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STRICT INEQUALITIES FOR THE TIME CONSTANT IN FIRST PASSAGE PERCOLATION

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In this work we are interested in the variations of the asymptotic shape in first passage percolation on $\mathbb{Z}^2$ according to the passage time distribution. Our main theorem extends a result proved by van den Berg and Kesten, which says that the time constant strictly decreases when the distribution of the passage time is modified in a certain manner (according to a convex order extending stochastic comparison). Van den Berg and Kesten's result requires, when the minimum r of the support of the passage time distribution is strictly positive, that the mass given to r is less than the critical threshold of an embedded oriented percolation model. We get rid of this assumption in the two-dimensional case, and to achieve this goal, we entirely determine the flat edge occurring when the mass given to r is greater than the critical threshold, as a functional of the asymptotic speed of the supercritical embedded oriented percolation process, and we give a related upper bound for the time constant.

1. Introduction. First passage percolation was introduced in 1965 by Hammersley and Welsh [9] as a stochastic model for a porous media. See [10] for a review on this subject. In this paper, we consider the grid $\mathbb{Z}^2$, included in the plane $\mathbb{R}^2$ endowed with the following norms:

$$\forall\,(x,y)\in\mathbb{R}^2,\quad \|(x,y)\|_1=|x|+|y| \quad\text{and}\quad \|(x,y)\|_\infty=\max\{|x|,|y|\}.$$

We denote by $\mathbb{N}=\{0,1,2,\dots\}$ the set of positive integers and by $\mathbb{N}^*$ the set of strictly positive integers. Two vertices of $\mathbb{Z}^2$ are said to be neighbors if their distance for $\|\cdot\|_1$ is equal to 1. The edges of the grid are the line segments between neighbor sites of $\mathbb{Z}^2$, and the set of all edges is denoted by $\mathbb{E}_2$. A path γ is a finite sequence of sites $(z_0,z_1,\dots,z_l)$ where two successive points are neighbors. The integer l is called the length of the path, and is denoted by $|\gamma|$. We now give to each edge $e\in\mathbb{E}_2$ a random passage time $t(e)$, by considering a family of independent and identically distributed nonnegative random variables $(t(e))_{e\in\mathbb{E}_2}$, with common distribution F. The travel time of a path γ is defined by $t(\gamma)=\sum_{e\in\gamma}t(e)$. The travel time between two sites x and y in $\mathbb{Z}^2$ is then the shortest travel time of all the paths with extremities x and y:

$$t(x,y)=\inf\{t(\gamma),\gamma\text{ path from }x\text{ to }y\}.$$

The paths for which this infimum is reached are called optimal paths (or t-optimal paths to underline the corresponding passage time). In the following, we are going

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to work on "the" optimal path, so we have to give a way to choose one of the optimal paths if there exist several. We choose an order on the edges of $\mathbb{Z}^2$, and we call the optimal path from x to y the smallest optimal path for the lexicographic order on the sequence of edges starting from x.

The definition of the passage time between two points can be extended to points in $\mathbb{R}^2$: If x and y are in $\mathbb{R}^2$, we define $t(x, y) = t(\tilde{x}, \tilde{y})$, where $\tilde{x}$ (resp., $\tilde{y}$) is the nearest neighbor of x (resp., y) in $\mathbb{Z}^2$. Possible indetermination can be dropped by choosing an order on the sites of $\mathbb{Z}^2$ and taking the smallest nearest neighbor for this order. First passage percolation studies in particular the evolution of the random subset of the points in $\mathbb{R}^2$ which are reached before time s, defined by $A(s) = \{z \in \mathbb{R}^2, t(0, z) \le s\}$. Subadditive ergodic techniques, developed by Richardson [14] and deepened by Cox and Durrett [2] gave the existence of a directed asymptotic speed,

$$(1) \quad \forall\, x \in \mathbb{R}^2,\ \exists\, \mu(x) \in \mathbb{R}^+ \quad \text{such that } \lim_{n \to +\infty} \frac{t(0, nx)}{n} = \mu(x) \text{ in probability.}$$

Note that if F has finite mean, then this convergence is also a.s. and in L^1. Moreover, this limit $\mu(x)$ is equal to $\inf_{n \in \mathbb{N}^*} \frac{Et(0,nx)}{n}$. This constant, for $x = (1, 0)$, is called the time constant, and is denoted by μ. The time constant μ is positive if and only if $F(0)$ is smaller than the critical percolation threshold p_c for Bernoulli percolation on the edges of $\mathbb{Z}^2$ ($p_c = 1/2$; see [6]). Under this assumption, $x \mapsto \mu(x)$ is a norm on $\mathbb{R}^2$ (see [10]). The ball with radius 1 for this norm is denoted by A and is called the asymptotic shape associated to F. It is a compact convex deterministic set, with the same symmetries as the lattice and which describes the evolution of the random set $A(n)$. The following result is known as the asymptotic shape theorem (Theorem 4 in [2]):

$$(2) \quad \forall\, \varepsilon > 0, \quad P\left(\text{for } n \text{ large enough, } (1 - \varepsilon)A \subset \frac{A(n)}{n} \subset (1 + \varepsilon)A\right) = 1.$$

Consequently, the evolution of the random set is at the first order in n entirely determined by the asymptotic shape A. The determination of A and μ as functionals of F is thus a fundamental but difficult problem in first passage percolation. Following van den Berg and Kesten in [17], we try in this paper to compare μ and $\tilde{\mu}$, the time constants respectively associated to two distributions F and $\tilde{F}$, when these distributions are comparable for the following order.

DEFINITION 1.1. Let F et $\tilde{F}$ be two distributions on $\mathbb{R}$. We say that $\tilde{F}$ is more variable than F, denoted by $\tilde{F} \succcurlyeq F$, if

$$\int \Phi \, d\tilde{F} \le \int \Phi \, dF$$

for every concave increasing function $\Phi : \mathbb{R} \to \mathbb{R}$ for which the two integrals converge absolutely. If, moreover, the two distributions are distinct, we say that $\tilde{F}$ is strictly more variable than F.

EXAMPLES. (i) If t is a random variable with finite mean on a probability space $(\Omega, \mathcal{F}, P)$ and if $\mathcal{G}$ is a σ-field included in $\mathcal{F}$, then the conditional expectation $E(t|\mathcal{G})$ of t with respect to $\mathcal{G}$ is less variable than t.

(ii) If F stochastically dominates $\tilde{F}$ then $\tilde{F}$ is more variable than F.

NOTATION. In the following, we consider two distributions F and $\tilde{F}$ on $\mathbb{R}^+$; t (resp., $\tilde{t}$) is a random variable with distribution F (resp., $\tilde{F}$), r (resp., $\tilde{r}$) is the infimum of the support of F (resp., $\tilde{F}$), μ (resp., $\tilde{\mu}$) is the time constant associated to F or t (resp., $\tilde{F}$ or $\tilde{t}$) by the first passage percolation model on $\mathbb{Z}^2$ and A (resp., $\tilde{A}$) denotes the asymptotic shape associated to F or t (resp., $\tilde{F}$ or $\tilde{t}$) by the first passage percolation model on $\mathbb{Z}^2$.

This order is well adapted to the comparison of time constants. Theorem 2.9 in [17] ensures that, when F and $\tilde{F}$ have finite means,

$$\text{if } \tilde{F} \succcurlyeq F \text{ then } \tilde{\mu} \leq \mu \text{ and } \tilde{A} \supset A. \tag{3}$$

One can naturally wonder if a discrepancy between two comparable distribution functions is transmitted to their respective time constants and our first result answers exhaustively the question of the strict comparison in dimension 2:

THEOREM 1.2. *Let F be a distribution on $\mathbb{R}^+$ such that $F(0) < p_c$. If $\tilde{F}$ is a distribution on $\mathbb{R}^+$ which is strictly more variable than F, then $\tilde{\mu} < \mu$.*

INTERPRETATION. Assume that t has unbounded support. Naïvely, one can think that the optimal paths tend to use edges with small passage times, and a natural question arises: Is there a threshold $T > r$ such that optimal paths, at least those with far enough away extremities, use only edges with passage time smaller than T? The continuity result of Cox and Kesten in [3] ensures that if μ_T is the time constant associated to the truncated passage time $t_T = t \wedge T$, then μ_T tends to μ when T goes to infinity, but by Theorem 1.2, $\mu_T < \mu$ for every T large enough. Thus, it is more efficient for the model to use a certain proportion of edges with large passage time than to try to avoid them.

Another way to illustrate this result is to consider a Bernoulli passage time of the form $t \sim p\delta_1 + (1-p)\delta_M$ where $p \in [0,1]$ and $M > 1$: even if p is close to 1 and M very large, an optimal path from 0 to $(n, 0)$ asymptotically uses, at least in mean, a positive proportion of edges with passage time M.

REMARKS. (i) In [17], the distribution functions are supposed to have finite mean. Here we get rid of this assumption.

(ii) The case $\inf \operatorname{supp} F = 0$ and $F(0) < p_c$ is treated on $\mathbb{Z}^d$ by van den Berg and Kesten [17]. The requirement $F(0) < p_c$ cannot be dropped because if $F(0) \geq p_c$ then $\mu = 0$ (see [10]). In the case $\inf \operatorname{supp} F = r > 0$ and $F(r) < \vec{p}_c$, where $\vec{p}_c$ is the critical percolation threshold for oriented percolation on $\mathbb{Z}^d$; van den Berg and Kesten proved this result for first passage percolation on $\mathbb{Z}^d$, but the assumption $F(r) < \overrightarrow{p}_c$, needed in their proof, is not a priori necessary. We get rid of this assumption, but only in dimension 2, because of the large deviations results we need for oriented percolation.

Thanks to the second remark, it only remains to prove Theorem 1.2 when $\inf \operatorname{supp} F = r > 0$ and $F(r) \geq \overrightarrow{p}_c$. Replacing $F(x)$ by $F(rx)$ (that means dividing the passage time t by r), we can suppose that $r = 1$. In the following, we will always work under these hypotheses:

$$\inf \operatorname{supp} F = 1 \quad \text{and} \quad F(1) = P(t = 1) \geq \overrightarrow{p}_c. \tag{4}$$

In this case, the passage time between two vertices x and y is greater or equal to the minimal number of edges of a path with extremities x and y, which is $\|x - y\|_1$, multiplied by the minimal accessible passage time, which is 1. Thus, by definition of the time constant,

$$\forall\, x \in \mathbb{R}^2, \quad \mu(x) \geq \|x\|_1 \quad \text{and} \quad A \subset \{x \in \mathbb{R}^2,\ \|x\|_1 \leq 1\}.$$

Now, if $P(t = 1) \geq \overrightarrow{p}_c$, unlike the case treated by van den Berg and Kesten, it may exist $x \in \mathbb{Z}^2$ and infinitely many $n \in \mathbb{N}^*$ such that $t(0, nx) = n\|x\|_1$, which implies $\mu(x) = \|x\|_1$. But for such an x, if $\tilde{F}$ is strictly more variable than F and verifies $\tilde{r} = 1$ and $\tilde{F}(1) = F(1)$, then $\tilde{\mu}(x) = \mu(x) = \|x\|_1$.

The first step of the proof of Theorem 1.2 is thus to determine, for a fixed distribution F satisfying (4), the directions x such that $\mu(x) = \|x\|_1$. Durrett and Liggett [5] showed for Richardson's model, a particular first passage percolation model on the sites of $\mathbb{Z}^2$, that under a similar condition the asymptotic shape has a flat edge corresponding to these directions. Our next result characterizes the flat edge occurring for the asymptotic shape in the general first passage percolation model on the edges of $\mathbb{Z}^2$ under hypotheses (4), as a functional of the asymptotic speed of an endowed supercritical oriented percolation model. Let us first introduce some notation: If $p \geq \overrightarrow{p}_c$, α_p denotes the asymptotic speed of the supercritical oriented percolation on the edges of $\mathbb{Z}^2$ with parameter p (see Durrett [4] for a review of this percolation model, and Section 3 of this paper for a reminder of its main notation and properties). M_p is the point in $\mathbb{R}^2$ with coordinates $(\frac{1}{2} + \frac{\alpha_p}{\sqrt{2}}, \frac{1}{2} - \frac{\alpha_p}{\sqrt{2}})$, N_p the one with coordinates $(\frac{1}{2} - \frac{\alpha_p}{\sqrt{2}}, \frac{1}{2} + \frac{\alpha_p}{\sqrt{2}})$ and $[M_p, N_p]$ denotes the segment line in $\mathbb{R}^2$ with extremities M_p and N_p. We can now state our flat edge result (see Figure 1).

THEOREM 1.3. *Let F be a distribution on $\mathbb{R}^+$ satisfying* (4), *and let A be the corresponding asymptotic shape obtained by first passage percolation on $\mathbb{Z}^2$. We set $p = F(1)$.*

(i) $A \subset \{x \in \mathbb{R}^2, \|x\|_1 \leq 1\}$.

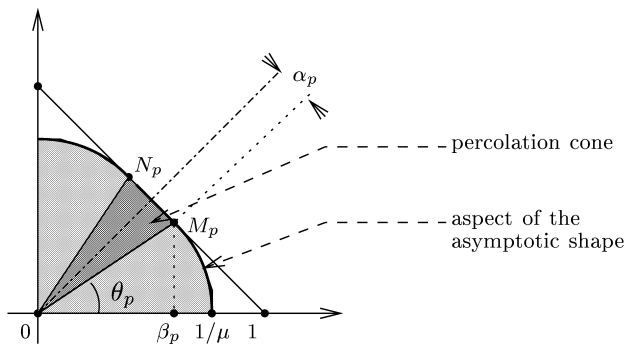

FIG. 1. *Aspect of the flat edge when* $\inf \operatorname{supp} F = 1$ *and* $F(1) = p > \vec{p_c}$.

(ii) *If* $p < \overrightarrow{p}_c$, *then* $A \subset \{x \in \mathbb{R}^2, \|x\|_1 < 1\}$.

(iii) *If* $p > \overrightarrow{p}_c$, *then* $A \cap \{x \in \mathbb{R}^+ \times \mathbb{R}^+, \|x\|_1 = 1\} = [M_p, N_p]$. *The segment* $[M_p, N_p]$ *is what we call the flat edge of the asymptotic shape* A.

(iv) *If* $p = \overrightarrow{p}_c$, *then* $A \cap \{x \in \mathbb{R}^+ \times \mathbb{R}^+, \|x\|_1 = 1\} = \{(1/2, 1/2)\}$.

REMARK. Note that this implies that the equality $\mu = 1$ cannot happen for the time constant unless $P(t = 1) = 1$.

If we note $\beta_p = 1/2 + \alpha_p/\sqrt{2}$, the convexity and symmetry of A imply that $1/\mu \geq \beta_p$. The next step is to prove that this inequality cannot be an equality. Indeed, if $1/\mu = \beta_p$, then μ only depends on the atom of F at 1, and for each $\tilde{F}$ such that $\tilde{r} = 1$ and $\tilde{F}(1) = F(1)$, we would have that $\tilde{\mu} = \mu$, even if $\tilde{F}$ is strictly more variable than F.

THEOREM 1.4. *Let* F *be a distribution on* $\mathbb{R}^+$ *such that* $\inf \operatorname{supp} F = 1$ *and* $\overrightarrow{p}_c \leq F(1) = p < 1$. *Then*

$$\frac{1}{\mu} > \frac{1}{2} + \frac{\alpha_p}{\sqrt{2}} = \beta_p.$$

REMARKS. (i) We do not know how to prove a similar result in higher dimension because of the "crossing paths" argument we use for the proof in the plane.

(ii) In [8], Häggström and Meester prove that any compact convex set whose interior is not empty and which is symmetric with respect to the origin can occur as the asymptotic shape for a first passage percolation model associated to stationary passage times. Here we can see that the convex set obtained as the convex hull of the two points M_p, N_p and their images by the symmetries with respect to the axes cannot be obtained as the asymptotic shape for a first passage percolation model associated to independent and identically distributed passage times.

(iii) Theorem 1.4 is a small step toward the proof of the strict convexity of the asymptotic shape in the direction of the axes. This convexity is a fundamental hypothesis to obtain estimations for shape fluctuations (see [11], [12] and [13]).

What can be said to compare two asymptotic shapes in other directions? Result 3 of van den Berg and Kesten on the time constant can immediately be extended in every direction. In the case $\inf \operatorname{supp} F = 1$ and $F(1) = p \geq \overrightarrow{p}_c$, denote by θ_p the angle between the first coordinate axis and (OM_p), and set $\mu_\theta = \mu(e^{i\theta})$ for $\theta \in [0, 2\pi[$. The proof of Theorem 1.4 can be adapted to show that the radius of the asymptotic shape in any direction $\theta \in [0, \theta_p[$ is strictly greater than the projection of M_p on $\mathbb{R}e^{i\theta}$ along the supporting hyperplane of A in the direction θ. Theorem 1.2 can then be extended in every direction which is not in the percolation cone. To avoid intricate geometrical considerations, we restrict ourselves to the proof in the first coordinate direction, but still give the following result (because of the symmetries of A, we can restrict ourselves to $\theta \in [0, \pi/4]$).

THEOREM 1.5. *Let F be a distribution on $\mathbb{R}^+$ such that $F(0) < p_c$ and $\tilde{F}$ be a distribution on $\mathbb{R}^+$ strictly more variable than F. We have*:

(i) *If $r = 0$, then $\forall\, \theta \in [0, \pi/4]$, $\tilde{\mu}_\theta < \mu_\theta$.*
(ii) *If $r > 0$ and $F(r) < \overrightarrow{p}_c$, then $\forall\, \theta \in [0, \pi/4]$, $\tilde{\mu}_\theta < \mu_\theta$.*
(iii) *If $r > 0$ and $F(r) \geq \overrightarrow{p}_c$ and $\tilde{r} < r$, then $\forall\, \theta \in [0, \pi/4]$, $\tilde{\mu}_\theta < \mu_\theta$.*
(iv) *If $r > 0$ and $F(r) \geq \overrightarrow{p}_c$ and $\tilde{r} = r$, then $\forall\, \theta \in [0, \theta_p[$, $\tilde{\mu}_\theta < \mu_\theta$ and $\forall\, \theta \in [\theta_p, \pi/4]$, $\tilde{\mu}_\theta = \mu_\theta = \frac{1}{r}\|e^{i\theta}\|_1$.*

EXAMPLE. The same results are still valid for first passage percolation on sites of $\mathbb{Z}^2$. We can then apply our results to Richardson's model (see [14]). Consider, for each $p \in [0, 1]$, a family $(t_p(z))_{z \in \mathbb{Z}^2}$ of independent and identically distributed random variables such that for every $k \in \mathbb{N}^*$, $P(t_p = k) = p(1 - p)^k$. Choose p and p' in $[0, 1]$ such that $p' > p$. Then, if $\overrightarrow{p}_c^{\,s}$ denotes the critical threshold for oriented site percolation on $\mathbb{Z}^2$, we have:

(i) If $p < \overrightarrow{p}_c^{\,s}$, then $\forall\, \theta \in [0, \pi/4]$, $\mu'_\theta < \mu_\theta$.
(ii) If $p \geq \overrightarrow{p}_c^{\,s}$, then $\forall\, \theta \in [0, \theta_p[$, $\mu'_\theta < \mu_\theta$ and $\forall\, \theta \in [\theta_p, \pi/4]$, $\mu'_\theta = \mu_\theta = \|e^{i\theta}\|_1$.

In particular, the time constant is a strictly decreasing function of the parameter p.

MAIN LINES OF THE PROOF OF THEOREM 1.2. Suppose that F satisfies (4), and to simplify take $\tilde{F}$ stochastically smaller than F and distinct from F. A coupling argument enables us to realize the passage times t and $\tilde{t}$, with respective distribution F and $\tilde{F}$, on the same space $(\Omega, \mathcal{F}, P)$, in a manner such that almost surely $\tilde{t} \leq t$. As $\tilde{F}$ is distinct from F, we can find $\eta > 0$ such that

$$P(\tilde{t} \leq t - \eta) > 0. \tag{5}$$

Remember that the time constant μ is obtained as the limit of the ratio $t(0,n)/n$. To compare μ and $\tilde{\mu}$, we want to find, along the t-optimal path γ_n between 0 and n, a number αn (with $\alpha > 0$ small enough) of edge-disjoint portions γ_n^i of γ_n such that

$$\tilde{t}(\gamma_n^i) \le t(\gamma_n^i) - \eta. \tag{6}$$

Thus $\tilde{t}(0,n) \le \tilde{t}(\gamma_n) \le t(0,n) - \alpha\eta n$. If we can obtain such an inequality on a set with probability tending to 1 as n goes to ∞, we will obtain the desired strict comparison.

Imagine now, to simplify, that $\tilde{r} = 1$ and $\tilde{F}(1) = F(1)$. We are going to construct the γ_n^i as crossings by γ_n of some rectangular boxes of size $N \times 3N$, where N will be chosen large enough by a renormalization process. The question is now, is it possible for γ_n to cross a box of width N using a passage time equal to $\|x - y\|_1$, where x and y are the extremities of the crossing? If "yes," then $\tilde{t}(x,y) = t(x,y) = \|x - y\|_1$, which implies $\tilde{t}(\gamma_n^i) = t(\gamma_n^i)$, and thus it will indeed be impossible to get (6).

By using large deviation results for supercritical oriented percolation, we will prove that if the extremity y of the crossing of a box of width N is not in the percolation cones issued from the other extremity x, then $t(x,y) \ge (1+\delta) \times \|x - y\|_1$, with a probability tending to 1 when N, and thus $\|x - y\|_1$, goes to infinity; this will be done in Proposition 3.1. The first application of this proposition will be the proof of Theorem 1.3. The second will be to define a coloring and to use a renormalization process to ensure that the overwhelming majority of crossings along γ_n that are not "in a percolation cone" will verify $t(x,y) \ge (1+\delta)\|x - y\|_1$, if x and y are the extremities of the crossing. On this event, we will modify the passage times configuration in the box to force a copy of γ_n to use an edge with passage times satisfying (5), and thus construct a crossing γ_n^i satisfying (6).

So it remains to prove that along γ_n, it is not possible to cross too many boxes of width N inside the percolation cones, and this will be ensured by Theorem 1.4.

The rest of the paper is organized as follows. In Section 2, we give a renormalization lemma; in Section 3, we study the coupling of first passage percolation and an embedded oriented percolation model and prove Proposition 3.1; in Section 4, we prove the flat edge result (Theorem 1.3); in Section 5, we prove Theorem 1.4. Section 6 is devoted to the coupling of two random variables t and $\tilde{t}$ with respective distribution F and $\tilde{F}$ on the same space $(\Omega, \mathcal{F}, P)$, when $\tilde{F}$ is more variable than F, and we show that we can get rid of the integrability assumption used in [17]. Section 7 gives the proof of Theorem 1.2 from an intermediate result, Proposition 7.6; finally, in Section 8, we prove Proposition 7.6.

2. A renormalization lemma. In this section, we define a renormalization grid and the main crossings of a path associated to this grid, and we give a renormalization lemma we will use several times in the proofs of our results.

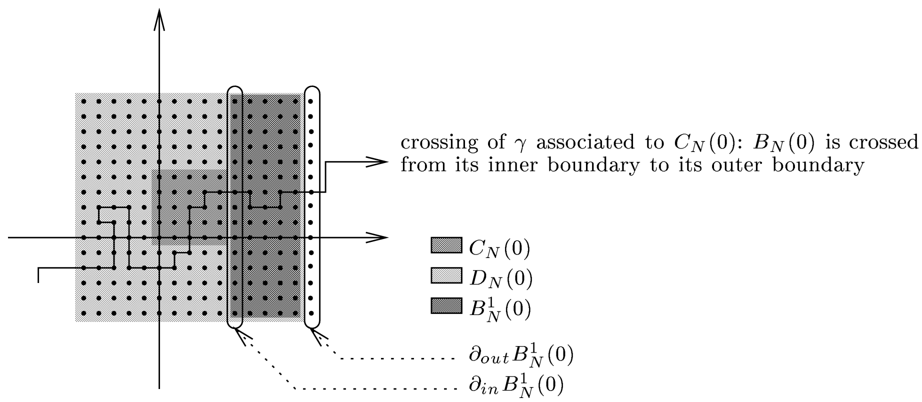

FIG. 2. *Elements of the renormalization grid for $N = 5$.*

A renormalization grid. Let N be a strictly positive integer. We introduce the following notations (see Figure 2).

C_N is the cube $[-1/2, N - 1/2]^2$. We call N-cubes the cubes $C_N(k) = kN + C_N$ obtained by translating C_N according to Nk with $k \in \mathbb{Z}^2$. The coordinates of k are called the coordinates of the N-cube $C_N(k)$. Note that N-cubes induce a partition of $\mathbb{Z}^2$.

D_N is the large cube $[-N - 1/2, 2N - 1/2]^2$, and the large cube $D_N(k)$ is obtained by translating D_N according to Nk with $k \in \mathbb{Z}^2$. The boundary of $D_N(k)$, denoted by $\partial D_N(k)$, is the set of sites outside $D_N(k)$ that have a neighbor in $D_N(k)$.

B_N is the rectangular box $[-1/2, N - 1/2] \times [-N - 1/2, 2N - 1/2]$. In the large cube $D_N(k)$, the N-cube $C_N(k)$ is surrounded by the four following N-boxes:

$$\begin{aligned}
B_N^1(k) &= Nk + (N, 0) + B_N;\\
B_N^2(k) &= Nk + (N, N) + e^{i(\pi/2)} B_N;\\
B_N^3(k) &= Nk - (N, 0) + B_N;\\
B_N^4(k) &= Nk - (N, 0) + e^{i(\pi/2)} B_N.
\end{aligned}$$

An edge is said to be in a subset E of $\mathbb{R}^2$ if at least one of its two extremities is in E. We now define the inner and outer boundaries of a N-box associated to a pair $(C_N(k), D_N(k))$ of cubes. Let us do this for $B_N^1(0)$ and extend the definition to other boxes by rotation and translation:

$$\begin{aligned}
\partial_{\mathrm{out}} B_N^1(0) &= \{(2N, y), y \in [-N, \dots, 2N - 1]\},\\
\partial_{\mathrm{in}} B_N^1(0) &= \{(N, y), y \in [-N, \dots, 2N - 1]\}.
\end{aligned}$$

Note that $\partial D_N(k)$ is the disjoint union of the sets $(\partial_{\mathrm{out}} B_N^i(k))_{1 \le i \le 4}$, and that a path entering in $C_N(k)$ and getting out of $D_N(k)$ has to cross one of the four N-boxes surrounding $C_N(k)$ in $D_N(k)$, from its inner boundary to its outer boundary. We can then define the crossing associated to a N-cube $C_N(k)$ (see Figure 2).

DEFINITION 2.1. Let $\gamma = (x_0, \dots, x_l)$ be a path such that $x_0 \in C_N(k)$ and $x_l \notin D_N(k)$. We set $j_f = \min\{0 \le k \le l,\ x_k \in \partial D_N(k)\}$. There exists a unique i such that $x_{j_f} \in B_N^i(k)$. Let then $j_0 = \max\{0 \le k \le j_f,\ x_k \notin B_N^i(k)\}$. The portion $(x_{j_0+1}, \dots, x_{j_f})$ of γ is the crossing of γ associated to $C_N(k)$.

Main crossings of a path. Let N be a strictly positive integer, x be a point in $\mathbb{Z}^2$ and γ be a path without any double point from 0 to x. We want to associate to γ a sequence of crossings of N-boxes (the main crossings of γ), in a way that two different crossings are edge-disjoint. Consider first the sequence $\sigma_0 = (k_1, \dots, k_{\tau_0})$ made of the coordinates of the N-cubes successively visited by γ. As the N-cubes induce a partition of $\mathbb{Z}^2$, this sequence is well defined, and has the following properties:

$$(P_0)\begin{cases} 0 \in C_N(k_1), \quad x \in C_N(k_{\tau_0}), \\ \forall\ 1 \le i \le \tau_0 - 1, \quad \|k_{i+1} - k_i\|_1 = 1. \end{cases}$$

But σ_0 can have double points; we remove them by the classical loop-removal process described in [7]. We thus obtain a sequence $\sigma_1 = (k_{\phi_1(1)}, \dots, k_{\phi_1(\tau_1)})$ extracted from σ_0, with the following properties:

$$(P_1)\begin{cases} 0 \in C_N(k_{\phi_1(1)}), \quad x \in C_N(k_{\phi_1(\tau_1)}), \\ \forall\ 1 \le i \le \tau_1 - 1, \quad \|k_{\phi_1(i+1)} - k_{\phi_1(i)}\|_1 = 1, \\ \sigma_1 \text{ has no double point.} \end{cases}$$

To every cube $C_N(k)$ in this sequence such that γ gets out of $D_N(k)$, that means for every N-cube in σ_1 with the possible exception of the seven last, we associate a crossing of a N-box in the following way: let z be the first point of γ to be in $C_N(k)$, and let z_2 be the first point of γ after z to be in $\partial D_N(k)$. Then the crossing associated to the N-cube $C_N(k)$ is the crossing of the portion of γ between z and z_2 associated to $C_N(k)$ in Definition 2.1.

The problem now is that two distinct cubes in σ_1 can have the same associated crossing. We have to extract a subsequence once again in order to obtain edge-disjoint crossings. Set $\phi_2(1) = 1$, and define ϕ_2 by induction,

$$\phi_2(i+1) = \inf\bigl\{j > \phi_2(i) \text{ such that } \|k_{\phi_1(j)} - k_{\phi_1\circ\phi_2(i)}\|_\infty > 1\bigr\} - 1$$

if the infimum exists, and let τ be the smallest index i for which $\phi_2(i+1)$ is not defined. Set $\phi = \phi_1 \circ \phi_2$; the elements of $\sigma = (k_{\phi(i)})_{1 \le i \le \tau}$ are called the main cubes of γ, and their associated crossings the main crossings of γ. This sequence has the following properties:

$$(P)\begin{cases} 0\in C_N(k_{\phi(1)}),\\ \|k_{\phi(\tau)}-k_{\tau_0}\|_\infty\le 1,\\ \forall\, 1\le i\le \tau-1,\quad \|k_{\phi(i+1)}-k_{\phi(i)}\|_\infty=1,\\ \text{the main crossings of } \gamma \text{ are edge-disjoint.}\end{cases}$$

Indeed, the first three properties are directly obtained from (P_1) and the construction of ϕ_2. Let us verify now that the main crossings of a path γ are edge-disjoint. Call, respectively, z, z_1 and z_2 the first point of γ in $C_N(k_{\phi(i)})$ and the extremities of the crossing associated to this N-cube. The index $\phi(i+1)$ is the index of the coordinates in $\mathbb{Z}^2$ of the N-cube containing z_2; thus, the crossing associated to $C_N(k_{\phi(i+1)})$ is a portion of γ which appears after z_2 in γ. As γ has no double point, the crossings associated to two adjacent cubes in σ are edge-disjoint. As the main crossings of γ appear along γ in the same order as the N-cubes they are associated to appear in σ, and as the path γ has no double points, the main crossings of γ are edge-disjoint.

From properties (P) we can deduce that for every x in $\mathbb{R}^2$, the number τ of main N-cubes of a path with no double point from 0 to x satisfies the following inequality:

$$\tau\ge\frac{\|x\|_\infty}{N}. \tag{7}$$

A renormalization lemma. Let $(\Omega,\mathcal{F},P)$ be a probability space and $(t(e))_{e\in\mathbb{E}_2}$ be a family of independent, identically distributed and nonnegative random variables. The following lemma is an adaptation of Lemma (5.2) in [17], and its proof is a standard Peierl's argument (see proof of (3.12) in [7]). We will thus not give any proof of it.

LEMMA 2.2. *For each $N\in\mathbb{N}^*$, we give to the N-cubes a random color, black or white, according to the values of the passage times in the initial model, such that*:

(i) *For each $N\in\mathbb{N}^*$, the colors of the N-cubes are identically distributed.*
(ii) *For each $N\in\mathbb{N}^*$, for each $k\in\mathbb{Z}^2$, the color of the N-cube $C_N(k)$ depends only on the passage times of the edges in $D_N(k)$.*
(iii) $\lim_{N\to+\infty} P(C_N(k)\text{ is black})=1$.

Then for every $\rho\in]0,1[$, there exists N_ρ such that for all $N\ge N_\rho$, there exist two strictly positive constants A and B such that for every $x\in\mathbb{R}^2$,

$$P\left(\begin{array}{c}\text{There exists a path } \gamma \text{ from } 0 \text{ to } x \text{ that, among}\\ \text{its } \tau \text{ main } N\text{-cubes, has less than } \rho\tau \text{ black cubes}\end{array}\right)\le A\exp(-B\|x\|_\infty). \tag{8}$$

EXAMPLE. Let F be a distribution on $\mathbb{R}^+$. Note $r = \inf \operatorname{supp} F$ and suppose that $r > 0$ and $F(r) < 1$. The time constant μ is clearly greater or equal to r. Durrett and Liggett [5] prove that for Richardson's model, the time constant is strictly greater than 1 by comparing this model with a branching random walk. The analogous result for first passage percolation can be obtained as a consequence of Theorem 1.3. By using the renormalization lemma, we can here prove directly that $\mu > r$. Indeed, take $0 < \delta < r$ such that $F(r + \delta) < 1$ and color the N-boxes in the following way. B_N is black if $\forall y \in [-N, \dots, 2N - 1]$, there exists $x \in [0, \dots, N - 1]$ such that the passage time of the edge with extremities (x, y) and $(x + 1, y)$ is greater or equal to $r + \delta$, and white otherwise. By rotation and translation, we extend this coloring to each N-box. The N-cube $C_N(k)$ is black if its four surrounding boxes in $D_N(k)$ are black, and white otherwise. This coloring clearly satisfies the hypotheses of the lemma. Choose $\rho \in]0, 1[$ and N given by the renormalization lemma, apply the renormalization lemma and (7) and conclude by noting that the passage time of a main crossing associated to a black N-cube is at least $Nr + \delta$. Thus $P(t(0, n) < nr + \delta\rho n/N) \le A \exp(-Bn)$, and by dividing by n and taking the limit when n goes to ∞, we get that $\mu(F) \ge r + \delta\rho/N > r$.

3. Coupling with oriented percolation. In this section, we couple our first passage percolation model with an embedded oriented percolation model. Let us first introduce the oriented percolation model and some notations. Our reference here is the paper of Durrett [4]; our notation and constants, in order to fit with our context, are slightly different from his. Consider the lattice $\mathbb{Z}^2$ and a parameter $p \in [0, 1]$. Each vertex z of $\mathbb{Z}^2$ has two oriented edges giving, respectively, access when they are open to $z + (1, 0)$ and $z + (0, 1)$. The edges can only be used in the way allowed by their orientation. We give to the set $\mathbb{E}_2^{\rightarrow}$ of these oriented edges a family $(\eta(e))_{e \in \mathbb{E}_2^{\rightarrow}}$ of independent and identically distributed random variables with law $p\delta_1 + (1 - p)\delta_0$ on a probability space $(\Omega, \mathcal{F}, P_p)$, where δ_x is Dirac's measure on x. The edge e is said to be open in η if $\eta(e) = 1$, and closed otherwise. Let us now introduce the following notation:

1. $z_1 \to z_2$ denotes the existence of an (oriented) open path between z_1 and z_2. For every $n \in \mathbb{N}$, ξ_n is the set of (x, y) in $\mathbb{N}^2$ such that $x + y = n$ and $0 \to (x, y)$. We define r_n as the signed Euclidean distance between $(\frac{n}{2}, \frac{n}{2})$ and the point in ξ_n with the largest first coordinate; the sign of r_n is chosen to be the same as the sign of the difference between the first coordinate of this point and $\frac{n}{2}$; r_n is only defined when $\xi_n \neq \varnothing$.
2. For every $n \in \mathbb{N}$, $\bar{\xi_n}$ is the set of (x, y) in $\mathbb{Z}^2$ such that $x + y = n$ and such that there exists $z \in \mathbb{N}$ with $(-z, z) \to (x, y)$. We define $\bar{r_n}$ as the signed Euclidean distance between $(\frac{n}{2}, \frac{n}{2})$ and the point in $\bar{\xi_n}$ with the largest first coordinate, with the same convention for the sign.
3. Ω_∞ is the event that for each n, ξ_n is not empty.

There exists a critical threshold $\overrightarrow{p}_c \in]0,1[$ such that if $p \le \overrightarrow{p}_c$ then $P(\Omega_\infty) = 0$ and if $p > \overrightarrow{p}_c$ then $P(\Omega_\infty) > 0$. In the case $p \ge \overrightarrow{p}_c$ we are interested in, there exists a constant $\alpha_p \in [0, \frac{1}{\sqrt{2}}]$ called the asymptotic speed such that

$$\lim_{n \to +\infty} \frac{r_n \mathbf{1}_{\Omega_\infty}}{n} = \alpha_p \mathbf{1}_{\Omega_\infty}, \qquad P_p\text{-a.s and in } L^1(P_p). \tag{9}$$

Define then $\beta_p = \frac{1}{2} + \frac{\alpha_p}{\sqrt{2}}$; call M_p the point with coordinates $(\beta_p, 1-\beta_p)$ and N_p the point with coordinates $(1-\beta_p, \beta_p)$. Call θ_p the angle between the first axis and (OM_p). The cone generated by the half-lines $[OM_p)$ and $[ON_p)$ is called the percolation cone issued of 0 (see Figure 1). The next proposition gives a lower bound for the passage time between 0 and a point "outside" the percolation cone. It will be used for the proofs of the flat edge result and the comparison theorem, where it will play the role of Lemma (5.5) in [17].

PROPOSITION 3.1. *Let F be a distribution on $\mathbb{R}^+$ such that* $\inf \operatorname{supp} F = 1$ *and $\overrightarrow{p}_c \le F(1) = p < 1$. For every $\varepsilon > 0$, there exist three strictly positive constants A, B and δ such that*:

$$\forall\, x \in \mathbb{N}^*, \forall\, y \in \mathbb{N} \text{ such that } \frac{y}{x} \le \frac{1-\beta_p-\varepsilon}{\beta_p+\varepsilon},$$
$$P\big(t\big(0,(x,y)\big) \le (1+\delta)(x+y)\big) \le A\exp(-Bx).$$

PROOF. Let F be a distribution satisfying the hypotheses of Proposition 3.1 and let $(t(e))_{e\in\mathbb{E}_2}$ be a family of independent and identically distributed random variables with distribution F on a probability space $(\Omega, \mathcal{F}, P)$. Choose $\varepsilon > 0$ and choose now the four following parameters $p_1, \delta_1, \varepsilon_1$ and ρ_1. We have to distinguish between two cases to choose first p_1 and δ_1.

(i) Suppose there exists $\eta > 0$ such that $F(1+\eta) = F(1)$. Choose then $p_1 = p$ and $\delta_1 > 0$ such that $\delta_1 < \min\{\eta, 1\}$.

(ii) Otherwise, there exists $\eta > 0$ such that F is strictly increasing and continuous on $[1, 1+\eta[$. By maybe decreasing η, suppose $\eta < 1$. Choose then $p_1 \in]p, F(1+\eta)[$ such that $\beta_{p_1} < \beta_p + \varepsilon$; this is possible thanks to the continuity of $p \mapsto \beta_p$ on $[\overrightarrow{p}_c, 1]$. Set $1 + \delta_1 = \inf\{x, F(x) \ge p_1\}$, thus $F(1+\delta_1) = p_1$ and $0 < \delta_1 < 1$.

In both cases, we have

$$\beta_{p_1} < \beta_p + \varepsilon, \tag{10}$$

$$0 < \delta_1 < 1 \quad \text{and} \quad F(1+\delta_1) = p_1. \tag{11}$$

We can then choose ε_1 and ρ_1 such that

$$\varepsilon_1 > 0 \quad \text{and} \quad \beta_{p_1} + \varepsilon_1 < \beta_p + \varepsilon, \tag{12}$$

$$\rho_1 < 1 \quad \text{and} \quad \rho_1 > \left(\frac{\beta_{p_1}+\varepsilon_1}{1-\beta_{p_1}-\varepsilon_1}\right)\left(\frac{1-\beta_p-\varepsilon}{\beta_p+\varepsilon}\right). \tag{13}$$

Now build a family of independent and identically distributed random variables $(\eta(e))_{e\in\vec{\mathbb{E}}_2}$ by setting $\eta(e)=1$ if $t(e)\le 1+\delta_1$, and $\eta(e)=0$ otherwise. Our first passage percolation model is thus coupled with an oriented percolation model with parameter p_1. Fix a pair of integers (x,y) such that $x>0$, $y\ge 0$ and $\frac{y}{x}\le\frac{1-\beta_p-\varepsilon}{\beta_p+\varepsilon}$. Choose an optimal path $\gamma_{x,y}$. The proof is now in two steps.

Step 1 (Renormalization). For every strictly positive integer N, we give to the N-boxes $B_N^i(k)$, defined in the previous section, a random color in the following way:

DEFINITION 3.2. The box $B_N^1(k)$, with $k\in\mathbb{Z}^2$, is said to be *black* if

$$\forall\, y=(y_1,y_2)\in\partial_{\text{in}}B_N^1(k),\ \forall\, z=(z_1,z_2)\in\partial_{\text{out}}B_N^1(k)$$
$$\text{such that } \left|\frac{z_2-y_2}{z_1-y_1}\right|\le\frac{1-\beta_{p_1}-\varepsilon_1}{\beta_{p_1}+\varepsilon_1}$$

every path γ from y to z, included in $B_N^1(k)$, verifies

$$t(\gamma)\ge\|z-y\|_1+\delta_1;$$

it is said to be *white* otherwise.

Using rotations and translations, we extend this coloring to all N-boxes. If the extremities y and z of the path γ satisfy the inequality of the definition, it is said to get out of the N-box between the percolation cones (see Figure 3).

A N-cube $C_N(k)$, with $k\in\mathbb{Z}^2$, is then said to be black if its four surrounding N-boxes $B_N^j(k)$, with $j\in\{1,2,3,4\}$, are black, and white otherwise. Let us check that this random coloring verifies the hypotheses of renormalization Lemma 2.2. The colors of the N-cubes are, as the colors of the N-boxes, clearly identically distributed, and by definition the color of $C_N(k)$ only depends on the passage

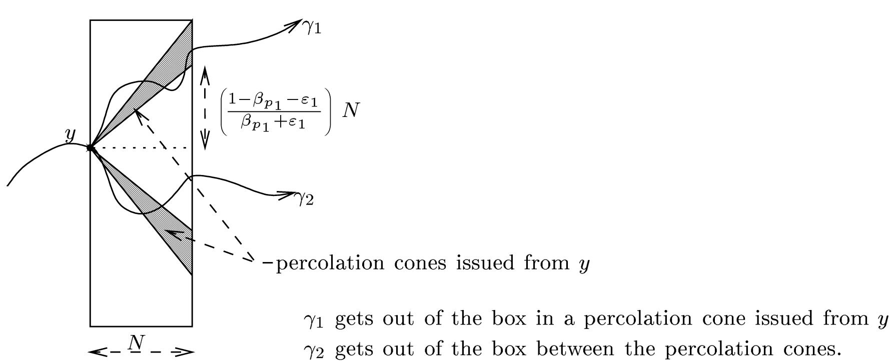

FIG. 3. *Percolation cones issued from a point y.*

times of the edges in $D_N(k)$. Let us now estimate the probability p_N that $C_N(0)$ is white and prove that

$$\lim_{N\to+\infty} p_N = 0. \tag{14}$$

It is clear that $p_N \le 4P(B_N^1(0)$ is white). Using translation invariance for our first passage percolation model, and noting that the number of vertices in $\partial_{\mathrm{in}} B_N^1(0)$ is equal to $3N$, we get

$$p_N \le 12NP\left(\begin{array}{l}\text{there exists } z \in \left[-N\left(\dfrac{1-\beta_{p_1}-\varepsilon_1}{\beta_{p_1}+\varepsilon_1}\right), \dots, N\left(\dfrac{1-\beta_{p_1}-\varepsilon_1}{\beta_{p_1}+\varepsilon_1}\right)\right] \\ \text{such that } t(0,(N,z)) < N+|z|+\delta_1\end{array}\right).$$

Now, if $t(0,(N,z)) < N+|z|+\delta_1$, as $\delta_1 < 1$, the number of edges of an optimal path from 0 to (N,z) must be equal to $N+|z|$. Moreover, the passage time of each of its edges must be less than $1+\delta_1$. This implies that there exists, in the coupled oriented percolation model with parameter p_1, an open (oriented) path from 0 to (N,z). Considering the sign of z, there are two possible orientations of the edges (to the right/to the top if $z \ge 0$ or to the right/to the bottom if $z \le 0$); thus

$$\begin{aligned} p_N &\le 24NP\left(\exists\, z \in \left[0,\dots,N\left(\frac{1-\beta_{p_1}-\varepsilon_1}{\beta_{p_1}+\varepsilon_1}\right)\right], (N,z)\in \xi_{N+z}\right) \\ &\le 24NP\left(\exists\, z \in \left[0,\dots,N\left(\frac{1-\beta_{p_1}-\varepsilon_1}{\beta_{p_1}+\varepsilon_1}\right)\right], r_{N+z} \ge \sqrt{2}\left(\frac{N-z}{2}\right)\right). \end{aligned}$$

But $N+z \le \frac{N}{\beta_{p_1}+\varepsilon_1}$ and $N-z \ge N(\frac{2(\beta_{p_1}+\varepsilon_1)-1}{\beta_{p_1}+\varepsilon_1})$ and so $\frac{N-z}{N+z} \ge 2(\beta_{p_1}+\varepsilon_1)-1 = \sqrt{2}\alpha_p + 2\varepsilon_1$. Therefore we have

$$p_N \le 24NP\left(\exists\, z \in \left[0,\dots,N\left(\frac{1-\beta_{p_1}-\varepsilon_1}{\beta_{p_1}+\varepsilon_1}\right)\right], \frac{r_{N+z}}{N+z} \ge \alpha_{p_1} + \sqrt{2}\varepsilon_1\right).$$

By large deviation results for oriented percolation (see [4]), there exist two strictly positive constants A and B such that

$$p_N \le 24N\left(\frac{1-\beta_{p_1}-\varepsilon_1}{\beta_{p_1}+\varepsilon_1}\right) A\exp(-BN).$$

This concludes the proof of (14).

We can then apply the renormalization Lemma 2.2 with the parameter ρ_1 we chose in (13). Let thus N be large enough to have estimation (8) with two strictly positive constants A and B. Choose finally $\delta > 0$ small enough to have

$$\left(\frac{\beta_{p_1}+\varepsilon_1}{1-\beta_{p_1}-\varepsilon_1}\right)\left(\frac{1-\beta_p-\varepsilon}{\beta_p+\varepsilon}\right) + 2\delta\left(\frac{\beta_{p_1}+\varepsilon_1}{1-\beta_{p_1}-\varepsilon_1} + \frac{N}{\delta_1}\right) < \rho_1. \tag{15}$$

The choice (13) we made for ρ_1 allows us to take such a δ. Let now $\sigma_{x,y} = (k_1,\dots,k_{\tau_{x,y}})$ be the sequence of the main N-cubes of the optimal path $\gamma_{x,y}$

defined in Section 2. Denote by $A_{x,y}$ the event that among the $\tau_{x,y}$ main N-cubes of $\gamma_{x,y}$, at most $\rho_1\tau_{x,y}$ cubes are black. Renormalization Lemma 2.2 ensures then that

$$P(A_{x,y}) \le A\exp(-Bx). \tag{16}$$

Step 2 (Estimation of the number of main crossings associated to a black cube, end of the proof). To each main N-cube of $\gamma_{x,y}$, with the possible exception of the last seven, a main crossing is associated. For each $i \in I = [1, \dots, \tau_{x,y} - 7]$, denote by $\gamma^i_{x,y}$ the main crossing associated to the N-cube $C_N(k_i)$, and denote by a_i, b_i its extremities in the order they appear along $\gamma_{x,y}$; let $\pi^i_{x,y}$ be the portion of $\gamma_{x,y}$ between b_i and a_{i+1}, with $a_{\tau_{x,y}-6} = (x, y)$. Set finally

$$I^b_{x,y} = \left\{ \begin{array}{c} i \in I \text{ such that } C_N(k_i) \text{ is black} \\ \text{and } \gamma^i_{x,y} \text{ gets out of its } N\text{-box between the percolation cones} \end{array} \right\},$$

$$I^c_{x,y} = \{ i \in I \text{ such that } C_N(k_i) \text{ is black and } i \notin I^b_{x,y} \}.$$

We are going to prove that if $t(0, (x, y)) \le (1 + \delta)(x + y)$ then

$$|I^b_{x,y}| + |I^c_{x,y}| < \rho_1\tau_{x,y}. \tag{17}$$

We omit the indices for the proof. Let us estimate first $|I^c|$. Note that if $i \in I^c$, then γ^i has at least $\frac{N}{\beta_{p_1}+\varepsilon_1}$ edges. The other portions γ^i with $i \notin I^c$ cross a N-box and admit at least N edges. Thus, we get

$$(1 + \delta)(x + y) \ge t\big(0, (x, y)\big) \ge |I^c| \frac{N}{\beta_{p_1} + \varepsilon_1} + (\tau - |I^c|)N.$$

This can be written in the following manner:

$$\begin{aligned} N|I^c|\left(\frac{1}{\beta_{p_1} + \varepsilon_1} - 1\right) &\le (1 + \delta)x\left(1 + \frac{y}{x}\right) - \tau N \qquad &&\text{but } x \le \tau N, \text{ so} \\ &\le \tau N\left((1 + \delta)\left(1 + \frac{y}{x}\right) - 1\right) &&\text{but } \frac{y}{x} \le 1, \text{ so} \\ &\le \tau N\left(2\delta + \frac{y}{x}\right) \\ &\le \tau N\left(2\delta + \frac{1 - \beta_p - \varepsilon}{\beta_p + \varepsilon}\right). \end{aligned}$$

We finally have

$$|I^c| \le \tau\left(\left(\frac{\beta_{p_1} + \varepsilon_1}{1 - \beta_{p_1} - \varepsilon_1}\right)\left(\frac{1 - \beta_p - \varepsilon}{\beta_p + \varepsilon}\right) + 2\delta\left(\frac{\beta_{p_1} + \varepsilon}{1 - \beta_{p_1} - \varepsilon}\right)\right).$$

Let us estimate now $|I^b|$. By the choice we made for the coloring, if $i \in I^b$, then $t(\gamma^i) \geq |\gamma^i| + \delta_1$. Thus we have the two following inequalities, in time and in first coordinate:

$$(1+\delta)(x+y) \geq \delta_1 |I^b| + \sum_{i \in I}(|\gamma^i| + |\pi^i|) \quad \text{and} \quad x + y \leq \sum_{i \in I}(|\gamma^i| + |\pi^i|).$$

By subtraction, we obtain that $2\delta\tau N \geq \delta(x+y) \geq \delta_1 |I^b|$, which means $|I^b| \leq 2\delta\tau N/\delta_1$. The choice we made for δ in (15) gives then the desired estimation (17).

To conclude, note that we have just proved that

$$\{t(0,(x,y)) \leq (1+\delta)(x+y)\} \subset A_{x,y},$$

where $A_{x,y}$ is the set introduced at the end of the first step. Equation (16) then gives two strictly positive constants A and B such that

$$P\bigl(t(0,(x,y)) \leq (1+\delta)(x+y)\bigr) \leq P(A_{x,y}) \leq A\exp(-Bx).$$

This ends the proof of Proposition 3.1. $\square$

4. Proof of the flat edge result. In this section we prove Theorem 1.3. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a family $(t(e))_{e \in \mathbb{E}_2}$ of independent and identically distributed random variables, with distribution F. Suppose $\inf \operatorname{supp} F = 1$ and $F(1) = p \geq \vec{p}_c$. Let A denote the asymptotic shape associated to F by first passage percolation.

Note that the first point of Theorem 1.3 is clear with the assumption $\inf \operatorname{supp} F = 1$. The second point is ensured by van den Berg and Kesten's comparison result in [17].

Let us prove that when $p \geq \vec{p}_c$, the segment line $[M_p, N_p]$ is included in A. In the case $p > \vec{p}_c$, this result is Lemma 6.13 in [10]. To deal with the case $p = \vec{p}_c$, note that van den Berg and Kesten's result ensures that if $p < \vec{p}_c$, then the asymptotic shape is included in the open unit ball for $\|\cdot\|_1$. The continuity of the asymptotic shape (see [1] and [2]) and the result for $p > \vec{p}_c$ ensures that $(1/2, 1/2)$ is in the asymptotic shape in the case $p = \vec{p}_c$.

It remains now to prove that the flat edge of A in the first quadrant is exactly $[M_p, N_p]$. The symmetry and convexity of A ensure that it is enough to prove

$$\forall\,(x,y) \in \mathbb{N}^2 \text{ such that } x > y \text{ and } \frac{(x,y)}{x+y} \notin [M_p, N_p], \qquad \mu\bigl((x,y)\bigr) > x + y.$$

If $(x,y) \in \mathbb{N}^2$ and $y < x$, note that $\frac{(x,y)}{x+y} \notin [M_p, N_p] \Leftrightarrow \frac{y}{x} < \frac{1-\beta_p}{\beta_p}$. Now choose $\varepsilon > 0$ such that $\frac{1-\beta_p-\varepsilon}{\beta_p+\varepsilon} - \frac{y}{x} > 0$. Proposition 3.1 says then that there exist three strictly positive constants A, B, and δ such that for each integer n,

$$P\bigl(t(0, n(x,y)) \leq n(1+\delta)(x+y)\bigr) \leq A\exp(-Bnx).$$

Let then $\eta > 0$ be such that $\eta < \delta(x+y)$, and let n be large enough to have $A\exp(-Bnx) < \frac{1}{2}$ and $P(t(0, n(x,y)) > n(\mu((x,y)) + \eta)) < \frac{1}{2}$. Then, on the intersection of the complementary sets of these two events (that has a strictly positive probability) we have

$$\mu\big((x,y)\big) + \eta \geq \frac{t(0, n(x,y))}{n} \geq (1+\delta)(x+y);$$

this means $\mu((x,y)) \geq (x+y) + (\delta(x+y) - \eta) > (x+y)$. This concludes the proof of Theorem 1.3. $\square$

5. Strict comparison between the time constant and $1/\beta_p$. This section is devoted to the proof of Theorem 1.4. We will once again need the embedded oriented percolation model we introduced in Section 3. Let us give some further notation:

1. $\tilde{\mathbb{Z}} = \{(-k,k), k \in \mathbb{Z}\}$, $\tilde{\mathbb{N}} = \{(-k,k), k \in \mathbb{N}\}$ and $S_0 = \{(1,-1), (2,-2)\}$.
2. If S is a subset of $\tilde{\mathbb{Z}}$, for every $n \in \mathbb{N}$, $\xi_n(S)$ is the set of vertices (x,y) in $\mathbb{Z}^2$ such that $x + y = n$ and such that there exists $s \in S$ with $s \to (x,y)$.
3. If S is a subset of $\tilde{\mathbb{Z}}$, for every $n \in \mathbb{N}$, $r_n(S)$ is the signed Euclidean distance between $(\frac{n}{2}, \frac{n}{2})$ and the point in $\xi_n(S)$ with the largest first coordinate, with the same sign convention as previously. If $\xi_n(S) = \varnothing$ we set $r_n(S) = -\infty$.

The proof of Theorem 1.4 is somewhat similar to the one of the strict increasing of the asymptotic speed in oriented percolation (see [4]). Thus, this proof is only valid in dimension 2. The idea is to use the monotony of $S \mapsto \xi_n(S)$ and the Markovian properties of $(\xi_n)_{n\in\mathbb{N}}$ to construct nonoriented bypasses, which use more edges than an oriented path but are, however, faster. We begin with the case $p > \vec{p}_c$.

PROOF OF THEOREM 1.4 WHEN $p > \vec{p}_c$. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a family $(t(e))_{e\in\mathbb{E}_2}$ of independent and identically distributed random variables with distribution F. Suppose that $\inf\operatorname{supp} F = 1$ and $1 > F(1) = p > \vec{p}_c$. Build, as before, the family $(\eta(e))_{e\in\vec{\mathbb{E}}_2}$ by setting $\eta(e) = 1$ if $t(e) = 1$, and $\eta(e) = 0$ otherwise, in order to couple our first passage percolation model with oriented percolation with parameter p.

Step 1 (Construction of the first stopping time and the first bypasses). Set $\xi_n^0 = \xi_n(\tilde{\mathbb{N}})$, and note that almost surely for every n, $\xi_n^0 \neq \varnothing$; set $r_n^0 = r_n(\tilde{\mathbb{N}})$ and let M_n^0 be the point in $\mathbb{R}^2$ with coordinates $(\frac{n}{2} + \frac{r_n^0}{\sqrt{2}}, \frac{n}{2} - \frac{r_n^0}{\sqrt{2}})$. Let τ_1 be the smallest n such that we have the following (C)-configuration around M_{2n-2}^0 (see Figure 4).

DEFINITION 5.1. We say that there is the (C)-configuration around the vertex (x,y) in $\mathbb{Z}^2$ if the edges between (x,y) and $(x+1,y)$, $(x+1,y)$ and $(x+1,y-1)$,

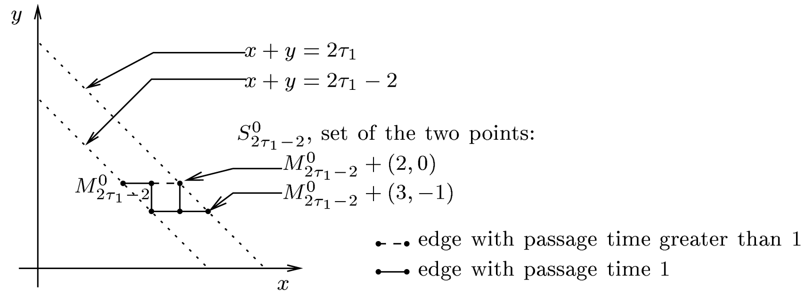

FIG. 4. (C)-*configuration around* $M^0_{2\tau_1-2}$.

$(x+1, y-1)$ and $(x+2, y-1)$, $(x+2, y-1)$ and $(x+3, y-1)$, $(x+2, y-1)$ and $(x+2, y)$ have passage time 1 and if the edge between $(x+1, y)$ and $(x+2, y)$ has a passage time strictly greater than 1.

Note that the probabilities that there is the (C)-configuration around the vertices (x, y) are all equal to $\rho = p^5(1-p) > 0$.

Look at what happens on the line $y = 2\tau_1 - x$. Points in $\xi^0_{2\tau_1}$ are linked to 0 by a path with $2\tau_1$ (oriented) edges, each of these edges having a passage time of 1. On the other hand, the two points $M^0_{2\tau_1-2} + (2, 0)$ and $M^0_{2\tau_1-2} + (3, -1)$ are also on this line, and are linked to 0 by a path with $2\tau_1 + 2$ (nonoriented) edges, each of these edges having a passage time of 1. We restart the process ξ^0_n from the time $2\tau_1$ by adding to $\xi^0_{2\tau_1}$ the set $S^0_{2\tau_1}$ composed of $M^0_{2\tau_1-2} + (2, 0)$ and $M^0_{2\tau_1-2} + (3, -1)$. The open paths from $M_{2\tau_1-2}$ to the two points in set $S^0_{2\tau_1}$ are what we call the two first bypasses. We set

$$\xi^1_n = \begin{cases} \xi^0_n, & \text{if } n \le 2\tau_1 - 1, \\ \xi_{n-2\tau_1}(\xi^0_{2\tau_1} \cup S^0_{2\tau_1}), & \text{if } n \ge 2\tau_1. \end{cases}$$

As before, let r^1_n be the signed Euclidean distance between $(\frac{n}{2}, \frac{n}{2})$ and the point in ξ^1_n with the largest first coordinate (same sign convention), and let M^1_n be the point with coordinates $(\frac{n}{2} + \frac{r^1_n}{\sqrt{2}}, \frac{n}{2} - \frac{r^1_n}{\sqrt{2}})$.

We want now to estimate the difference between Er^1_{2n} and Er^0_{2n}:

$$Er^1_{2n} - Er^0_{2n} \ge 2\sqrt{2}P(\tau_1 \le n). \tag{18}$$

The idea is to use the translation invariance of the model, and the monotony of $B \mapsto \xi_n(B)$. We admit here the following result, analogous to (13) in [4]: If A and B are two infinite sets in $\tilde{\mathbb{N}}$ such that $B \subset A$, then for every integer n we have

$$E\big(r_n(B \cup S_0) - r_n(B)\big) \ge E\big(r_n(A \cup S_0) - r_n(A)\big) \ge 2\sqrt{2}.$$

Now, by definition of ξ_n^1 and τ_1, we have

$$\begin{aligned} Er_{2n}^1 - Er_{2n}^0 &= E\big((r_{2n}^1 - r_{2n}^0)\mathbf{1}_{\tau_1 \le n}\big) \\ &= E\big((r_{2n-2\tau_1}(\xi_{2\tau_1}^0 \cup S_{2\tau_1}) - r_{2n-2\tau_1}(\xi_{2\tau_1}^0))\mathbf{1}_{\tau_1 \le n}\big). \end{aligned}$$

The Markov property and the previous inequality give then the desired inequality (18).

Step 2 (Iteration of the construction). We define in the same manner a sequence of stopping times $(\tau_k)_{k\in\mathbb{N}^*}$ by letting τ_{k+1} be the smallest $n \ge \tau_k + 1$ such that there is the (C)-configuration from M_{2n-2}^k. We call $S_{2\tau_{k+1}}^k$ the subset of $\tilde{\mathbb{Z}}$ composed of the two points $M_{2\tau_{k+1}-2}^k + (2, 0)$ and $M_{2\tau_{k+1}-2}^k + (3, -1)$). Set

$$\xi_n^{k+1} = \begin{cases} \xi_n^k, & \text{if } n \le 2\tau_{k+1} - 1, \\ \xi_{n-2\tau_{k+1}}(\xi_{2\tau_{k+1}}^0 \cup S_{2\tau_{k+1}}^k), & \text{if } n \ge 2\tau_{k+1}. \end{cases}$$

As before, r_n^{k+1} is the signed Euclidean distance between $(\frac{n}{2}, \frac{n}{2})$ and the point in ξ_n^{k+1} with the largest first coordinate (same sign convention), and M_n^{k+1} is the point with coordinates $(\frac{n}{2} + \frac{r_n^{k+1}}{\sqrt{2}}, \frac{n}{2} - \frac{r_n^{k+1}}{\sqrt{2}})$. Call N_n the number of k such that $\tau_k \le n$. Then, an iteration of the previous step gives us

$$Er_{2n}^{N_{2n}} - Er_{2n}^0 \ge 2\sqrt{2}EN_{2n}. \tag{19}$$

Indeed, note that $N_{2n} \le n$; we have then $Er_{2n}^{N_{2n}} - Er_{2n}^0 = Er_{2n}^n - Er_{2n}^0 = \sum_{k=1}^n E(r_{2n}^k - r_{2n}^{k-1})$. Using (18), we get $Er_{2n}^{N_{2n}} - Er_{2n}^0 \ge \sum_{k=1}^n 2\sqrt{2}P(\tau_k \le 2n) \ge 2\sqrt{2}EN_{2n}$. This ends the proof of (19).

Step 3 (Comparison of the modified process $r_{2n}^{N_{2n}}$ and the first passage percolation model). Note that $M_{2n}^{N_{2n}}$ is linked to a point in $\tilde{\mathbb{N}}$ by a path with exactly $2n + 2N_{2n}$ edges with passage time 1. Denote by $0 \to D_{2n}$ the event that there exists on open (oriented) path from the origin to a point in the line $D_{2n} = \{(x, y) \in \mathbb{Z}^2, x + y = 2n\}$. On this event, the point $M_{2n}^{N_{2n}}$ is linked to the origin by a path with at most $2n + 2N_{2n}$ edges with passage time 1 (because we work in dimension 2). Let $b(x_0)$ be the first time any point in the line $\{x = x_0\}$ is reached by first passage percolation beginning at the origin. We can obtain the time constant μ as the following limit (Theorem 6 in [2]):

$$\frac{b(n)}{n} \to \mu, \qquad P\text{-a.s. as } n \to +\infty. \tag{20}$$

The previous remark leads us to

$$\mathbf{1}_{\{0\to D_{2n}\}} b\left(n + \frac{r_{2n}^{N_{2n}}}{\sqrt{2}}\right) \le \mathbf{1}_{\{0\to D_{2n}\}}(2n + 2N_{2n}), \tag{21}$$

This can also be written: a.s. for n large enough,

$$\mathbf{1}_{\{0\to D_{2n}\}} \frac{n + r_{2n}^{N_{2n}}/\sqrt{2}}{2n} \le \mathbf{1}_{\{0\to D_{2n}\}} \left(1 + \frac{N_{2n}}{n}\right) \frac{n + r_{2n}^{N_{2n}}/\sqrt{2}}{b(n + r_{2n}^{N_{2n}}/\sqrt{2})}. \tag{22}$$

Remember that $p > \overrightarrow{p}_c$, and thus r_{2n}^0 goes almost surely to infinity with n. Let us now study the means of the two sides of (22).

First, as N_{2n} is the number of successes among n independent trials with probability of success ρ, we have

$$\frac{N_{2n}}{n} \le 1 \quad \text{and} \quad \frac{N_{2n}}{n} \to \rho \quad \text{a.s. and in } L^1. \tag{23}$$

Then, by definition, $b(x_0) \ge (\inf \operatorname{supp} F) \cdot x_0 = x_0$. Note then that $r_{2n}^{N_{2n}} \ge r_{2n}^0$; by using (9), composed limits and (20), we get

$$\frac{n + r_{2n}^{N_{2n}}/\sqrt{2}}{b(n + r_{2n}^{N_{2n}}/\sqrt{2})} \le 1 \quad \text{and its a.s. limit is } \frac{1}{\mu}. \tag{24}$$

Let us now look at the convergence of the mean of the right-hand side in (22). $\mathbf{1}_{\{0\to D_{2n}\}}$ tends almost surely to the indicator function of the event "the cluster containing 0 in the endowed oriented percolation model is infinite"; this event is denoted by $\{0 \to \infty\}$. Then (23), (24) and the dominated convergence theorem lead to

$$\text{the mean of the right-hand side in (22) tends to } \frac{1+\rho}{\mu} P(0 \to \infty). \tag{25}$$

We are going now to give a lower bound for the expectation of the left member in (22). On $[1, +\infty)^{\mathbb{Z}^2}$, the maps $\mathbf{1}_{\{0\to D_{2n}\}}$ and $r_{2n}^{N_{2n}}$ are cylinder nonincreasing functions. The FKG inequality ensures then

$$E\left(\mathbf{1}_{\{0\to D_{2n}\}} \frac{n + r_{2n}^{N_{2n}}/\sqrt{2}}{2n}\right) \ge P(0 \to D_{2n}) E\left(\frac{n + r_{2n}^{N_{2n}}/\sqrt{2}}{2n}\right).$$

With (19), this gives

$$E\left(\mathbf{1}_{\{0\to D_{2n}\}} \frac{n + r_{2n}^{N_{2n}}/\sqrt{2}}{2n}\right) \ge P(0 \to D_{2n}) \left(\frac{1}{2} + \frac{1}{\sqrt{2}} E\left(\frac{r_{2n}^0}{2n}\right) + E\left(\frac{N_{2n}}{n}\right)\right). \tag{26}$$

We can now come back to (22). Applying (25), (26), (9), (23) and taking limits, we get

$$\left(\frac{1}{2} + \frac{1}{\sqrt{2}}\alpha_p + \rho\right) P(0 \to \infty) \le \frac{1+\rho}{\mu} P(0 \to \infty).$$

Since we are supposing $p > \overrightarrow{p}_c$, $P(0 \to \infty) > 0$, and we get

$$\frac{1}{\mu} \ge \frac{\beta_p + \rho}{1 + \rho}. \tag{27}$$

To end the proof of Theorem 1.4 in the case $p > \overrightarrow{p}_c$, note that the asymptotic speed in oriented percolation strictly increases as a function of the parameter p, and thus if $p < 1$, $\alpha_p < \frac{1}{\sqrt{2}}$. Thus $\beta_p = \frac{1}{2} + \frac{\alpha_p}{\sqrt{2}} < 1$, and, as $\rho > 0$, $\frac{\beta_p+\rho}{1+\rho} > \beta_p$. Theorem 1.4 is then a direct consequence of (27).

PROOF OF THEOREM 1.4 WHEN $p = \overrightarrow{p}_c$. In this case, we cannot use the fact that the probability for the cluster containing the origin in the endowed oriented percolation model to be infinite is strictly positive. The idea is, however, to use the same technique: we say that the edges with passage time 1 are open, but we allow some other edges to be open too, in order to have a supercritical endowed oriented percolation model. Let $\varepsilon > 0$ and $K > 1$ be such that $F(K) > \overrightarrow{p}_c + \varepsilon$. Note that we can decrease ε, keep K constant and still have this property. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $(t(e))_{e \in \mathbb{E}_2}$ be a family of independent and identically distributed random variables, with distribution F, and suppose $\inf \operatorname{supp} F = 1$ and $F(1) = \overrightarrow{p}_c$. We build, as previously, a family $(\eta(e))_{e \in \overrightarrow{\mathbb{E}}_2}$ of independent and identically distributed random variables in the following manner:

1. Let $(\psi(e))_{e \in \overrightarrow{\mathbb{E}}_2}$ be a family of independent and identically distributed random variables, with distribution $q\delta_1 + (1-q)\delta_0$, where q will be chosen later; moreover, choose this family to be independent from $(t(e))_{e \in \mathbb{E}_2}$.
2. If $t(e) = 1$ then set $\eta(e) = 1$.
3. If $t(e) > K$ then set $\eta(e) = 0$.
4. If $1 < t(e) \le K$, then set $\eta(e) = \psi(e)$.

The random variables $(\eta(e))_{e \in \overrightarrow{\mathbb{E}}_2}$ are independent, identically distributed, take their values in $\{0, 1\}$, and

$$\begin{aligned} P\bigl(\eta(e) = 1\bigr) &= P\bigl(t(e) = 1\bigr) + P\bigl(1 < t(e) \le K \text{ and } \psi(e) = 1\bigr) \\ &= \overrightarrow{p}_c + q\bigl(F(K) - \overrightarrow{p}_c\bigr). \end{aligned}$$

The parameter q is now chosen to have $P(\eta(e) = 1) = \overrightarrow{p}_c + \varepsilon$; this means $q = \frac{\varepsilon}{F(K) - \overrightarrow{p}_c}$. We thus endowed an oriented percolation model with parameter $p = \overrightarrow{p}_c + \varepsilon$, in our first passage percolation model. We can then proceed to the same construction as in the supercritical case, and with the same notations we see that on the event $0 \to D_{2n}$, $M_{2n}^{N_{2n}}$ is linked to the origin by a path, denoted by π_{2n}, with at most $2n + 2N_{2n}$ edges with passage times taking their values between 1 and K. To define properly π_{2n}, choose an order on the edges of $\mathbb{Z}^2$ and take π_{2n} the smallest such path for the associated lexicographic order. Equation (21) becomes here

$$\mathbf{1}_{\{0 \to D_{2n}\}} b\left(n + \frac{r_{2n}^{N_{2n}}}{\sqrt{2}}\right) \le \mathbf{1}_{\{0 \to D_{2n}\}} \sum_{e \in \pi_{2n}} t(e),$$

which can be rewritten: a.s. for n large enough,

$$\mathbf{1}_{\{0\to D_{2n}\}}\frac{n+r_{2n}^{N_{2n}}/\sqrt{2}}{2n}\le \mathbf{1}_{\{0\to D_{2n}\}}\left(\frac{1}{2n}\sum_{e\in\pi_{2n}}t(e)\right)\frac{n+r_{2n}^{N_{2n}}/\sqrt{2}}{b(n+r_{2n}^{N_{2n}}/\sqrt{2})}.$$

Note, moreover, that

$$\frac{1}{2n}\sum_{e\in\pi_{2n}}t(e)=\frac{|\pi_{2n}|}{2n}\cdot\frac{1}{|\pi_{2n}|}\sum_{e\in\pi_{2n}}t(e)\le\left(1+\frac{N_{2n}}{n}\right)\cdot\frac{1}{|\pi_{2n}|}\sum_{e\in\pi_{2n}}t(e).$$

This finally gives

$$\begin{aligned}&\mathbf{1}_{\{0\to D_{2n}\}}\frac{n+r_{2n}^{N_{2n}}/\sqrt{2}}{2n}\\ &\qquad\le \mathbf{1}_{\{0\to D_{2n}\}}\left(1+\frac{N_{2n}}{n}\right)\left(\frac{\sum_{e\in\pi_{2n}}t(e)}{|\pi_{2n}|}\right)\frac{n+r_{2n}^{N_{2n}}/\sqrt{2}}{b(n+r_{2n}^{N_{2n}}/\sqrt{2})}.\end{aligned}\tag{28}$$

The study of the convergence of the means of the two sides of this equation is nearly the same as for (22). Note that the term with the sum in the right-hand side member is the only term that is different and needs a new study; by a direct adaptation of the law of large numbers for arrays of independent random variables in L^4, we can prove that

$$\frac{1}{|\pi_{2n}|}\sum_{e\in\pi_{2n}}t(e)\le K\quad\text{and tends a.s. to } E\bigl(t(e)|\eta(e)=1\bigr).\tag{29}$$

In the same manner as for (27), by applying the dominated convergence theorem for the right-hand side member, we get

$$(\beta_p+\rho)P_p(0\to+\infty)\le\frac{1+\rho}{\mu}E\bigl(t(e)|\eta(e)=1\bigr)P_p(0\to+\infty).$$

Remember that $p=\overrightarrow{p}_c+\varepsilon>\overrightarrow{p}_c$, and thus $P_p(0\to+\infty)>0$, so we have

$$\frac{1}{\mu}\ge\frac{\beta_p+\rho}{1+\rho}\cdot\frac{1}{E(t(e)|\eta(e)=1)}.$$

Remember that $p=\overrightarrow{p}_c+\varepsilon$, that $\rho=p^5(1-p)$, and that we can bring ε to 0 without changing K. The continuity of $p\mapsto\beta_p$ ensures that β_p tends to $1/2$, ρ tends to $\rho_c=\overrightarrow{p}_c^{\,5}(1-\overrightarrow{p}_c)>0$ and $E(t(e)|\eta(e)=1)$ tends to 1, so

$$\frac{1}{\mu}\ge\lim_{\varepsilon\to0}\frac{\beta_p+\rho}{1+\rho}\cdot\frac{1}{E(t(e)|\eta(e)=1)}=\frac{\frac{1}{2}+\rho_c}{1+\rho_c}>\frac{1}{2}.$$

This ends the proof of Theorem 1.4. $\square$

6. Coupling of two passage times with comparable distributions. The first step to prove Theorem 1.2 is to couple the two passage times on the same probability space. The next lemma is an extension of Theorem 2.6 in [17] for the case of random variables which are not supposed to have finite means.

LEMMA 6.1. *Let F and $\tilde{F}$ be two distributions on $\mathbb{R}^+$. Then $\tilde{F}$ is more variable than F if and only if there exists a pair of random variables $(t, \tilde{t})$ on a probability space $(\Omega_0, \mathcal{F}_0, P_0)$ with marginal distributions F and $\tilde{F}$ such that*

$$F\text{-a.s. for } y, \qquad E_0(\tilde{t}|t = y) \le y. \tag{30}$$

Note first that if $\tilde{F}$ has infinite mean, then the conditional expectation $E(\tilde{t}|t = y)$ is a priori a random variable with values in $\mathbb{R}^+ \cup \{+\infty\}$. This result ensures that it is, moreover, almost surely finite.

PROOF. As a direct consequence of Jensen's inequality for conditional expectations, condition (30) is sufficient.

To prove that condition (30) is also necessary, we are going to construct t and $\tilde{t}$ as the first and second coordinates in $(\mathbb{R}^+)^2$ under a certain probability measure λ on $(\mathbb{R}^+)^2$. Let us introduce the following notation. Let μ and ν be two probability measures on $\mathbb{R}^+$ endowed with the Borel σ-field. Denote respectively by C_b and D_b the sets of bounded continuous functions on $\mathbb{R}^+$ and $(\mathbb{R}^+)^2$. On $(\mathbb{R}^+)^2$, p_1 and p_2 denote the first and second coordinate applications. Let Π be the set of all Borel probability measures on $(\mathbb{R}^+)^2$, endowed with the topology of weak convergence. Let Λ be a nonempty closed convex subset of Π. In this context, Strassen gives the following coupling result ([16], Theorem 7).

THEOREM 6.2. *A necessary and sufficient condition for the existence of a probability measure $\lambda \in \Lambda$ such that $\mu = \lambda \circ p_1^{-1}$ and $\nu = \lambda \circ p_2^{-1}$ is that*

$$\forall\, y, z \in C_b, \quad \int y\, d\mu + \int z\, d\nu \le \sup_{\gamma \in \Lambda} \int (y \circ p_1 + z \circ p_2)\, d\gamma. \tag{31}$$

To prove Lemma 6.1, we are going to apply this result to the set Λ of Borel probability measures λ on $(\mathbb{R}^+)^2$ such that for every positive $y \in C_b$ and for every concave increasing and positive $z \in C_b$,

$$\int (y \circ p_1)(z \circ p_2)\, d\lambda \le \int (y \circ p_1)(z \circ p_1)\, d\lambda. \tag{32}$$

Step 1. Let us prove that if $\lambda \in \Lambda$, then $E_\lambda(p_2|p_1) \le p_1$, λ-p.s.

The probability measure $\lambda_1 = \lambda \circ p_1^{-1}$ on $\mathbb{R}^+$ is regular: we can build, for every Borel set B in $\mathbb{R}^+$, a sequence of continuous functions $y_n : \mathbb{R}^+ \to [0, 1]$, that goes λ_1-p.s. to the indicator function of B. Equation (32) and the dominated convergence theorem ensure then that for every concave increasing and positive $z \in C_b$ and for every Borel set B in $\mathbb{R}^+$,

$$\int (\mathbf{1}_B \circ p_1)(z \circ p_2)\, d\lambda \leq \int (\mathbf{1}_B \circ p_1)(z \circ p_1)\, d\lambda. \tag{33}$$

Define $z_n(t) = t \wedge n$ for $t \in \mathbb{R}^+$. The map z_n is then continuous, bounded, concave and increasing. Moreover, the sequence $(z_n)_{n\in\mathbb{N}}$ is nondecreasing; (33) and the monotone convergence theorem ensure then that for every Borel set B in $\mathbb{R}^+$,

$$\int (\mathbf{1}_B \circ p_1)(p_2)\, d\lambda \leq \int (\mathbf{1}_B \circ p_1)(p_1)\, d\lambda.$$

This implies that for every event A in the σ-field generated by p_1,

$$\int \mathbf{1}_A p_2\, d\lambda \leq \int \mathbf{1}_A p_1\, d\lambda,$$

and thus $E_\lambda(p_2|p_1) \leq p_1$, λ-p.s.

Step 2. Let us verify that Λ is closed, convex and nonempty in Π.

The facts that that Λ is closed and convex for the topology of the weak convergence is easily verified.

The set Λ is nonempty because it contains every probability measure that puts mass only on the first diagonal; for these measures, the inequality is in fact an equality.

Step 3. Let us check inequality (31) by choosing for μ and ν the probability measures on $\mathbb{R}^+$, respectively, associated to F and $\tilde{F}$.

Let y and z be in C_b. Let z_0 be the smallest concave increasing function on $\mathbb{R}^+$ that is greater or equal to z (take for z_0 the infimum of increasing affine functions on $\mathbb{R}^+$ that are greater or equal to z). The map z is bounded, thus so is z_0. We then have

$$\begin{aligned}
\int y\, dF + \int z\, d\tilde{F} &\leq \int y\, dF + \int z_0\, d\tilde{F} \\
&\leq \int (y + z_0)\, dF \qquad \text{as } \tilde{F} \text{ is more variable than } F \\
&\leq \sup_{s\in\mathbb{R}^+} \bigl(y(s) + z_0(s)\bigr).
\end{aligned}$$

Let then $r < \sup_{s\in\mathbb{R}^+}(y(s) + z_0(s))$; we have to prove that $r < \sup_{\gamma\in\Lambda} \int (y \circ p_1 + z \circ p_2)\, d\gamma$. Choose s such that $r < y(s) + z_0(s)$. Denote by Λ_t the set of probability measures on $\mathbb{R}^+$ whose mean is less or equal to t, and introduce

$$z_1(t) = \sup_{\alpha\in\Lambda_t} \int z\, d\alpha.$$

The map z_1 is clearly increasing and greater or equal to z. Let us prove it is also concave. Let $\alpha_s \in \Lambda_s$, $\alpha_t \in \Lambda_t$ and $\gamma \in [0,1]$. The probability measure $\alpha = \gamma\alpha_s + (1-\gamma)\alpha_t$ is in $\Lambda_{\gamma s+(1-\gamma)t}$, and

$$\gamma \int z\,d\alpha_s + (1-\gamma)\int z\,d\alpha_t = \int z\,d\alpha \le z_1\bigl(\gamma s + (1-\gamma)t\bigr).$$

By taking the supremum for $\alpha_s \in \Lambda_s$ and $\alpha_t \in \Lambda_t$, we get the concavity inequality for z_1.

The map z_1 is thus greater or equal to z_0, so $r < y(s) + z_0(s) \le y(s) + z_1(s)$. Consequently, there exists $\alpha \in \Lambda_s$ such that

$$r < y(s) + \int z\,d\alpha = \int (y \circ p_1 + z \circ p_2)\,d\gamma \qquad \text{for } \gamma = \delta_s \times \alpha.$$

It remains now to check that the probability measure γ we have just built on $(\mathbb{R}^+)^2$ is in Λ. Let $y \in C_b$ be positive and $z \in C_b$ be concave increasing and positive,

$$\int (y \circ p_1)(z \circ p_2)\,d\gamma = y(s)\int z\,d\alpha,$$
$$\int (y \circ p_1)(z \circ p_1)\,d\gamma = y(s)z(s).$$

However, $\alpha \in \Lambda_s$ and z is concave and increases, therefore

$$z(s) \ge \int z\,d\alpha.$$

The maps y and z being positive, we can conclude that (32) is verified. So γ is in Λ, and

$$\forall\, y, z \in C_b, \int y\,dF + \int z\,d\tilde{F} \le \int (y \circ p_1 + z \circ p_2)\,d\gamma,$$

thus inequality (31) is verified.

Step 4 (Conclusion). We can apply Theorem 6.2: There exists a probability measure λ on $(\mathbb{R}^+)^2$ such that under λ, p_1 and p_2, respectively, admit F and $\tilde{F}$ as distribution functions, and, by Step 1, $E_\lambda(p_2|p_1) \le p_1$ λ-p.s. This ends the proof of Lemma 6.1. □

As a consequence of this coupling, we can compare the time constants and the asymptotic shapes associated to two comparable passage times.

THEOREM 6.3. *Let F and $\tilde{F}$ be two distributions on $\mathbb{R}^+$ such that $\tilde{F}$ is more variable than F. Denote, respectively, by μ, $\tilde{\mu}$ the time constants and A, $\tilde{A}$ the asymptotic shapes associated to F, $\tilde{F}$; then $\mu \ge \tilde{\mu}$ and $A \subset \tilde{A}$.*

PROOF. Van den Berg and Kesten have already proved this result for passage times with finite means (see [17], Theorem 2.9). By truncating the distributions F and $\tilde{F}$ at T, where $T > 0$, we get that $\mu_T \geq \tilde{\mu}_T$. Cox and Durrett's continuity result enables us to take the limit whens T goes to infinity in the inequality, and thus to obtain the desired result. The inclusion for the asymptotic shapes can be proved in the same manner. □

7. Proof of the comparison theorem. In this section, we prove Theorem 1.2 in the case $\inf \operatorname{supp} F = 1$ and $F(1) \geq \overrightarrow{p}_c$. Remember that the cases $\inf \operatorname{supp} F = 0$ or $\inf \operatorname{supp} F = 1$ and $F(1) < \overrightarrow{p}_c$ are treated by van den Berg and Kesten [17]. The point in their paper is that under these hypotheses, there cannot exist too many oriented portions of length N along an optimal path from 0 to n, using only edges with passage time 1, if N is large enough. In the case $F(1) \geq \overrightarrow{p}_c$, it is Theorem 1.4 that allows us to show that such oriented portions cannot occur too often. Indeed, large deviations for supercritical oriented percolation ensure that they use a too large number of vertical edges to be fast enough. We will follow the lines of the proof in [17].

Let F be distribution function on $\mathbb{R}^+$ such that $\inf \operatorname{supp} F = 1$ and $\overrightarrow{p}_c \leq F(1) < 1$. Let $\tilde{F}$ be a distribution function on $\mathbb{R}^+$ such that $\tilde{F}$ is strictly more variable than F. Couple these two distribution functions on the same space $(\Omega_0, \mathcal{F}_0, P_0)$ as in Lemma 6.1 in a pair $(t, \tilde{t})$ of random variables with respective distribution functions F and $\tilde{F}$.

The first step is to remark that, as in Lemma 4.5 in [17], it is enough to prove Theorem 1.2 under the additional hypothesis

$$P_0(\tilde{t} > t) > 0.$$

Then, Lemma 4.8 in [17] gives the following properties of the coupling.

LEMMA 7.1. *Under the additional hypothesis $P_0(\tilde{t} > t) > 0$, there exist an integer $k > 0$, strictly positive real numbers s, q, η and a Borel set I_0 included in $[1, \infty[$ such that*

$$F([1, \inf I_0]) > 0, \quad F(I_0) > 0 \quad \textit{and} \quad F([\sup I_0, \infty[) > 0, \tag{34}$$

$$\forall\, y \in I_0, \qquad P_0(\tilde{t} > y + s \mid t = y) \geq q, \tag{35}$$

$$\begin{gathered} \forall\, y_1, \ldots, y_k, y_1', \ldots, y_{k+2}' \in I_0, \\ \sum_{i=1}^{k} y_i < \sum_{i=1}^{k+2} y_i' \quad \textit{and} \quad \sum_{i=1}^{k} (y_i + s) > \eta + \sum_{i=1}^{k+2} y_i'. \end{gathered} \tag{36}$$

In the following, we will always work with these hypotheses and we fix k, s, q, η, I_0 as in the lemma. We realize the two families of independent and identically distributed random variables $(t(e))_{e\in\mathbb{E}_2}$ and $(\tilde{t}(e))_{e\in\mathbb{E}_2}$ with respective common distribution functions F and $\tilde{F}$ on the product space $\Omega_0^{\mathbb{E}_2}$.

In order to compare the time constants associated to t and $\tilde{t}$, let us introduce two new passage times: $\hat{t}$, obtained as a random convex combination between t and $\tilde{t}$, and $\check{t}$, which is the conditional expectation of $\hat{t}$ with respect to the σ-field generated by t. Consider thus $(\xi(e))_{e\in\mathbb{E}_2}$, a family of independent and identically distributed random variables; choose them to be independent from $(t(e))_{e\in\mathbb{E}_2}$ and $(\tilde{t}(e))_{e\in\mathbb{E}_2}$, and such that $P(\xi(e)=1)=P(\xi(e)=0)=1/2$. Define then

$$\hat{t}(e)=\xi(e)\tilde{t}(e)+\bigl(1-\xi(e)\bigr)t(e), \tag{37}$$

$$\check{t}(e)=E\bigl(\hat{t}(e)|t(e)\bigr). \tag{38}$$

Denote by γ_n, $\tilde{\gamma}_n$, $\hat{\gamma}_n$ and $\check{\gamma}_n$ the optimal paths from 0 to n, respectively, associated to passage times t, $\tilde{t}$, $\hat{t}$ and $\check{t}$. Let us compare now this times for the "more variable" order.

LEMMA 7.2. *$\tilde{t}\succcurlyeq\hat{t}\succcurlyeq\check{t}\succcurlyeq t$ and $\tilde{\mu}\le\hat{\mu}\le\check{\mu}\le\mu$.*

PROOF. The inequalities for the time constants are a consequence of Theorem 6.3 and of the inequalities for the passage times. The distribution $\hat{F}$ of $\hat{t}$ is given by $\hat{F}=\frac{1}{2}F+\frac{1}{2}\tilde{F}$. As $\tilde{F}$ is more variable than F, it is also more variable than $\hat{F}$. The comparison between $\hat{t}$ and $\check{t}$ is a consequence of the definition of the order and of Jensen's property for conditional expectation. Finally, $\check{t}=E(\hat{t}|t)=E(\xi\tilde{t}+(1-\xi)t|t)=\frac{1}{2}E(\tilde{t}|t)+\frac{1}{2}t\le t$. Lemma 6.1 ensures then that $\check{t}\succcurlyeq t$. $\square$

Let us give the following definitions.

DEFINITION 7.3. A pair (π^+,π^-) of paths is said to be feasible for γ_n if:

(i) π^+ and π^- have the same extremities and are edge disjoint;
(ii) $|\pi^+|=k$ and $|\pi^-|=k+2$;
(iii) $\forall\, e\in\pi^+\cup\pi^-$, $t(e)\in I_0$;
(iv) π^+ is a portion of γ_n.

Note that the two first points in this definition are geometrical considerations. The two last points only depend on the values of $(t(e))_{e\in\mathbb{E}_2}$, and thus, for a given pair (π^+,π^-) of paths, the event $\{(\pi^+,\pi^-)$ is feasible for $\gamma_n\}$ is in the σ-field $\mathcal{G}$ generated by $(t(e))_{e\in\mathbb{E}_2}$.

DEFINITION 7.4. A feasible pair is advantageous if, moreover,

(i) $\forall\, e \in \pi^+$, $\tilde{t}(e) > t(e) + s$ and $\xi(e) = 1$,
(ii) $\forall\, e \in \pi^-$, $\xi(e) = 0$.

We can compare the probabilities of the events for a pair to be feasible and to be advantageous:

LEMMA 7.5. *If $\mathcal{G}$ is the σ-field generated by $(t(e))_{e\in\mathbb{E}_2}$, then*

$$P\big((\pi^+,\pi^-) \textit{ is advantageous for } \gamma_n|\mathcal{G}\big) \geq \frac{q^k}{2^{2k+2}}\mathbf{1}_{\{(\pi^+,\pi^-) \textit{is feasible for } \gamma_n\}}.$$

PROOF.

$$\begin{aligned}
&P\big((\pi^+,\pi^-) \text{ is advantageous for } \gamma_n|\mathcal{G}\big)\\
&\quad= E\Bigg[\mathbf{1}_{\{(\pi^+,\pi^-)\text{is feasible for } \gamma_n\}}\Bigg(\prod_{e\in\pi^+}\mathbf{1}_{\{\tilde{t}(e)\geq t(e)+s\}}\mathbf{1}_{\{\xi(e)=1\}}\Bigg)\\
&\qquad\qquad \cdots\times\Bigg(\prod_{e\in\pi^-}\mathbf{1}_{\{\xi(e)=0\}}\Bigg)\Bigg|\mathcal{G}\Bigg],
\end{aligned}$$

but with the previous remark, $\mathbf{1}_{\{(\pi^+,\pi^-)\text{is feasible for } \gamma_n\}}$ is in $\mathcal{G}$, so

$$\begin{aligned}
&= \mathbf{1}_{\{(\pi^+,\pi^-)\text{is feasible for } \gamma_n\}}\\
&\quad \cdots\times E\Bigg[\Bigg(\prod_{e\in\pi^+}\mathbf{1}_{\{\tilde{t}(e)\geq t(e)+s\}}\mathbf{1}_{\{\xi(e)=1\}}\Bigg)\Bigg(\prod_{e\in\pi^-}\mathbf{1}_{\{\xi(e)=0\}}\Bigg)\Bigg|\mathcal{G}\Bigg],
\end{aligned}$$

but every $\xi(e_0)$ is independent from $(t(e))_{e\in\mathbb{E}_2\setminus\{e_0\}}$, from every $\tilde{t}(e)$ and from $(\xi(e))_{e\in\mathbb{E}_2\setminus\{e_0\}}$, so

$$= \frac{1}{2^{2k+2}}\mathbf{1}_{\{(\pi^+,\pi^-) \text{ is feasible for } \gamma_n\}}E\Bigg(\prod_{e\in\pi^+}\mathbf{1}_{\{\tilde{t}(e)\geq t(e)+s\}}\Bigg|\mathcal{G}\Bigg),$$

but every $\tilde{t}(e_0)$ is independent from $(t(e))_{e\in\mathbb{E}_2\setminus\{e_0\}}$ and from $(\tilde{t}(e))_{e\in\mathbb{E}_2\setminus\{e_0\}}$; moreover, $(t(e))_{e\in\mathbb{E}_2}$ are independent, thus

$$= \frac{1}{2^{2k+2}}\mathbf{1}_{\{(\pi^+,\pi^-) \text{ is feasible for } \gamma_n\}}\prod_{e\in\pi^+}E\big(\mathbf{1}_{\{\tilde{t}(e)\geq t(e)+s\}}|t(e)\big),$$

Equation (35) finally gives the desired result. $\square$

The point of the proof of Theorem 1.2 is to prove the following proposition, which gives the existence of a certain number of feasible pairs along γ_n.

PROPOSITION 7.6. *For every* $\varepsilon > 0$, *there exists a strictly positive constant* D *such that for every integer* n *large enough*, *there exists a deterministic sequence* $((\pi_{i,n}^+, \pi_{i,n}^-))_{i\in\mathbb{N}}$ *of pairs of paths such that*:

(i) *For all* $i \in \mathbb{N}$, $(\pi_{i,n}^+, \pi_{i,n}^-)$ *satisfies the first two points of the definition of the feasible pair*;

(ii) *If* $i \neq j$ *then* $(\pi_{i,n}^+ \cup \pi_{i,n}^-)$ *and* $(\pi_{j,n}^+ \cup \pi_{j,n}^-)$ *are edge-disjoint*;

(iii) $\sum_{i\in\mathbb{N}} P((\pi_{i,n}^+, \pi_{i,n}^-)$ *is a feasible pair for* γ_n *and* $t(0,n) \leq n(\mu+\varepsilon)) \geq Dn$.

Let us prove first that Theorem 1.2 is a consequence of this proposition. We delay the proof of Proposition 7.6 to the next section.

PROOF OF THE COMPARISON THEOREM 1.2 FROM PROPOSITION 7.6. Let F and $\tilde{F}$ be two distributions on $\mathbb{R}^+$ satisfying the hypotheses given at the beginning of this section. Suppose that Proposition 7.6 is proved. Let γ_n be the t-optimal path between 0 and n. Consider the family $((\pi_{i,n}^+, \pi_{i,n}^-))_{i\in\mathbb{N}}$ of feasible pairs given by Proposition 7.6. Replace, if $(\pi_{i,n}^+, \pi_{i,n}^-)$ is advantageous, the portion $\pi_{i,n}^+$ of γ_n by $\pi_{i,n}^-$. This gives a new path γ_n^- between 0 and n, thanks to the fact that $\pi_{i,n}^+$ and $\pi_{i,n}^-$ have the same extremities. The distinct modifications along γ_n are compatible as two distinct feasible pairs given by Proposition 7.6 are edge-disjoint. Thus

$$
\begin{aligned}
\hat{t}(\gamma_n) - \hat{t}(\gamma_n^-) &= \sum_{i\in\mathbb{N}} \left(\sum_{e\in\pi_{i,n}^+} \hat{t}(e) - \sum_{e\in\pi_{i,n}^-} \hat{t}(e) \right) \mathbf{1}_{\{(\pi_{i,n}^+,\pi_{i,n}^-) \text{ is advantageous for } \gamma_n\}} \\
&= \sum_{i\in\mathbb{N}} \left(\sum_{e\in\pi_{i,n}^+} \tilde{t}(e) - \sum_{e\in\pi_{i,n}^-} t(e) \right) \mathbf{1}_{\{(\pi_{i,n}^+,\pi_{i,n}^-) \text{ is advantageous for } \gamma_n\}} \\
&\qquad \text{by definition of } \hat{t} \text{ and of ``advantageous,''} \\
&\geq \sum_{i\in\mathbb{N}} \left(\sum_{e\in\pi_{i,n}^+} (t(e)+s) - \sum_{e\in\pi_{i,n}^-} t(e) \right) \mathbf{1}_{\{(\pi_{i,n}^+,\pi_{i,n}^-) \text{ is advantageous for } \gamma_n\}} \\
&\qquad \text{by definition of ``advantageous,''} \\
&\geq \eta \sum_{i\in\mathbb{N}} \mathbf{1}_{\{(\pi_{i,n}^+,\pi_{i,n}^-) \text{ is advantageous for } \gamma_n\}} \qquad \text{by (36).}
\end{aligned}
\tag{39}
$$

By conditioning with respect to $\mathcal{G}$, we get

$$
\begin{aligned}
\check{t}(0,n) = \check{t}(\check{\gamma}_n) \leq \check{t}(\gamma_n^-) &= E(\hat{t}(\gamma_n^-)|\mathcal{G}) \qquad \text{by definition of } \check{\gamma}_n, \\
&\leq E\left(\hat{t}(\gamma_n) - \eta \sum_i \mathbf{1}_{\{(\pi_{n,i}^+,\pi_{n,i}^-) \text{ is advantageous for } \gamma_n\}} \middle| \mathcal{G} \right) \qquad \text{by (39),}
\end{aligned}
$$

$$\leq \sum_{e\in\gamma_n} E(\hat{t}(e)|\mathcal{G}) - \eta \sum_i E\Big(\mathbf{1}_{\{(\pi^+_{n,i},\pi^-_{n,i}) \text{ is advantageous for } \gamma_n\}}\Big|\mathcal{G}\Big).$$

But $E(\hat{t}(e)|\mathcal{G}) = E(\hat{t}(e)|t(e))$ because of the independence property of the distinct edges; $E(\hat{t}(e)|t(e)) \leq t(e)$ by the coupling given in Lemma 6.1, and by bounding the right-hand side with Lemma 7.5, we get

$$\begin{aligned} \check{t}(0,n) &\leq \sum_{e\in\gamma_n} t(e) - \frac{\eta q^k}{2^{2k+2}} \sum_i \mathbf{1}_{\{(\pi^+_{n,i},\pi^-_{n,i}) \text{ is feasible for } \gamma_n\}} \\ &= t(0,n) - \frac{\eta q^k}{2^{2k+2}} \sum_i \mathbf{1}_{\{(\pi^+_{n,i},\pi^-_{n,i}) \text{ is feasible for } \gamma_n\}}. \end{aligned} \tag{40}$$

It is not possible here to take expectations as t is not supposed to have finite mean. But we are going to see that the sum in the right-hand side is greater than $D'n$, for a well-chosen D', on an event with a large enough probability. Let $\varepsilon > 0$ be small enough to have Proposition 7.6. Set $\Omega_{n,i} = \{(\pi^+_{n,i}, \pi^-_{n,i})$ is feasible for $\gamma_n\}$. Proposition 7.6 gives the existence of $D > 0$ such that for n large enough,

$$Dn \leq \sum_i P\big(\Omega_{n,i} \text{ and } t(0,n) \leq n(\mu+\varepsilon)\big)$$

and let $D' > 0$ be such that $D' < \min\{D, (\mu+\varepsilon)/k\}$ (remember that k is the length of the first path in the definition of an advantageous pair). Set $\Omega_n = \{\sum_i \mathbf{1}_{\Omega_{n,i}} \geq D'n\}$. Then

$$\begin{aligned} Dn &\leq \sum_i P\big(\Omega_{n,i} \text{ and } t(0,n) \leq n(\mu+\varepsilon)\big) \\ &= E\Big(\mathbf{1}_{t(0,n)\leq n(\mu+\varepsilon)} \sum_i \mathbf{1}_{\Omega_{n,i}}\Big) \\ &= E\Big(\Big(\mathbf{1}_{t(0,n)\leq n(\mu+\varepsilon)} \sum_i \mathbf{1}_{\Omega_{n,i}}\Big)\mathbf{1}_{\Omega_n} + \Big(\mathbf{1}_{t(0,n)\leq n(\mu+\varepsilon)} \sum_i \mathbf{1}_{\Omega_{n,i}}\Big)\mathbf{1}_{\Gamma\Omega_n}\Big). \end{aligned} \tag{41}$$

On $\Gamma\Omega_n$, the complementary event of Ω_n, $\sum_i \mathbf{1}_{\Omega_{n,i}} \leq D'n$, and on Ω_n, using that $t(0,n) \leq n(\mu+\varepsilon)$, we can bound the number of feasible pairs with disjoint support in γ_n by $n(\mu+\varepsilon)/k$. It is to obtain this upper bound that we add, in Proposition 7.6, the condition $t(0,n) \leq n(\mu+\varepsilon)$. Indeed, if t has not finite mean, it is not possible to obtain estimates of the type $P(t(0,n) \leq n(\mu+\varepsilon)) \leq A\exp(-Bn)$, which could enable us to omit the complementary event in the passage to the limit. We thus get

$$\begin{aligned} Dn \leq \frac{n(\mu+\varepsilon)}{k} P(\Omega_n) + D'n\big(1 - P(\Omega_n)\big) \quad &\Leftrightarrow \\ P(\Omega_n) \geq \frac{D - D'}{(\mu+\varepsilon)/k - D'} = D'' > 0.& \end{aligned}$$

Now choose $\varepsilon' > 0$ and $D''' > 0$ such that

$$\varepsilon' < D'\eta q^k/2^{2k+3} \quad \text{and} \quad D''' < D''/2, \tag{42}$$

and take n large enough for Proposition 7.6 to hold and to have

$$\max\left\{P\left(\left|\frac{\check{t}(0,n)}{n} - \check{\mu}\right| \geq \varepsilon'\right), P\left(\left|\frac{t(0,n)}{n} - \mu\right| \geq \varepsilon'\right)\right\} \leq D''',$$

which is possible by (1). Then

$$P\left(\frac{\check{t}(0,n)}{n} \geq \check{\mu} - \varepsilon' \text{ and } \frac{t(0,n)}{n} \leq \mu + \varepsilon'\right) \geq 1 - 2D''' > 1 - D'' \geq P(\Omega \backslash \Omega_n).$$

So there exists a set Ω'_n, with strictly positive probability, such that on Ω'_n, we have

$$\frac{\check{t}(0,n)}{n} \geq \check{\mu} - \varepsilon', \frac{t(0,n)}{n} \leq \mu + \varepsilon' \quad \text{and} \quad \sum_i \mathbf{1}_{\Omega_{n,i}} \geq D'n.$$

Put this in (40), and on Ω'_n get

$$\begin{aligned}\check{\mu} - \varepsilon' &\leq \frac{\check{t}(0,n)}{n} \leq \frac{t(0,n)}{n} - \frac{\eta q^k}{2^{2k+2}n} \sum_i \mathbf{1}_{\{(\pi_i^+,\pi_i^-) \text{ is feasible for } \gamma_n\}} \qquad \text{by (40)} \\ &\leq \mu + \varepsilon' - \frac{\eta q^k D'}{2^{2k+2}}.\end{aligned}$$

This means that $\check{\mu} \leq \mu + 2\varepsilon' - \frac{\eta q^k D'}{2^{2k+2}}$. The choice (42) for ε' finally gives the result. $\square$

8. Existence of feasible pairs and proof of Proposition 7.6. The idea of the proof is to use once again a renormalization process. Theorem 1.4 ensures that on the event $\{t(0,n) \leq n(\mu + \varepsilon_1)\}$, a large enough proportion of the main crossings of the t-optimal path γ_n between 0 and n must get out of their box between the two percolation cones. A renormalization process based on Proposition 7.6 gives then that with a probability that tends to 1 when n goes to infinity, the path γ_n has a number of main crossings (a_i, b_i) such that

$$t(a_i, b_i) \geq (1 + \delta)\|b_i - a_i\|_1$$

that is proportional to n. The exceeding amount of time $\delta\|b_i - a_i\|_1$ enables us to construct, with positive probability, a feasible pair in the crossed box for a copy t^* of t. We will restrict ourselves to prove Proposition 7.6 under the additional hypothesis "the support of F is not bound." This case is easier, and the other case can also be proved in the same manner, as in [17]. Let F and $\tilde{F}$ be two distributions satisfying the hypotheses given at the beginning of the previous section, and suppose that the support of F is not bound. Choose parameters ε_1, ε, ρ_1 and ρ_2 as follows:

$$\mu + \varepsilon_1 < \frac{1}{\beta_p}, \tag{43}$$

$$0 < \varepsilon < \varepsilon_1, \tag{44}$$

$$1 > \rho_1 > \frac{\mu + \varepsilon}{\mu + \varepsilon_1}, \tag{45}$$

$$0 < \rho_2 < \frac{\rho_1(\mu + \varepsilon_1) - (\mu + \varepsilon)}{\rho_1(\varepsilon_1 + \varepsilon)}. \tag{46}$$

The first inequality is possible by Theorem 1.4, and the other choices are enabled by the previous ones.

Step 1 (Renormalization). For each $N \in \mathbb{N}^*$, consider the renormalization grid induced by the N-cubes $\{C_N(k) = Nk + C_N, k \in \mathbb{Z}^2\}$ with $C_N = [-1/2, N - 1/2]^2$. Choose a sequence $(\nu(N))_{N \in \mathbb{N}}$ such that

$$\lim_{N \to +\infty} P\left(\sum_{e \in B_N^1(0)} t(e) \geq \nu(N)\right) = 0. \tag{47}$$

We give to each N-box $B_N^i(k)$ a random color depending on a parameter $\delta > 0$ that will be chosen later.

DEFINITION 8.1. The box $B_N^1(k)$, with $k \in \mathbb{Z}^2$, is said to be black if and only if it satisfies the three following properties:

(C1) $\forall y \in \partial_{\mathrm{in}} B_N^1(k), \forall z \in \partial_{\mathrm{out}} B_N^1(k)$, $\forall$ path γ from y to z, included in $B_N^1(k)$, $t(\gamma) \leq N(\mu + \varepsilon_1) \Rightarrow t(\gamma) \geq (1 + \delta)\|z - y\|_1$.

(C2) $\forall y \in \partial_{\mathrm{in}} B_N^1(k), \forall z \in \partial_{\mathrm{out}} B_N^1(k)$, $\forall$ path γ from y to z, included in $B_N^1(k)$, $t(\gamma) \geq N(\mu - \varepsilon)$.

(C3) $\sum_{e \in B_N^1(k)} t(e) \leq \nu(N)$.

It is said to be white otherwise. This definition is extended to other boxes by rotations. A N-cube $C_N(k)$, with $k \in \mathbb{Z}^2$, is said to be black if the four boxes $B_N^j(k)$ with $j \in \{1, 2, 3, 4\}$ that surround it in $D_N(k)$ are black, and white otherwise.

REMARK. If an optimal path between 0 and n crosses a black N-box in a "typical time," that is, a time less than $N(\mu + \varepsilon_1)$, then under (C1) it cannot do it by using an oriented path only made of edges with passage time 1.

We will see that the number of N-boxes crossed by such a path in a typical time is, with high probability, proportional to n. Let us verify that this coloring satisfies the conditions of renormalization Lemma 2.2. It is clear that the colors of the different cubes are identically distributed, and that the color of $C_N(k)$ only depends on the passage times of the edges in $D_N(k)$. Let us now estimate the probability $p_N(\delta)$ for $C_N(0)$ to be white, and prove that

$$\exists\, \delta > 0 \quad \text{such that} \lim_{N\to+\infty} p_N(\delta) = 0. \tag{48}$$

It is clear that $p_N(\delta) \le 4P(B_N^1(0) \text{ is white})$ and that the probability for $B_N^1(0)$ to be white is bounded by the sum of the probabilities for each condition (C1), (C2) and (C3) to fail. The probability for (C3) to fail tends to 0 when N goes to infinity because of the choice (47) we made for the sequence $(\nu_N)_{N\in\mathbb{N}}$. The probability for (C2) to fail is bounded by $(3N+1)P(b(N) \le N(\mu-\varepsilon))$, where $b(N)$ is the first time when the plane $x = N$ is reached by the first passage percolation model starting at 0. A classical large deviation result (see [10]) ensures now that $P(b(N) \le N(\mu-\varepsilon))$ decreases exponentially in N, and so $P(\text{(C2) fails})$ tends to 0 when N goes to infinity.

Let us now consider $\delta > 0$:

$$\begin{aligned} P\big(\text{(C1) fails}\big) &\le \sum_{y\in\partial_{\text{in}}B_N^1(0)} \sum_{\substack{z\in\partial_{\text{out}}B_N^1(0),\\ \|z-y\|_1\le(\mu+\varepsilon_1)N}} P\big(t(y,z) \le (1+\delta)\|z-y\|_1\big) \\ &\le 2(3N+1) \sum_{z\in\{N\}\times[0,\dots,(\mu+\varepsilon_1-1)N]} P\big(t(0,z) \le (1+\delta)\|z\|_1\big). \end{aligned}$$

Denote by (x,y) the coordinates of such a point z, then $x = N$ and $y \in [0,\dots,(\mu+\varepsilon_1-1)N]$. So $0 \le y/x \le \mu+\varepsilon_1-1$. The choice (43) for ε_1 gives the existence of $\varepsilon_2 > 0$ such that

$$\mu+\varepsilon_1-1 < \frac{1-\beta_p-\varepsilon_2}{\beta_p+\varepsilon_2}.$$

Proposition 3.1 applied with ε_2 gives us the existence of three positive constants A, B and δ such that $P(\text{(C1) fails}) \le 2(3N+1)((\mu+\varepsilon_1-1)N+1)A\exp(-BN)$. By choosing such a δ, we prove (48).

Choose now $\delta > 0$ satisfying (48). We can then apply the renormalization lemma with parameter ρ_1 we took in (45). Let N be large enough to have (8) with positive A and B. By increasing N if necessary, we can suppose that

$$N \ge k+2 \quad \text{and} \quad N \ge \frac{k \sup I_0}{\delta}. \tag{49}$$

For each n, let γ_n be the t-optimal path between 0 and $(n,0)$, and let $\sigma_n = (k_1,\dots,k_{\tau_n})$ be the sequence of its main cubes, as defined in Section 2. Denote by A_n the event that among these τ_n main cubes, at most $\rho_1\tau_n$ cubes are black. With Lemma 2.2 we have

$$P(A_n) \le A\exp(-Bn). \tag{50}$$

Step 2 (Evaluation of the number of good crossings). To each of the τ_n main cubes of γ_n, with the possible exception of the last seven, a main crossing is associated. Define now the good and bad cubes:

DEFINITION 8.2. An N-cube is said to be good for the path γ if and only if it is a main cube of γ, it is black and the passage time of the main crossing associated to it is less than $N(\mu+\varepsilon_1)$; it is said to be bad otherwise.

For every i in $I=[1,\dots,\tau_n-7]$, denote by γ_n^i the main crossing associated to the main cube $C_N(k_i)$ and set

$$I_n^g=\{i\in I, C_N(k_i) \text{ is good for } \gamma_n\}, \tag{51}$$

$$I_n^b=\{i\in I, C_N(k_i) \text{ is black and bad for } \gamma_n\}. \tag{52}$$

Note that if $i\in I_n^g$, then $C_N(k_i)$ is black and so $t(\gamma_n^i)\geq N(\mu-\varepsilon)$, and that if $i\in I_n^b$, then $C_N(k_i)$ is black and bad for γ_n and so $t(\gamma_n^i)\geq N(\mu+\varepsilon_1)$. Define ΓA_n as the complementary event of A_n introduced at the previous step and

$$B_n=\{t(\gamma_n)\leq n(\mu+\varepsilon)\} \quad \text{and} \quad \beta_n=\frac{\text{number of good cubes for } \gamma_n}{\text{number of black main cubes for } \gamma_n}.$$

Then we can estimate the number of good cubes of a path.

LEMMA 8.3. *For every n large enough, on $\Gamma A_n\cap B_n$, the optimal path γ_n has at least $\rho_1\rho_2\tau_n$ good cubes.*

PROOF. We have the following inequalities in time:

$$\begin{aligned} n(\mu+\varepsilon)&\geq t(\gamma_n)\\ &\geq \sum_{i\in I_n^g} t(\gamma_n^i)+\sum_{i\in I_n^b} t(\gamma_n^i)\\ &\geq \beta_n(\rho_1\tau_n-7)(\mu-\varepsilon)N+(1-\beta_n)(\rho_1\tau_n-7)(\mu+\varepsilon_1)N. \end{aligned}$$

Remembering that $\tau_n\geq\frac{n}{N}$, we get $\beta_n\geq\frac{1}{\varepsilon_1+\varepsilon}(\mu+\varepsilon_1-\frac{1}{\rho_1-7N/n}(\mu+\varepsilon))$. This quantity tends, as n goes to $+\infty$, to $\frac{\rho_1(\mu+\varepsilon_1)-(\mu+\varepsilon)}{\rho_1(\varepsilon_1+\varepsilon)}$. Thus for n large enough, with the choice (46) we made for ρ_2, we have $\beta_n\geq\rho_2$. So on $\Gamma A_n\cap B_n$, the path γ_n has at least $\rho_1\rho_2\tau_n$ good cubes. $\square$

LEMMA 8.4. *There exists a constant $D'>0$ such that for n large enough, there exists a deterministic family $\mathcal{B}^n$ of disjoint N-boxes such that*

$$\sum_{B\in\mathcal{B}^n} P\left(\begin{array}{c} t(0,n)\leq n(\mu+\varepsilon),\ B \text{ is black,}\\ \gamma_n \text{ crosses } B \text{ in a time less than } N(\mu+\varepsilon_1)\end{array}\right)\geq D'n.$$

PROOF. We can find six families of boxes $(\mathcal{B}_i)_{1\le i\le 6}$ such that each N-box is in one and only one family and such that two boxes in the same family are disjoint. The previous lemma can be written in the following manner:

$$\left(\sum_{1\le i\le 6}\sum_{B\in\mathcal{B}_i}\mathbf{1}_{\{B\text{ is black}\}}\mathbf{1}_{\{\gamma_n\text{ crosses }B\text{ in a time less than }N(\mu+\varepsilon_1)\}}\mathbf{1}_{B_n}\right)\mathbf{1}_{\Gamma A_n}$$
$$\ge \rho_1\rho_2\tau_n\mathbf{1}_{\Gamma A_n}\mathbf{1}_{B_n}.$$

Take expectations and use the fact that $\tau_n \le \frac{n}{N}$,

$$\sum_{1\le i\le 6}\sum_{B\in\mathcal{B}_i}P\left(\begin{array}{c} t(0,n)\le n(\mu+\varepsilon),\ B\text{ is black},\\ \gamma_n\text{ crosses }B\text{ in a time less than }N(\mu+\varepsilon_1)\end{array}\right)$$
$$\ge \rho_1\rho_2\frac{n}{N}P(\Gamma A_n\cap B_n).$$

Equation (50) ensures that $P(\Gamma A_n)$ tends to 1 and equation (1) that $P(B_n)$ tends to 1 as n goes to infinity. This implies, for n large enough, the existence of a family $\mathcal{B}^n = \mathcal{B}_{i_0}$ satisfying the conditions of the lemma with $D' = \frac{\rho_1\rho_2}{12N}$. $\square$

Step 3 (Construction of feasible pairs along good crossings). Let n be large enough and let B be a box in the family given by Lemma 8.4. Suppose that γ_n crosses B in a time less than $(\mu+\varepsilon_1)N$ and that this crossing, denoted by $\gamma_{n|B}$, is a main crossing of γ_n. Denote by y and z the extremities of the restriction $\gamma_{n|B}$ of the path γ_n to the box B. We build a pair of paths (π^+,π^-) in the following manner (see Figure 5). Suppose that B is a box of the type $B_N^1(k)$; this means there exists $h\in\mathbb{Z}$ such that $z = y+(N,h)$. Let π be the direct path between y and z; this means the path composed of the three segment lines: $[y, y+(N-1,0)]$, $[y+(N-1,0), z-(1,0)]$ and $[z-(1,0), z]$. Call π^+ the portion of π between $y+(1,0)$ and $y+(k+1,0)$. The choice (49) for N ensures that π^+ is well defined as a portion of π. Build then π^- as the bypass of π^+ composed with

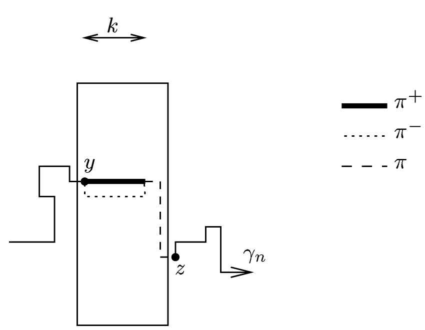

FIG. 5. *Construction of a feasible pair.*

the three portions $[y+(1,0), y+(1,-1)]$, $[y+(1,-1), y+(k+1,-1)]$ and $[y+(k+1,-1), y+(k+1,0)]$; if y is at the bottom of the box, construct the bypass π^-, by symmetry, above π^+ to be sure that it is inside the box. This construction can be extended by rotation to all boxes. Consider a new family $(t^*(e))_{e\in\mathbb{E}_2}$ of passage time such that for every edge outside B, t^* is equal to t and such that for every e in B, $t^*(e)$ is an independent copy of $t(e)$ and is independent of all the other passage times. For every n, denote by γ_n^* the t^*-optimal path between 0 and n. Set

$$\Gamma_1 = \left\{ \begin{array}{c} t(0,n) \le n(\mu+\varepsilon),\ B \text{ is black,} \\ \gamma_n \text{ crosses } B \text{ in a time less than } (\mu+\varepsilon_1)N \end{array} \right\},$$

$$\Gamma_2 = \{\forall\, e \in \pi^+ \cup \pi^-, t^*(e) \in I_0\},$$

$$\Gamma_3 = \{\forall\, e \in \pi \backslash \pi^+, t^*(e) = 1\},$$

$$\Gamma_4 = \{\forall\, e \in B\backslash(\pi \cup \pi^-), t^*(e) \ge \nu(N)\},$$

$$\Gamma = \bigcap_{1\le i\le 4} \Gamma_i.$$

By construction of t^* and the choice (34) for I_0, there exists a constant $\kappa > 0$ depending only on N, and in particular independent of n, of B in $\mathcal{B}^n$ and of the positions of the extremities y and z of $\gamma_{n|B}$, such that

$$P(\Gamma_2 \cap \Gamma_3 \cap \Gamma_4 | t(e), e \in \mathbb{E}_2) \ge \kappa. \tag{53}$$

We can now prove the following.

LEMMA 8.5. *Let B be a box in $\mathcal{B}^n$, then*

$$P\begin{pmatrix} B \textit{ contains a feasible} \\ \textit{pair for } \gamma_n \textit{ and} \\ t(0,n) \le n(\mu+\varepsilon) \end{pmatrix} \ge \kappa P \begin{pmatrix} B \textit{ is black}, \gamma_n \textit{ crosses } B \\ \textit{in a time less than } (\mu+\varepsilon_1)N, \\ \textit{and } t(0,n) \le n(\mu+\varepsilon) \end{pmatrix}.$$

PROOF. We are going to show in fact that on Γ, the pair (π^+, π^-) is feasible for γ_n^* and $t^*(0,n) \le n(\mu+\varepsilon)$. Let us first prove that on Γ, the t^*-passage time between 0 and n is strictly smaller than the t-passage time. Denote by γ_n^+ the path obtained by replacing in γ_n the crossing $\gamma_{n|B}$ of the box B by the path π. By definition of the coloring and of a good cube for γ_n, $t(\gamma_{n|B}) \ge (1+\delta)\|z-y\|_1$. By construction of π and definition of Γ_2 and Γ_3, $t^*(\pi) \le \|z-y\|_1 + k \sup I_0$. Thus

$$\begin{aligned} t(\gamma_n) - t^*(\gamma_n^+) &= t(\gamma_{n|B}) - t^*(\pi) \\ &\ge (1+\delta)\|z-y\|_1 - \|z-y\|_1 - k\sup I_0 \\ &\ge \delta N - k \sup I_0 > 0 \qquad \text{by our choice (49) for } N. \end{aligned}$$

But $t^*(0,n) \le t^*(\gamma_n^+)$ and so $t^*(0,n) < t(\gamma_n) = t(0,n)$. Note that this implies in particular that on Γ, $t^*(0,n) \le n(\mu+\varepsilon)$. Let us show now that on Γ, the path π is a portion of γ_n^*; this will imply that the pair (π^+,π^-) is feasible for γ_n^*. As t and t^* are equal outside B and as $t^*(0,n) < t(0,n)$, γ_n^* must use at least one edge in B. But on $\Gamma_1 \cap \Gamma_4$, for every path γ using an edge f in $B \setminus (\pi \cup \pi^-)$, $t^*(\gamma) > t(\pi)$, because $t(f)$ is already bigger, by the coloring and the choice (47) we made for $(\nu_N)_{N\in\mathbb{N}}$, than $t(\pi)$. The optimal path γ_n^* can then only use edges in $\pi \cup \pi^-$, and thus enter B by y and leave it by z, or conversely. By the choice (36) we made for I_0, we have moreover that

$$\sum_{e\in\pi^+} t^*(e) < \sum_{e\in\pi^-} t^*(e)$$

and thus π^+ is a portion of γ_n^*. So on Γ, (π^+,π^-) is feasible for γ_n^*. But now,

$$\begin{aligned} P\begin{pmatrix} B \text{ contains a feasible pair} \\ \text{for } \gamma_n, \text{ and } t(0,n) \le n(\mu+\varepsilon)\end{pmatrix} &= P\begin{pmatrix} B \text{ contains a feasible pair} \\ \text{for } \gamma_n^*, \text{ and } t^*(0,n) \le n(\mu+\varepsilon)\end{pmatrix} \\ &\ge P(\Gamma) = \kappa P(\Gamma_1) \quad \text{by the previous lemma.}\end{aligned}$$

This ends the proof of Lemma 8.5. $\square$

Last step (End of proof). Let n be large enough. Let us arrange the boxes of $\mathcal{B}^n$ in a sequence $(B_{n,i})_i$. There exists only a finite number m, depending on the size N of the box, of possible configurations for a feasible pair (π^+,π^-) in a box B. So for every box $B_{n,i}$, we can choose a pair $(\pi^+_{n,i},\pi^-_{n,i})$ such that

$$P\begin{pmatrix} (\pi^+_{n,i},\pi^-_{n,i}) \text{ is feasible for } \gamma_n \\ \text{and } t(0,n) \le n(\mu+\varepsilon)\end{pmatrix} \ge \frac{1}{m} P\begin{pmatrix} B_{n,i} \text{ contains a feasible pair} \\ \text{for } \gamma_n \text{ and } t(0,n) \le n(\mu+\varepsilon)\end{pmatrix}.$$

Let us then verify that the family $((\pi^+_{n,i},\pi^-_{n,i}))_{i\in\mathbb{N}}$ satisfy the conditions of Proposition 7.6. The two first points are trivial by the construction of $\mathcal{B}^n$ and by the choice of $((\pi^+_{n,i},\pi^-_{n,i}))_{i\in\mathbb{N}}$. Let us check the third point: using Lemmas 8.5 and 8.4, we get

$$\begin{aligned} \sum_i P\begin{pmatrix} (\pi^+_{n,i},\pi^-_{n,i}) \text{ is feasible} \\ \text{for } \gamma_n \text{ and} \\ t(0,n) \le n(\mu+\varepsilon)\end{pmatrix} &\ge \frac{1}{m}\sum_i P\begin{pmatrix} B_{n,i} \text{ contains a feasible} \\ \text{pair for } \gamma_n \\ \text{and } t(0,n) \le n(\mu+\varepsilon)\end{pmatrix} \\ &\ge \frac{\kappa}{m}\sum_i P\begin{pmatrix} B_{n,i} \text{ is black, } \gamma_n \text{ crosses } B_{n,i} \\ \text{in a time less than } (\mu+\varepsilon_1)N, \\ \text{and } t(0,n) \le n(\mu+\varepsilon)\end{pmatrix} \\ &\ge \frac{\kappa D' n}{m}.\end{aligned}$$

This ends the proof of Proposition 7.6. $\square$

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