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STOCHASTIC CALCULUS FOR BROWNIAN MOTION ON A BROWNIAN FRACTURE

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In this paper, we give a pathwise development of stochastic integrals with respect to iterated Brownian motion. We also provide a detailed analysis of the variations of iterated Brownian motion. These variations are linked to Brownian motion in random scenery and iterated Brownian motion itself.

1. Introduction and preliminaries. Heat flow on a fractal $\mathfrak{F}$ has been the subject of recent and vigorous investigations. See, for example, the survey article [2]. As in the classical studies of heat flow on smooth manifolds (see [16]), a probabilistic interpretation of such problems comes from the description and analysis of the "canonical stochastic process" on $\mathfrak{F}$, which is usually called Brownian motion on $\mathfrak{F}$. One of the many areas of applications is heat flow along fractures. In this vein, see [14, 17, 20, 21, 22, 26, 39]. These articles start with an idealized fracture (usually a simple geometric construct, such as a comb) and proceed to the construction and analysis of Brownian motion on this fracture. Let us attack the problem from a different point of view; namely, rather than considering a fixed idealized fracture, let us begin with a random idealization of a fracture. Let $\mathfrak{R}$ be a vertically homogeneous, two-dimensional, rectangular medium with sides parallel to the axes. Then a left-to-right random fracture on $\mathfrak{R}$ would resemble the graph of a one-dimensional Brownian motion. (To make this more physically sound, one needs some mild conditions on the local growth of the fracture together with the invariance principle of Donsker.) Approximating the Brownian graph by the paths of random walks and applying Donsker's invariance principle once more, it is reasonable to suppose that Brownian motion on a Brownian fracture is described by the stochastic process $\{(Y_t, Z_t)\colon t \geq 0\}$, where $Y = \{Y_t\colon t \geq 0\}$ is a one-dimensional Brownian motion and $Z = \{Z_t\colon t \geq 0\}$ is an associated iterated Brownian motion. To construct Z, let $X^+ = \{X_t^+ : t \geq 0\}$ and $X^- = \{X_t^- : t \geq 0\}$ be independent one-dimensional Brownian motions which are independent of Y, as well. Here, and throughout this paper, we assume that all Brownian motions start at the origin. Let $X = \{X_u\colon u \in \mathbb{R}\}$ be

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the two-sided Brownian motion given by

$$X_u = \begin{cases} X^+(u), & \text{if } u \geq 0, \\ X^-(-u), & \text{if } u < 0. \end{cases}$$

Then the iterated Brownian motion Z is defined as

$$Z_t = X(Y_t)$$

for $t \geq 0$. As is customary, given a function f, we will freely interchange between the notation $f(t)$ and f_t for typographical ease.

This model for Brownian motion on a Brownian fracture appears earlier (in a slightly different form) in [13]. The model is supported by the results of [11], which demonstrate that iterated Brownian motion arises naturally as the (weak) limit of reflected Brownian motion in an infinitesimal fattening of the graph of a Brownian motion.

Recently, iterated Brownian motion and its cognates have been the subject of various investigations. See, for example, [1, 4, 5, 8–11, 12, 13, 15, 23–25, 29, 30, 38, 40]. In addition to its relation to heat flow on fractures, iterated Brownian motion has a loose connection with the parabolic operator $\frac{1}{8}\Delta^2 - \partial/\partial t$. See [19] for more on this relationship.

In this paper, we are concerned with developing a stochastic calculus for iterated Brownian motion, and the key step in our analysis is a construction of stochastic integral processes of form $\int_0^t f(Z_s)\, dZ_s$, where f is from a class of smooth functions. Since Z is not a semimartingale, such a construction is necessarily nontrivial. (A folk theorem of Dellacherie essentially states that for $\int H\, dM$ to exist as an "integral" for a large class of H's, M need necessarily be a semimartingale.) Our construction of $\int_0^t f(Z_s)\, dZ_s$ is reminiscent of the integrals of Stratonovich and Lebesgue. For each nonnegative integer n, we divide space into an equipartition of mesh size $2^{-n/2}$. According to the times at which the Brownian motion Y is in this partition, we obtain an induced random partition $\{T_{k,n}\colon 1 \leq k \leq 2^n t\}$ of the time interval $[0, t]$. One of the useful features of this random partition is that it uniformly approximates the common dyadic partition $\{k2^{-n}\colon 1 \leq k \leq 2^n t\}$. Upon developing the partition, we show that

$$\int_0^t f(Z_s)\, dZ_s = \lim_{n \to \infty} \sum_{1 \leq k \leq 2^n t} f\left(\frac{Z(T_{k+1,n}) + Z(T_{k,n})}{2} \right) (Z(T_{k+1,n}) - Z(T_{k,n}))$$

exists with probability 1, and can be explicitly identified in terms of familiar processes. This material is developed in Section 2. The use of the midpoint rule in defining the stochastic integral is significant: the midpoint rule is a symmetric rule, and symmetry will play an important role in our analysis. As we will show later in this section, the analogous partial sum process based on the right-hand rule does not converge.

We have argued, based on Donsker's invariance principle, that iterated Brownian motion is a reasonable candidate for the canonical process on a Brownian fracture. This viewpoint is strengthened by our results in the

remainder of this paper, which are concerned with the variations of iterated Brownian motion. For a smooth function f and a positive integer j, let

$$V_n^{(j)}(f,t) = \sum_{1\le k\le 2^n t} f\left(\frac{Z(T_{k+1,n}) + Z(T_{k,n})}{2}\right)(Z(T_{k+1,n}) - Z(T_{k,n}))^j.$$

When $f \equiv 1$, we will write $V_n^{(j)}(t)$ in place of $V_n^{(j)}(1,t)$. We call $V_n^{(j)}(t)$ the jth *variation* of Z. A more traditional definition of the variation of iterated Brownian motion has been studied in [9]. Using the random partitions, we extend the results of [9]: we prove that with probability 1,

$$\lim_{n\to\infty} 2^{-n/2} V_n^{(2)}(f,t) = \int_0^t f(Z_s)\,ds,$$

$$\lim_{n\to\infty} V_n^{(3)}(f,t) = 0$$

and

$$\lim_{n\to\infty} V_n^{(4)}(f,t) = 3\int_0^t f(Z_s)\,ds.$$

We also study refinements of these strong limit theorems. In essence, we show that appropriately normalized versions of $V_n^{(2)}(t) - 2^{n/2}t$ and $V_n^{(4)}(t) - 3t$ converge in distribution to Kesten and Spitzer's Brownian motion in random scenery [27], while an appropriately normalized version of $V_n^{(3)}(t)$ converges in distribution to iterated Brownian motion itself. Indeed, it can be shown that, after suitable normalization, each even-ordered variation converges in distribution to Brownian motion in random scenery while each odd-ordered variation converges in distribution to iterated Brownian motion. Our results on the variation of iterated Brownian motion appear in Sections 3 and 4 of this paper.

Our analysis of the variation of iterated Brownian motion indicates the failure of the right-hand rule in defining the stochastic integral. If f is sufficiently smooth and has enough bounded derivatives, then, by Taylor's theorem,

$$\sum_{1\le k\le 2^n t} f(Z(T_{k,n}))(Z(T_{k+1,n}) - Z(T_{k,n}))$$
$$= V_n^{(1)}(f,t) - \tfrac{1}{2}V_n^{(2)}(f',t) + \tfrac{1}{8}V_n^{(3)}(f'',t) - \tfrac{1}{48}V_n^{(4)}(f''',t) + o(1),$$

where $o(1) \to 0$ almost surely and in $L^2(\mathbb{P})$ as $n \to \infty$. It follows that

$$\lim_{n\to\infty}\left(\sum_{1\le k\le 2^n t} f(Z(T_{k,n}))(Z(T_{k+1,n}) - Z(T_{k,n})) + \tfrac{1}{2}V_n^{(2)}(f',t)\right)$$
$$= \int_0^t f(Z_s)\,dZ_s - \tfrac{1}{16}\int_0^t f'''(Z_s)\,ds.$$

Consequently, the right-hand rule process will converge if and only if the associated quadratic variation process converges; however, the quadratic variation process diverges whenever f' is a positive function, to name an obvious, but by no means exceptional, case.

Our construction of $\int_0^t f(Z_s)\,dZ_s$ is performed pathwise and relies heavily on excursion theory for Brownian motion. It is interesting that a simplified version of our methods yields an excursion-theoretic construction of ordinary Itô integral processes of the type $\int_0^t f(Y_s)\,dY_s$ for Brownian motion Y (see Section 5 for these results). While stochastic calculus treatments of excursion theory have been carried out in the literature (see, for example, [37]), ours appears to be the first attempt in the reverse direction.

A general pathwise approach to integration, based on a construction of Lévy-type stochastic areas, is carried out in [35]; it would be interesting to see the connection between these results and ours.

We conclude this section by defining some notation which will be used throughout this paper. For any array $\{a_{j,n}\colon j \in \mathbb{Z},\ n \geq 0\}$, we define $\Delta a_{j,n} = a_{j+1,n} - a_{j,n}$. Whenever a process U has local times, we will denote them by $L_t^x(U)$. This means that for any Borel function f and all $t \geq 0$,

$$\int_0^t f(U_s)\,ds = \int_{-\infty}^{\infty} f(x) L_t^x(U)\,dx$$

almost surely. We will write $\mathbb{I}\{A\}$ for the indicator of a Borel set A. In other words, viewed as a random variable,

$$\mathbb{I}\{A\}(\omega) = \begin{cases} 1, & \text{if } \omega \in A, \\ 0, & \text{if } \omega \notin A. \end{cases}$$

Let $C^2(\mathbb{R})$ denote the set of all twice continuously differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$. For $f \in C^2(\mathbb{R})$, let

$$S(f) = \sup_x \left(|f(x)| + |f'(x)| + |f''(x)|\right) \tag{1.1}$$

and let $C_b^2(\mathbb{R})$ denote the collection of all $f \in C^2(\mathbb{R})$ for which $S(f) < \infty$.

For each integer j and each nonnegative integer n, let $r_{j,n} = j2^{-n/2}$. Recalling that X is a two-sided Brownian motion, let

$$\begin{aligned} X_{j,n} &= X(r_{j,n}), \\ M_{j,n} &= \frac{X(r_{j+1,n}) + X(r_{j,n})}{2}. \end{aligned} \tag{1.2}$$

Finally, for any $p > -1$, let μ_p denote the absolute pth moment of a standard normal distribution, that is,

$$\mu_p = (2\pi)^{-1/2} \int_{-\infty}^{\infty} |x|^p \exp(-x^2/2)\,dx = \pi^{-1/2} 2^{p/2} \Gamma\left(\frac{p+1}{2}\right). \tag{1.3}$$

2. The stochastic integral. In this section we will define a stochastic integral with respect to iterated Brownian motion. For each $t > 0$, we will construct a sequence of partitions $\{T_{k,n}\colon 0 \leq k \leq [2^n t]\}$ of the integral $[0,t]$ along with the partial sum process

$$V_n^{(1)}(f,t) = \sum_{k=0}^{[2^n t]-1} f\left(\frac{Z(T_{k+1,n}) + Z(T_{k,n})}{2}\right)\left(Z(T_{k+1,n}) - Z(T_{k,n})\right)$$

converges almost surely and in $L^2(\mathbb{P})$ as $n \to \infty$, for each $f \in C_b^2(\mathbb{R})$. The limiting random variable is properly called the stochastic integral of $f \circ Z$ with respect to Z over the interval $[0, t]$ and will be denoted by $\int_0^t f(Z_s)\, dZ_s$. Our point of departure from the classical development of the stochastic integral is that the partitioning members $T_{k,n}$ are random variables, which we define presently. For each integer $n \geq 0$ and each integer j, recall that $r_{j,n} = j2^{-n/2}$ and let

$$\mathscr{D}_n = \{r_{j,n} : j \in \mathbb{Z}\}.$$

To define the elements of the nth partition, let $T_{0,n} = 0$ and, for each integer $k \geq 1$, let

$$T_{k,n} = \inf\{s > T_{k-1,n} : Y_s \in \mathscr{D}_n \setminus \{Y(T_{k-1,n})\}\}.$$

For future reference we observe that the process $\{Y(T_{n,k}) : k \geq 0\}$ is a simple symmetric random walk on $\mathscr{D}_n$.

Here is the main result of this section.

THEOREM 2.1. *Let $t > 0$ and $f \in C_b^2(\mathbb{R})$. Then*

$$V_n^{(1)}(f, t) \to \int_0^{Y_t} f(X_s)\, dX_s + \tfrac{1}{2}\operatorname{sgn}(Y_t) \int_0^{Y_t} f'(X_s)\, ds$$

almost surely and in $L^2(\mathbb{P})$ as $n \to \infty$.

We have used the following natural definition for two-sided stochastic integrals: for $u \in \mathbb{R}$, let

$$\int_0^u f(X_s)\, dX_s = \begin{cases} \displaystyle\int_0^u f(X_s^+)\, dX_s^+, & \text{if } u \geq 0, \\ \displaystyle\int_0^{-u} f(X_s^-)\, dX_s^-, & \text{if } u < 0, \end{cases}$$

whenever the Itô integrals on the right exist.

REMARK 2.2.1. For $f \in C_b^2(\mathbb{R})$ and $u \in \mathbb{R}$, let

$$\langle f, X \rangle(u) = \int_0^u f(X_s)\, dX_s + \tfrac{1}{2}\operatorname{sgn}(u) \int_0^u f'(X_s)\, ds.$$

Then $\{\langle f, X \rangle(u) : u \in \mathbb{R}\}$ is the two-sided Stratonovich integral process of the integrand $f \circ X$. Thus Theorem 2.1 asserts that

$$\int_0^t f(Z_s)\, dZ_s = \langle f, X \rangle(Y_t).$$

In other words, stochastic integration with respect to Z is invariant under the natural composition map $(X, Y) \mapsto Z$.

Before proceeding to the proof of Theorem 2.1, a few preliminary remarks and observations are in order. First, we will demonstrate that the random partition $\{T_{k,n} : k \in \mathbb{Z}\}$ approximates the dyadic partition $\{k/2^n : k \in \mathbb{Z}\}$ as n

tends to infinity.

LEMMA 2.2. *Let $t > 0$. Then*

$$\sup_{0 \le s \le t} |T_{[2^n s], n} - s| \to 0$$

almost surely and in $L^2(\mathbb{P})$ as $n \to \infty$.

PROOF. By the strong Markov property, $\{\Delta T_{j,n}: j \ge 0\}$ is an i.i.d. sequence of random variables. Moreover, by Brownian scaling, $\Delta T_{1,n}$ has the same distribution as $2^{-n} T_{1,0}$. By Itô's formula, $\exp(\lambda Y_t - \lambda^2 t/2)$ is a mean-one martingale. Thus, by Doob's optional sampling theorem,

$$\mathbb{E}\big(\exp(-\lambda^2 T_{1,0}/2)\big) = (\cosh(\lambda))^{-1}.$$

It follows that $\mathbb{E}(T_{1,0}) = 1$, $\mathbb{E}(T_{1,0}^2) = 5/3$; consequently, $\operatorname{var}(T_{1,0}) = 2/3$. Thus, by Brownian scaling,

$$\mathbb{E}(\Delta T_{1,n}) = 2^{-n} \quad \text{and} \quad \operatorname{var}(\Delta T_{1,n}) = \tfrac{2}{3} 2^{-2n}. \tag{2.1}$$

Given $0 \le s \le t$,

$$\begin{aligned} \sup_{0 \le s \le t} |T_{[2^n s], n} - s| &\le \sup_{0 \le s \le t} \left| T_{[2^n s], n} - [2^n s] 2^{-n} \right| + \sup_{0 \le s \le t} |[2^n s] 2^{-n} - s| \\ &\le \max_{1 \le k \le [2^n t]} \left| T_{k,n} - \mathbb{E}(T_{k,n}) \right| + 2^{-n}. \end{aligned}$$

Since

$$T_{k,n} - \mathbb{E}(T_{k,n}) = \sum_{j=0}^{k-1} \big(\Delta T_{j,n} - \mathbb{E}(\Delta T_{j,n}) \big),$$

we have, by Doob's maximal inequality and (2.1),

$$\begin{aligned} \mathbb{E}\left[\max_{1 \le k \le [2^n t]} |T_{k,n} - \mathbb{E}(T_{k,n})|^2 \right] &\le 4 \sum_{j=0}^{[2^n t]-1} \operatorname{var}\big(\Delta T_{j,n}\big) \\ &= O(2^{-n}). \end{aligned}$$

In summary,

$$\left\| \sup_{0 \le s \le t} |T_{[2^n s], n} - s| \right\|_2 = O(2^{-n/2}), \tag{2.2}$$

which demonstrates the $L^2(\mathbb{P})$ convergence in question. The almost-sure convergence follows from applications of Markov's inequality and the Borel–Cantelli lemma. □

For each $n \ge 0$, let

$$\begin{aligned} \tau_n &= \tau(n,t) = T_{[2^n t], n}, \\ j^* &= j^*(n,t) = 2^{n/2} Y(\tau_n). \end{aligned}$$

Notice that $r_{j^*,n} = Y(\tau_n)$, in keeping with the notation that we have already developed.

LEMMA 2.3. *Let $t > 0$. Then, as $n \to \infty$:*

(a) $\|Y(\tau_n) - Y(t)\|_2 = O(2^{-n/8})$;
(b) $\|(\operatorname{sgn}(Y(t)) - \operatorname{sgn}(Y(\tau_n)))(|Y(t)| + |Y(\tau_n)|)\|_2 = O(2^{-n/64})$.

PROOF. For each integer $n \geq 1$, let $\varepsilon_n = \|\tau_n - t\|_2^{1/2}$. From (2.2),

$$\varepsilon_n = O(2^{-n/4}). \tag{2.3}$$

Observe that

$$\begin{aligned}\mathbb{E}\big(|Y(\tau_n) - Y(t)|^2\big) &= \mathbb{E}\big(|Y(\tau_n) - Y(t)|^2 \mathbb{1}(|\tau_n - t| \leq \varepsilon_n)\big) \\ &\quad + \mathbb{E}\big(|Y(\tau_n) - Y(t)|^2 \mathbb{1}(|\tau_n - t| > \varepsilon_n)\big) \\ &= A_n + B_n,\end{aligned}$$

with obvious notation. By (2.3) and the elementary properties of Brownian motion,

$$\begin{aligned}A_n &\leq 2\mathbb{E}\left[\sup_{t \leq s \leq t + \varepsilon_n} |Y(s) - Y(t)|^2\right] \\ &= 2\varepsilon_n \mathbb{E}\left[\sup_{0 \leq s \leq 1} |Y(s)|^2\right] \\ &= O(2^{-n/4}).\end{aligned}$$

Concerning B_n, observe that $\{2^{n/2} Y(T_{k,n}): k \geq 0\}$ is a simple symmetric random walk on $\mathbb{Z}$. As such,

$$\mathbb{E}\Big[\big(2^{n/2} Y(\tau_n)\big)^4\Big] = 3[2^n t]^2 - 2[2^n t].$$

It follows that $\{\|Y(\tau_n)\|_4: n \geq 0\}$ is a bounded sequence. By the Hölder, Minkowski and Markov inequalities,

$$\begin{aligned}B_n &\leq \|Y(\tau_n) - Y(t)\|_4^2 \sqrt{\mathbb{P}(|\tau_n - t| > \varepsilon_n)} \\ &\leq \big(\|Y(\tau_n)\|_4 + \|Y(t)\|_4\big)^2 \frac{\|\tau_n - t\|_2}{\varepsilon_n} \\ &= O(\varepsilon_n) \\ &= O(2^{-n/4}),\end{aligned}$$

which proves (a).

For each integer $n \geq 1$, let $\delta_n = \|Y(t) - Y(\tau_n)\|_2^{1/2}$, noting that $\delta_n = O(2^{-n/16})$. By elementary considerations,

$$|\operatorname{sgn}(Y(t)) - \operatorname{sgn}(Y(\tau_n))| \leq 2\mathbb{1}(|Y(t)| < \delta_n) + 2\mathbb{1}(|Y(t) - Y(\tau_n)| \geq \delta_n).$$

Consequently,

$$\begin{aligned}\|\operatorname{sgn}(Y(t)) - \operatorname{sgn}(Y(\tau_n))\|_4 &\leq 2\|\mathbb{1}(|Y(t)| < \delta_n)\|_4 \\ &\quad + 2\|\mathbb{1}(|Y(t) - Y(\tau_n)| \geq 2\delta_n)\|_4.\end{aligned}$$

We will obtain bounds for the terms on the right.

Since $t > 0$, $|Y(t)|$ has a bounded density function. In particular,

$$\mathbb{P}\{|Y(t)| < \delta_n\} \le \sqrt{\frac{2}{\pi t}}\,\delta_n,$$

which shows that $\|\mathbb{I}(|Y(t)| < \delta_n)\|_4 = O(\delta_n^{1/4}) = O(2^{-n/64})$.

By Markov's inequality,

$$\mathbb{P}\{|Y(t) - Y(\tau_n)| \ge \delta_n\} \le \frac{\|Y(t) - Y(\tau_n)\|_2}{\delta_n^2} = \delta_n,$$

which shows that $\|\mathbb{I}(|Y(t) - Y(\tau_n)| \ge \delta_n)\|_4 = O(2^{-n/64})$. In summary,

$$\|\operatorname{sgn}(Y(t)) - \operatorname{sgn}(Y(\tau_n))\|_4 = O(2^{-n/64}). \tag{2.4}$$

Finally, by the Hölder and Minkowski inequalities,

$$\begin{aligned}&\|(\operatorname{sgn}(Y(t)) - \operatorname{sgn}(Y(\tau_n)))(|Y(t)| + |Y(\tau_n)|)\|_2\\ &\quad\le \|\operatorname{sgn}(Y(t)) - \operatorname{sgn}(Y(\tau_n))\|_4\,(\|Y(t)\|_4 + \|Y(\tau_n)\|_4).\end{aligned}$$

As we have already observed, $\{\|Y(\tau_n)\|_4\colon n \ge 0\}$ is a bounded sequence; thus, item (b) of this lemma follows from (2.4), as was to be shown. □

We will adopt the following notation and definitions. For each integer $n \ge 0$, $j \in \mathbb{Z}$, and real number $t \ge 0$, let

$$U_{j,n}(t) = \sum_{k=0}^{[2^n t]-1} \mathbb{I}\{Y(T_{k,n}) = r_{j,n},\, Y(T_{k+1,n}) = r_{j+1,n}\}, \tag{2.5}$$

$$D_{j,n}(t) = \sum_{k=0}^{[2^n t]-1} \mathbb{I}\{Y(T_{k,n}) = r_{j+1,n},\, Y(T_{k+1,n}) = r_{j,n}\}. \tag{2.6}$$

Thus, $U_{j,n}(t)$ and $D_{j,u}(t)$ denote the number of upcrossings and downcrossings of the interval $[r_{j,n}, r_{j+1,n}]$ within the first $[2^n t]$ steps of the random walk $\{Y(T_{k,n})\colon k \ge 0\}$, respectively.

As is customary, we will say that $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ is *symmetric* provided that

$$\varphi(x, y) = \varphi(y, x)$$

for all $x, y \in \mathbb{R}$. We will say that φ is *skew symmetric* provided that

$$\varphi(x, y) = -\varphi(y, x)$$

for all $x, y \in \mathbb{R}$. Recalling (1.2), we state and prove a useful real-variable lemma.

LEMMA 2.4. *If φ is symmetric, then*

$$\sum_{k=0}^{[2^n t]-1} \varphi(Z(T_{k,n}), Z(T_{k+1,n})) = \sum_{j\in\mathbb{Z}} \varphi(X_{j,n}, X_{j+1,n})\{U_{j,n}(t) + D_{j,n}(t)\}.$$

If φ is skew symmetric, then

$$\sum_{k=0}^{[2^n t]-1} \varphi(Z(T_{k,n}), Z(T_{k+1,n})) = \sum_{j\in\mathbb{Z}} \varphi(X_{j,n}, X_{j+1,n})\{U_{j,n}(t) - D_{j,n}(t)\}.$$

PROOF. Since each step of the random walk $\{Y(T_{k,n})\colon k \geq 0\}$ is either an upcrossing or a downcrossing of some interval $[r_{j,n}, r_{j+1,n}]$, $j \in \mathbb{Z}$, it follows that

$$1 = \sum_{j\in\mathbb{Z}} \Big(\mathbb{I}\big\{Y(T_{k,n}) = r_{j,n},\, Y(T_{k+1,n}) = r_{j+1,n}\big\} + \mathbb{I}\big\{Y(T_{k,n}) = r_{j+1,n},\, Y(T_{k+1,n}) = r_{j,n}\big\}\Big).$$

Consequently,

$$\begin{aligned}
&\sum_{k=0}^{[2^n t]-1} \varphi(Z(T_{k,n}), Z(T_{k+1,n})) \\
&\quad = \sum_{j\in\mathbb{Z}} \sum_{k=0}^{[2^n t]-1} \varphi(Z(T_{k,n}), Z(T_{k+1,n})) \\
&\qquad \times \Big(\mathbb{I}\big\{Y(T_{k,n}) = r_{j,n},\, Y(T_{k+1,n}) = r_{j+1,n}\big\} \\
&\qquad\quad + \mathbb{I}\big\{Y(T_{k,n}) = r_{j+1,n},\, Y(T_{k+1,n}) = r_{j,n}\big\}\Big).
\end{aligned}$$

Observe that from (2.5) and (2.6),

$$\begin{aligned}
&\sum_{k=0}^{[2^n t]-1} \varphi(Z(T_{k,n}),\, Z(T_{k+1,n}))\mathbb{I}\big\{Y(T_{k,n}) = r_{j,n},\, Y(T_{k+1,n}) = r_{j+1,n}\big\} \\
&\quad = \varphi\big(X_{j,n}, X_{j+1,n}\big) U_{j,n}(t), \\
&\sum_{k=0}^{[2^n t]-1} \varphi(Z(T_{k,n}),\, Z(T_{k+1,n}))\mathbb{I}\big\{Y(T_{k,n}) = r_{j+1,n}, Y(T_{k+1,n}) = r_{j,n}\big\} \\
&\quad = \varphi\big(X_{j+1,n}, X_{j,n}\big) D_{j,n}(t).
\end{aligned}$$

The remainder of the argument follows from the definitions of symmetric and skew symmetric. □

Our next result will be used in conjunction with the decomposition developed in Lemma 2.3; its proof is easily obtained by observing that the upcrossings and downcrossings of the interval $[r_{j,n}, r_{j+1,n}]$ alternate.

LEMMA 2.5. *Let $t > 0$. For each $j \in \mathbb{Z}$,*

$$U_{j,n}(t) - D_{j,n}(t) = \begin{cases} \mathbb{I}(0 \leq j < j^*), & \text{if } j^* > 0, \\ 0, & \text{if } j^* = 0, \\ -\mathbb{I}(j^* \leq j < 0), & \text{if } j^* < 0. \end{cases}$$

We will need a set of auxiliary processes. For $s \geq 0$, let

$$\tilde{X}_s^{\pm} = X^{\pm}(r_{j,n}) \quad \text{when } r_{j,n} \leq s < r_{j+1,n}.$$

For $s \in \mathbb{R}$, let

$$\tilde{X}_s = \begin{cases} \tilde{X}_s^+, & \text{if } s \geq 0, \\ \tilde{X}_{-s}^-, & \text{if } s < 0. \end{cases}$$

We will adopt the following conventions: given $u \in \mathbb{R}$, let

$$\int_0^u f(\tilde{X}_s)\, dX_s = \begin{cases} \int_0^u f(\tilde{X}_s^+)\, dX_s^+, & \text{if } u \geq 0, \\ \int_0^{-u} f(\tilde{X}_s^-)\, dX_s^-, & \text{if } u < 0, \end{cases}$$

whenever the integrals on the right are defined. Due to the definition of $\{\tilde{X}_s\colon s \in \mathbb{R}\}$,

$$\int_0^{r_{k,n}} f(\tilde{X}_s)\, dX_s = \begin{cases} \sum_{j=0}^{k-1} f(X_{j,n}^+)\, \Delta X_{j,n}^+, & \text{if } k > 0, \\ 0, & \text{if } k = 0, \\ \sum_{j=0}^{|k|-1} f(X_{j,n}^-)\, \Delta X_{j,n}^-, & \text{if } k < 0. \end{cases} \tag{2.7}$$

Similarly, through consideration of the cases,

$$\operatorname{sgn}(r_{k,n}) \int_0^{r_{k,n}} f(\tilde{X}_s)\, ds = \begin{cases} \sum_{j=0}^{k-1} f(X_{j,n}^+)\, \Delta r_{j,n}, & \text{if } k > 0, \\ 0, & \text{if } k = 0, \\ \sum_{j=0}^{|k|-1} f(X_{j,n}^-)\, \Delta r_{j,n}, & \text{if } k < 0. \end{cases} \tag{2.8}$$

It will be preferable to rewrite the results of (2.8) in a modified form. For $k > 0$, observe that

$$\begin{aligned} \operatorname{sgn}(r_{k,n}) \int_0^{r_{k,n}} f(\tilde{X}_s)\, ds = {} & \sum_{j=0}^{k-1} f(X_{j,n}^+)(\Delta X_{j,n}^+)^2 \\ & + \sum_{j=0}^{k-1} f(X_{j,n}^+)\left(\Delta r_{j,n} - (\Delta X_{j,n}^+)^2\right). \end{aligned} \tag{2.9}$$

The obvious modifications can be made for the case $k < 0$.

PROOF OF THEOREM 2.1. Recall (1.1)–(1.3). For each integer $n \geq 0$, let

$$\tilde{V}_n^{(1)}(f,t) = \int_0^{r_{j^*,n}} f(\tilde{X}_s)\, dX_s + \tfrac{1}{2}\operatorname{sgn}(r_{j^*,n}) \int_0^{r_{j^*,n}} f'(\tilde{X}_s)\, ds,$$

$$\hat{V}_n^{(1)}(f,t) = \int_0^{Y(\tau_n)} f(X_s)\, dX_s + \tfrac{1}{2}\operatorname{sgn}(Y(\tau_n)) \int_0^{Y(\tau_n)} f'(X_s)\, ds,$$

$$V^{(1)}(f,t) = \int_0^{Y_t} f(X_s)\, dX_s + \tfrac{1}{2}\operatorname{sgn}(Y_t) \int_0^{Y_t} f'(X_s)\, ds.$$

In this notation, we need to show that $V_n^{(1)}(f,t) \to V^{(1)}(f,t)$ almost surely and in $L^2(\mathbb{P})$ as $n \to \infty$. To this end,

$$\begin{aligned}\|V_n^{(1)}(f,t) - V^{(1)}(f,t)\|_2 \le{}& \|V_n^{(1)}(f,t) - \tilde{V}_n^{(1)}(f,t)\|_2 \\ &+ \|\tilde{V}_n^{(1)}(f,t) - \hat{V}_n^{(1)}(f,t)\|_2 \\ &+ \|\hat{V}_n^{(1)}(f,t) - V^{(1)}(f,t)\|_2.\end{aligned}$$

We will estimate each of the terms on the right in order. We will begin by expressing $V_n^{(1)}(f,t)$ in an alternate form. We will place a $\pm$ superscript on $M_{j,n}$ whenever the underlying Brownian motion is so signed. Since the function

$$\varphi(x,y) = f\left(\frac{y+x}{2}\right)(y-x)$$

is skew symmetric, by Lemma 2.4,

$$V_n^{(1)}(f,t) = \sum_{j\in\mathbb{Z}} f(M_{j,n})\Delta X_{j,n}\big(U_{j,n}(t) - D_{j,n}(t)\big).$$

In light of Lemma 2.5, there will be three cases to consider, according to the sign of j^*. If $j^* = 0$, then $U_{j,n}(t) - D_{j,n}(t) = 0$ and, consequently, $V_n^{(1)}(f,t) = 0$. If $j^* > 0$, then $U_{j,n}(t) - D_{j,n}(t) = 1$ for $0 \le j \le j^* - 1$ and 0 otherwise; consequently,

$$V_n^{(1)}(f,t) = \sum_{j=0}^{j^*-1} f(M_{j,n}^+)\Delta X_{j,n}^+.$$

If, however, $j^* < 0$, then $U_{j,n}(t) - D_{j,n}(t) = -1$ for $j^* \le j \le -1$ and 0 otherwise, consequently,

$$\begin{aligned}V_n^{(1)}(f,t) &= -\sum_{j=j^*}^{-1} f\left(\frac{X_{j+1,n} + X_{j,n}}{2}\right)(X_{j+1,n} - X_{j,n}) \\ &= \sum_{j=j^*}^{-1} f\left(\frac{X_{-j-1,n}^- + X_{-j,n}^-}{2}\right)(X_{-j,n}^- - X_{-j-1,n}^-) \\ &= \sum_{j=0}^{|j^*|-1} f(M_{j,n}^-)\Delta X_{j,n}^-.\end{aligned}$$

In summary,

$$V_n^{(1)}(f,t) = \begin{cases} \displaystyle\sum_{j=0}^{j^*-1} f(M_{j,n}^+)\Delta X_{j,n}^+, & \text{if } j^* > 0, \\ 0, & \text{if } j^* = 0, \\ \displaystyle\sum_{j=0}^{|j^*|-1} f(M_{j,n}^-)\Delta X_{j,n}^-, & \text{if } j^* < 0. \end{cases} \tag{2.10}$$

By combining (2.7), (2.9) and (2.10),

$$V_n^{(1)}(f,t) - \tilde{V}_n^{(1)}(f,t) = A_n + B_n,$$

where

$$A_n = \begin{cases} \sum_{j=0}^{j^*-1} \left[f\left(M_{j,n}^+\right) - f\left(X_{j,n}^+\right) - \frac{1}{2} f'\left(X_{j,n}^+\right) \Delta X_{j,n}^+ \right] \Delta X_{j,n}^+, & \text{if } j^* > 0, \\ 0, & \text{if } j^* = 0, \\ \sum_{j=0}^{|j^*|-1} \left[f\left(M_{j,n}^-\right) - f\left(X_{j,n}^-\right) - \frac{1}{2} f'\left(X_{j,n}^-\right) \Delta X_{j,n}^- \right] \Delta X_{j,n}^-, & \text{if } j^* < 0, \end{cases}$$

and

$$B_n = \begin{cases} \frac{1}{2} \sum_{j=0}^{j^*-1} f'\left(X_{j,n}^+\right) \left(\left(\Delta X_{j,n}^+\right)^2 - \Delta r_{j,n} \right), & \text{if } j^* > 0, \\ 0, & \text{if } j^* = 0, \\ \frac{1}{2} \sum_{j=0}^{|j^*|-1} f'\left(X_{j,n}^-\right) \left(\left(\Delta X_{j,n}^-\right)^2 - \Delta r_{j,n} \right), & \text{if } j^* < 0. \end{cases}$$

By Taylor's theorem,

$$\left| f\left(M_{j,n}^{\pm}\right) - f\left(X_{j,n}^{\pm}\right) - \tfrac{1}{2} f'\left(X_{j,n}^{\pm}\right) \Delta X_{j,n}^{\pm} \right| \le \tfrac{1}{8} S(f) |\Delta X_{j,n}^{\pm}|^2.$$

Hence,

$$|A_n| \le \begin{cases} \frac{1}{8} S(f) \sum_{j=0}^{j^*-1} |\Delta X_{j,n}^+|^3, & \text{if } j^* > 0, \\ 0, & \text{if } j^* = 0, \\ \frac{1}{8} S(f) \sum_{j=0}^{|j^*|-1} |\Delta X_{j,n}^-|^3, & \text{if } j^* < 0. \end{cases}$$

However, for any integer m, we have, by the triangle inequality and Brownian scaling,

$$\begin{aligned} \left\| \sum_{j=0}^{|m|-1} |\Delta X_{j,n}^{\pm}|^3 \right\|_2 &\le |m| \, \| \left(\Delta X_{0,n}^{\pm}\right)^3 \|_2 \\ &= |m| \, \mu_6^{1/2} 2^{-3n/4}. \end{aligned}$$

Since the random variable j^* is independent of X, by conditioning on the value of j^* and applying the above inequality,

$$\|A_n\|_2 \le \tfrac{1}{8} S(f) \, \mu_6^{1/2} 2^{-3n/4} \mathbb{E}(|j^*|).$$

Since $\{2^{n/2} Y(T_{k,n}) : k \ge 0\}$ is a simple symmetric random walk on $\mathbb{Z}$, it follows that, for each $t > 0$,

$$\mathbb{E}(|j^*|) \le \|j^*\|_2 = \sqrt{[2^n t]} = O(2^{n/2}). \tag{2.11}$$

Consequently,

$$\|A_n\|_2 = O(2^{-n/4}). \tag{2.12}$$

Let us turn our attention to the analysis of B_n. For each $j \in \mathbb{Z}$, let

$$\varepsilon_{j,n}^{\pm} = \left(\Delta X_{j,n}^{\pm}\right)^2 - \Delta r_{j,n}.$$

Observe that $\mathbb{E}(\varepsilon_{j,n}^{\pm}) = 0$ and $\operatorname{var}(\varepsilon_{j,n}^{\pm}) = 2^{-n}\operatorname{var}(X(1)^2)$. Let $m \in \mathbb{Z}$. Since the random variables $\{f'(X_{j,n}^{\pm})\varepsilon_{j,n}^{\pm}\colon 0 \le j \le |m|-1\}$ are pairwise uncorrelated and since $\varepsilon_{j,n}^{\pm}$ is independent of $f'(X_{j,n}^{\pm})$, it follows that

$$\begin{aligned}\operatorname{var}\left(\tfrac{1}{2}\sum_{j=0}^{|m|-1} f'\left(X_{j,n}^{\pm}\right)\varepsilon_{j,n}^{\pm}\right) &= \tfrac{1}{4}\sum_{j=0}^{|m|-1}\mathbb{E}\left(f'\left(X_{j,n}^{\pm}\right)^2\right)\operatorname{var}\left(\varepsilon_{j,n}^{\pm}\right)\\ &\le C_1|m|2^{-n},\end{aligned}$$

where $C_1 = S(f)^2\operatorname{var}(X(1))^2/4$. Arguing as above, since j^* is independent of X, it follows that

$$\begin{aligned}\|B_n\|_2^2 &\le C_1 2^{-n}\mathbb{E}(|j^*|)\\ &= O(2^{-n/2}).\end{aligned}$$

We have used (2.11) to arrive at this last estimate. This estimate, in conjunction with (2.12), yields

$$\|V_n^{(1)}(f,t) - \tilde{V}_n^{(1)}(f,t)\|_2 = O(2^{-n/4}). \tag{2.13}$$

Recalling that $r_{j^*,n} = Y(\tau_n)$, we have

$$\begin{aligned}\left|\tilde{V}_n^{(1)}(f,t) - \hat{V}_n^{(1)}(f,t)\right| \le{}& \left|\int_0^{r_{j^*,n}}\left(f(X_s) - f\left(\tilde{X}_s\right)\right)dX_s\right|\\ &+ \tfrac{1}{2}\left|\int_0^{r_{j^*,n}}\left(f(X_s) - f\left(\tilde{X}_s\right)\right)ds\right|.\end{aligned}$$

For $j > 0$,

$$\begin{aligned}&\mathbb{E}\left[\left(\int_0^{r_{j,n}}\left(f(X_s) - f\left(\tilde{X}_s\right)\right)dX_s\right)^2\right]\\ &\quad= \int_0^{r_{j,n}}\mathbb{E}\left[\left|f\left(\tilde{X}_s^+\right) - f\left(X_s^+\right)\right|^2\right]ds\\ &\quad\le S(f)^2\int_0^{r_{j,n}}\mathbb{E}\left[\left|X_s^+ - \tilde{X}_s^+\right|^2\right]ds\\ &\quad= S(f)^2\sum_{k=0}^{j-1}\int_{r_{k,n}}^{r_{k+1,n}}\mathbb{E}\left[\left|X^+(s) - X^+(r_{k,n})\right|^2\right]ds\\ &\quad= S(f)^2 j\int_0^{r_{1,n}} s\,ds\\ &\quad= \tfrac{1}{2}S(f)^2 j r_{1,n}^2.\end{aligned}$$

A similar argument handles the case $j < 0$, and in general

$$\mathbb{E}\left[\left(\int_0^{r_{j,n}}\left(f(X_s) - f\left(\tilde{X}_s\right)\right) dX_s\right)^2\right] \le \tfrac{1}{2}S(f)^2\,|j|\,2^{-n}. \tag{2.14}$$

Since j^* is independent of the X, by conditioning on the value of j^* and applying (2.14),

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^{r_{j^*,n}}\left(f\left(\tilde{X}_s\right) dX_s - f\left(\tilde{X}_s\right)\right) dX_s\right)^2\right] &= \tfrac{1}{2}S(f)^2 2^{-n}\mathbb{E}(|j^*|)\\ &= O(2^{-n/2}),\end{aligned} \tag{2.15}$$

where we have used (2.11) to obtain this last estimate. Similarly, for any integer $j > 0$,

$$\begin{aligned}\left\|\int_0^{r_{j,n}}\left(f(X_s) - f\left(\tilde{X}_s\right)\right) ds\right\|_2 &\le \int_0^{r_{j,n}}\left\|f(X_s^+) - f\left(\tilde{X}_s^+\right)\right\|_2 ds\\ &\le S(f)\int_0^{r_{j,n}}\|X_s^+ - \tilde{X}_s^+\|_2\, ds\\ &= S(f)\sum_{k=0}^{j-1}\int_{r_{k,n}}^{r_{k+1,n}}\|X^+(s) - X^+(r_{k,n})\|_2\, ds\\ &= S(f)j\int_0^{r_{1,n}}\sqrt{s}\, ds\\ &= \tfrac{2}{3}S(f)jr_{1,n}^{3/2}.\end{aligned}$$

A similar proof handles the case $j < 0$, and, in general,

$$\mathbb{E}\left[\left(\int_0^{r_{j,n}}\left(f(X_s)\, ds - f\left(\tilde{X}_s\right)\right) ds\right)^2\right] \le \tfrac{4}{9}S(f)^2 j^2 2^{-3n/2}. \tag{2.16}$$

Since j^* is independent of X, by conditioning on the value of j^* and applying (2.16),

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^{r_{j^*,n}}\left(f(X_s)\, ds - f\left(\tilde{X}_s\right)\right) ds\right)^2\right] &\le \tfrac{4}{9}S(f)^2 2^{-3n/2}\mathbb{E}\left((j^*)^2\right)\\ &= O(2^{-n/2}).\end{aligned} \tag{2.17}$$

We have used (2.11) to obtain this last estimate. From (2.15) and (2.17),

$$\|\tilde{V}_n^{(1)}(f,t) - \hat{V}_n^{(1)}(f,t)\|_2 = O(2^{-n/4}). \tag{2.18}$$

Recalling that $r_{j^*,n} = Y(\tau_n)$, we have

$$\begin{aligned}|\hat{V}_n^{(1)}(f,t) - V^{(1)}(f,t)| \le &\left|\int_{Y(\tau_n)}^{Y(t)} f(X_s)\,dX_s\right| \\ &+ \left|\operatorname{sgn}(Y(t))\int_0^{Y(t)} f'(X_s)\,ds \right. \\ &\left. - \operatorname{sgn}(Y(\tau_n))\int_0^{Y(\tau_n)} f'(X_s)\,ds\right|.\end{aligned}$$

Let $a, b \in \mathbb{R}$. Then, by the Itô isometry,

$$\begin{aligned}\mathbb{E}\left[\left(\int_a^b f(X_s)\,dX_s\right)^2\right] &= \int_{a\wedge b}^{b\vee a} \mathbb{E}\big(f^2(X_s)\big)\,ds \\ &\le S(f)^2|b-a|.\end{aligned}$$

Since X and Y are independent, by item (a) of Lemma 2.3,

$$\begin{aligned}\mathbb{E}\left[\left(\int_{Y(\tau_n)}^{Y(t)} f(X_s)\,dX_s\right)^2\right] &\le S(f)^2\mathbb{E}(|Y(t) - Y(\tau_n)|) \\ &= O(2^{-n/8}).\end{aligned} \tag{2.19}$$

By consideration of the cases,

$$\left|\operatorname{sgn}(Y(t))\int_0^{Y(t)} f'(X_s)\,ds - \operatorname{sgn}(Y(\tau_n))\int_0^{Y(\tau_n)} f'(X_s)\,ds\right|$$

is bounded by

$$\begin{aligned}&\left|\int_{Y(\tau_n)}^{Y(t)} f'(X_s)\,ds\right| \\ &\quad + \tfrac{1}{2}|\operatorname{sgn}(Y(t)) - \operatorname{sgn}(Y(\tau_n))|\left|\int_0^{Y(t)} f'(X_s)\,ds + \int_0^{Y(\tau_n)} f'(X_s)\,ds\right|.\end{aligned}$$

However, by an elementary bound on the integral and item (a) of Lemma 2.3,

$$\begin{aligned}\left\|\int_{Y(\tau_n)}^{Y_t} f'(X_s)\,ds\right\|_2 &\le S(f)\|Y(t) - Y(\tau_n)\|_2 \\ &= O(2^{-n/8}).\end{aligned} \tag{2.20}$$

Finally, note that

$$\begin{aligned}&|\operatorname{sgn}(Y(t)) - \operatorname{sgn}(Y(\tau_n))|\left|\int_0^{Y(t)} f'(X_s)\,ds + \int_0^{Y(\tau_n)} f'(X_s)\,ds\right| \\ &\quad \le S(f)|\operatorname{sgn}(Y(t)) - \operatorname{sgn}(Y(\tau_n))|\,(|Y(t)| + |Y(\tau_n)|).\end{aligned}$$

Therefore, by Lemma 2.3(b),

$$\left\|\left(\operatorname{sgn}(Y(t)) - \operatorname{sgn}(Y(\tau_n))\right)\left(\int_0^{Y(t)} f'(X_s)\,ds + \int_0^{Y(\tau_n)} f'(X_s)\,ds\right)\right\|_2 = O(2^{-n/64}). \tag{2.21}$$

From (2.19), (2.20) and (2.21),

$$\|\hat{V}_n^{(1)}(f,t) - V^{(1)}(f,t)\|_2 = O(2^{-n/64}). \tag{2.22}$$

Combining (2.13), (2.18) and (2.22), it follows that

$$\|V_n^{(1)}(f,t) - V^{(1)}(f,t)\|_2 = O(2^{-n/64}),$$

which yields the $L^2(\mathbb{P})$ convergence in question. The almost sure convergence follows from applications of Markov's inequality and the Borel–Cantelli lemma. □

3. The quadratic variation of iterated Brownian motion. Given an integer $n \geq 0$ and a real number $t > 0$, let

$$V_n^{(2)}(t) = \sum_{k=0}^{[2^n t]-1} \left(Z(T_{k+1,n}) - Z(T_{k,n})\right)^2,$$

$$V_n^{(2)}(f,t) = \sum_{k=0}^{[2^n t]-1} f\left(\frac{Z(T_{k+1,n}) + Z(T_{k,n})}{2}\right)\left(Z(T_{k+1,n}) - Z(T_{k,n})\right)^2.$$

In this section, we will examine both strong and weak limit theorems associated with these quadratic variation processes. Our first result is the strong law of large numbers for $V_n^{(2)}(f,t)$.

THEOREM 3.1. *Let $t > 0$ and $f \in C_b^2(\mathbb{R})$. Then,*

$$2^{-n/2} V_n^{(2)}(f,t) \to \int_0^t f(Z_s)\,ds$$

almost surely and in $L^2(\mathbb{P})$ as $n \to \infty$.

As a corollary, we have $2^{-n/2}V_n^{(2)}(t) \to t$ almost surely and in $L^2(\mathbb{P})$ as $n \to \infty$. Our next result examines the deviations of the centered process $(2^{-n/2}V_n^{(2)}(t) - t)$ and connects the quadratic variation of iterated Brownian motion with the stochastic process called Brownian motion in random scenery, first described and studied in [27]. Since the introduction of this model, various aspects of Brownian motion in random scenery have been investigated in [7, 31–34, 36].

We will use the following notation in the sequel. Let $D_{\mathbb{R}}[0,1]$ denote the space of real-valued functions on $[0,1]$ which are right continuous and have left-hand limits. Given random elements $\{T_n\colon n \geq 0\}$ and T in $D_{\mathbb{R}}[0,1]$, we will write $T_n \Rightarrow T$ to denote the convergence in distribution of $\{T_n\colon n \geq 0\}$ to T

(see [6], Chapter 3). Let $\{B_1(t)\colon t \in \mathbb{R}\}$ be a two-sided Brownian motion, let $\{B_2(t)\colon t \geq 0\}$ denote an independent standard Brownian motion, and let

$$\mathscr{G}(t) = \int_{\mathbb{R}} L_t^x(B_2)\, B_1(dx).$$

The process $\{\mathscr{G}(t)\colon t \geq 0\}$ is called a Brownian motion in random scenery. Our next result states that $V_n^{(2)}(t)$, suitably normalized, converges in $D_{\mathbb{R}}[0,1]$ to $\mathscr{G}(t)$.

THEOREM 3.2. *As* $n \to \infty$,

$$\frac{2^{n/4}}{\sqrt{2}}\left(2^{-n/2}V_n^{(2)}(t) - t\right) \Rightarrow \mathscr{G}(t).$$

We will prove these theorems in order, but first we will develop several lemmas pertaining to the local time of Brownian motion.

LEMMA 3.3. *For real numbers* $p, q > 0$,

$$\sum_{j\in\mathbb{Z}} \|L_t^{r_{j,n}}(Y)\|_p^q = O(2^{n/2}).$$

PROOF. We will use the following notation: given $x \in \mathbb{R}$, let

$$\tau_x = \inf\{s \geq 0\colon Y_s = x\}. \tag{3.1}$$

Let $C = \mathbb{E}((L_1^0)^p)$. Then, from the strong Markov property, elementary properties of the local time process, the reflection principle, and a standard Gaussian estimate, it follows that

$$\begin{aligned}
\mathbb{E}\left(\left(L_t^x(Y)\right)^p\right) &= \int_0^t \mathbb{E}\left(\left(L_t^x(Y)\right)^p \mid \tau_x = s\right) d\mathbb{P}(\tau_x \leq s) \\
&= \int_0^t \mathbb{E}\left(\left(L_{t-s}^0(Y)\right)^p\right) d\mathbb{P}(\tau_x \leq s) \\
&\leq \mathbb{E}\left(\left(L_t^0(Y)\right)^p\right)\mathbb{P}(\tau_x \leq t) \\
&= 2Ct^{p/2}\mathbb{P}(Y_t \geq |x|) \\
&\leq 2Ct^{p/2}\exp(-x^2/(2t)).
\end{aligned}$$

Consequently,

$$\int_{\mathbb{R}} \|L_t^x(Y)\|_p^q\, dx < \infty.$$

Since the mapping $x \mapsto \|L_t^x(Y)\|_p^q$ is uniformly continuous,

$$\lim_{n\to\infty} \sum_{j\in\mathbb{Z}} \|L_t^{r_{j,n}}(Y)\|_p^q 2^{-n/2} = \lim_{n\to\infty} \sum_{j\in\mathbb{Z}} \|L_t^{r_{j,n}}(Y)\|_p^q \Delta r_{j,n} = \int_{-\infty}^{\infty} \|L_t^x(Y)\|_p^q\, dx.$$

It follows that

$$\sum_{j\in\mathbb{Z}} \|L_t^{r_{j,n}}(Y)\|_p^q = O(2^{n/2}),$$

which proves the lemma in question. □

LEMMA 3.4. *Let $a, b \in \mathbb{R}$ with $ab \geq 0$. Then there exists a positive constant μ, independent of a, b, and t, such that*

$$\|L_t^b(Y) - L_t^a(Y)\|_2 \leq \mu\sqrt{|b-a|}\, t^{1/4} \exp\bigl(-a^2/(4t)\bigr).$$

PROOF. Let $c \in \mathbb{R}$ and $t \geq 0$. From [37], Theorem 1.7, page 210 and its proof, there exists a constant γ, independent of c and t, such that

$$\mathbb{E}\Bigl(\bigl(L_t^c(Y) - L_t^0(Y)\bigr)^2\Bigr) \leq \gamma|c|t^{1/2}. \tag{3.2}$$

By symmetry, it is enough to consider the case $0 \leq a < b$. Recall (3.1). By the strong Markov property, Brownian scaling, the reflection principle, item (3.2), and a standard estimate,

$$\begin{aligned}
\mathbb{E}\Bigl(\bigl(L_t^b(Y) - L_t^a(Y)\bigr)^2\Bigr) &= \int_0^a \mathbb{E}\Bigl(\bigl(L_t^b(Y) - L_t^a(Y)\bigr)^2 \mid \tau_a = s\Bigr)\, d\mathbb{P}(\tau_a \leq s) \\
&= \int_0^a \mathbb{E}\Bigl(\bigl(L_{t-s}^{b-a}(Y) - L_{t-s}^0(Y)\bigr)^2\Bigr)\, d\mathbb{P}(\tau_a \leq s) \\
&\leq \gamma(b-a)t^{1/2}\mathbb{P}(\tau_a \leq t) \\
&\leq \gamma(b-a)t^{1/2}\exp\bigl(-a^2/(2t)\bigr).
\end{aligned}$$

The desired result follows upon taking square roots and setting $\mu = \gamma^{1/2}$. □

What follows is an immediate application of the preceeding lemma.

LEMMA 3.5. *Let $t > 0$. In the notation of* (1.2),

$$\sum_{j\in\mathbb{Z}} f(M_{j,n}) L_t^{r_{j,n}}(Y) \Delta r_{j,n} \to \int_0^t f(Z_s)\, ds$$

almost surely and in $L^2(\mathbb{P})$ as $n \to \infty$.

PROOF. By the occupation times formula,

$$\begin{aligned}
\int_0^t f(Z_s)\, ds &= \int_{-\infty}^{\infty} f(X_u) L_t^u(Y)\, du \\
&= \sum_{j\in\mathbb{Z}} \int_{r_{j,n}}^{r_{j+1,n}} f(X_u) L_t^u(Y)\, du.
\end{aligned}$$

It follows that

$$\begin{aligned}(3.3)\qquad &\left\| \int_0^t f(Z_s) - \sum_{j\in\mathbb{Z}} f(M_{j,n}) L_t^{r_{j,n}}(Y) \Delta r_{j,n} \right\|_2 \\ &\qquad \le \sum_{j\in\mathbb{Z}} \int_{r_{j,n}}^{r_{j+1,n}} \| f(X_u) L_t^u(Y) - f(M_{J,n}) L_t^{r_{j,n}}(Y) \|_2 \, du.\end{aligned}$$

Since $f \in C_b^2(\mathbb{R})$ and X is independent of Y,

$$\begin{aligned}(3.4)\qquad &\| f(X_u) L_t^u(Y) - f(M_{j,n}) L_t^{r_{j,n}}(Y) \|_2 \\ &\qquad \le S(f) \big(\| L_t^u(Y) - L_t^{r_{j,n}}(Y) \|_2 + \| X_u - M_{j,n} \|_2 \| L_t^{r_{j,n}}(Y) \|_2 \big).\end{aligned}$$

However, by Lemma 3.4,

$$\| L_t^u(Y) - L_t^{r_{j,n}}(Y) \|_2 \le C \sqrt{\Delta r_{j,n}} \exp\left(- \frac{\big(r_{j,n} \wedge r_{j+1,n}\big)^2}{(4t)} \right),$$

where C depends only upon t. By the integral test, the sums,

$$\sum_{j\in\mathbb{Z}} \exp\left(- \frac{\big(r_{j,n} \wedge r_{j+1,n}\big)^2}{(4t)} \right) \Delta r_{j,n},$$

are bounded in n. Thus,

$$\begin{aligned}(3.5)\qquad \sum_{j\in\mathbb{Z}} \int_{r_{j,n}}^{r_{j+1,n}} \| L_t^u(Y) - L_t^{r_{j,n}}(Y) \|_2 \, du &= O\Big(\sqrt{\Delta r_{j,n}} \Big) \\ &= O(2^{-n/4}).\end{aligned}$$

For $r_{j,n} \le u \le r_{j+1,n}$, $\| X_u - M_{j,n} \|_2 = \sqrt{\Delta r_{j,n}}$. Thus, by Lemma 3.3,

$$(3.6)\qquad \sum_{j\in\mathbb{Z}} \int_{r_{j,n}}^{r_{j+1,n}} \| X_u - M_{j,n} \|_2 \, \| L_t^{r_{j,n}}(Y) \|_2 \, du = O(2^{-n/4}).$$

Combining (3.3), (3.4), (3.5) and (3.6) we see that

$$\left\| \int_0^t f(Z_s) \, ds - \sum_{j\in\mathbb{Z}} f(M_{j,n}) L_t^{r_{j,n}}(Y) \Delta r_{j,n} \right\|_2 = O(2^{-n/4}).$$

This demonstrates the convergence in $L^2(\mathbb{P})$. By applications of Markov's inequality and the Borel–Cantelli lemma, this convergence is almost sure, as well. □

Our next result is from [28], Theorem 1.4 and its proof. See [3] for a related but slightly weaker version in $L^p(\mathbb{P})$.

LEMMA 3.6. *There exists a positive random variable* $K \in L^8(\mathbb{P})$ *such that, for all* $j \in \mathbb{Z}$, $n \geq 0$ *and* $t \geq 0$,

$$\left| U_{j,n}(t) - \frac{2^{n/2}}{2} L_t^{r_{j,n}}(Y) \right| \leq Kn2^{n/4} \sqrt{L_t^{r_{j,n}}(Y)},$$

$$\left| D_{j,n}(t) - \frac{2^{n/2}}{2} L_t^{r_{j,n}}(Y) \right| \leq Kn2^{n/4} \sqrt{L_t^{r_{j,n}}(Y)}.$$

PROOF OF THEOREM 3.1. Since the mapping

$$\varphi(x,y) = f\left(\frac{y+x}{2}\right)(y-x)^2$$

is symmetric, by Lemma 2.4,

$$\begin{aligned} 2^{-n/2} V_n^{(2)}(f,t) &= \sum_{j \in \mathbb{Z}} 2^{-n/2} f(M_{j,n}) (\Delta X_{j,n})^2 \big(U_{j,n}(t) + D_{j,n}(t)\big) \\ &= A_n + B_n + C_n, \end{aligned}$$

where

$$\begin{aligned} A_n &= \sum_{j \in \mathbb{Z}} 2^{-n/2} f(M_{j,n}) (\Delta X_{j,n})^2 \big(U_{j,n}(t) + D_{j,n}(t) - 2^{n/2} L_t^{r_{j,n}}(Y)\big), \\ B_n &= \sum_{j \in \mathbb{Z}} f(M_{j,n}) \Big((\Delta X_{j,n})^2 - \mathbb{E}\Big((\Delta X_{j,n})^2\Big)\Big) L_t^{r_{j,n}}(Y), \\ C_n &= \sum_{j \in \mathbb{Z}} f(M_{j,n}) L_t^{r_{j,n}}(Y) \Delta r_{j,n}. \end{aligned}$$

By Lemma 3.6, since $f \in C_b^2(\mathbb{R})$,

$$|A_n| \leq S(f) n 2^{-n/4} \sum_{j \in \mathbb{Z}} (\Delta X_{j,n})^2 K \sqrt{L_t^{r_{j,n}}(Y)}.$$

Since X is independent of Y, by Hölder's inequality, for each $j \in \mathbb{Z}$,

$$\left\| (\Delta X_{j,n})^2 K \sqrt{L_t^{r_{j,n}}(Y)} \right\|_2 \leq \left\| (\Delta X_{1,n})^2 \right\|_2 \|K\|_4 \left\| L_t^{r_{j,n}}(Y) \right\|_2^{1/2}.$$

By scaling, $\|(\Delta X_{j,n})^2\|_2 = 2^{-n/2}\sqrt{\mu_4}$. Hence, by the triangle inequality and Lemma 3.3,

$$\begin{aligned} \|A_n\|_2 &\leq n 2^{-3n/4} \|K\|_4 \sqrt{\mu_4} \sum_{j \in \mathbb{Z}} \|L_t^{r_{j,n}}(Y)\|_2^{1/2} \\ &= O(n 2^{-n/4}), \end{aligned}$$

which shows that $A_n \to 0$ in $L^2(\mathbb{P})$ as $n \to \infty$. By Markov's inequality and the Borel–Cantelli lemma, the convergence is almost sure, as well.

Let

$$X_{j,n}^* = \begin{cases} X_{j,n}, & \text{if } j \geq 0, \\ X_{j+1,n}, & \text{if } j < 0. \end{cases}$$

Then we may write $B_n = B_n^{(1)} + B_n^{(2)}$, where

$$B_n^{(1)} = \sum_{j\in\mathbb{Z}} \big(f(M_{j,n}) - f(X_{j,n}^*)\big)\Big((\Delta X_{j,n})^2 - \mathbb{E}\Big((\Delta X_{j,n})^2\Big)\Big)L_t^{r_{j,n}}(Y),$$

$$B_n^{(2)} = \sum_{j\in\mathbb{Z}} f(X_{j,n}^*)\Big((\Delta X_{j,n})^2 - \mathbb{E}\Big((\Delta X_{j,n})^2\Big)\Big)L_t^{r_{j,n}}(Y).$$

By noting that $|M_{j,n} - X_{j,n}^*| = \frac{1}{2}|\Delta X_{j,n}|$, we see that

$$|B_n^{(1)}| \le \tfrac{1}{2}S(f)\sum_{j\in\mathbb{Z}} |\Delta X_{j,n}|\Big((\Delta X_{j,n})^2 - \mathbb{E}\Big((\Delta X_{j,n})^2\Big)\Big)L_t^{r_{j,n}}.$$

Since X and Y are independent,

$$\begin{aligned}\|B_n^{(1)}\|_2 &\le \tfrac{1}{2}S(f)\sum_{j\in\mathbb{Z}} \|\Delta X_{j,n}\|_4 \Big\|(\Delta X_{j,n})^2 - \mathbb{E}\Big((\Delta X_{j,n})^2\Big)\Big\|_4 \|L_t^{r_{j,n}}(Y)\|_2\\ &= O(2^{-n/4}).\end{aligned}$$

We have used Brownian scaling and Lemma 3.3 to obtain this last estimate.

Observe that the collection $\{f(X_{j,n}^*)((\Delta X_{j,n})^2 - \mathbb{E}((\Delta X_{j,n})^2)): j\in\mathbb{Z}\}$ is centered and pairwise uncorrelated. Since X and Y are independent,

$$\begin{aligned}\operatorname{var}\big(B_n^{(2)}\big) = \sum_{j\in\mathbb{Z}} &\|f(X_{j,n}^*)\|_2^2 \|L_t^{r_{j,n}}(Y)\|_2^2\\ &\times \operatorname{var}\Big((\Delta X_{j,n})^2 - \mathbb{E}\Big((\Delta X_{j,n})^2 - \mathbb{E}\Big((\Delta X_{j,n})^2\Big)\Big).\end{aligned}$$

Since f is bounded, by Brownian scaling,

$$\operatorname{var}\Big((\Delta X_{j,n})^2 - \mathbb{E}\Big((\Delta X_{j,n})^2\Big)\Big) = O(2^{-n}).$$

Therefore,

$$\|B_n^{(2)}\|_2 = O(2^{-n/4}).$$

In summary, $\|B_n\|_2 = O(2^{-n/4})$, which shows that $B_n \to 0$ in $L^2(\mathbb{P})$ as $n\to\infty$. By applications of Markov's inequality and the Borel–Cantelli lemma, this convergence is almost sure, as well.

Finally, by Lemma 3.5, $C_n \to \int_0^t f(Z_s)\,ds$ almost surely and in $L^2(\mathbb{P})$ as $n\to\infty$, which proves the theorem in question. □

We turn our attention to the proof of Theorem 3.2. In preparation for the proof of this result, we will prove several lemmas. For each integer j, each nonnegative integer n and each positive real number t, let

$$\mathcal{Q}_{j,n}(t) = 2^{-n/2}\big(U_{j,n}(t) + D_{j,n}(t)\big).$$

LEMMA 3.7. *For each* $t \ge 0$,

$$\sum_{j\in\mathbb{Z}} \mathbb{E}\big(|\mathcal{Q}_{j,n}(t)|^3\big) = O(2^{n/2}).$$

PROOF. By the triangle inequality and a standard convexity argument, it follows that

$$\mathbb{E}\Big(|\mathcal{Q}_{j,n}(t)|^3\Big) \le 4\mathbb{E}\Big(|L_t^{r_{j,n}}(Y)|^3\Big) + 4\mathbb{E}\Big(|\mathcal{Q}_{j,n}(t) - L_t^{r_{j,n}}(Y)|^3\Big).$$

By Lemma 3.3,

$$\sum_{j\in\mathbb{Z}} \mathbb{E}\Big(|L_t^{r_{j,n}}(Y)|^3\Big) = O(2^{n/2}).$$

By Lemma 3.6,

$$\big|\mathcal{Q}_{j,n}(t) - L_t^{r_{j,n}}(Y)\big|^3 \le K^3 n^3 2^{-3n/4} \big(L_t^{r_{j,n}}(Y)\big)^{3/2}.$$

By Hölder's inequality,

$$\mathbb{E}\Big(\big|\mathcal{Q}_{j,n}(t) - L_t^{r_{j,n}}(Y)\big|^3\Big) \le \|K\|_6^3 n^3 2^{-3n/4} \big\|L_t^{r_{j,n}}(Y)\big\|_3^{3/2}.$$

From Lemma 3.3, it follows that

$$\sum_{j\in\mathbb{Z}} \mathbb{E}\Big(\big|\mathcal{Q}_{j,n}(t) - L_t^{r_{j,n}}(Y)\big|^3\Big) = O(n^3 2^{-n/4}).$$

This proves the lemma in question. □

LEMMA 3.8. *For each pair of nonnegative real numbers s and t,*

$$\lim_{n\to\infty} \sum_{j\in\mathbb{Z}} \mathbb{E}\Big(\big|\mathcal{Q}_{j,n}(s)\mathcal{Q}_{j,n}(t) - L_s^{r_{j,n}}(Y) L_t^{r_{j,n}}(Y)\big|\Big) 2^{-n/2} = 0.$$

PROOF. We have the decomposition

$$\begin{aligned}
&\big|\mathcal{Q}_{j,n}(s)\mathcal{Q}_{j,n}(t) - L_s^{r_{j,n}}(Y) L_t^{r_{j,n}}(Y)\big| \\
&\quad\le \big|\big(\mathcal{Q}_{j,n}(s) - L_s^{r_{j,n}}(Y)\big)\big(\mathcal{Q}_{j,n}(t) - L_t^{r_{j,n}}(Y)\big)\big| \\
&\qquad + \big|\big(\mathcal{Q}_{j,n}(s) - L_s^{r_{j,n}}(Y)\big)\big| L_t^{r_{j,n}}(Y) \\
&\qquad + \big|\big(\mathcal{Q}_{j,n}(t) - L_t^{r_{j,n}}(Y)\big)\big| L_s^{r_{j,n}}(Y).
\end{aligned}$$

By Lemma 3.6,

$$\begin{aligned}
&\big|\big(\mathcal{Q}_{j,n}(s) - L_s^{r_{j,n}}(Y)\big)\big(\mathcal{Q}_{j,n}(t) - L_t^{r_{j,n}}(Y)\big)\big| \\
&\quad\le K^2 n^2 2^{-n/2} \sqrt{L_s^{r_{j,n}}(Y)}\, \sqrt{L_t^{r_{j,n}}(Y)}\,.
\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}
&\mathbb{E}\Big(\big|\big(\mathcal{Q}_{j,n}(s) - L_s^{r_{j,n}}(Y)\big)\big(\mathcal{Q}_{j,n}(t) - L_t^{r_{j,n}}(Y)\big)\big|\Big) \\
&\quad\le \|K\|_6^2 n^2 2^{-n/2} \|L_s^{r_{j,n}}(Y)\|_{3/2}^{1/3}\, \|L_t^{r_{j,n}}(Y)\|_{3/2}^{1/3}.
\end{aligned}$$

By applications of the Cauchy–Schwarz inequality and Lemma 3.3,

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \mathbb{E}\Big(\Big|\big(\mathfrak{L}_{j,n}(s) - L_s^{r_{j,n}}(Y)\big)\big(\mathfrak{L}_{j,n}(t) - L_t^{r_{j,n}}(Y)\big)\Big|\Big)2^{-n/2}&\\ \le \|K\|_6^2 n^2 2^{-n} \sqrt{\sum_{j\in\mathbb{Z}} \|L_s^{r_{j,n}}(Y)\|_{3/2}^{2/3}} \sqrt{\sum_{j\in\mathbb{Z}} \|L_t^{r_{j,n}}(Y)\|_{3/2}^{2/3}}&\\ = O(n^2 2^{-n/2}).&\end{aligned}$$

The remaining two terms can be handled similarly. □

LEMMA 3.9. *Let* $0 \le s \le t \le 1$. *Then*

$$\sum_{j\in\mathbb{Z}} L_s^{r_{j,n}}(Y) L_t^{r_{j,n}}(Y) \Delta r_{j,n} \to \int_{\mathbb{R}} L_s^x(Y) L_t^x(Y)\, dx$$

in $L^1(\mathbb{P})$ *as* $n \to \infty$.

PROOF. We have

$$\int_{\mathbb{R}} L_s^x(Y) L_t^x(Y)\, dx = \sum_{j\in\mathbb{Z}} \int_{r_{j,n}}^{r_{j+1,n}} L_s^x(Y) L_t^x(Y)\, dx.$$

From this it follows that

$$\begin{aligned}\left\| \int_{\mathbb{R}} L_s^x(Y) L_t^x(Y)\, dx - \sum_{j\in\mathbb{Z}} L_s^{r_{j,n}}(Y) L_t^{r_{j,n}}(Y) \Delta r_{j,n} \right\|_1&\\ \le \sum_{j\in\mathbb{Z}} \int_{r_{j,n}}^{r_{j+1,n}} \|L_s^x(Y) L_t^x(Y) - L_s^{r_{j,n}}(Y) L_t^{r_{j,n}}(Y)\|_1\, dx.&\end{aligned}$$

By the Hölder and Minkowski inequalities,

$$\begin{aligned}\|L_s^x(Y) L_t^x(Y) - L_s^{r_{j,n}}(Y) L_t^{r_{j,n}}(Y)\|_1 \le{}& \|L_t^x(Y) - L_t^{r_{j,n}}(Y)\|_2 \|L_s^x(Y)\|_2\\ &+ \|L_s^x(Y) - L_s^{r_{j,n}}(Y)\|_2 \|L_t^{r_{j,n}}(Y)\|_2.\end{aligned}$$

Since $s, t \in [0,1]$, it follows that $\|L_s^x(Y)\|_2$ and $\|L_t^x(Y)\|_2$ are bounded by 1. Therefore, by Lemma 3.4 and Jensen's inequality, there exists a universal constant C such that

$$\|L_s^x(Y) L_t^x(Y) - L_s^{r_{j,n}}(Y) L_t^{r_{j,n}}(Y)\|_1 \le C\sqrt{\Delta r_{j,n}} \exp\left(- \frac{(r_{j,n} \wedge r_{j+1,n})^2}{4} \right).$$

By the integral test, the sums,

$$\sum_{j\in\mathbb{Z}} \exp\left(- \frac{(r_{j,n} \wedge r_{j+1,n})^2}{4} \right) \Delta r_{j,n},$$

are bounded in n. Since $\sqrt{\Delta r_{j,n}} = 2^{-n/4}$,

$$\left\| \int_{\mathbb{R}} L_s^x(Y) L_t^x(Y)\, dx - \sum_{j \in \mathbb{Z}} L_s^{r_{j,n}}(Y) L_t^{r_{j,n}}(Y) \Delta r_{j,n} \right\|_1 = O(2^{-n/4}).$$

This proves the lemma. □

Given a function f defined on $[0,1]$ and $\delta > 0$, let

$$\omega(f, \delta) = \sup_{\substack{0 \le s,t \le 1 \\ |s-t| < \delta}} |f(t) - f(s)|.$$

LEMMA 3.10. *There exists a positive real number c such that*

$$\mathbb{E}(\omega(L^a(Y), \delta)) \le c\left(\sqrt{\delta \ln(1/\delta)} \wedge |a| \exp(-a^2/4)\right)$$

for $a \in \mathbb{R}$ and $\delta \in (0, 1/2)$.

PROOF. Since local time at a is nondecreasing in the time variable,

$$\omega(L^a(Y), \delta) \le L_1^a(Y).$$

Consequently, by Lemma 3.4, there exists a positive real number c_1 such that

$$\begin{aligned} \mathbb{E}(\omega(L^a(Y), \delta)) &\le \mathbb{E}(L_1^a(Y)) \\ &\le \|L_1^a(Y)\|_2 \\ &\le c_1 |a| \exp(-a^2/4) \end{aligned}$$

for $a \in \mathbb{R}$ and $\delta > 0$.

By Tanaka's formula,

$$L_t^a(Y) = |Y_t - a| - |a| - \int_0^t \operatorname{sgn}(Y_r - a)\, dY_r.$$

Hence, for $0 \le s < t \le 1$,

$$L_t^a(Y) - L_s^a(Y) \le |Y_t - Y_s| - \int_s^t \operatorname{sgn}(Y_r - a)\, dY_r.$$

By Lévy's characterization theorem (see [37], page 141), $t \mapsto \int_0^t \operatorname{sgn}(Y_r - a)\, dY_r$ is a standard Brownian motion. By Lévy's theorem on the modulus of continuity of Brownian motion (see [37], page 29), there exists a positive real number c_2 such that

$$\mathbb{E}(\omega(L^a(Y), \delta)) \le c_2 \sqrt{\delta \log(1/\delta)}$$

for $a \in \mathbb{R}$ and $\delta \in (0, 1/2)$. Letting $c = c_1 \wedge c_2$,

$$\mathbb{E}(\omega(L^a(Y), \delta)) \le c\left(\sqrt{\delta \ln(1/\delta)} \wedge |a| \exp(-a^2/4)\right),$$

as was to be shown. □

PROOF OF THEOREM 3.2. For each integer j and each positive integer n, let

$$\varepsilon_{j,n} = \frac{2^{n/2}}{\sqrt{2}}\Big((\Delta X_{j,n})^2 - \mathbb{E}\Big((\Delta X_{j,n})^2\Big)\Big).$$

For each n, the random variables $\{\varepsilon_{j,n}: j \in \mathbb{Z}\}$ are independent and identically distributed. A scaling argument shows that $\varepsilon_{j,n}$ is distributed as $\varepsilon = (X_1^2 - 1)/\sqrt{2}$ for all admissible integers j and n. Let ϕ denote the characteristic function of ε. Since $\mathbb{E}(\varepsilon) = 0$ and $\mathbb{E}(\varepsilon^2) = 1$, we have, as $z \to 0$,

$$\log \phi(z) = -\frac{z^2}{2} + O(z^3).$$

Thus, there exist $\gamma > 0$ and $0 < \delta \le 1$ such that

$$\left|\log \phi(z) + \frac{z^2}{2}\right| \le \gamma|z|^3, \tag{3.7}$$

for all $|z| \le \delta$.

By Lemma 2.4 and the definition of $\{\mathcal{Q}_{j,n}(t): j \in \mathbb{Z}\}$,

$$2^{-n/2}V_n^{(2)}(t) = \sum_{j\in\mathbb{Z}} (\Delta X_{j,n})^2 \mathcal{Q}_{j,n}(t).$$

Noting that $\mathbb{E}((\Delta X_{j,n})^2) = \Delta r_{j,n} = 2^{-n/2}$,

$$2^{-n/2}V_n^{(2)}(t) = \sum_{j\in\mathbb{Z}} \sqrt{2}\,\varepsilon_{j,n}\mathcal{Q}_{j,n}(t)2^{-n/2} + 2^{-n}\sum_{j\in\mathbb{Z}}\big(U_{j,n}(t) + D_{j,n}(t)\big).$$

Concerning this last term on the right,

$$2^{-n}\sum_{j\in\mathbb{Z}}\big(U_{j,n}(t) + D_{j,n}(t)\big) = 2^{-n}[2^n t] = t + O(2^{-n}),$$

since the number of upcrossings and downcrossings of all the intervals $[r_{j,n}, r_{j+1,n}]$ by the random walk is equal to the number of steps taken by this same random walk. It follows that

$$\frac{2^{n/4}}{\sqrt{2}}\big(V_n^{(2)}(t) - t\big) = \sum_{j\in\mathbb{Z}} \varepsilon_{j,n}\mathcal{Q}_{j,n}(t)2^{-n/4} + O(2^{-3n/4}).$$

Letting

$$\mathcal{G}_n(t) = \sum_{j\in\mathbb{Z}} \varepsilon_{j,n}\mathcal{Q}_{j,n}(t)2^{-n/4},$$

it is enough to show that

$$\mathcal{G}_n \Rightarrow \mathcal{G}. \tag{3.8}$$

First we will demonstrate the convergence of the finite-dimensional distributions and then we will give the tightness argument.

Let $0 \le t_1 < t_2 < \cdots < t_m \le 1$ and let $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}$. To demonstrate the convergence of the finite-dimensional distributions, it is enough to show that

$$\mathbb{E}\left[\exp\left(i\sum_{k=1}^m \lambda_k \mathcal{G}_n(t_k)\right)\right] \to \mathbb{E}\left[\exp\left(i\sum_{k=1}^m \lambda_k \mathcal{G}(t_k)\right)\right], \tag{3.9}$$

as $n \to \infty$. For simplicity, let,

$$a_{j,n} = 2^{-n/4} \sum_{k=1}^{m} \lambda_k \mathcal{Q}_{j,n}(t_k),$$

$$\tilde{a}_{j,n} = 2^{-n/4} \sum_{k=1}^{m} \lambda_k L^{r_{j,n}}_{t_k}(Y).$$

We have the following:

$$\left| \mathbb{E}\left[\exp\left(i \sum_{k=1}^{m} \lambda_k \mathcal{G}_n(t_k) \right) \right] - \mathbb{E}\left[\exp\left(i \sum_{k=1}^{m} \lambda_k \mathcal{G}(t_k) \right) \right] \right| \le A_n + B_n + C_n,$$

where

$$A_n = \left| \mathbb{E}\left[\exp\left(i \sum_{k=1}^{m} \lambda_k \mathcal{G}_n(t_k) \right) \right] - \mathbb{E}\left[\exp\left(-\tfrac{1}{2} \sum_{j \in \mathbb{Z}} a_{j,n}^2 \right) \right] \right|,$$

$$B_n = \left| \mathbb{E}\left[\exp\left(-\tfrac{1}{2} \sum_{j \in \mathbb{Z}} a_{j,n}^2 \right) \right] - \mathbb{E}\left[\exp\left(-\tfrac{1}{2} \sum_{j \in \mathbb{Z}} \tilde{a}_{j,n}^2 \right) \right] \right|,$$

$$C_n = \left| \mathbb{E}\left[\exp\left(-\tfrac{1}{2} \sum_{j \in \mathbb{Z}} \tilde{a}_{j,n}^2 \right) \right] - \mathbb{E}\left[\exp\left(i \sum_{k=1}^{m} \lambda_k \mathcal{G}(t_k) \right) \right] \right|.$$

We will estimate each term in turn.

Observe that

$$\sum_{k=1}^{m} \lambda_k \mathcal{G}_n(t_k) = \sum_{j \in \mathbb{Z}} \varepsilon_{j,n} a_{j,n}.$$

Let $\mathcal{Y}$ denote the σ-algebra generated by $\{Y_t\colon t \ge 0\}$ and observe that the random variables $\{a_{j,n}\colon j \in \mathbb{Z}\}$ are $\mathcal{Y}$-measurable. Thus,

$$\mathbb{E}\left[\exp\left(i \sum_{k=1}^{m} \lambda_k \mathcal{G}_n(t_k) \right) \right] = \mathbb{E}\left\{ \mathbb{E}\left[\prod_{j \in \mathbb{Z}} e^{i a_{j,n} \varepsilon_{j,n}} \,\middle|\, \mathcal{Y} \right] \right\}$$

$$= \mathbb{E}\left[\prod_{j \in \mathbb{Z}} \phi(a_{j,n}) \right].$$

Assuming that $\sum_{j \in \mathbb{Z}} |a_{j,n}|^3 \le \delta^3$, we have, by (3.7),

$$\sum_{j \in \mathbb{Z}} \left| \log \phi(a_{j,n}) - \tfrac{1}{2} a_{j,n}^2 \right| \le \gamma \sum_{j \in \mathbb{Z}} |a_{j,n}|^3.$$

From this it follows that

$$\left| \prod_{j \in \mathbb{Z}} \phi(a_{j,n}) - \exp\left(-\tfrac{1}{2} \sum_{j \in \mathbb{Z}} a_{j,n}^2 \right) \right| \le \gamma e^{\gamma} \sum_{j \in \mathbb{Z}} |a_{j,n}|^3.$$

Since

$$\left| \prod_{j\in\mathbb{Z}} \phi(a_{j,n}) - \exp\Big(-\tfrac{1}{2} \sum_{j\in\mathbb{Z}} a_{j,n}^2 \Big) \right| \le 2,$$

we may conclude that

$$\left| \prod_{j\in\mathbb{Z}} \phi(a_{j,n}) - \exp\Big(-\tfrac{1}{2} \sum_{j\in\mathbb{Z}} a_{j,n}^2 \Big) \right| \le \gamma e^{\gamma} \sum_{j\in\mathbb{Z}} |a_{j,n}|^3 + 2\mathbb{I}\Big(\sum_{j\in\mathbb{Z}} |a_{j,n}|^3 > \delta^3 \Big).$$

Upon taking expectations and applying Markov's inequality,

$$A_n \le C \sum_{j\in\mathbb{Z}} \mathbb{E}\big(|a_{j,n}|^3\big),$$

where $C = (\gamma e^{\gamma} + 2\delta^{-3})$. However, by a convexity argument and Lemma 3.7,

$$\begin{aligned} \sum_{j\in\mathbb{Z}} E\big(|a_{j,n}|^3\big) &\le m^2 2^{-3n/4} \sum_{k=1}^{m} |\lambda_k|^3 \sum_{j\in\mathbb{Z}} \mathbb{E}\big(\mathfrak{Q}_{j,n}^3(t_k)\big) \\ &= O(2^{-n/4}), \end{aligned}$$

which shows that $A_n \to 0$ as $n \to \infty$.

Note that

$$\begin{aligned} B_n &\le \tfrac{1}{2} \sum_{j\in\mathbb{Z}} \mathbb{E}\big(|a_{j,n}^2 - \tilde{a}_{j,n}^2|\big) \\ &\le \tfrac{1}{2} \sum_{k=1}^{m} \sum_{l=1}^{m} |\lambda_k||\lambda_l| \sum_{j\in\mathbb{Z}} \mathbb{E}\big(|\mathfrak{Q}_{j,n}(t_k)\mathfrak{Q}_{j,n}(t_l) - L_{t_k}^{r_{j,n}}(Y) L_{t_l}^{r_{j,n}}(Y)|\big) 2^{-n/2}. \end{aligned}$$

By Lemma 3.8, we see that $B_n \to 0$.

Finally, observe that

$$\mathbb{E}\left[\exp\left(i \sum_{k=1}^{m} \lambda_k \mathcal{G}(t_k) \right) \right] = \mathbb{E}\left[\exp\left(-\tfrac{1}{2} \int_{\mathbb{R}} \left(\sum_{k=1}^{m} \lambda_k L_{t_k}^x(Y) \right)^2 dx \right) \right].$$

Thus,

$$\begin{aligned} C_n &\le \tfrac{1}{2}\mathbb{E}\left[\left| \sum_{j\in\mathbb{Z}} \tilde{a}_{j,n}^2 - \int_{\mathbb{R}} \left(\sum_{k=1}^{m} \lambda_k L_{t_k}^x(Y) \right)^2 dx \right| \right] \\ &\le \tfrac{1}{2} \sum_{k=1}^{m} \sum_{l=1}^{m} |\lambda_k||\lambda_l| \left\| \sum_{j\in\mathbb{Z}} \mathfrak{Q}_{j,n}(t_k)\mathfrak{Q}_{j,n}(t_l)\Delta r_{j,n} - \int_{\mathbb{R}} L_{t_k}^x(Y) L_{t_l}^x(Y)\, dx \right\|_1. \end{aligned}$$

By Lemma 3.9, $C_n \to 0$, which, in conjunction with the above, verifies (3.9).

To demonstrate tightness, observe that

$$\omega(\mathcal{G}_n, \delta) \le \sum_{j\in\mathbb{Z}} 2^{-n/4} |\varepsilon_{j,n}| \omega\big(\mathfrak{Q}_{j,n}, \delta\big),$$

from which it follows that

$$\begin{aligned}\mathbb{E}\big(\omega(\mathscr{G}_n,\delta)\big) &\le \sum_{j\in\mathbb{Z}} 2^{-n/2}\mathbb{E}\big(|\varepsilon_{j,n}|\big)\mathbb{E}\Big(\omega\big(\mathfrak{L}_{j,n},\delta\big)\Big)\\ &\le \sum_{j\in\mathbb{Z}} 2^{-n/2}\mathbb{E}\Big(\omega\big(\mathfrak{L}_{j,n},\delta\big)\Big).\end{aligned} \tag{3.10}$$

By Lemma 3.6 and the fact that the local time is nondecreasing in the time variable,

$$\Big|\omega\big(\mathfrak{L}_{j,n},\delta\big)-\omega\big(L^{r_{j,n}}(Y),\delta\big)\Big| \le 2Kn2^{-n/4}\sqrt{L_1^{r_{j,n}}(Y)}\,.$$

Thus, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\mathbb{E}\Big(\omega\big(\mathfrak{L}_{j,n},\delta\big)\Big) &\le E\big(\omega\big(L^{r_{j,n}}(Y),\delta\big)\big) + \frac{2n}{2^{n/4}}\|K\|_2\,\|L_1^{r_{j,n}}(Y)\|_1^{1/2}\\ &= E\big(\omega\big(L^{r_{j,n}}(Y),\delta\big)\big) + A(n),\end{aligned}$$

with obvious notation.

By Lemma 3.3,

$$\sum_{j\in\mathbb{Z}} 2^{-n/2}A_n = O(n2^{-n/4}). \tag{3.11}$$

Given $\delta\in(0,1/2)$, let us partition the integers into two sets J_1 and J_2, where

$$\begin{aligned}J_1 &= \big\{j\in\mathbb{Z}\colon |j|\le \delta^{-1/4}2^{n/2}\big\},\\ J_2 &= \big\{j\in\mathbb{Z}\colon |j|> \delta^{-1/4}2^{n/2}\big\}.\end{aligned}$$

Then, by Lemma 3.10,

$$\begin{aligned}\sum_{j\in J_1} 2^{-n/2}\mathbb{E}\big(\omega\big(L^{r_{j,n}},\delta\big)\big) &\le c|J_1|2^{-n/2}\sqrt{\delta\log(1/\delta)}\\ &\le 2c\delta^{1/4}\sqrt{\log(1/\delta)}\,.\end{aligned} \tag{3.12}$$

However, recalling that $\Delta r_{j,n}=2^{-n/2}$ and applying Lemma 3.10,

$$\begin{aligned}\sum_{j\in J_2} 2^{-n/2}\mathbb{E}\big(\omega\big(L^{r_{j,n}},\delta\big)\big) &\le c\sum_{j\in J_2}|r_{j,n}|\exp\big(-r_{j,n}^2/4\big)\Delta r_{j,n}\\ &\sim 2c\int_{\delta^{-1/4}}^{\infty}|x|\exp\big(-x^2/4\big)\,dx.\end{aligned} \tag{3.13}$$

Combining (3.10), (3.11), (3.12) and (3.13), we obtain the requisite tightness. This demonstrates (3.8), and the theorem is proved. □

4. Higher-order variation. In this section, we will examine strong and weak limit theorems for the tertiary and quartic variation of iterated Brownian motion. Let us begin by recalling a theorem, essentially due to [8].

PROPOSITION 4.1. *Let $t \geq 0$ and $p > 0$. The following hold in $L^p(\mathbb{P})$:*

(a) $$\sum_{k=0}^{[2^{n/2}t]} \left(Z(r_{k+1,n}) - Z(r_{k,n})\right)^3 \to 0;$$

(b) $$\sum_{k=0}^{[2^{n/2}t]} \left(Z(r_{k+1,n}) - Z(r_{k,n})\right)^4 \to 3t.$$

Our next two theorems generalize the above, but along our random partitions. Given an integer $n \geq 0$ and a real number $t > 0$, let

$$V_n^{(3)}(f,t) = \sum_{k=0}^{[2^n t]-1} f\left(\frac{Z(T_{k+1,n}) + Z(T_{k,n})}{2}\right)\left(Z(T_{k+1,n}) - Z(T_{k,n})\right)^3,$$

$$V_n^{(4)}(f,t) = \sum_{k=0}^{[2^n t]-1} f\left(\frac{Z(T_{k+1,n}) + Z(T_{k,n})}{2}\right)\left(Z(T_{k+1,n}) - Z(T_{k,n})\right)^4.$$

When $f \equiv 1$, we will write $V_n^{(3)}(t)$ and $V_n^{(4)}(t)$ in place of $V_n^{(3)}(f,t)$ and $V_n^{(4)}(f,t)$, respectively. Our first result is a strong limit theorem for the tertiary variation of iterated Brownian motion and is related to Theorem 2.1 and Proposition 4.1(a).

THEOREM 4.2. *Let $t > 0$ and let $f \in C_b^2(\mathbb{R})$. Then*

$$V_n^{(3)}(f,t) \to 0$$

almost surely and in $L^2(\mathbb{P})$ as $n \to \infty$.

Our next result is a strong limit theorem for the quartic variation of iterated Brownian motion and is related to Theorem 3.1 and Proposition 4.1(b).

THEOREM 4.3. *Let $t > 0$ and let $f \in C_b^2(\mathbb{R})$. Then*

$$V_n^{(4)}(f,t) \to 3\int_0^t f(Z_s)\,ds$$

almost surely and in $L^2(\mathbb{P})$ as $n \to \infty$.

As corollaries to Theorem 4.2 and Theorem 4.3, $V_n^{(3)}(t) \to 0$ and $V_n^{(4)}(t) \to 3t$ almost surely and in $L^2(\mathbb{P})$. Our next two results concern the deviations of $V_n^{(3)}(t)$ and $V_n^{(4)}(t) - 3t$: we will demonstrate that $V_n^{(3)}(t)$ and $V_n^{(4)}(t) - 3t$, suitably normalized, converge in distribution to an iterated Brownian motion and to Brownian motion in random scenery, respectively. As in Section 3, let $\{B_1(u)\colon u \in \mathbb{R}\}$ denote a standard two-sided Brownian motion and let $\{B_2(t)\colon t \geq 0\}$ denote an independent standard Brownian motion. Observe that $\{B_1 \circ B_2(t)\colon t \geq 0\}$ is an iterated Brownian motion and that

$$\mathcal{G}(t) = \int_{\mathbb{R}} L_t^x(B_2)\,B_1(dx)$$

is a Brownian motion in random scenery.

THEOREM 4.4. *As* $n \to \infty$,

$$\frac{2^{n/2}}{\sqrt{15}} V_n^{(3)}(t) \Rightarrow B_1 \circ B_2(t).$$

THEOREM 4.5. *As* $n \to \infty$,

$$\frac{2^{n/4}}{\sqrt{96}} \left(V_n^{(4)}(t) - 3t\right) \Rightarrow \mathcal{G}(t).$$

We will prove these theorems in order.

PROOF OF THEOREM 4.2. Since the mapping

$$\varphi(x, y) = f\left(\frac{y + x}{2}\right)(y - x)^3$$

is skew symmetric, by Lemma 2.4,

$$V_n^{(3)}(f, t) = \sum_{j \in \mathbb{Z}} f\left(M_{j,n}\right)\left(\Delta X_{j,n}\right)^3 \left(U_{j,n}(t) - D_{j,n}(t)\right).$$

By Lemma 2.5 and by following the argument preceding (2.10), we obtain

$$V_n^{(3)}(f, t) = \begin{cases} \sum_{j=0}^{j^*-1} f\left(M_{j,n}^+\right)\left(\Delta X_{j,n}^+\right)^3, & \text{if } j^* > 0, \\ 0, & \text{if } j^* = 0, \\ \sum_{j=0}^{|j^*|-1} f\left(M_{j,n}^-\right)\left(\Delta X_{j,n}^-\right)^3, & \text{if } j^* < 0. \end{cases}$$

However, for any integer m, by the triangle inequality, the boundedness of f and Brownian scaling,

$$\begin{aligned} \left\| \sum_{j=0}^{|m|-1} f\left(M_{j,n}^{\pm}\right)\left(\Delta X_{j,n}^{\pm}\right)^3 \right\|_2 &\le S(f) \left\| \left(\Delta X_{0,n}^{\pm}\right)^3 \right\|_2 |m| \\ &= S(f)|m| \mu_6^{1/2} 2^{-3n/4}. \end{aligned}$$

Since the random variable j^* is independent of X, by conditioning on the value of j^* and applying the above inequality,

$$\begin{aligned} \mathbb{E}\left(\left(V_n^{(3)}(f, t)\right)^2\right) &\le S(f)^2 \mu_6 2^{-3n/2} \mathbb{E}\left(\left(j^*\right)^2\right) \\ &= O(2^{-n/2}). \end{aligned}$$

We have used (2.11) to obtain this last estimate. This demonstrates the $L^2(\mathbb{P})$-convergence in question. By applications of Markov's inequality and the Borel–Cantelli lemma, the convergence is almost sure, as well. □

PROOF OF THEOREM 4.3. Since the mapping

$$\varphi(x, y) = f\left(\frac{y + x}{2}\right)(y - x)^4$$

is symmetric, by Lemma 2.4,

$$\begin{aligned} V_n^{(4)}(f,t) &= \sum_{j\in\mathbb{Z}} f(M_{j,n})(\Delta X_{j,n})^4\big(U_{j,n}(t) + D_{j,n}(t)\big) \\ &= A_n + B_n + C_n, \end{aligned}$$

where

$$\begin{aligned} A_n &= \sum_{j\in\mathbb{Z}} f(M_{j,n})(\Delta X_{j,n})^4\big(U_{j,n}(t) + D_{j,n}(t) - 2^{n/2}L_t^{r_{j,n}}(Y)\big), \\ B_n &= \sum_{j\in\mathbb{Z}} f(M_{j,n})\Big((\Delta X_{j,n})^4 - \mathbb{E}\Big((\Delta X_{j,n})^4\Big)\Big)2^{n/2}L_t^{r_{j,n}}(Y), \\ C_n &= \sum_{j\in\mathbb{Z}} 3f(M_{j,n})L_t^{r_{j,n}}(Y)\Delta r_{j,n}. \end{aligned}$$

Since $f \in C_b^2(\mathbb{R})$, by Lemma 3.6,

$$|A_n| \le S(f)n2^{n/4}\sum_{j\in\mathbb{Z}} (\Delta X_{j,n})^4 K\sqrt{L_t^{r_{j,n}}(Y)}\,.$$

Since X is independent of Y, by Hölder's inequality,

$$\Big\|(\Delta X_{j,n})^4 K\sqrt{L_t^{r_{j,n}}(Y)}\Big\|_2 \le \|(\Delta X_{1,n})^4\|_2\,\|K\|_4\,\|L_t^{r_{j,n}}(Y)\|_2^{1/2}$$

for each $j \in \mathbb{Z}$. By scaling, $\|(\Delta X_{j,n})^4\|_2 = 2^{-n}\sqrt{\mu_8}$. Hence, by the triangle inequality and Lemma 3.3,

$$\begin{aligned} \|A_n\|_2 &\le S(f)n2^{-3n/4}\|K\|_4\sqrt{\mu_8}\sum_{j\in\mathbb{Z}}\|L_t^{r_{j,n}}(Y)\|_2^{1/2} \\ &= O(n2^{-n/4}), \end{aligned}$$

which shows that $A_n \to 0$ in $L^2(\mathbb{P})$ as $n \to \infty$. By Markov's inequality and the Borel–Cantelli lemma, the convergence is almost sure, as well.

Let

$$X_{j,n}^* = \begin{cases} X_{j,n}, & \text{if } j \ge 0, \\ X_{j+1,n}, & \text{if } j < 0. \end{cases}$$

Then we may write $B_n = B_n^{(1)} + B_n^{(2)}$, where

$$\begin{aligned} B_n^{(1)} &= \sum_{j\in\mathbb{Z}} \big(f(M_{j,n}) - f(X_{j,n}^*)\big)\Big((\Delta X_{j,n})^4 - \mathbb{E}\Big((\Delta X_{j,n})^4\Big)\Big)2^{n/2}L_t^{r_{j,n}}(Y), \\ B_n^{(2)} &= \sum_{j\in\mathbb{Z}} f(X_{j,n}^*)\Big((\Delta X_{j,n})^4 - \mathbb{E}\Big((\Delta X_{j,n})^4\Big)\Big)2^{n/2}L_t^{r_{j,n}}(Y). \end{aligned}$$

Noting that $|M_{j,n} - X_{j,n}^*| = \frac{1}{2}|\Delta X_{j,n}|$,

$$|B_n^{(1)}| \le \tfrac{1}{2}S(f)2^{n/2}\sum_{j\in\mathbb{Z}} |\Delta X_{j,n}|\Big((\Delta X_{j,n})^4 - \mathbb{E}\Big((\Delta X_{j,n})^4\Big)\Big)L_t^{r_{j,n}}(Y).$$

Since X and Y are independent,

$$\|B_n^{(1)}\|_2 \le \tfrac{1}{2}S(f)2^{n/2}\sum_{j\in\mathbb{Z}}\|\Delta X_{j,n}\|_4\left\|\left(\Delta X_{j,n}\right)^4-\mathbb{E}\left(\left(\Delta X_{j,n}\right)^4\right)\right\|_4\|L_t^{r_{j,n}}(Y)\|_2$$
$$= O(2^{-n/4}).$$

We have used Brownian scaling and Lemma 3.3 to obtain this last estimate.

Observe that the collection $\{f(X^*_{j,n})((\Delta X_{j,n})^4-\mathbb{E}((\Delta X_{j,n})^4)):\ j\in\mathbb{Z}\}$ is centered and pairwise uncorrelated. Since X and Y are independent,

$$\operatorname{var}\left(B_n^{(2)}\right)=2^n\sum_{j\in\mathbb{Z}}\|f(X^*_{j,n})\|_2^2\,\|L_t^{r_{j,n}}(Y)\|_2^2\operatorname{var}\left(\left(\Delta X_{j,n}\right)^4-\mathbb{E}\left(\left(\Delta X_{j,n}\right)^4\right)\right).$$

By Brownian scaling,

$$\operatorname{var}\left(\left(\Delta X_{j,n}\right)^4-\mathbb{E}\left(\left(\Delta X_{j,n}\right)^4\right)\right)=O(2^{-2n}).$$

Therefore,

$$\|B_n^{(2)}\|_2=O(2^{-n/4}).$$

In summary, $\|B_n\|_2=O(2^{-n/4})$, which shows that $B_n\to 0$ in $L^2(\mathbb{P})$ as $n\to\infty$. By applications of Markov's inequality and the Borel–Cantelli lemma, this convergence is almost sure, as well.

Finally, by Lemma 3.5, $C_n\to 3\int_0^t f(Z_s)\,ds$ almost surely and in $L^2(\mathbb{P})$ as $n\to\infty$. This proves Theorem 4.3. □

PROOF OF THEOREM 4.4. Since the mapping

$$\varphi(x,y)=(y-x)^3$$

is skew symmetric, by Lemma 2.4,

$$V_n^{(3)}(t)=\sum_{j\in\mathbb{Z}}\left(\Delta X_{j,n}\right)^3\left(U_{j,n}(t)-D_{j,n}(t)\right).$$

From Lemma 2.5 and some algebra, it follows that

$$V_n^{(3)}(t)=\begin{cases}\displaystyle\sum_{j=0}^{j^*-1}\left(\Delta X^+_{j,n}\right)^3, & \text{if } j^*>0,\\ 0, & \text{if } j^*=0,\\ \displaystyle\sum_{j=0}^{|j^*|-1}\left(\Delta X^-_{j,n}\right)^3, & \text{if } j^*<0.\end{cases}\tag{4.1}$$

For each $j\in\mathbb{Z}$ and each integer $n\ge 0$,

$$\operatorname{var}\left(\left(\Delta X_{j,n}\right)^3\right)=15\cdot 2^{-3n/2}.$$

Let

$$\varepsilon^{\pm}_{j,n}=\frac{1}{\sqrt{15}}2^{3n/4}\left(\Delta X^{\pm}_{j,n}\right)^3.\tag{4.2}$$

A scaling argument shows that, for each n, the random variables $\{\varepsilon_{j,n}^{\pm}: j \geq 0\}$ are independent and identically distributed as $\varepsilon = X_1^3/\sqrt{15}$. For future reference, let us note that $\mathbb{E}(\varepsilon) = 0$ and $\mathbb{E}(\varepsilon^2) = 1$. For each $t \geq 0$, let

$$X_n^{\pm}(t) = \begin{cases} 2^{-n/4} \displaystyle\sum_{j=0}^{[2^{n/2}t]-1} \varepsilon_{j,n}^{\pm}, & \text{if } t \geq 2^{-n/2}, \\ 0, & \text{if } 0 \leq t < 2^{-n/2}. \end{cases}$$

For $u \in \mathbb{R}$, let

$$X_n(u) = \begin{cases} X_n^+(u), & \text{if } u \geq 0, \\ X_n^-(u), & \text{if } u < 0. \end{cases}$$

In order that we may emphasize their dependence upon n and t, recall that

$$\tau_n = \tau(n,t) = T_{[2^n t],n},$$
$$j^* = j^*(n,t) = 2^{n/2} Y(\tau_n).$$

For $t \in [0,1]$, let

$$Y_n(t) = Y(\tau(n,t)).$$

We observe that

$$\frac{1}{\sqrt{15}} 2^{n/2} V_n^{(3)}(t) = X_n \circ Y_n(t). \tag{4.3}$$

Let $D_{\mathbb{R}}[0,\infty)$ denote the space of all real-valued functions on $[0,\infty)$ which are right continuous and have left limits. Given a function $g: \mathbb{R} \to \mathbb{R}$, let us define $g^+, g^-: [0,\infty) \to \mathbb{R}$ accordingly: for each $t \geq 0$, let $g^+(t) = g(t)$ and let $g^-(t) = g(-t)$. Let

$$D_{\mathbb{R}}^*(\mathbb{R}) = \{g: \mathbb{R} \to \mathbb{R}: g^+ \in D_{\mathbb{R}}[0,\infty) \text{ and } g^- \in D_{\mathbb{R}}[0,\infty)\}.$$

Let q denote the usual Skorohod metric on $D_{\mathbb{R}}[0,\infty)$ (cf. [18], page 117). Then we can define a metric q^* on $D_{\mathbb{R}}^*(\mathbb{R})$ as follows: given $f, g \in D_{\mathbb{R}}^*(\mathbb{R})$, let

$$q^*(f,g) = q(f^+, g^+) + q(f^-, g^-).$$

So defined, $(D_{\mathbb{R}}^*(\mathbb{R}), q^*)$ is a complete separable metric space. Moreover, $\{g_n\}$ converges to g in $D_{\mathbb{R}}^*(\mathbb{R})$ if and only if $\{g_n^+\}$ and $\{g_n^-\}$ converge to g^+ and g^- in $D_{\mathbb{R}}[0,\infty)$, respectively. By Donsker's theorem, $X_n^+ \Rightarrow B_1^+$ and $X_n^- \Rightarrow B_1^-$ in $D_{\mathbb{R}}[0,\infty)$ consequently,

$$X_n \Rightarrow B_1 \quad \text{in } D_{\mathbb{R}}^*(\mathbb{R}). \tag{4.4}$$

By another application of Donsker's theorem,

$$Y_n \Rightarrow B_2 \quad \text{in } D_{\mathbb{R}}([0,1]). \tag{4.5}$$

From (4.4) and (4.5), the independence of X and Y and the independence of B_1 and B_2, it follows that

$$(X_n, Y_n) \Rightarrow (B_1, B_2) \quad \text{in } D_{\mathbb{R}}^*(\mathbb{R}) \times D_{\mathbb{R}}([0,1]).$$

Since $(x, y) \in D^*_{\mathbb{R}}(\mathbb{R}) \times D_{\mathbb{R}}([0,1]) \mapsto D_{\mathbb{R}}([0,1]) \ni x \circ y$ is measurable and since $B_1 \circ B_2$ is continuous, it follows that

$$X_n \circ Y \Rightarrow B_1 \circ B_2 \quad \text{in } D_{\mathbb{R}}([0,1]).$$

Recalling (4.3), this proves the theorem. □

PROOF OF THEOREM 4.5. For each integer j and each positive integer n, let

$$\varepsilon_{j,n} = \frac{2^n}{\sqrt{96}}\Big(\big(\Delta X_{j,n}\big)^4 - \mathbb{E}\Big(\big(\Delta X_{j,n}\big)^4\Big)\Big).$$

For each n, the random variables $\{\varepsilon_{j,n}\colon j \in \mathbb{Z}\}$ are independent and identically distributed. A scaling argument shows that $\varepsilon_{j,n}$ is distributed as $\varepsilon = (X_1^4 - 3)/\sqrt{96}$ for all admissible integers j and n. For future reference, we note that $\mathbb{E}(\varepsilon) = 0$ and $\mathbb{E}(\varepsilon^2) = 1$.

By Lemmas 2.4 and 2.5,

$$\begin{aligned} V_n^{(4)}(t) &= \sum_{j\in\mathbb{Z}} \big(\Delta X_{j,n}\big)^4\big(U_{j,n}(t) + D_{j,n}(t)\big) \\ &= \sum_{j\in\mathbb{Z}} \sqrt{96}\,2^{-n/2}\,\varepsilon_{j,n}\mathcal{Q}_{j,n}(t) + 3\cdot 2^{-n}\sum_{j\in\mathbb{Z}}\big(U_{j,n}(t) + D_{j,n}(t)\big). \end{aligned}$$

Arguing as in the proof of Theorem 3.2,

$$3\cdot 2^{-n}\sum_{j\in\mathbb{Z}}\big(U_{j,n}(t) + D_{j,n}(t)\big) = 3t + O(2^{-n}).$$

From this it follows that

$$2^{-n/4}\big(V_n^{(4)}(t) - 3t\big) = O(2^{-3n/4}) + \sum_{j\in\mathbb{Z}} \varepsilon_{j,n}\mathcal{Q}_{j,n}(t)2^{-n/4}.$$

As was shown in the proof of Theorem 3.2,

$$\sum_{j\in\mathbb{Z}} \varepsilon_{j,n}\mathcal{L}_{j,n}(t)2^{-n/4} \Rightarrow \mathcal{G}(t).$$

This completes the proof. □

5. An excursion-theoretic construction of the Itô integral. In this section we show that the Itô integral process $\int_0^t f(Y_r)\,dY_r$ can be developed by means of the random partitions defined in Section 2. For each integer $n \geq 0$ and $k \in \mathbb{Z}$, let

$$Y_{k,n} = Y(T_{k,n}).$$

We offer the following theorem.

THEOREM 5.1. *Let $t > 0$ and let $f \in C_b^2(\mathbb{R})$. Then*

$$\sum_{k=0}^{[2^n t]-1} f(Y_{k,n})\Delta Y_{k,n} \to \int_0^t f(Y_r)\,dY_r$$

almost surely and in $L^2(\mathbb{P})$, as $n \to \infty$.

We will need the following lemma, which is a simple consequence of the mean value theorem for integrals.

LEMMA 5.2. *Let $a, b \in \mathbb{R}$, $a < b$, and let $f \in C_b^2(\mathbb{R})$. Let*

$$a = u_0 < u_1 < \cdots < u_{n-1} < u_n = b$$

be a partition of $[a, b]$. Then

$$\left| \int_a^b f(s)\, ds - \int_{k=0}^{n-1} f(u_k) \Delta u_k \right| \le S(f)|b-a| \max_{0 \le k \le n-1} \{|\Delta u_k|\}.$$

PROOF OF THEOREM 5.1. By the proof of Lemma 2.4,

$$\begin{aligned} \sum_{k=0}^{[2^n t]-1} f(Y_{k,n}) \Delta Y_{k,n} &= \sum_{j \in \mathbb{Z}} \left[f(r_{j,n}) \Delta r_{j,n} U_{j,n}(t) - f(r_{j+1,n}) \Delta r_{j,n} D_{j,n}(t) \right] \\ &= \sum_{j \in \mathbb{Z}} f(r_{j,n}) \Delta r_{j,n} \big(U_{j,n}(t) - D_{j,n}(t) \big) \\ &\quad - \sum_{j \in \mathbb{Z}} \big(f(r_{j+1,n}) - f(r_{j,n}) \big) \Delta r_{j,n} D_{j,n}(t) \\ &= I_n - II_n, \end{aligned}$$

in the obvious notation. We will show that

$$I_n \to \int_0^{Y_t} f(u)\, du, \tag{5.1}$$

$$II_n \to \tfrac{1}{2} \int_0^t f'(Y_u)\, du \tag{5.2}$$

almost surely and in $L^2(\mathbb{P})$ as $n \to \infty$.

First observe that

$$\begin{aligned} \left| I_n - \int_0^{Y_t} f(u)\, du \right| &\le \left| \int_0^{Y(\tau_n)} f(u)\, du - \int_0^{Y_t} f(u)\, du \right| + \left| I_n - \int_0^{Y(\tau_n)} f(u)\, du \right| \\ &= A_n + B_n, \end{aligned}$$

in the obvious notation.

By Lemma 2.3 and an elementary bound, it follows that

$$\begin{aligned} \|A_n\|_2 &\le S(f) \|Y(t) - Y(\tau_n)\|_2 \\ &= O(2^{-n/8}). \end{aligned} \tag{5.3}$$

By Lemma 2.5,

$$I_n = \begin{cases} \displaystyle\sum_{j=0}^{j^*-1} f(r_{j,n})\, \Delta r_{j,n}, & \text{if } j^* > 0, \\ 0, & \text{if } j^* = 0, \\ \displaystyle -\sum_{j=j^*}^{-1} f(r_{j,n})\, \Delta r_{j,n}, & \text{if } j^* < 0. \end{cases}$$

Thus, by Lemma 5.2,

$$B_n \le S(f)\,|Y(\tau_n)|\,2^{-n/2}.$$

In the proof of Lemma 2.3, it was shown that $\{\|Y(\tau_n)\|\colon n \ge 0\}$ is bounded; thus,

$$\|B_n\| = O(2^{-n/2}). \tag{5.4}$$

Combining (5.3) and (5.4),

$$I_n \to \int_0^{Y_t} f(u)\,du,$$

in $L^2(\mathbb{P})$ as $n \to \infty$. By Markov's inequality and the Borel–Cantelli lemma, this convergence is almost sure, as well. This verifies (5.1).

Observe that

$$\begin{aligned} II_n &= \sum_{j\in\mathbb{Z}} \bigl(f(r_{j+1,n}) - f(r_{j,n}) - f'(r_{j,n})\Delta r_{j,n}\bigr)\Delta r_{j,n} D_{j,n}(t) \\ &\quad + \sum_{j\in\mathbb{Z}} f'(r_{j,n})(\Delta r_{j,n})^2 D_{j,n}(t) \\ &= A_n + B_n, \end{aligned}$$

using obvious notation. By Taylor's theorem,

$$\begin{aligned} |A_n| &\le \tfrac{1}{2} S(f) 2^{-3n/2} \sum_{j\in\mathbb{Z}} D_{j,n}(t) \\ &= O(2^{-n/2}). \end{aligned} \tag{5.5}$$

We have used the fact that $\sum_{j\in\mathbb{Z}} D_{j,n}(t) \le [2^n t]$ to obtain this last bound. Observe that

$$\left| B_n - \tfrac{1}{2}\int_{\mathbb{R}} f'(u) L_t^u(Y)\,du \right| \le B_n^{(1)} + \tfrac{1}{2} B_n^{(2)},$$

where

$$B_n^{(1)} = S(f) 2^{-n} \sum_{j\in\mathbb{Z}} \left| D_{j,n}(t) - \frac{2^{n/2}}{2} L_t^{r_{j,n}}(Y) \right|$$

and

$$B_n^{(2)} = \left| \sum_{j\in\mathbb{Z}} f'(r_{j,n}) L_t^{r_{j,n}}(Y)\Delta r_{j,n} - \int_{\mathbb{R}} f'(u) L_t^u(Y)\,du \right|.$$

By Lemma 3.6,

$$B_n^{(1)} \le S(f) n 2^{-3n/4} \sum_{j\in\mathbb{Z}} K\sqrt{L_t^{r_{j,n}}(Y)}\,.$$

Consequently, by the Minkowski and Hölder inequalities,

$$\begin{aligned} \|B_n^{(1)}\|_2 &\le S(f) n 2^{-3n/4} \sum_{j\in\mathbb{Z}} \|K\|_4 \,\|L_t^{r_{j,n}}(Y)\|_2^{1/2} \\ &= O(n 2^{-n/4}). \end{aligned} \tag{5.6}$$

We have used Lemma 3.3 to obtain this last bound.

As in the proof of Proposition 3.3,

$$B_n^{(2)} \le S(f) \sum_{j \in \mathbb{Z}} \int_{r_{j,n}}^{r_{j+1,n}} |L_t^u(Y) - L_t^{r_{j,n}}(Y)|\, du.$$

Consequently, by symmetry,

$$\|B_n^{(2)}\|_2 \le 2S(f) \sum_{j \ge 0} \int_{r_{j,n}}^{r_{j+1,n}} \|L_t^u(Y) - L_t^{r_{j,n}}(Y)\|_2\, du.$$

By Lemmas 3.4,

$$\|L_t^u(Y) - L_t^{r_{j,n}}(Y)\|_2 \le C\sqrt{\Delta r_{j,n}}\, \exp\bigl(-r_{j,n}^2/2\bigr).$$

Thus,

$$\|B_n^{(2)}\|_2 \le 2S(f)C2^{-n/4} \sum_{j \ge 0} \exp\bigl(-r_{j,n}^2/2\bigr)\, \Delta r_{j,n} \tag{5.7}$$

$$= O(2^{-n/4}).$$

Combining (5.6) and (5.7), we see that,

$$II_n \to \tfrac{1}{2} \int_{\mathbb{R}} f'(u) L_t^u(Y)\, du,$$

in $L^2(\mathbb{P})$ as $n \to \infty$. By Markov's inequality and the Borel–Cantelli lemma, this convergence is almost sure, as well. By the occupation times formula, this verifies (5.2).

We can now complete the proof. By (5.1) and (5.2),

$$\sum_{k=0}^{[2^n t]} f(Y_{k,n})\, \Delta Y_{k,n} \to \int_0^{Y_t} f(u)\, du - \tfrac{1}{2} \int_0^t f'(Y_u)\, du \tag{5.8}$$

almost surely and in $L^2(\mathbb{P})$ as $n \to \infty$. Let $F(t) = \int_0^t f(u)\, du$ and apply Itô's formula to $F(Y_t)$ to see that the right-hand side of (5.8) is another way to write $\int_0^t f(Y_s)\, dY_s$. This proves the theorem. □

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