. Each $\mathbb{Q}_r$ can be solved by standard software packages like the MATLAB LMI toolbox. The resulting sequence $\{\sup \mathbb{Q}_r\}$ provides better and better upper bounds on $\sup_{\mu \in \Phi} \mu(S)$. Under some additional assumption on S (but with no convexity assumption) satisfied in most cases of interest, the sequence $\{\sup \mathbb{Q}_r\}$ converges to a tight upper bound. In fact, as shown on a few examples, a tight upper bound may be obtained at the first relaxation, even for nonconvex and nonconnected sets S. If in addition, one wishes to restrict μ to have its support in some semialgebraic Borel set K, one may easily add related necessary conditions stated in terms of linear matrix inequalities (LMI).

In two special cases of importance, a tight upper bound is obtained directly at a single LMI relaxation: The first is when one considers the real line $\mathbb{R}$ and S is an interval $[a, b]$ (here, we cannot invoke the theory of Chebyshev systems because of the indicator function $\mathbb{1}_S$). A tight upper bound is achieved even if γ is not in the interior of the moment space. The second is when γ contains only first and second moments (not necessarily all of them) and S is defined by linear and/or quadratic concave polynomials. The first LMI relaxation provides a tight upper bound as in the case of the algorithm of [3].

Interestingly enough, this methodology permits one to get some insight on the impact of the moment conditions on upper bounds on $\mu(S)$ and on the sensitivity of the bound with respect to a change in the definition of the set S. When solvable, each $\max \mathbb{Q}_r$ not only provides an upper bound on $\sup_{\mu \in \Phi} \mu(S)$ but also provides an upper bound on $\sup_{\mu \in \Phi} \mu(S')$ for sets S' which can be much larger or much smaller than S, obtained by relaxing or enforcing some conditions $g_k(x) \geq 0$ in the definition of S. For instance, with $n = 1$, $S = [a, b]$ and two moment conditions $\{1, \gamma_1, \gamma_2\}$, with $\gamma_1 > b$ (resp. $\gamma_1 < a$), the tight upper bound $\sup_{\mu \in \Phi} \mu(S)$ is the same for sets $S' := [-\infty, b]$ or $S' := [b - \varepsilon, b]$ (resp. $S' := [a, \infty)$ or $S' := [a, a + \varepsilon]$). Only under more than four moment conditions, a tight upper bound will be more discriminating. This is because a quadratic nonnegative univariate polynomial minimized outside $[a, b]$ necessarily attains its minimum on $[a, b]$ at either a or b only, and a cubic univariate polynomial cannot be nonnegative.

The same remark is valid for arbitrary n, and only first- and second-order moment conditions.

For clarity of exposition, all of Section 5 is devoted to the proofs of the main theorems.

2. Notation and definitions. For any two real-valued symmetric matrices A, B, the notation $\langle A, B\rangle$ stands for the usual scalar product trace(AB), whereas the notation $A \succeq B$ (resp. $A \succ B$) stands for $A - B$ positive semidefinite (resp. positive definite). For a vector y, the notation y' stands for the transpose vector of y.

Let $\mathcal{B}$ be the usual Borel σ-field of $\mathbb{R}^n$ and let $\mathcal{M}$ be the set of finite signed Borel measures on $\mathcal{B}$, with $\mathcal{M}^+$ its positive cone. Let Γ be a finite subset of $\mathbb{N}^n$ and let $\{\gamma_\alpha\}_{\alpha \in \Gamma}$ be a given family of scalars (with $\gamma_0 = 1$). Let $\Phi := \{\mu \in \mathcal{M}^+ \mid \int x^\alpha \, d\mu = \gamma_\alpha,\ \alpha \in \Gamma\}$; that is, Φ is the set of probability measures (in short, p.m.) on $\mathcal{B}$ that satify the moment conditions $\int x^\alpha \, d\mu = \gamma_\alpha$ for all $\alpha \in \Gamma$.

We want to approximate

$$\sup_{\mu \in \Phi} \mu(B) \quad \text{and} \quad \inf_{\mu \in \Phi} \mu(B) \tag{2.1}$$

for a given semialgebraic compact set $S \in \mathcal{B}$, defined by polynomial inequalities; that is,

$$S := \{x \in \mathbb{R}^n \mid g_i(x) \geq 0,\ i = 1, \dots, m\}, \tag{2.2}$$

where $g_i(x) : \mathbb{R}^n \to \mathbb{R}$ is a polynomial for all $i = 1, \dots, m$. Let

$$v(x) = 1, x_1, x_2, \dots, x_n, x_1^2, \dots, x_n^{2r}, \tag{2.3}$$

be a basis [of dimension denoted by $s(2r)$] for the vector space $\mathcal{A}_{2r}$ of real-valued polynomials $p(x) : \mathbb{R}^n \to \mathbb{R}$ of degree at most $2r$, and write

$$x \mapsto p(x) := \sum_\alpha p_\alpha x^\alpha = \langle p, v(x)\rangle,$$

where $p = \{p_\alpha\} \in \mathbb{R}^{s(2r)}$ is the coefficient vector of $p(x)$ in the basis (2.3).

Moment matrix. Given an $s(2r)$-sequence $\{y_\alpha\}$, let $M_r(y)$ be the *moment-like* matrix of dimension $s(r)$, with rows and columns labelled by (2.3). For instance, for illustration purposes, and for clarity of exposition, consider the two-dimensional case. The moment matrix $M_r(y)$ is the block matrix $\{M_{i,j}(y)\}_{0\le i,j\le r}$ defined by

$$M_{i,j}(y) = \begin{bmatrix} y_{i+j,0} & y_{i+j-1,1} & \cdots & y_{i,j} \\ y_{i+j-1,1} & y_{i+j-2,2} & \cdots & y_{i-1,j+1} \\ \cdots & \cdots & \cdots & \cdots \\ y_{j,i} & y_{i+j-1,1} & \cdots & y_{0,i+j} \end{bmatrix}. \tag{2.4}$$

To fix ideas, when $n = 2$ and $r = 2$, one obtains

$$M_2(y) = \left[\begin{array}{c|cc|ccc} y_0 & y_{1,0} & y_{0,1} & y_{2,0} & y_{1,1} & y_{0,2} \\ \hline y_{1,0} & y_{2,0} & y_{1,1} & y_{3,0} & y_{2,1} & y_{1,2} \\ y_{0,1} & y_{1,1} & y_{0,2} & y_{2,1} & y_{1,2} & y_{0,3} \\ \hline y_{2,0} & y_{3,0} & y_{2,1} & y_{4,0} & y_{3,1} & y_{2,2} \\ y_{1,1} & y_{2,1} & y_{1,2} & y_{3,1} & y_{2,2} & y_{1,3} \\ y_{0,2} & y_{1,2} & y_{0,3} & y_{2,2} & y_{1,3} & y_{0,4} \end{array}\right].$$

If y is the sequence of moments of some measure μ_y, $M_r(y)$ is called a *moment matrix*, and one may define a bilinear form $\langle .,\rangle_y : \mathcal{A}_r \times \mathcal{A}_r \to \mathbb{R}$ by

$$\langle q(x), p(x)\rangle_y = \langle q, M_r(y)p\rangle = \int q(x)p(x)\mu_y(dx). \tag{2.5}$$

This bilinear form also defines a positive semidefinite form on $\mathcal{A}_r$ since

$$\langle q(x), q(x)\rangle_y = \int q(x)^2\mu_y(dx) \ge 0. \tag{2.6}$$

It is well known that $M_i(y) \succeq 0$ is a necessary condition for y to be a truncated moment sequence.

Localizing matrix. Given a polynomial $\theta(x) : \mathbb{R}^n \to \mathbb{R}$ with coefficient vector θ, introduce the matrix $M_r(\theta y)$ obtained from $M_r(y)$ as follows. Let $\alpha(i, j)$ be the index of $y_\alpha = M_r(i, j)$. Then,

$$M_r(\theta y)(i, j) = \sum_\alpha \theta_\alpha y_{\alpha(i,j)+\alpha}.$$

For instance, with $x \mapsto \theta(x) := a - x_1^2 - x_2^2$,

$$M_1(\theta y) = \begin{bmatrix} a - y_{20} - y_{02} & ay_{10} - y_{30} - y_{12} & ay_{01} - y_{21} - y_{03} \\ ay_{10} - y_{30} - y_{12} & ay_{20} - y_{40} - y_{22} & ay_{11} - y_{31} - y_{13} \\ ay_{01} - y_{21} - y_{03} & ay_{11} - y_{31} - y_{13} & ay_{02} - y_{22} - y_{04} \end{bmatrix}.$$

In Curto and Fialkow [5],$M_r(\theta y)$ is called a *localizing matrix*. Again, if y is the sequence of moments of a measure μ_y, then for every polynomial $q(x) \in \mathcal{A}_r$, with coefficient vector q,

$$\langle q, M_r(\theta y)q\rangle = \int \theta(x)q(x)^2\mu_y(dx), \tag{2.7}$$

so that if μ_y is concentrated on the set $\{\theta(x) \geq 0\}$, one must have $M_r(\theta y) \succeq 0$.

Before proceding further, we make the following remark.

REMARK 2.1. Introduce the following (linear) optimization problem

$$\mathbb{P} \mapsto \begin{cases} \sup\limits_{\mu \geq 0} \mu(S), \\ \displaystyle\int x^\alpha \, d\mu = \gamma_\alpha \qquad \forall \alpha \in \Gamma, \end{cases} \tag{2.8}$$

and its dual

$$\mathbb{D} \mapsto \begin{cases} \inf\limits_{\lambda} \displaystyle\sum_{\alpha \in \Gamma} \lambda_\alpha \gamma_\alpha, \\ \displaystyle\sum_{\alpha \in \Gamma} \lambda_\alpha x^\alpha \geq \mathbb{1}_S(x) \qquad \forall x \in \mathbb{R}^n \end{cases} \tag{2.9}$$

and we recall that if γ_α is an interior point of the moment space

$$\left\{\left(\mu(S), \int x^\alpha \, d\mu, \alpha \in \Gamma\right) \Big| \, \mu \in \mathcal{M}^+\right\},$$

then $\sup \mathbb{P} = \inf \mathbb{D}$ and in fact, in our case, since the primal problem is bounded, we also have $\sup \mathbb{P} = \min \mathbb{D}$ (see, e.g., [7] and the duality theorem in [15], page 812). Observe that this duality result does not depend on the structure of S. It happens as soon as $\mathbb{P}$ has an admissible solution, no matter what S is. In particular, S does not need to be convex.

Hence, an optimal solution of $\mathbb{D}$ is a nonnegative polynomial $p^*(x)$ of degree $\max_{\alpha \in \Gamma} \sum_{i=1}^n \alpha_i$, with $p^*(x) \geq 1$ on S. If the primal problem $\mathbb{P}$ is solvable, then by complementarity slackness, at an optimal solution μ^*, we must have $\mu^*(p^*(x) = \mathbb{1}_S(x)) = 1$.

Hence, by weak duality, solving $\mathbb{D}$ provides an upper bound on $\mu(S)$ and a tight bound if strong duality holds. However, as noticed in [3], solving $\mathbb{D}$ is equivalent (under certain conditions) to solving a separation problem (see Grötschel, Lovász and Schrijver [6]), which, except in a few (but interesting) special cases, is a NP-hard problem. Observe that in case of a duality gap between $\mathbb{P}$ and $\mathbb{D}$, and if a sequence of values of relaxations of $\mathbb{P}$ converges to $\sup \mathbb{P}$, the bounds obtained from the relaxations will become strictly better than any feasible solution of $\mathbb{D}$. We provide below such a sequence of LMI relaxations, and in general, a tight upper bound is obtained at a particular relaxation.

3. Upper bounds on $\mu(S)$. In this section we present a numerical procedure to approximate a tight upper bound on $\mu(S)$ where S is the semialgebraic set defined in (2.2).

Let $\{\gamma_\alpha\}$, $\alpha \in \Gamma$ be a given finite sequence of moments, that is, there is a measure μ on $\mathcal{B}$ such that

$$\int x^\alpha \, d\mu(x) = \gamma_\alpha \qquad \forall \alpha \in \Gamma.$$

For convenience, it is assumed that $\gamma_0 = 1$; that is, Φ is a subset of probability measures. We first state the following result.

PROPOSITION 3.1. *If γ is a (finite) vector of moments then for every p.m. $\mu \in \Phi$, there is a p.m. $\nu \in \Phi$ with all its moments finite, that is,*

$$\int x^\alpha \, d\nu(x) = \gamma_\alpha \qquad \forall \alpha \in \Gamma \quad \textit{and} \quad \left| \int x^\alpha \, d\nu(x) \right| < \infty \qquad \forall \alpha,$$

and with $\nu(S) = \mu(S)$.

PROOF. Let $\mu \in \Phi$ be fixed arbitrary. As $\{\mathbb{1}_S(x), \{x^\alpha\}_{\alpha\in\Gamma}\}$ is a finite family of real-valued measurable functions, each of them integrable with respect to μ, there exists a p.m. ν with finite support and such that

$$\int \mathbb{1}_S \, d\nu = \int \mathbb{1}_S \, d\mu \quad \text{and} \quad \int x^\alpha \, d\nu = \int x^\alpha \, d\mu = \gamma_\alpha \qquad \forall \alpha \in \Gamma;$$

that is, $\nu \in \Phi$ and $\nu(S) = \mu(S)$ (see, e.g., Theorem 2.1.1 in [1]). Moreover, as ν is finitely supported, all its moments are finite. $\square$

If odd (resp. even), let $2d_k - 1$ (resp. $2d_k$) be the degree of the polynomial $g_k(x)$ involved in the definition of the set S, and write

$$M_r(y) = \sum_{|\alpha|\le 2r} y_\alpha B_\alpha, \qquad M_{r-d_k}(g_k y) = \sum_{|\alpha|\le 2r} y_\alpha C_\alpha^k, \tag{3.1}$$

for appropriate symmetric matrices $\{B_\alpha, C_\alpha^k\}$. For $r \ge \max_k d_k$, consider the following psd program:

$$\mathbb{Q}_r \to \begin{cases} \sup y_0, \\ M_r(y) \succeq 0, \\ M_r(z) \succeq 0, \\ M_{r-d_k}(g_k y) \succeq 0, & k = 1, \dots, m, \\ y_\alpha + z_\alpha = \gamma_\alpha & \forall \alpha \in \Gamma, \end{cases} \tag{3.2}$$

where the moment-like matrices $M_r(y), M_r(z)$ and the localizing matrices $M_{r-d_k}(g_k y)$ have been defined in the previous section. The interpretation of $\mathbb{Q}_r$ is as follows. The constraints of $\mathbb{Q}_r$ state:

1. Necessary conditions for the variables $\{y_\alpha, z_\alpha\}$ to be moments of some measures μ and ν, respectively, with μ having its support contained in S;
2. That $\int x^\alpha (d(\mu+\nu)) = \gamma_\alpha$ for all $\alpha \in \Gamma$ (in particular, as $1 = \gamma_0$, $\mu+\nu$ is a probability measure).

The dual of $\mathbb{Q}_r$ is the psd program

$$\mathbb{Q}_r^* \to \begin{cases} \inf\limits_{X,Z,\{W_k\}\succeq 0,\lambda} \sum\limits_{\alpha\in\Gamma} \lambda_\alpha \gamma_\alpha, \\ \langle X, B_\alpha\rangle - \lambda_\alpha \mathbb{1}_\Gamma(\alpha) + \sum\limits_{k=1}^{m} \langle W_k, C_\alpha^k\rangle = \begin{cases} -1, & \text{if } \alpha = 0, \\ 0, & \text{otherwise,} \end{cases} \\ \langle Z, B_\alpha\rangle - \lambda_\alpha \mathbb{1}_\Gamma(\alpha) = 0. \end{cases} \tag{3.3}$$

Denote by $\sup \mathbb{Q}_r$ and $\inf \mathbb{Q}_r^*$ the optimal values of $\mathbb{Q}_r$ and $\mathbb{Q}_r^*$, respectively (and $\max \mathbb{Q}_r$ and $\min \mathbb{Q}_r$ if the optimal value is attained).

The interpretation of the dual psd program $\mathbb{Q}_r^*$ is as follows. Let (X, Z, W, λ) be a feasible solution of $\mathbb{Q}_r^*$, and write the first constraint as

$$\mathbb{1}_{\alpha=0} + \langle X, B_\alpha\rangle + \sum_{k=1}^{m} \langle W_k, C_\alpha^k\rangle = \lambda_\alpha \mathbb{1}_\Gamma(\alpha) \qquad \forall \alpha. \tag{3.4}$$

Using spectral decomposition, write $X = \sum_i q_i q_i'$ and $W_k = \sum_l v_{kl} v_{kl}'$, $k = 1, \dots, m$. Then, from (3.4), the polynomial $x \mapsto p(x) = \sum_{\alpha\in\Gamma} \lambda_\alpha x^\alpha$ satisfies

$$p(x) = 1 + \sum_i q_i(x)^2 + \sum_{k=1}^{m} g_k(x) \sum_l v_{kl}(x)^2,$$

with the polynomials $\{q_i(x), v_{kl}(x)\}$ having respective coefficient vectors $\{q_i, v_{kl}\}$. Therefore, $p(x) \geq 1$ on S. Similarly, with $Z = \sum_j p_j p_j'$ and from the second constraint,

$$\langle Z, B_\alpha\rangle - \lambda_\alpha \mathbb{1}_\Gamma(\alpha) = 0,$$

it also follows that $p(x)$ is a sum of squares $\sum_j p_j(x)^2$, and hence, nonnegative. In other words, from a feasible solution of $\mathbb{Q}_r^*$, we exhibit a polynomial

$$p(x) := \sum_{\alpha\in\Gamma} \lambda_\alpha x^\alpha \geq \mathbb{1}_S(x);$$

that is, $p(x)$ is a feasible solution of $\mathbb{D}$ in Remark 2.1. That $\mathbb{D}$ is a relaxation of $\mathbb{Q}_r^*$ follows from the fact that in $\mathbb{Q}_r^*$, $p(x)$ is required to be a *sum of squares*, whereas in $\mathbb{D}$, $p(x)$ is only required to be *nonnegative*. Therefore, we must have $\inf \mathbb{D} \leq \inf \mathbb{Q}_r^*$. In the multivariate case, and in contrast to the univariate case, not every nonnegative polynomial is a sum of squares.

PROPOSITION 3.2. (a) *Let S have a nonempty interior. If there is some $\mu \in \Phi$ with a strictly positive density with respect to the Lebesgue measure and with all* *moments finite, then each psd program $\mathbb{Q}_r^*$ is solvable and there is no duality gap, that is,* $\sup \mathbb{Q}_r = \min \mathbb{Q}_r^*$ *for all r.*

(b) *If $\mathbb{Q}_r^*$ has a strictly admissible solution $X, Z, W_k \succ 0$, then $\mathbb{Q}_r$ is solvable and there is no duality gap; that is,* $\max \mathbb{Q}_r = \inf \mathbb{Q}_r^*$.

PROOF. (a) Let $\mu \in \Phi$ be a p.m. equivalent to the Lebesgue measure Λ (noted $\mu \sim \Lambda$). Write

$$\mu(B) = \varphi(B) + \psi(B) \qquad \forall B \in \mathcal{B},$$

with

$$\varphi(B) := \mu(B \cap S); \quad \psi(B) := \mu(B \cap S^c) \qquad \forall B \in \mathcal{B}.$$

Denote by y and z the vector of moments up to order $2r$ of the measures φ and ψ, respectively. From the definition of φ and ψ, it follows that $M_r(y), M_r(z) \succeq 0$ and $M_{r-d_k} g_k(y) \succeq 0$, so that the pair (y, z) is a feasible solution of $\mathbb{Q}_r$. From $\mu \sim \Lambda$, we must have $M_r(y), M_r(z) \succ 0$ and $M_r(g_k y) \succ 0$ (as $\mathrm{int}(S) \neq \emptyset$). Therefore, $\mathbb{Q}_r$ has a strictly admissible solution. Morevover, $\mathbb{Q}_r^*$ has always a feasible solution. Take $X = 0$, $W_k = 0$ for all $k = 1, \dots, m$, $Z = ee'$ [with $e = (1, 0, \dots, 0)$), $\lambda_\alpha = 0$ for all $0 \neq \alpha \in \Gamma$, and $\lambda_0 = 1$. As both solutions have finite values, from a standard result in convex optimization, there is no duality gap and $\mathbb{Q}_r^*$ is solvable.

(b) Let $\mu \in \Phi$ be arbitrary. From Proposition 3.1 we may assume that μ is finitely supported, so let φ and ψ be as in the proof of (a) [except that now we only have $M_r(y), M_r(z) \succeq 0$ and $M_{r-d_k}(g_k y) \succeq 0$] so that (y, z) is an admissible solution of $\mathbb{Q}_r$. The rest is a simple consequence of the same standard result in convex optimisation invoked in (a). $\square$

We now provide a sequence of upper bounds. Let $d := \max_k d_k$.

THEOREM 3.3. (a) *As $r \to \infty$,*

$$\sup \mathbb{Q}_r \downarrow \rho^* \geq \sup_{\mu \in \Phi} \mu(S). \tag{3.5}$$

(b) *In addition, assume that $\mathbb{Q}_r$ is solvable, and let (y^*, z^*) be an optimal solution of $\mathbb{Q}_r$. If*

$$\mathrm{rank}\, M_r(y^*) = \mathrm{rank}\, M_{r-d}(y^*) \quad \textit{and} \quad \mathrm{rank}\, M_r(z^*) = \mathrm{rank}\, M_{r-1}(z^*)$$

then

$$\max \mathbb{Q}_r = \max_{\mu \in \Phi} \mu(S). \tag{3.6}$$

A detailed proof is postponed until Section 5. Observe that in Theorem 3.3(a) we do not assume that the vector γ of moment conditions is in the interior of the moment space. In this case, $\inf \mathbb{Q}_r^*$ might provide an upper bound strictly larger than $\sup \mathbb{Q}_r$. Hence, if there is a duality gap between $\mathbb{P}$ and $\mathbb{D}$, the upper bounds $\{\sup \mathbb{Q}_r\}$ might be strictly better than those obtained from $\inf \mathbb{D}$ (and of course from $\inf \mathbb{Q}_r^*$). To illustrate Theorem 3.3, consider the following examples.

EXAMPLE 1. Let $n = 2$ and let

$$\gamma := (1, 20, 20, 500, 390, 500) = (1, \gamma_{10}, \gamma_{01}, \gamma_{20}, \gamma_{11}, \gamma_{02})$$

be the vector of first- and second-order moment conditions and let S be the ball $x^2 + y^2 \leq 1$. Solving $\mathbb{Q}_1$ yields the optimal value 0.1079 and an optimal solution y, the vector of moments of the Dirac measure at the point $(\sqrt{2}/2, \sqrt{2}/2)$ on S with mass 0.1079. Thus, at the first relaxation, we have $\max \mathbb{Q}_1 = \max_{\mu\in\Phi} \mu(S) = 0.1079$. If we instead consider the ellipsoid $x^2/2 + y^2 \leq 1$ we now obtain $\max \mathbb{Q}_1 = \max_{\mu\in\Phi} \mu(S) = 0.10944$ with optimal solution y, the Dirac measure at the point $(1.1454, 0.5865)$ on S, with mass 0.10944. That in this case, the relaxation $\mathbb{Q}_1$ is enough to get a tight upper bound is consistent with Theorem 3.7 below.

EXAMPLE 2. Now, consider the more interesting case where γ contains higher order moments. For instance, let

$$\begin{aligned}\gamma &:= (1, 20, 20, 500, 390, 500, 251000, 251000)\\ &= (1, \gamma_{10}, \gamma_{01}, \gamma_{20}, \gamma_{11}, \gamma_{02}, \gamma_{40}, \gamma_{04}).\end{aligned}$$

The first relaxation to consider is $\mathbb{Q}_2$ since we have fourth-order moments. For the case of the sphere, solving $\mathbb{Q}_2$ yields an optimal value 0.003274 and an optimal solution y, the vector of moments of Dirac measure at the point $(\sqrt{2}/2, \sqrt{2}/2)$ on S. Thus, at the first relaxation, we already have $\max \mathbb{Q}_2 = \max_{\mu\in\Phi} \mu(S) = 0.003274$. Observe how the upper bound has decreased significantly with only two other fourth-order moment conditions.

With S the ellipsoid $x^2/2 + y^2 \leq 1$ and the same moment conditions, we obtain $\sup \mathbb{Q}_2 = 0.003284$ and an optimal solution y, the vector of moments of the Dirac measure at the point $(1.2766, 0.4303)$ on S. Again, at the first relaxation, we already have $\max \mathbb{Q}_2 = \max_{\mu\in\Phi} \mu(S)$. If one puts very large fourth-order moments, one nearly retrieves the upper bound 0.10944 obtained with only second-order moments, showing that the four-moment conditions have almost no influence in this case.

Different runs also confirmed that adding a third moment condition to the second-order conditions (and with no fourth-order condition) does not improve the bound. This is because a cubic polynomial $p(x)$ cannot be nonnegative. Thus, the coefficients λ_α of the cubic terms in an optimal solution $p(x)$ of $\mathbb{D}$, are all zero.

EXAMPLE 3. Finally, consider the case of a nonconvex and not connected set S defined by $x^2/2+y^2 \leq 1$ and $x^2+y^2/2 \geq 1$. The set S being the intersection of an ellipsoid S_1 with the complement of another ellipsoid S_2 that intersects S_1, consists of two disconnected nonconvex sets. Let

$$\gamma := (1,0,0,20,0,20,500,500) = (1,\gamma_{10},\gamma_{01},\gamma_{20},\gamma_{11},\gamma_{02},\gamma_{40},\gamma_{04}).$$

Solving $\mathbb{Q}_2$ yields the optimal value 0.2111 and an optimal solution y, the vector of moments of the measure

$$0.2111\left[\frac{1}{4}\sum_{i=1}^{4}\delta_{x_i}\right],$$

where the x_i are the intersection points of the two ellipsoids defining the set S. As this measure is supported in S, it follows that $\max \mathbb{Q}_2 = \max_{\mu\in\Phi}\mu(S)$. If $\gamma_{02} = 10$ instead of 20, one also obtains an optimal solution y, moments of a measure equally supported on S at points $\pm(\sqrt{2},0)$ with mass, the optimal value, $\max \mathbb{Q}_2 = 0.23585$. If $\gamma_{11} = 5$ instead of 0, we obtain the same solution. With S defined by $x^2/100 + y^2 \leq 1$ and $x^2/50 + y^2/2 \geq 1$, and the same moment conditions, we also obtain $\max \mathbb{Q}_2 = \max_{\mu\in\Phi}\mu(S)$.

COROLLARY 3.4. *Let $\mathbb{Q}_r^*$ be solvable, and let $(X, Z, \{W_k\}, \lambda)$ be an optimal solution with $W_k = 0$ for some indices $k \in I \subset [1,\dots,m]$. Then with $S' := \{g_k(x) \geq 0,\ k \notin I\} \supset S$, one has*

$$\sup_{\mu\in\Phi}\mu(S) \leq \sup_{\mu\in\Phi}\mu(S') \leq \inf \mathbb{Q}_r^*.$$

PROOF. Let $\mathbb{Q}_r^*(S)$ [resp. $\mathbb{Q}_r^*(S')$], be the dual problem $\mathbb{Q}_r^*$ associated with S (S', respectively). As every admissible solution of $\mathbb{Q}_r^*(S')$ is admissible for $\mathbb{Q}_r^*(S)$, it follows that $\inf \mathbb{Q}_r^*(S') \geq \inf \mathbb{Q}_r^*(S)$. But since the optimal solution of $\mathbb{Q}_r^*(S)$ in Corollarly 3.4 is also admissible for $\mathbb{Q}_r^*(S')$ (as $W_k = 0$ whenever $k \in I$), the result follows. $\square$

REMARK 3.5. If in $\mathbb{P}$, one wishes to restrict the measure μ to have its support in a semialgebraic set $\Theta \supset S$ [as, e.g., $(\mathbb{R}^n)^+$] defined by polynomial inequalities $\theta_j(x) \geq 0,\ j = 1,\dots,p$, it suffices to include in the psd program $\mathbb{Q}_r$, the additional LMI inequalities $M_r(\theta_j z) \succeq 0$, $j = 1,\dots,p$. Indeed, if one writes $\mu = \varphi + \psi$, the latter constraints are necessary for ψ to have its support contained in Θ (whereas $\mathbb{Q}_r$ already contains necessary conditions for φ to have its support contained in $S \subset \Theta$).

Before proceding further, we consider two special cases of importance.

3.1. *The real line case.* In this section we consider uper bounds on a p.m. μ on the Borel sets of the real line $\mathbb{R}$. In this case, we slightly modify the psd program $\mathbb{Q}_r$. Introduce the matrices

$$M_r(y) := \begin{bmatrix} y_0 & y_1 & y_2 & \cdots & y_r \\ y_1 & y_2 & \cdots & \cdots & y_{r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_r & y_{r+1} & \cdots & y_{2r-1} & y_{2r} \end{bmatrix},$$

$$B_r(y) := \begin{bmatrix} y_1 & y_2 & \cdots & y_{r+1} \\ y_2 & y_3 & \cdots & y_{r+2} \\ \cdots & \cdots & \cdots & \cdots \\ y_{r+1} & y_{r+1} & \cdots & y_{2r+1} \end{bmatrix},$$

$$C_r(y) := \begin{bmatrix} y_2 & y_3 & \cdots & y_{r+1} \\ y_3 & y_4 & \cdots & y_{r+2} \\ \cdots & \cdots & \cdots & \cdots \\ y_{r+1} & y_{r+2} & \cdots & y_{2r} \end{bmatrix}.$$

The Hankel matrix $M_r(y)$ is just the moment-like matrix introduced before when $n = 1$. With $[a, b]$ compact, the psd program $\mathbb{Q}_r$ now reads

$$\mathbb{Q}_r \rightarrow \begin{cases} \sup y_0, \\ M_r(y) \succeq 0, \\ M_r(z) \succeq 0, \\ (a+b)B_{r-1}(y) \succeq abM_{r-1}(y) + C_r(y), \\ y_\alpha + z_\alpha = \gamma_\alpha \qquad \forall \alpha \in \Gamma. \end{cases} \tag{3.7}$$

THEOREM 3.6. *Let $n = 1$ (univariate case) and let S be the interval $[a, b] \subset \mathbb{R}$. Let the vector of moments γ be such that $|\alpha| \leq 2r$ for all $\alpha \in \Gamma$. Then,*

$$\sup \mathbb{Q}_r = \sup_{\mu \in \Phi} \mu(S); \tag{3.8}$$

that is, $\sup \mathbb{Q}_r$ is a tight upper bound.

Let $[a, b] \subset \mathbb{R}^+$ and include the additional constraint $B_{r-1}(z) \succeq 0$ in $\mathbb{Q}_r$. Then:

$$\sup \mathbb{Q}_r = \sup_{\mu \in \Phi, \mu(\mathbb{R}^+)=1} \mu(S). \tag{3.9}$$

The proof is postponed to Section 5.2. Observe that there is *no* assumption on γ (except it must be a vector of moments). The result follows from the fact that in the univariate case, $M_r(y)$, $M_r(z)$ can be shown to be moment matrices of some measures μ and ν, respectively, with the support of μ contained in $[a, b]$. The additional constraint $B_{r-1}(z) \succeq 0$ will ensure that we restrict ourselves to measures μ with support on the positive half-line.

Theorem 3.6 provides a simple and easy way to compute bounds on an interval $[a, b]$. It is specially interesting when one has at least four moment conditions.

Indeed, otherwise, with only two moment conditions $(1, \gamma_1, \gamma_2)$, the tight upper bound (here $\sup \mathbb{Q}_1$) does not depend on a if $\gamma_1 > b$ (resp. b if $\gamma_1 < a$). Hence the tight upper bound would be the same for $S' := (-\infty, b]$ or $S' := [a, \infty)$ if $\gamma_1 \notin [a, b]$ (see Corollary 3.4). This is because an optimal solution of $\mathbb{D}$ is a quadratic polynomial $p(x) \geq \mathbb{1}_S(x)$ and an optimal solution μ of $\mathbb{P}$ satisfies $\mu(p(x) = \mathbb{1}_{[a,b]}(x)) = 1$. As $p(x)$ is nonnegative quadratic and must vanish at some point $\xi \notin [a, b]$, it necessarily attains its minimum on $[a, b]$ at either a or b. This can also be deduced from a direct inspection of the formula in [3] which, for example, in the case $\gamma_1 > b$ gives $\sup_{\mu \in \Phi} \mu([a, b]) = (1 + (\gamma_1 - b)/\gamma_2^2)^{-1}$ (see Theorem 15.3.3 in [3]).

If $p(x)$ has degree at least four, then $p(x)$ may attain its global minimum at two points outside $[a, b]$, and a global minimum 1 on S, at both points a and b (cf. Example 4) or a global minimum at some point outside $[a, b]$ and a global minimum on $[a, b]$ at two points of $[a, b]$ (cf. Example 5).

EXAMPLE 4. Let γ be the vector of moment conditions $(1, 2, 10, 15, 150)$, and $[a, b] = [1, 3]$. With only two conditions (γ_1, γ_2), $\sup_{\mu \in \Phi} \mu([a, b]) = 1$ for $\gamma_1 \in [a, b]$. With the four conditions, $\max_{\mu \in \Phi}(\mu([a, b]) = \max \mathbb{Q}_2 = 0.8815$ with

$$
\begin{aligned}
y &= (0.8815, 2.4502, 7.1564, 21.2749, 63.6303) \\
&= 0.8815 \cdot (1, 2.7796, 8.1183, 24.1344, 72.1828)
\end{aligned}
$$

and $(1, 2.7796, 8.1183, 24.1344, 72.1828)$ is the vector of the first four moments of the measure

$$\varphi := \alpha \delta_{\{1\}} + (1 - \alpha)\delta_{\{3\}} \qquad \text{with } \alpha = 0.1102,$$

finitely supported in $[1, 3]$ at the points $\{1\}$ and $\{3\}$. Thus the four-degree polynomial $p(x)$, optimal solution of $\mathbb{D}$, vanishes at two points outside $[1, 3]$ and takes its minimum value 1 on $[a, b]$ at the points 1 and 3.

EXAMPLE 5. Let γ be the vector of moment conditions $(1, 3.5, 15, 70, 550)$, and $[a, b] = [-100, 3]$. Solving $\mathbb{Q}_2$ yields $\max \mathbb{Q}_2 = 0.888$ with optimal value

$$y = 0.888 \cdot (1, 2.9683, 9.3841, 20.9023, 172.4339),$$

and one may show that $(1, 2.9683, 9.3841, 20.9023, 172.4339)$ is the vector of the first four moments of the measure

$$\varphi := \alpha \delta_{\{-15.1349\}} + (1 - \alpha)\delta_{\{3\}} \qquad \text{with } \alpha = 0.0017.$$

An optimal solution of $\mathbb{D}$ is a four-degree polynomial that vanishes at $x = 7.7156$ and is minimized at $x = -15.1349$ and $x = 3$ on $[-100, 3]$.

In both cases, the optimal value is sensitive to both parameters a, b because there are more than three moment conditions.

3.2. *The case of second-order moment conditions.* We now consider the case where the vector of moment conditions γ contains only first- and second-order moments, but not necessarily all of them. The polynomials defining the set S are assumed to be linear and/or quadratic polynomials; that is, S is the intersection of half-spaces and/or ellipsoids.

In this case, for $r = 1$, the dual $\mathbb{Q}_1^*$ in (3.3) has the special form,

$$\mathbb{Q}_1^* \to \begin{cases} \inf\limits_{X,Z,w_k \geq 0,\lambda} \sum\limits_{\alpha\in\Gamma} \lambda_\alpha \gamma_\alpha, \\ \langle X, B_\alpha\rangle - \lambda_\alpha \mathbb{1}_\Gamma(\alpha) + \sum\limits_{k=1}^{m} w_k (g_k)_\alpha = \begin{cases} -1, & \text{if } \alpha = 0, \\ 0, & \text{otherwise,} \end{cases} \\ \langle Z, B_\alpha\rangle - \lambda_\alpha \mathbb{1}_\Gamma(\alpha) = 0, \end{cases} \tag{3.10}$$

where $\{(g_k)_\alpha\}$ is the coefficient vector of the linear or quadratic polynomial $g_k(x)$, $k = 1, \ldots, m$.

THEOREM 3.7. *Let S be compact, convex, with nonempty interior, and let the polynomials $g_k(x)$ defining S be linear and/or quadratic. Let γ contain only moments up to order 2. If γ is in the interior of the moment space then*

$$\sup \mathbb{Q}_1 = \min \mathbb{Q}_1^* = \sup_{\mu\in\Phi} \mu(S), \tag{3.11}$$

that is, $\min \mathbb{Q}_1^*$ is a tight upper bound.

If in addition, γ contains all the second-order marginal moments, then $\min \mathbb{Q}_1^* = \max \mathbb{Q}_1$.

The first statement is already proved for more general convex sets S in [3]. However, for more general convex sets S, one has to invoke a separation algorithm of Grötschel, Lovsz and Schrijver as indicated in [3]. The second statement provides a simple condition to ensure the solvability of $\mathbb{Q}_1$.

Observe that in the case of Theorem 3.7, an optimal solution of $\mathbb{D}$ is a quadratic nonnegative (hence convex) polynomial $p(x)$. Therefore, it must have a global minimum $x_0 \notin S$ and attains its minimum on S at some point ξ of S. Only those constraints $g_k(x)$ binding at ξ [i.e., such that $g_k(\xi) = 0$] are relevant. Indeed, let $S' \supset S$ be the larger set defined by only those $g_k(x)$ binding at ξ. By inspection of the proof in Section 5, $p(x)$ is still an admissible solution of $\mathbb{D}$ (for the problem with S' instead of S). As $\inf \mathbb{Q}_1^*(S') \geq \inf \mathbb{Q}_1^*(S)$ (there are less variables w_k), the equality follows (see also Corollary 3.4).

For the same reasons, the smaller set $S' \subset S$ defined by $g_k(x) \geq g_k(\xi)$ will also have the same tight upper bound.

We now return to the general case. Theorem 3.3 provides a sequence of upper bounds. In some cases, it provides a tight upper bound at some relaxation but this is not guaranteed. To get that in the limit, this sequence of upper bounds provides a tight upper bound, we make the following assumption on the set S.

ASSUMPTION 3.8. *The set S is compact with nonempty interior and there exists a real-valued polynomial $u(x):\mathbb{R}^n \to \mathbb{R}$ such that $\{u(x) \geq 0\}$ is compact, and*

$$u(x) = u_0(x) + \sum_{k=1}^{r} g_i(x)u_i(x) \qquad \forall\, x \in \mathbb{R}^n, \tag{3.12}$$

where all the polynomials $u_i(x)$, $i = 0, \dots, r$, are sums of squares.

This assumption is satisfied in a number of interesting cases. For example, when $\{g_i(x) \geq 0\}$ is compact for some index i, or when all the polynomials $g_i(x)$ are linear and S is compact. The latter case is of great importance as one is often interested in rectangles; that is, when S is the rectangle $a_i \leq x_i \leq b_i$, $i = 1, \dots, n$. Another case of importance is when S is an ellipsoid (or an intersection of ellipsoids).

The reason why we introduce Assumption 3.8 is because if a polynomial $p(x):\mathbb{R}^n \to \mathbb{R}$ is strictly positive on S which satisfies Assumption 3.8, then it can be written as a weighted sum of squares; that is,

$$p(x) = q(x) + \sum_{k=1}^{m} g_k(x)v_k(x),$$

with $\{q(x), v_k(x)\}$ polynomials that are sums of squares (see, e.g., Putinar [13], Jacobi [8], Jacobi and Prestel [9]). This property will be used to prove that with $\varepsilon > 0$ fixed, arbitrary, from every polynomial $p(x)$, optimal solution of $\mathbb{D}$, one may construct an admissible solution of $\mathbb{Q}_r^*$ (provided r is sufficiently large), with value bounded from above by $\min \mathbb{D} + \varepsilon$.

Let $\theta(x):\mathbb{R}^n \to \mathbb{R}$ be the quadratic polynomial $A(r) - \|x\|^2$ with $A(r) > 0$ and $\lim_{r\to\infty} A(r) = +\infty$, say, for instance, $A(r) = r$, and write

$$M_r(\theta z) = \sum_{\alpha \leq 2r} z_\alpha D_\alpha$$

for appropriate symmetric matrices D_α.

Consider the psd programs (with $r \geq \max_k d_k$),

$$\mathbb{Q}_r \to \begin{cases} \sup y_0, \\ M_r(y) \succeq 0, \\ M_r(z) \succeq 0, \\ M_{r-1}(\theta z) \succeq 0, \\ M_{r-d_k}(g_k y) \succeq 0, \qquad k = 1, \dots, m, \\ y_\alpha + z_\alpha = \gamma_\alpha, \qquad \alpha \in \Gamma. \end{cases} \tag{3.13}$$

The dual psd program of $\mathbb{Q}_r$ is the psd program

$$(3.14)\quad \mathbb{Q}_r^* \to \begin{cases} \inf\limits_{X,Z,W_k,\Delta\succeq 0} \sum\limits_{\alpha\in\Gamma} \lambda_\alpha \gamma_\alpha, \\ \langle X, B_\alpha\rangle - \lambda_\alpha \mathbb{1}_\Gamma(\alpha) + \sum\limits_{k=1}^{m} \langle W_k, C_\alpha^k\rangle = \begin{cases} -1, & \text{if } \alpha = 0, \\ 0, & \text{otherwise,} \end{cases} \\ \langle Z, B_\alpha\rangle + \langle \Delta, D_\alpha\rangle - \lambda_\alpha \mathbb{1}_\Gamma(\alpha) = 0, \qquad |\alpha| \le 2r. \end{cases}$$

The interpretation of the psd programs $\mathbb{Q}_r$ is similar to the one previously introduced, except that now, the measure ν with associated moment vector z is required to have its support contained in the ball $B_r := \{\|x\|^2 \le A(r)\}$. The reason why we introduce this additional constraint is because, as B_r satisfies Assumption 3.8, every polynomial $p(x) : \mathbb{R}^n \to \mathbb{R}$ strictly positive on B_r, can be written

$$p(x) = q(x) + \theta(x) g(x),$$

with both $q(x)$ and $g(x)$ sums of squares. We will use this property for the polynomial $p(x)$ in Remark 2.1.

THEOREM 3.9. *Let S be a compact semialgebraic set with nonempty interior, that sastisfies Assumption* 3.8 *and assume that the vector γ is in the interior of the moment space. Then as $r \to \infty$,*

$$(3.15)\qquad \sup \mathbb{Q}_r \to \rho^* = \sup_{\mu\in\Phi} \mu(S).$$

In addition, if $\sup \mathbb{P} = \max \mathbb{P}$, *then* $\sup \mathbb{Q}_r \ge \rho^*$ *for all r sufficiently large.*

Theorem 3.9 permits to get an upper bound, arbitrary close to the tight upper bound ρ^*. Note that S is *not* required to be convex.

REMARK 3.10. Under the assumptions stated in Theorem 3.9, $\sup \mathbb{P} = \min \mathbb{D}$. Therefore, every optimal solution of $\mathbb{D}$ is a polynomial $p(x)$ that satisfies $p(x) \ge \mathbb{1}_S(x)$. As B_r and S are both compact sets that satisfy Assumption 3.8, every polynomial $q(x)$ strictly positive, and such that $q(x) - 1 > 0$ on S can be written

$$q(x) = \begin{cases} 1 + v(x) + \sum\limits_{k=1}^{m} g_k(x) h_k(x), \\ \text{and } w(x) \end{cases}$$

with $v(x), h(x), w(x)$ all sums of squares. As $p(x) \ge 0$ and $p(x) - 1 \ge 0$ on S, it is often possible to write $p(x)$ as above (despite the nonstrict positivity). In that case, one may construct from $p(x)$ an admissible solution to $\mathbb{Q}_r^*$ with value $\min \mathbb{D}$, so that $\min \mathbb{D} = \min \mathbb{Q}_r^* = \sup \mathbb{Q}_r$; that is, a tight upper bound is attained at a particular relaxation. Otherwise, we only have the asymptotic result (3.15).

One may also relax the interior point condition on γ by the weaker condition "$\mathbb{P}$ is solvable."

4. Lower bound on $\mu(S)$. In this section, we provide psd programs that yield a sequence of lower bounds that converges to a tight lower bound.

We cannot simply replace "max" by "min" in the psd program $\mathbb{Q}_r$ previously defined. Indeed, if we decompose μ into the sum $\varphi + \psi$, with the support of φ contained in S, we now must impose ψ to have its support in S^c, otherwise we would obtain $\min \mathbb{Q}_r = 0$ with $\varphi \equiv 0$ (as soon as $0 \in S$).

But, since the lower bound case amounts to maximize $\mu(S^c)$, and as S^c can be expressed as the union $\bigcup_i \Omega_i$ of semialgebraic sets Ω_i (not necessarily disjoint), one still may use similar arguments as in the maximizing case. The measure μ is now written as the sum $\varphi + \sum_i \psi_i$, with the support of each measure ψ_i contained in Ω_i. The resulting psd program $\mathbb{Q}_r$ is more complicated since it involves a moment-vector y^i, a moment-like matrix $M_r(y^i)$ and a localizing matrices $M_{r-d_k^i}(g_k^i y^i)$ for each ψ_i (when Ω_i is defined by $\{g_k^i(x) \geq 0,\ k = 1, \dots, p_i\}$). Then one has the analogue of Theorem 3.3 for maximizing $\mu(S^c)$; that is,

$$\sup \mathbb{Q}_r \downarrow \rho^* \geq \max_{\mu \in \Phi} \mu(S^c) = 1 - \min_{\mu \in \Phi} \mu(S).$$

EXAMPLE 6. Consider the case where S^c is the set defined by $x^2/100 + y^2 \geq 1$ and the moment vector γ is given by

$$\gamma := (1, 0, 0, 0.1, 0, 0.1, 0.5, 0.5) = (1, \gamma_{10}, \gamma_{01}, \gamma_{20}, \gamma_{11}, \gamma_{02}, \gamma_{40}, \gamma_{04}).$$

As S^c is defined by a single polynomial $g_k(x) \geq 0$, $\mathbb{Q}_r$ has exactly the same form as in the maximizing case.

Solving $\mathbb{Q}_2$ yields an optimal value of 0.1010, whereas solving $\mathbb{Q}_3$ yields also an optimal value 0.1010 but with no minimizer (the y_i's are unbounded). To get a bounded approximate solution, we penalize with ε very small, say $\varepsilon := 10^{-6}$, the higher marginal moments of order 6. With this modified criterion, $\mathbb{Q}_3$ yields the same value 0.1010 with an optimal solution y such that $\operatorname{rank} M_3(y) = \operatorname{rank} M_2(y) = 4$. One observes that y is indeed the moment vector of a measure ψ with support contained in S^c. In fact, y is the vector of moments of the measure ψ with mass 0.101 and supported on the points $\pm(\sqrt{0.9901}, 0)$ and $\pm(0, \sqrt{0.9901})$ on S^c. Thus, $\max \mathbb{Q}_3 = \rho^* = \max_{\mu \in \Phi} \mu(S^c)$. Observe also that the bound $\sup \mathbb{Q}_2$ was already very good.

However, in order to ensure $\rho^* = \max_{\mu \in \Phi} \mu(S^c)$, we cannot invoke Theorem 3.9 for the set S^c does not satisfy Assumption 3.8 (as S^c is not compact). Therefore, if one wishes to get a sequence of lower bounds that converges to a tight lower bound, as in the maximizing case, one needs to address the minimizing case $\min_{\mu \in \Phi} \mu(S)$ directly, instead of considering $\max_{\mu \in \Phi} \mu(S^c)$.

The resulting psd programs in the lower bound case slightly differ from the preceding $\mathbb{Q}_r$ in Theorem 3.3. Indeed, in the minimizing case, if we write μ as the sum $\varphi + \psi$ of two measures φ, ψ, with φ having its support contained in S, we now must impose that the support of ψ is contained in S^c (which is not necessary in the maximizing case). Therefore, we need to introduce additional constraints. For simplicity of exposition, we restrict it to the case where S is an ellipsoid.

Therefore, let S be defined by $\theta(x) \geq 0$ where $\theta(x)$ is a strictly concave quadratic polynomial, and with $\varepsilon > 0$ fixed, let $\theta_r(x)$ be the polynomial $x \mapsto \theta(x) + \varepsilon(r)$ and $\theta^r(x)$ be the polynomial $x \mapsto \varepsilon(r)^{-1} + \theta(x)$, where $\varepsilon(r) > 0$ and $\varepsilon(r) \downarrow 0$ as $r \to \infty$.

For $r \geq 1$, introduce the psd programs

$$\mathbb{Q}_r \to \begin{cases} \inf y_0, \\ M_r(y), M_{r-1}(\theta y) \succeq 0, \\ M_r(z), M_{r-1}(-\theta_r z), M_{r-1}(\theta^r z) \succeq 0, \\ y_\alpha + z_\alpha = \gamma_\alpha \qquad \forall \alpha \in \Gamma. \end{cases} \tag{4.1}$$

We write

$$M_r(\theta y) = \sum_\alpha y_\alpha C_\alpha, \quad M_r(-\theta_r z) = \sum_\alpha z_\alpha E_\alpha \quad \text{and} \quad M_r(\theta^r z) = \sum_\alpha z_\alpha F_\alpha.$$

The dual psd program of $\mathbb{Q}_r$ reads

$$\mathbb{Q}_r^* \to \begin{cases} \sup\limits_{X,V,W_1,W_2 \succeq 0} \sum\limits_{\alpha \in \Gamma} \lambda_\alpha \gamma_\alpha, \\ \langle X, B_\alpha \rangle + \langle V, C_\alpha \rangle + \mathbb{1}_\Gamma(\alpha)\lambda_\alpha = \begin{cases} 1, & \text{if } \alpha = 0, \\ 0, & \text{otherwise}, \end{cases} \\ \langle Z, B_\alpha \rangle + \langle W_1, E_\alpha \rangle + \langle W_2, F_\alpha \rangle + \mathbb{1}_\Gamma(\alpha)\lambda_\alpha = 0 \qquad |\alpha| \leq 2r. \end{cases} \tag{4.2}$$

The LMI constraints of the psd program $\mathbb{Q}_r$ state:

1. Necessary conditions for the variables $\{y_\alpha, z_\alpha\}$ to be moments of some measures φ and ψ, respectively.
2. Necessary conditions for the measures φ, ψ to have their support in $\{\theta(x) \geq 0\}$ and $\{-\varepsilon(r)^{-1} \leq \theta(x) \leq -\varepsilon(r)\}$, respectively.
3. That $\int x^\alpha d(\varphi + \psi) = \gamma_\alpha$ for all $\alpha \in \Gamma$.

THEOREM 4.1. *Let S be the ellipsoid $\{\theta(x) \geq 0\}$ with nonempty interior. Assume that γ is in the interior of the moment space. Then, as $r \to \infty$,*

$$\inf \mathbb{Q}_r \to \rho^* = \inf_{\mu \in \Phi} \mu(S). \tag{4.3}$$

In addition, if $\inf \mathbb{P} = \min \mathbb{P}$, *then* $\inf \mathbb{Q}_r \leq \rho^*$ *for all r sufficiently large.*

5. Proofs.

PROOF OF THEOREM 3.3. The first statement follows from the fact that each $\mathbb{Q}_r$ has an admissible solution and from an admissible solution (y, z) of $\mathbb{Q}_{r+1}$, one may construct by truncation of the vectors y and z, and admissible solution of $\mathbb{Q}_r$ with same value. As the optimal values $\sup \mathbb{Q}_r$ are bounded from above,

$\sup \mathbb{Q}_r \downarrow \rho^*$. That $\rho^* \geq \sup_{\mu \in \Phi} \mu(S)$ follows from Proposition 3.1. Indeed, from Proposition 3.1, there is a measure $\nu \in \Phi$ with all its moments finite, and with $\nu(S) = \mu(S)$. Let ν_1 (resp. ν_2) be the restriction of ν to S (resp. S^c). Let y (resp. z) be the vector of moments of ν_1 (resp. ν_2) up to order $2r$ (guaranteed to exist). It is immediate that (y, z) is admissible for $\mathbb{Q}_r$, with value $y_0 = \nu_1(X) = \nu(S) = \mu(S)$. Hence $\sup \mathbb{Q}_r \geq \sup_{\mu \in \Phi} \mu(S)$.

Next, assume that $\mathbb{Q}_r$ is solvable. Let (y^*, z^*) be an optimal solution of $\mathbb{Q}_r$. If $\operatorname{rank} M_r(y^*) = \operatorname{rank} M_{r-d}(y^*)$ (recall $d := \max_k d_k$), it follows that $M_r(y^*)$ is a so-called *flat positive extension* of $M_{r-d}(y^*)$ from which it follows that both $M_r(y^*)$ and $M_{r-d}(y^*)$ are moment matrices. Moreover, as $M_{r-d_k}(g_k y^*) \succeq 0$ for all $k = 1, \dots, m$, y^* is the moment vector of an atomic measure φ supported in S. (Theorem 1.1 in [5], is stated in dimension 2 for the complex plane, but is valid for n real or complex variables; see comments on page 2 in [5].) With the same argument, $M_r(z^*)$ is a moment matrix. Therefore, there exist two measures φ, ψ, with φ supported in S, and such that

$$\int x^\alpha d\varphi(x) = y^*_\alpha; \quad \int x^\alpha d\psi(x) = z^*_\alpha \qquad \forall \alpha \text{ with } \sum_{i=1}^n \alpha_i \leq 2r.$$

In addition, $y^*_\alpha + z^*_\alpha = \gamma_\alpha$ for all $\alpha \in \Gamma$. Therefore, $\mu := \varphi + \psi \in \Phi$, and from

$$\mu(S) \geq \varphi(S) = y^*_0 = \max \mathbb{Q}_r \geq \sup_{\nu \in \Phi} \nu(S),$$

it follows that $y^*_0 = \max_{\mu \in \Phi} \mu(S)$. $\square$

PROOF OF THEOREM 3.6. Let $\sup \mathbb{Q}_r = \rho^*$. As in the proof of Theorem 3.3, $\mathbb{Q}_r$ is a relaxation of $\mathbb{P}$ for all r, and we have $\rho^* \geq \sup_{\mu \in \Phi} \mu(S)$. Now, fix $\varepsilon > 0$ arbitrary and let (y, z) be an admissible solution of $\mathbb{Q}_r$ with $y_0 \geq \rho^* - \varepsilon$. As $n = 1$, $M_r(y)$ and $M_r(z)$ are moment matrices. Let φ, ψ be measures associated with y and z, respectively, and let $\mu := \varphi + \psi$. From the condition

$$(a + b) B_{r-1}(y) \succeq ab M_{r-1}(y) \succeq C_{r-1}(y),$$

it follows that φ is supported on $[a, b]$. (It follows from a result of Krein and Nudel'man [11]; see, e.g., Remark 4.4 in Curto and Fialkow [4].) Therefore, $y_0 = \varphi(S) \geq \rho^* - \varepsilon$. Moreover, the measure $\mu = \varphi + \psi$ satisfies $y_\alpha + z_\alpha = \gamma_\alpha$ for all $\alpha \in \Gamma$; that is, $\int x^\alpha \, d\mu = \gamma_\alpha$ for all $\alpha \in \Gamma$. In addition, as $\mu(S) \geq \varphi(S) = y_0$, we obtain $\mu(S) \geq \rho^* - \varepsilon$, and the result follows since ε was arbitrary. $\square$

PROOF OF THEOREM 3.7. Let $\rho^* := \sup_{\mu \in \Phi} \mu(S)$. We have already seen that $\sup \mathbb{Q}_1 \geq \rho^*$ (as the psd programs $\mathbb{Q}_r$ are relaxations of $\mathbb{P}$ for all r). Moreover, as the vector γ is in the interior of the moment space, there is no duality gap between $\mathbb{P}$ and $\mathbb{D}$ in Remark 2.1.

Let $p(x) \geq \mathbb{1}_S(x)$ be a polynomial, optimal solution of $\mathbb{D}$ in Remark 2.1 and write $p(x) = \sum_{\alpha \in \Gamma} \lambda_\alpha x^\alpha$. Hence, as $\sum_{\alpha \in \Gamma} \lambda_\alpha \gamma_\alpha = \rho^* = \inf \mathbb{P}$, it suffices to show that from $p(x)$, one may exhibit an optimal solution (X, Z, w_k, λ) of $\mathbb{Q}_1^*$ with value $\sum_{\alpha\in\Gamma} \lambda_\alpha \gamma_\alpha = \rho^*$. We will use the fact that, as $p(x)$ is of degree 2 and is nonnegative, it is a sum of squares. Therefore, $p(x) = \sum_j q_j(x)^2$ for some linear polynomials $\{q_j(x)\}$. Let Z be the symmetric matrix $\sum_j q_j q_j'$, with $\{q_j\}$ the vectors of coefficients of the polynomials $\{q_j(x)\}$ in the basis (2.3). Hence,

$$\langle Z, B_\alpha\rangle = \lambda_\alpha \mathbb{1}_\Gamma(\alpha) \qquad \forall \alpha.$$

Moreover, let x^* be a global minimizer of $p(x)$ on S (guaranteed to exist as S is compact), with associated Kuhn–Tucker multipliers $w_k \geq 0$ (guaranteed to exist as S is convex and Slater's condition holds). Let $L(x, w) := p(x) - \sum_{k=1}^m w_k g_k(x)$ be the associated Lagrangian. As $L(x, w)$ is a quadratic convex polynomial and x^* is minimizer, we have

$$L(x, w) - L(x^*, w) = \langle x - x^*, H(x - x^*)\rangle,$$

with $H := \nabla_{xx}^2 L(x, \lambda) \succeq 0$. Equivalently, as $L(x^*, w) = p(x^*) = 1$,

$$p(x) - 1 = \langle x - x^*, H(x - x^*)\rangle + \sum_{k=1}^m w_k g_k(x) \qquad \forall x \in \mathbb{R}^n. \tag{5.1}$$

As $\langle x - x^*, H(x - x^*)\rangle$ is a nonnegative quadratic polynomial, it is a sum of squares of linear polynomials $\{h_j(x)\}$. Let $X := \sum h_j h_j'$ with $\{h_j\}$ being the vectors of coefficients of the linear polynomials $h_j(x)$ in the basis (2.3). From (5.1) it follows that

$$\langle X, B_\alpha\rangle + \sum_{k=1}^m \langle w_k (g_k)_\alpha\rangle - \lambda_\alpha \mathbb{1}_\Gamma(\alpha) = \begin{cases} -1, & \text{if } \alpha = 0 \\ 0, & \text{otherwise,} \end{cases}$$

and $\langle Z, B_\alpha\rangle - \lambda_\alpha \mathbb{1}_\Gamma(\alpha) = 0$, that is, (X, Z, w_k, λ) is an admissible solution of $\mathbb{Q}_1^*$, with value $\sum_{\alpha\in\Gamma} \lambda_\alpha \gamma_\alpha = \rho^*$.

We next prove that when γ contains all the second-order marginal moments, $\mathbb{Q}_1$ is solvable. Let $w_k > 0$ be fixed arbitrary. Choose a sufficiently large scalar $M > 0$ to ensure that

$$M + M \sum_j x_j^2 - \sum_{k=1}^m w_k g_k(x) - 1 > 0$$

[remember that the $g_k(x)$ are concave quadratic]. Therefore,

$$h(x) := M + M \sum_{j=1}^n x_j^2 - \sum_{k=1}^m w_k g_k(x) - 1 = \sum_j h_j(x)^2,$$

for some linear polynomials $\{h_j(x)\}$, with $X := \sum_j h_j h_j' \succ 0$. Let $p(x) := M + M \sum_{j=1}^m x_j^2$ so that

$$p(x) = M\left(1 + \sum_j p_j(x)^2\right),$$

with $p_j(x) := x_j^2$ for all j, and $Z := \sum_j p_j p_j' \succ 0$. We thus have

$$p(x) - 1 = \sum_j h_j(x)^2 + \sum_{k=1}^m w_k g_k(x).$$

As γ contains all the second-order marginal moments, $p(x) = \sum_{\alpha\in\Gamma} p_\alpha x^\alpha$ for some vector $\{p_\alpha\}$. Moreover, it easily follows that with the above definitions of X, Z, w_k, p and $X, Z \succ 0$, $w_k > 0$, (X, Z, w_k, p) is a strictly admissible solution of $\mathbb{Q}_1^*$, and as $\mathbb{Q}_1$ has a feasible solution, it follows that $\sup \mathbb{Q}_1 = \max \mathbb{Q}_1$, by a standard duality result in convex optimization. $\square$

PROOF OF THEOREM 3.9. Let $\varepsilon > 0$ be fixed, and let $\mu \in \Phi$ be such that $\mu(S) \geq \rho^* - \varepsilon$, with $\rho^* = \sup \mathbb{P}$. From Proposition 3.1, there is a p.m. $\nu \in \Phi$, finitely supported and such that $\nu(S) = \mu(S) \geq \rho^* - \varepsilon$. Let φ, ψ be the restriction of ν to S and S^c, respectively. As ν is finitely supported, and $A(r) \to \infty$, there is some r_0 such that the support of ψ is contained in $B_r = \{\|x\|^2 \leq A(r)\}$ for all $r \geq r_0$. Therefore, with (y, z) the respective vectors of moments of φ, ψ, up to order $2r$, we have

$$M_r(y) \succeq 0,\ M_r(z) \succeq 0,\ M_{r-1}(\theta z) \succeq 0,\ M_{r-d_k}(g_k y) \succeq 0, \qquad k = 1, \ldots, m,$$

for all $r \geq r_0$. Hence, (y, z) is an admissible solution of $\mathbb{Q}_r$ for all $r \geq r_0$, and thus $\sup \mathbb{Q}_r \geq \rho^* - \varepsilon$.

Next, from Remark 2.1, $\sup \mathbb{P} = \min \mathbb{D}$. Let $p(x) \geq \mathbb{1}_S(x)$ be an optimal solution of $\mathbb{D}$, and write

$$p(x) = \sum_{\alpha\in\Gamma} p_\alpha x^\alpha.$$

As S is a compact set that satisfies Assumption 3.8, and $p(x) - 1 + \varepsilon > 0$ on S, the polynomial

$$p'(x) := p(x) + \varepsilon = \varepsilon + \sum_{\alpha\in\Gamma} p_\alpha x^\alpha = \sum_{\alpha\in\Gamma} p'_\alpha x^\alpha$$

can be written

$$p'(x) - 1 = \sum_j q_j(x)^2 + \sum_{k=1}^m g_k(x) \sum_j v_{kl}(x)^2, \tag{5.2}$$

for some polynomials $\{q_j(x), v_{kl}(x)\}$. Let r_1 be such that

$$\sup_j \deg q_j(x)^2 \leq 2r_1 \quad \text{and} \quad \sup_{k,l} 2d_k + \deg v_{kl}(x)^2 \leq 2r_1.$$

Similarly, as $p(x)$ is nonnegative, $p'(x) > 0$ on B_r, and thus, $p'(x)$ can be written

$$p'(x) = \sum_j p_j(x)^2 + \theta(x) \sum_i v_i(x)^2. \tag{5.3}$$

Let r_2 be such that

$$\sup_j \deg p_j(x)^2 \leq 2r_2 \quad \text{and} \quad 2 + \sup_i \deg v_i(x)^2 \leq 2r_2,$$

and let $r = \max[r_0, r_1, r_2]$. Let $\{q_j, p_j\}$ be the coefficient vectors of the polynomials $\{q_j(x), p_j(x)\}$ in the basis of dimension $s(r)$; let $\{v_i\}$ be the coefficient vectors of the polynomials $\{v_i(x)\}$ in the basis of dimension $s(r-1)$, and let $\{v_{kl}\}$ be the coefficient vectors of the polynomials $\{v_{kl}(x)\}$ in the basis of dimension $s(r-d_k)$, $k = 1, \ldots, m$. Define the matrices

$$X := \sum_j q_j q_j'; \quad Z := \sum_j p_j p_j'; \quad \Delta := \sum_i v_i v_i'; \quad W_k := \sum_l v_{kl} v_{kl}', \ k = 1, \ldots, m.$$

From (5.2) and (5.3) it follows that

$$\langle X, B_\alpha\rangle + \sum_{k=1}^m \langle W_k, C_\alpha^k\rangle - p_\alpha' = \begin{cases} -1, & \text{if } \alpha = 0, \\ 0, & \text{otherwise,} \end{cases} \tag{5.4}$$

and

$$\langle Z, B_\alpha\rangle + \langle \Delta, D_\alpha\rangle - p_\alpha' = 0 \qquad \forall \alpha; \tag{5.5}$$

that is, (X, Z, W_k, Δ, p') is an admissible solution of $\mathbb{Q}_r^*$ with value

$$\sum_{\alpha\in\Gamma} p_\alpha' \gamma_\alpha = \varepsilon + \sum_{\alpha\in\Gamma} p_\alpha \gamma_\alpha = \varepsilon + \rho^*.$$

By weak duality $\sup \mathbb{Q}_r \leq \rho^* + \varepsilon$. In addition, since $\sup \mathbb{Q}_r \geq \rho^* - \varepsilon$, it follows that

$$\rho^* + \varepsilon \geq \sup \mathbb{Q}_r \geq \rho^* - \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, the result follows.

Finally, if $\sup \mathbb{P} = \max \mathbb{P}$, there is a p.m. $\mu \in \Phi$ with $\mu(S) = \rho^*$, and with arguments already developed, one may assume that μ is finitely supported. In particular, the support of μ is contained in B_r for all r sufficiently large. The vectors y, z of the moments up to order $2r$ of the restrictions of μ to S and S^c respectively, are feasible for $\mathbb{Q}_r$, from which $\sup \mathbb{Q}_r \geq \rho^*$ follows. □

PROOF OF THEOREM 4.1. The proof mimics that of Theorem 3.9. Let $\rho^* := \inf \mathbb{P}$ where now

$$\mathbb{P} \to \inf_{\mu\in\Phi} \mu(S) \quad \text{and} \quad \mathbb{D} \to \left\{ \sup \sum_{\alpha\in\Gamma} \lambda_\alpha \gamma_\alpha \,\middle|\, \sum_{\alpha\in\Gamma} \lambda_\alpha x^\alpha \leq \mathbb{1}_S(x), \ \forall x \in \mathbb{R}^n \right\}.$$

With the same arguments as in the proof of Theorem 3.9, one easily proves that with $\varepsilon > 0$ fixed, arbitrary, there is some r_0 such that $\inf \mathbb{Q}_r \leq \rho^* + \varepsilon$ for all $r \geq r_0$. Indeed, let $\mu \in \Phi$ be such that $\mu(S) \leq \rho^* + \varepsilon$. From Proposition 3.2, there is a p.m. $\nu \in \Phi$ finitely supported and with $\nu(S) = \mu(S) \le \rho^* + \varepsilon$. Let φ, ψ be the restrictions of ν to S and S^c, respectively. As ψ is finitely supported, S is closed and $\varepsilon(r) \downarrow 0$ as $r \to \infty$, we must have $-\varepsilon(r)^{-1} \le \theta(x) \le -\varepsilon(r)$ for all x in the support of ψ, as soon as $r \ge r_0$, for some r_0. Therefore, with (y, z) the vectors of moments up to order r of φ and ψ, respectively, we have $M_r(y), M_r(z) \succeq 0$ and $M_{r-1}(\theta y) \succeq 0$ as well as $M_{r-1}(\theta_r(z), M_{r-1}(\theta^r(z) \succeq 0$, so that (y, z) is admissible for $\mathbb{Q}_r$ with value $y_0 = \varphi(S) = \nu(S) \le \rho^* + \varepsilon$.

From the hypothesis on γ, there is no duality gap between $\mathbb{D}$ and $\mathbb{P}$ and $\mathbb{P}$ is solvable. Therefore, let $p(x) = \sum_{\alpha\in\Gamma} p_\alpha x^\alpha$ be an optimal solution of $\mathbb{P}$ with value $\rho^* = \inf_{\mu\in\Phi} \mu(S)$.

We next show that we can construct an admissible solution of $\mathbb{Q}_r^*$ with value

$$\sum_{\alpha\in\Gamma} p_\alpha \gamma_\alpha - \varepsilon = \rho^* - \varepsilon,$$

provided r is sufficiently large. As $\inf \mathbb{Q}_r \le \rho^* + \varepsilon$ for all $r \ge r_0$, the result will follow from a simple application of weak duality between $\mathbb{Q}_r^*$ and $\mathbb{Q}_r$.

From $-p(x) \ge -\mathbb{1}_S(x)$ it follows that with

$$p'(x) := -p(x) + \varepsilon = \sum_{\alpha\in\Gamma} p'_\alpha x^\alpha,$$

the polynomial $p'(x)$ satisfies

$$p'(x) + 1 > 0 \quad \text{on } S \quad \text{and} \quad p'(x) > 0 \quad \text{on } S^c.$$

Moreover, S satisfies Assumption 3.8, since $\{\theta(x) \ge 0\}$ is compact. The same is true for the set $S' := \{-\varepsilon^{-1} \le \theta(x) \le -\varepsilon\} \subset S^c$. Hence, as $p'(x) + 1 > 0$ on S, it can be written

$$p'(x) + 1 = \sum_j q_j(x)^2 + \theta(x) \sum_k v_k(x)^2, \tag{5.6}$$

for some polynomials $\{q_j(x), v_k(x)\}$. Similarly, as $p'(x) > 0$ on S', it can be written

$$p'(x) = \sum_j u_j(x)^2 + \theta_\varepsilon(x) \sum_k w_k^1(x)^2 + \theta^\varepsilon(x) \sum_k w_k^2(x)^2 \tag{5.7}$$

for some polynomials $\{u_j(x), w_k^1(x), w_k^2(x)\}$.

Let r_1 be the maximum degree of the polynomials $\{q_j(x), u_j(x)\}$ and r_2 the maximum degree of the polynomials $\{v_k(x), w_k^1(x), w_k^2(x)\}$, and let $r := \max[r_0, r_1, r_2 + 1]$. Let $\{q_j, u_j, v_k, w_k^1, w_k^2\}$ be the coefficient vectors in the basis of dimension $s(r)$ of the polynomials $\{q_j(x), u_j(x), v_k(x), w_k^1(x), w_k^2(x)\}$, respectively.

Introduce the matrices

$$X := \sum_j q_j q_j', \qquad V := \sum_k v_k v_k'$$

and

$$Z := \sum_j u_j u_j'; \qquad W_1 := \sum_k w_k^1 (w_k^1)', \qquad W_2 := \sum_k w_k^2 (w_k^2)'.$$

From (5.6) it follows that

$$\langle X, B_\alpha \rangle + \langle V_k, C_\alpha \rangle - p'_\alpha = \begin{cases} 1, & \text{if } \alpha = 0, \\ 0, & \text{otherwise,} \end{cases}$$

and from (5.7) it follows that

$$\langle Z, B_\alpha \rangle + \langle W^1, E_\alpha \rangle + \langle W^2, F_\alpha \rangle - p'_\alpha = 0 \qquad \forall \alpha \le 2r,$$

which proves that (X, Z, V, W^1, W^2, p') is an admissible solution of $\mathbb{Q}_r^*$ with value

$$\sum_{\alpha \in \Gamma} -p'_\alpha \gamma_\alpha = -\varepsilon \gamma_0 + \sum_{\alpha \in \Gamma} p_\alpha \gamma_\alpha = -\varepsilon + \rho^*.$$

Therefore, as $\inf \mathbb{Q}_r \le \rho^* + \varepsilon$, from weak duality, one obtains

$$\rho^* - \varepsilon \le \inf \mathbb{Q}_r \le \rho^* + \varepsilon.$$

As ε was arbitrary, the result follows.

Finally, the last statement in Theorem 4.1 can be proved with the same arguments as in the proof of the analogue statement in Theorem 3.9. □

REFERENCES

[1] ANASTASSIOU, G. A. (1993). *Moments in Probability Theory and Approximation Theory*. Longman, New York.

[2] BERTSIMAS, D. and POPESCU, I. (1998, rev. 2001). Optimal inequalities in probability theory. A convex optimization approach. INSEAD working paper TM62, Fontainebleau, France.

[3] BERTSIMAS, D. and SETHURAMAN, J. (2000). Moment problems and semidefinite optimization. In *Handbook of Semidefinite Programming* (R. Saigal, L. Vandenberghe and H. Wolkowicz, eds.) 311–339. Kluwer, Boston.

[4] CURTO, R. E. and FIALKOW, L. A. (1991). Recursiveness, positivity, and truncated moment problems. *Houston J. Math.* **17** 603–635.

[5] CURTO, R. E. and FIALKOW, L. A. (2000). The truncated complex K-moment problem. *Trans. Amer. Math. Soc.* **352** 2825–2855.

[6] GRÖTSCHEL, M., LOVÁSZ, L. and SCHRIJVER, A. (1988). *Geometric Algorithms and Combinatorial Optimization*. Springer, Berlin.

[7] ISII, K. (1963). On sharpness of Tchebycheff-type inequalities. *Ann. Inst. Statist. Math.* **14** 185–197.

[8] JACOBI, T. (2001). A representation theorem for certain partially ordered commutative rings. *Math. Z.* **237** 259–273.

[9] JACOBI, T. and PRESTEL, A. (2001). Distinguished representations of strictly positive polynomials. *J. Reine Angew. Math.* **532** 223–235.

[10] JOHNSON, M. A. and TAAFFE, M. R. (1993). Tchebycheff systems for probabilistic analysis. *Amer. J. Math. Management Sci.* **13** 83–111.

[11] KREIN, M. and NUDEL'MAN, A. (1977). *The Markov Moment Problem and Extremal Problems*. Amer. Math. Soc., Providence, RI.

[12] LASSERRE, J. B. (2001). Global optimization with polynomials and the problem of moments. *SIAM J. Optim.* **11** 796–817.

[13] PUTINAR, M. (1993). Positive polynomials on compact semi-algebraic sets. *Indiana Univ. Math. J.* **42** 969–984.

[14] SCHMÜDGEN, K. (1991). The K-moment problem for compact semi-algebraic sets. *Math. Ann.* **289** 203–206.

[15] SMITH, J. E. (1995). Generalized Chebychev inequalitites: Theory and applications in decision analysis. *Oper. Res.* **43** 807–825.

[16] WHITT, W. (1984). On approximations for queues, I. Extremal distributions. *AT&T Bell Labs Tech. J.* **63** 115–138.

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