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STOCHASTIC APPROXIMATION ALGORITHMS WITH CONSTANT STEP SIZE WHOSE AVERAGE IS COOPERATIVE[1]

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We consider stochastic approximation algorithms with constant step size whose average ordinary differential equation (ODE) is cooperative and irreducible. We show that, under mild conditions on the noise process, invariant measures and empirical occupations measures of the process weakly converge (as the time goes to infinity and the step size goes to zero) toward measures which are supported by stable equilibria of the ODE. These results are applied to analyzing the long-term behavior of a class of learning processes arising in game theory.

0. Introduction. *Stochastic approximation algorithms with constant step size* are discrete time stochastic processes whose general form can be written as

$$X_{n+1}^{\varepsilon} - X_n^{\varepsilon} = \varepsilon f(X_n^{\varepsilon}, \xi_{n+1}), \tag{1}$$

where X_n^{ε} lives in $\mathbf{R}^m$, $\{\xi_n\}_{n\in\mathbf{N}}$ is a stochastic process, f is a suitable function and ε a small positive parameter (the step size).

Processes described by (1) appear in a large variety of domains such as system identification or control theory; they encompass several models of learning and adaptive behavior in neural network, game theory and elsewhere.

To analyze the asymptotic behavior of (1) it is often convenient to introduce an ordinary differential equation (ODE)

$$\frac{dx}{dt} = F(x) \tag{2}$$

obtained from (1) by suitable averaging. This method, called the *method of ordinary differential equation*, was introduced by Ljung (1977) and widely studied thereafter [see, e.g., Kushner and Clark (1978), Benveniste, Métivier and Priouret (1990), Duflo (1997)]. Until recently, however, most of the work in this direction has assumed the simplest dynamics for F (for example that F is the negative of the gradient of a cost function), and little attention has been paid to dynamical systems issues.

Recent works by Benaïm (1996a, b), Benaïm and Hirsch (1995, 1996), Duflo (1996) and Fort and Pages (1997) have shown how the long-term behavior of stochastic approximation algorithms can be precisely related to the long-

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term behavior of the associated ODE with a great deal of generality beyond gradients or other dynamically simple systems.

The present paper is a contribution along this line of research. It is devoted to a particular but fairly broad class of algorithms, namely those which are associated to a *cooperative and irreducible* ODE (defined below). This condition occurs naturally in various situations; an example from game theory will be discussed in the paper.

Briefly speaking, our main result is that when F is cooperative and irreducible, weak* limits of the empirical occupation measure of X_n^ε (obtained as $n \to \infty$, $\varepsilon \to 0$) have their supports in the set of stable equilibria of (2) even though certain trajectories of (2) may have arbitrary complicated behavior (periodic, quasiperiodic, chaotic...). Random perturbations of dynamical systems with complicated dynamics have often been considered for hyperbolic or axiom A systems [see, e.g., Kifer (1988)]. However, cooperative vector fields are usually not axiom A and their limit sets cannot be expressed as a finite union of basic sets.

The key to our results are rough large deviation properties for (1) combined with some geometric properties of attractors and attractor-free sets of cooperative vector fields.

Section 1 states the hypotheses and the main results. These hypotheses are discussed in Section 2 and are shown to be satisfied for a large class of processes. Geometric properties of the supports of limiting measures of (1) are proved in Section 3. The main results are proved in Section 4. Section 5 is an application to a class of learning processes which are associated to repeated games of coordination.

TERMINOLOGY. A (Borel) measure μ on $\mathbf{R}^m$ is a *weak* limit point* of a set $\mathscr{M}$ of probability measures on $\mathbf{R}^m$ if there is a sequence $\{\mu_n\}$ in $\mathscr{M}$ such that for every bounded continuous function $f\colon \mathbf{R}^m \to \mathbf{R}$,

$$\lim_{n\to\infty} \int_{\mathbf{R}^m} f\, d\mu_n = \int_{\mathbf{R}^m} f\, d\mu.$$

In other words, μ is a limit point of $\mathscr{M}$ in the weak* topology on the space of measures.

A family $\mathscr{J}$ of Borel probability measures on $\mathbf{R}^m$ is *tight* if for every $\eta > 0$ there exists a compact set $K \subset \mathbf{R}^m$ such that $\mu(K) \geq 1 - \eta$ for all $\mu \in \mathscr{J}$. By the Prohorov theorem, a tight family is relatively compact for the topology of weak* convergence.

The indicator function of a set B is defined as usual by $\mathbf{1}_B(x) = 1$ if $x \in B$ and $\mathbf{1}_B(x) = 0$ otherwise.

1. Hypotheses and main results. In this paper we are concerned with three entities.

1. A family of discrete-time stochastic processes $X^\varepsilon = \{X_n^\varepsilon\}_{n\in\mathbf{N}}$ parameterized by $\varepsilon > 0$, defined on a probability space $\{\Omega, \mathscr{F}, \mathbf{P}\}$ and taking values in $\mathbf{R}^m$. [While we do not assume that $\{X_n^\varepsilon\}$ is given by (1), our results are motivated by such processes.]

For each ε we denote by $\overline{X}^{\varepsilon}$: $\mathbf{R}_+ \to \mathbf{R}^m$ the continuous time *interpolated process* defined by piecewise linear interpolation of X_n^{ε} and step size ε. That is, $\overline{X}^{\varepsilon}(n\varepsilon) = X_n^{\varepsilon}$. We let $X^{\varepsilon, x}$ denote the process with initial condition $X_0^{\varepsilon} = x$ and $\overline{X}^{\varepsilon, x}$ denote the associated interpolated process.

2. A C^1 vector field F: $\mathbf{R}^m \to \mathbf{R}^m$ generating the solution flow

$$\Phi\colon \mathbf{R} \times \mathbf{R}^m \to \mathbf{R}^m,$$
$$(t, x) \to \Phi(t, x) = \Phi_t(x),$$

where $t \mapsto \Phi_t(x_0)$ is the solution to (2) such that $x(0) = x_0$. In practice $F(x)$ is closely related to the family $\{X_n^{\varepsilon}\}$. Our assumptions will imply, for example, that for all $\delta > 0$,

$$\lim_{\varepsilon \to 0} \mathbf{P}\Big\{ \sup_{0 \le t \le T} ||\overline{X}^{\varepsilon, x}(t) - \Phi_t(x)|| \ge \delta \Big\} = 0.$$

We refer to F as the *mean field* associated to $\{X_n^{\varepsilon}\}$.

3. Two measurable maps L^- and L^+:

$$L^{\sigma}\colon T\mathbf{R}^m \equiv \mathbf{R}^m \times \mathbf{R}^m \to [0, \infty] = \mathbf{R}_+ \cup \{\infty\}, \qquad \sigma \in \{-, +\}$$

such that $L^{\sigma}(x, v) = 0$ if and only $v = F(x)$. We call L^- a *lower gauge* and L^+ an *upper gauge* for the vector field F.

Let $C_T(\mathbf{R}^m)$ denote the set of continuous functions h: $[0, T] \to \mathbf{R}^m$ endowed with the topology of uniform convergence induced by the uniform norm: $||h|| = \sup_{0 \le t \le T} ||h(t)||$. Given $x \in \mathbf{R}^m$ and $T > 0$ we construct the *action functionals*

$$\mathscr{L}_{x, T}^{\sigma}\colon C_T(\mathbf{R}^m) \to [0, \infty], \qquad \sigma \in \{-, +\},$$

defined by

$$\mathscr{L}_{x, T}^{\sigma}(h) = \int_0^T L^{\sigma}(h(t), h'(t))\, dt$$

if h is absolutely continuous and $h(0) = x$ and $\mathscr{L}_{x, T}^{\sigma}(h) = \infty$ otherwise. This measures the deviation of the path $h(t)$ from the trajectory $\Phi_t(x)$ of F.

We call $\mathscr{L}_{x, T}^{\sigma}$ a *rate function* [Varadhan (1984)] provided the set of paths

$$\big\{h \in C_T(\mathbf{R}^m)\colon \mathscr{L}_{x, T}^{\sigma}(h) \le s\big\}$$

is compact for every finite $s \ge 0$.

For every Borel set $\mathscr{A} \subset C_T(\mathbf{R}^m)$ we let

$$\mathscr{L}_{x, T}^{\sigma}(\mathscr{A}) = \inf\big\{\mathscr{L}_{x, T}^{\sigma}(h)\colon h \in \mathscr{A}\big\}$$

and $\mathscr{L}_{x, T}^{\sigma}(\varnothing) = \infty$.

We now state our assumptions concerning $\{X_n^{\varepsilon}\}$, F and $\mathscr{L}_{x, T}^{\sigma}$.

HYPOTHESIS 1.1. *There exists a metric space M, a family of Markov chains $\{Z_n^{\varepsilon}\}_{n \in \mathbf{N}}$ taking values in M indexed by $\varepsilon > 0$ and a Borel measurable map Π: $M \to \mathbf{R}^m$ such that $X_n^{\varepsilon} = \Pi(Z_n^{\varepsilon})$.*

Concerning F we always assume the following.

HYPOTHESIS 1.2. *F is cooperative, irreducible and dissipative.*

Cooperative means

$$\frac{\partial F_i}{\partial x_j}(x) \geq 0 \quad \text{for all } x \in \mathbf{R}^m \text{ and } i \neq j,$$

and *irreducible* means that the Jacobian matrix $DF(x)$ is irreducible for all $x \in \mathbf{R}^m$. Recall that a $m \times m$ matrix $D = (D_{i,j})$ is irreducible if the directed graph with vertices $\{1, \dots, m\}$ and oriented edges (i, j) for $D_{i,j} \neq 0$ is connected by directed paths.

The vector field F is *dissipative* if there exists a compact invariant set $\Lambda \subset \mathbf{R}^m$, called the *global attractor*, such that for every compact set $K \subset \mathbf{R}^m$,

$$\lim_{t \to \infty} \operatorname{dist}(\Phi_t(x), \Lambda) = 0$$

uniformly in $x \in K$, where dist denotes the distance from a point to a set.

We now introduce a nondegeneracy condition which has the consequence that the process $t \to \overline{X}^{\varepsilon, x}(t)$ has a nonzero probability of deviating from the mean trajectory $t \to \Phi_t(x)$.

DEFINITION 1.3 Nondegeneracy condition. Let $K \subset \mathbf{R}^m$ be a compact invariant set. The lower gauge L^- is *nondegenerate at* K (with respect to F) if there exists a neighborhood U of K and real numbers $r_0 > 0$, $\rho > 1$ such that

$$\sup_{x \in U,\, v \in \mathbf{R}^m,\, 0 < ||v|| \leq r_0} \frac{L^-(x, F(x) + v)}{||v||^\rho} < \infty. \tag{3}$$

Our main assumption, expressed in Hypothesis 1.4(iii) below, is that $\overline{X}^\varepsilon$ satisfies upper and lower *large deviation principles*, locally uniform in the initial state.

HYPOTHESIS 1.4.

(i) *L^- is nondegenerate on the global attractor Λ of F.*
(ii) *$\mathscr{L}^\sigma_{x,T}$ is a rate function for every $x \in \mathbf{R}^m$, $T > 0$ and $\sigma \in \{-, +\}$.*
(iii) *For every Borel set $\mathscr{A} \subset C_T(\mathbf{R}^m)$ and compact set $K \subset \mathbf{R}^m$, the following estimates hold*:

(a) *There exists $a \in K$ such that*

$$\liminf_{\varepsilon \to 0} \varepsilon \Big[\sup_{x \in K} \log \mathbf{P}(\overline{X}^{\varepsilon, x} \in \mathscr{A}) \Big] \geq -\mathscr{L}^-_{a,T}(\operatorname{int}(\mathscr{A})).$$

(b) *There exists $b \in K$ such that*

$$\limsup_{\varepsilon \to 0} \varepsilon \Big[\sup_{x \in K} \log \mathbf{P}(\overline{X}^{\varepsilon, x} \in \mathscr{A}) \Big] \leq -\mathscr{L}^+_{b,T}(\operatorname{clos}(\mathscr{A})).$$

Main Results. We denote the *equilibrium set* of the vector field F by

$$\mathscr{E} = \big\{ p \in \mathbf{R}^m \colon F(p) = 0 \big\}.$$

An equilibrium p is termed:

1. *Stable* if for each neighborhood V of p there exists a neighborhood $V_1 \subset V$ of p such that $\Phi_t(V_1) \subset V$ for all $t \geq 0$.
2. *Asymptotically stable* if there exists a neighborhood N of p such that

$$\lim_{t\to\infty} \operatorname{dist}(\Phi_t(x), p) = 0$$

   uniformly in $x \in N$.
3. *Unstable* if p is not stable.

The set of stable equilibria is denoted by $\mathscr{E}_{\text{stab}}$.

We now state our main result.

THEOREM 1.5. *Assume Hypotheses* 1.1, 1.2 *and* 1.4. *Let $\mathscr{I}^\varepsilon$ denote the set of invariant probability measures of the Markov process $\{Z_n^\varepsilon\}_{n\in\mathbf{N}}$. Suppose the family*

$$\mathscr{J} = \left\{\mu^\varepsilon = \nu^\varepsilon \circ \Pi^{-1} \colon \nu^\varepsilon \in \mathscr{I}^\varepsilon,\ \varepsilon > 0\right\}$$

is tight. Let $\mu = \lim_{\varepsilon_i \to 0} \mu^{\varepsilon_i}$ be a weak limit point of $\mathscr{J}$. Let $H \subset \mathbf{R}^m$ be any connected component of the support of μ. Then H is contained in a simply ordered arc* (*possibly degenerate*) *of stable equilibria of F. If $\mathscr{E}_{\text{stab}}$ is finite or F is real analytic*, *then H reduces to an asymptotically stable equilibrium.*

This theorem has the important consequence—made precise by the next corollary—that as $\varepsilon \to 0$, the process $\{X_n^\varepsilon\}_{n\in\mathbf{N}}$ tends under reasonable conditions to spend most of the time in the neighborhood of the stable equilibria.

The *empirical occupation measure* of the process $\{X_n^\varepsilon\}$ is the random measure τ_n^ε defined by

$$\tau_n^\varepsilon(A) = \frac{1}{n+1} \sum_{i=0}^{n} \mathbf{1}_A(X_i^\varepsilon)$$

for every Borel set $A \subset \mathbf{R}^m$.

Let $C_b(M)$ be the set of real-valued bounded continuous functions defined on M. For $f \in C_b(M)$ let

$$P^\varepsilon f(z) = \mathbf{E}_z(f(Z_1^\varepsilon)) = \mathbf{E}(f(Z_1^\varepsilon) | Z_0^\varepsilon = z).$$

The Markov process $\{Z_n^\varepsilon\}_{n\in\mathbf{N}}$ is called *Feller* if the operator P^ε maps $C_b(M)$ into itself. For instance, the process given by (1) when $\{\xi_n\}$ are independent identically distributed random variables is clearly Feller provided $x \mapsto f(x, \xi)$ is continous for almost all ξ.

COROLLARY 1.6. *Suppose that the assumptions of Theorem* 1.5 *hold. Suppose furthermore that for every $\varepsilon > 0$*:

(a) *The process $\{Z_n^\varepsilon\}_{n\in\mathbf{N}}$ is Feller.*
(b) *The map $\Pi\colon M \to \mathbf{R}^m$ is continuous.*
(c) *The sequence of random measures $\{\tau_n^\varepsilon\}_{n\in\mathbf{N}}$ is almost surely tight.*

Let $U \subset \mathbf{R}^m$ be a neighborhood of the set of stable equilibria of F. Then

$$\lim_{\varepsilon\to 0}\liminf_{n\to\infty} \tau_n^\varepsilon(U) = 1$$

with probability 1.

PROOF. Given $\omega \in \Omega$, $\liminf_{n\to\infty} \tau_n^\varepsilon(U) = \lim_{n_i\to\infty} \tau_{n_i}^\varepsilon(U)$ for some subsequence $\{n_i\}$ (depending on ω) with $n_i \to \infty$ as $i \to \infty$. By tightness of $\{\tau_n^\varepsilon\}_{n\in\mathbf{N}}$ we can suppose that $\{\tau_{n_i}^\varepsilon\}_{i\in\mathbf{N}}$ converges (for almost all ω) for the topology of weak* convergence, toward some probability measure τ^ε. The condition that $\{Z_n^\varepsilon\}_{n\in\mathbf{N}}$ is Feller implies that τ^ε is almost surely an invariant probability measure of $\{Z_n^\varepsilon\}_{n\in\mathbf{N}}$ [see, e.g., Duflo (1996), I.IV.19, page 20]. Now, let g: $\mathbf{R}^m \to [0,1]$ be a continuous function which is 1 on $\mathbf{R}^m \setminus U$ and is zero on a neighborhood $U' \subset U$ of the stable equilibria. Then

$$\begin{aligned}\lim_{n_i\to\infty} \tau_{n_i}^\varepsilon(\mathbf{R}^m \setminus U) &\le \lim_{n_i\to\infty}\int_M (g\circ\Pi)(z)\tau_{n_i}^\varepsilon(dz) = \int_M (g\circ\Pi)\tau^\varepsilon(dz)\\ &= \int_{\mathbf{R}^m} g(x)\mu^\varepsilon(dz)\end{aligned}$$

with $\mu^\varepsilon(\cdot) = \tau^\varepsilon(\Pi^{-1}(\cdot))$, and Theorem 1.5 implies that $\lim_{\varepsilon\to 0}\int_{\mathbf{R}^m} g(x)\mu^\varepsilon(dx) = 0$. □

REMARK 1.7. If $\mathscr{E}$ is finite or F is real analytic, Corollary 1.6 holds where U denotes any neighborhood of the set of *asymptotically* stable equilibria.

2. Discussion of hypotheses.

Tightness assumptions. The tightness assumptions in Theorem 1.5 and Corollary 1.6 are automatically satisfied when M and $\Pi(M)$ are compact. If M is not compact, criteria based on the existence of a suitable Lyapounov function are particulary useful. The following proposition due to Fort and Pages (1996) and Duflo (1996) gives a practical criterion well suited to stochastic approximations with constant step size. For more details and further results, we refer the reader to Section 1 of Fort and Pages (1996).

PROPOSITION 2.1 [Fort and Pages (1996) and Duflo (1996)]. *Suppose that for every $\varepsilon > 0$:*

(a) *The process $\{Z_n^\varepsilon\}_{n\in\mathbf{N}}$ is Feller.*
(b) *There exists a function H: $M \to \mathbf{R}_+$ (called a Lyapounov function) such that*

for all $R \ge 0$ the set $K_R = \{z \in M\colon H(z) \le R\}$ is compact and there exists $0 < \alpha(\varepsilon) < 1$ and $\beta(\varepsilon) \ge 0$ such that $P^\varepsilon H \le \alpha(\varepsilon)H + \beta(\varepsilon)$.

Then:

(i) *Assumption* (c) *of Corollary* 1.6 *is satisfied and the set $\mathscr{I}^\varepsilon$ of invariant probability measures for $\{Z_n^\varepsilon\}_{n\in\mathbf{N}}$ is nonempty.*

(ii) *If there exists* $\varepsilon_0 > 0$ *such that*

$$\sup_{0<\varepsilon\leq\varepsilon_0} \frac{\beta(\varepsilon)}{1-\alpha(\varepsilon)} < \infty,$$

then the family

$$\mathscr{I} = \bigcup_{\varepsilon_0 \geq \varepsilon > 0} \mathscr{I}^\varepsilon$$

is tight.

(iii) *If* Π: $M \to \mathbf{R}^m$ *is continuous, the tightness of* $\mathscr{I}$ *in* (ii) *implies the tightness of the family* $\mathscr{J}$ *defined in Theorem* 1.5.

PROOF. Conclusion (i) is proved in Duflo [(1996), Proposition 1.III.14]. Conclusions (ii) and (iii) follow from Propositions 1 and 2 of Fort and Pages (1996). If Π: $M \to \mathbf{R}^m$ is continuous, the tightness of $\mathscr{I}$ implies that the family $\{\mu^\varepsilon = \nu^\varepsilon(\Pi^{-1}),\ \nu^\varepsilon \in \mathscr{I}\}$ is relatively compact. It is then tight by the Prohorov theorem. □

Large deviation assumptions. Precise large deviation theorems for the dynamical system (1) have been proved under various assumptions on the noise process $\{\xi_n\}$ and the function f by several authors including Azencott and Ruget (1977), Freidlin (1978), Freidlin and Wentzell (1984) and Dupuis (1988). The recent book by Dupuis and Ellis (1997) provides a comprehensive and unified introduction to this literature. For readers' convenience we briefly describe here the general model considered in Dupuis and Ellis [(1997), Chapters 5 and 6], generalizing the work of Azencott and Ruget (1977).

A *Borel vector field* is a measurable map v: $\mathbf{R}^m \to \mathbf{R}^m$. We let $\chi(\mathbf{R}^m)$ denote the space of Borel vector fields. It is equipped with the σ algebra generated by the projections $\{\eta_x\colon x \in \mathbf{R}^m\}$ where η_x: $v \in \chi(\mathbf{R}^m) \to v(x) \in \mathbf{R}^m$.

Let $\mathscr{P}(\mathbf{R}^m)$ denote the space of probability measures on $\mathbf{R}^m$ endowed with the topology of weak* convergence and let μ be a continuous function

$$\mu\colon \mathbf{R}^m \to \mathscr{P}(\mathbf{R}^m),$$

$$x \to \mu_x.$$

Let $\{v_n\}_{n\in\mathbf{N}}$ be a sequence of i.i.d. random variables defined on some probability space $(\Omega, \mathscr{F}, \mathbf{P})$ taking value in $\chi(\mathbf{R}^m)$ such that for all $x \in \mathbf{R}^m$ and every Borel set $B \subset \mathbf{R}^m$

$$\mathbf{P}(v_n(x) \in B) = \mu_x(B).$$

The function μ being given, it is easy to construct such a sequence of random vector fields [see, e.g., Azencott and Ruget (1977)]. However it is clear that μ does not characterize the law of $\{v_n\}$.

Consider a family $\{X_n^\varepsilon\}$ of processes defined on $\mathbf{R}^m$, parametrized by $\varepsilon > 0$, satisfying recursion of the form

$$X_{n+1}^\varepsilon - X_n^\varepsilon = \varepsilon v_n(X_n^\varepsilon), \tag{4}$$

where $X_0^\varepsilon = x \in \mathbf{R}^m$. Since the random vector fields $\{v_n\}$ are i.i.d., the process $\{X_n^\varepsilon\}_{n\in\mathbf{N}}$ is a homogeneous Markov chain.

The *mean field* associated to (4) is the vector field F: $\mathbf{R}^m \to \mathbf{R}^m$ defined by

$$F(x) = \int_{\mathbf{R}^m} v\mu_x(dv).$$

For each $x \in \mathbf{R}^m$ we let S_x denote the support of μ_x, $\mathrm{conv}(S_x)$ the convex hull of S_x and $\mathrm{ri}(\mathrm{conv}(S_x))$ the relative interior of $\mathrm{conv}(S_x)$. The cumulant generating function of μ_x is defined as

$$H(x,\alpha) = \log\biggl(\int_{\mathbf{R}^m} \exp\langle\alpha, v\rangle \mu_x(dv)\biggr)$$

for all $\alpha \in \mathbf{R}^m$.

The next theorem follows from Theorem 6.3.3 of Dupuis and Ellis (1997).

THEOREM 2.2 (Dupuis and Ellis). *Suppose that*:

(a) *The sets* $\mathrm{ri}(\mathrm{conv}(S_x))$ *are independent of* $x \in \mathbf{R}^m$.
(b) $0 \in \mathrm{ri}(\mathrm{conv}(S_x))$.
(c) *For each* $\alpha \in \mathbf{R}^m$, $\sup_{x\in\mathbf{R}^m} H(x,\alpha) < \infty$.

Then the process $\{X_n^\varepsilon\}$ *satisfies Hypotheses* 1.4(ii) *and* (iii) *with the gauges* $L^+(x,v) = L^-(x,v) = L(x,v)$ *where* $L(x,\cdot)$ *is the the Legendre transform of* $H(x,\cdot)$. *That is*,

$$L(x,v) = \sup_{\alpha\in\mathbf{R}^m}(\langle\alpha, v\rangle - H(x,\alpha)).$$

PROOF. Let $\mathscr{F} \subset C_T(\mathbf{R}^m)$ be a closed set and $K \subset \mathbf{R}^m$ a compact set. Given any $C > 0$,

$$\sup_{x\in K}\mathbf{P}(\overline{X}^{\varepsilon,x} \in \mathscr{F}) \le \mathbf{P}(\overline{X}^{\varepsilon,x(\varepsilon)} \in \mathscr{F}) + e^{-C/\varepsilon}$$

for some $x(\varepsilon) \in K$. Since $\log(u+v) \le \log(2) + \sup(\log(u), \log(v))$ we have

$$\limsup_{\varepsilon\to 0}\sup_{x\in K}\varepsilon\log(\mathbf{P}(\overline{X}^{\varepsilon,x} \in \mathscr{F})) \le \sup\Bigl(-C, \limsup_{\varepsilon\to 0}\varepsilon\log(\mathbf{P}(\overline{X}^{\varepsilon,x(\varepsilon)} \in \mathscr{F}))\Bigr).$$

Let $\varepsilon_k \to 0$ be such that

$$\limsup_{\varepsilon\to 0}\varepsilon\log(\mathbf{P}(\overline{X}^{\varepsilon,x(\varepsilon)} \in \mathscr{F})) = \lim_{\varepsilon_k\to 0}\varepsilon_k\log(\mathbf{P}(\overline{X}^{\varepsilon_k,x(\varepsilon_k)} \in \mathscr{F})).$$

By compactness of K we can suppose that $x(\varepsilon_k) \to x*$ for some $x* \in K$. Therefore the *uniform* Laplace principle proved in Dupuis and Ellis [(1997), Theorem 6.3.3, page 165], implies that $\{\overline{X}^{\varepsilon_k,x(\varepsilon_k)}\}_{k\in\mathbf{N}}$ satisfies the Laplace principle on $C_T(\mathbf{R}^m)$ with rate function $\mathscr{L}_{x*,T}$. Since the Laplace principle and the large deviation principle are equivalents [Dupuis and Ellis (1997), Theorems

1.2.1 and 1.2.3] $\{\overline{X}^{\varepsilon_k, x(\varepsilon_k)}\}_{k\in\mathbf{N}}$ satisfies the large deviation principle with rate function $\mathscr{L}_{x*, T}$. Therefore

$$\limsup_{\varepsilon\to 0}\sup_{x\in K}\varepsilon\log(\mathbf{P}(\overline{X}^{\varepsilon, x}\in\mathscr{F}))\le\sup(-C, -\mathscr{L}_{x*, T}(\mathscr{F})). \tag{5}$$

If $\inf_{x\in K}\mathscr{L}_{x,T}(\mathscr{F})<\infty$, choose $C>\inf_{x\in K}\mathscr{L}_{x,T}(\mathscr{F})$ and $a\in K$ such that $\mathscr{L}_{a, T}(\mathscr{F})\le\inf(C, \mathscr{L}_{x*, T}(\mathscr{F})$. It then follows from (5) that

$$\limsup_{\varepsilon\to 0}\sup_{x\in K}\varepsilon\log(\mathbf{P}(\overline{X}^{\varepsilon, x}\in\mathscr{F}))\le -\mathscr{L}_{a, T}(\mathscr{F}).$$

If $\inf_{x\in K}\mathscr{L}_{x, T}(\mathscr{F})=\infty$, it suffices to let $C\to\infty$ in inequality (5). This proves that Hypothesis 1.4(iii)(b) is satisfied. The proof of Hypothesis 1.4(iii)(a) is similar. □

Conditions (a), (b), (c) of Theorem 2.2 are easily verified but the rate functional is usually difficult to compute. The next proposition gives rough estimates of this functional and provides a sufficient condition ensuring that the nondegeneracy condition (Definition 1.3) holds.

Let $\mathscr{F}_n$ denote the σ field generated by the random variables $\{v_0, v_1, \dots, v_{n-1}\}$ and let

$$U^\varepsilon_{n+1}=v_n(X^\varepsilon_n)-F(X^\varepsilon_n).$$

It is clear that U^ε_n is measurable with respect to $\mathscr{F}_n$ and satisfies

$$\mathbf{E}(U^\varepsilon_{n+1}|\mathscr{F}_n)=0.$$

Let $g_+\colon\mathbf{R}_+\to\mathbf{R}_+$ be a C^2 nonnegative convex function with $g_+(0)=0$ [hence $g'_+(0)=0$]. We say that $\{U^\varepsilon_n\}$ *is of type* g_+ if for all $\theta\in\mathbf{R}^m$,

$$\mathbf{E}(\exp(\langle\theta, U^\varepsilon_{n+1}\rangle)|\mathscr{F}_n))\le\exp(g_+(||\theta||)).$$

For instance, if $||U^\varepsilon_n||\le M$ for all $n\in\mathbf{N}$, then it is well known that $\{U^\varepsilon_n\}$ is of type g_+ with $g_+(u)=M^2u^2/2$.

The *Legendre transform* of g_+ is the function $g^*_+\colon\mathbf{R}\to\mathbf{R}_+\cup\{\infty\}$ defined by

$$g^*_+(x)=\sup_{t\in\mathbf{R}_+}tx-g_+(t).$$

It is a nonnegative strictly convex function [meaning that it is strictly convex on the interval $\mathrm{Dom}\,(g^*_+)=\{x\in\mathbf{R}\colon g^*_+(x)<\infty\}$] and vanishes at the origin.

PROPOSITION 2.3. *Suppose that* $\{U^\varepsilon_n\}$ *is of type* g_+.

(i) *Then* $\{X^\varepsilon_n\}$ *satisfies Hypothesis* 1.4(b) *with the upper gauge*

$$L^+(x, v)=g^*_+(||v-F(x)||).$$

(ii) *Suppose furthermore that the eigenvalues of the covariance matrices*

$$\mathbf{E}((U^{\varepsilon}_{n+1})(U^{\varepsilon}_{n+1})^T|\mathscr{F}_n)$$

are bounded below by some positive constant γ^2. *Then there exists a* C^2 *strictly convex function* $g_-\colon \mathbf{R}_+ \to \mathbf{R}_+$ *with* $g_-(0) = g'_-(0) = 0$ *and* $g''_-(0) > 0$ *such that* $\{X^{\varepsilon}_n\}$ *satisfies Hypothesis* 1.4(a) *with the lower gauge*

$$L^-(x, v) = g^*_-(||v - F(x)||) = \frac{1}{2g''_-(0)}||v - F(x)||^2 + o(||v - F(x)||^2).$$

(iii) *If* $\sup_{n,\varepsilon} ||U^{\varepsilon}_n|| \le M$, *for* g_+ *and* g_- *we can choose the functions*

$$g_+(u) = \frac{M^2}{2}u^2,$$
$$g_-(u) = \frac{\gamma^2}{2M}\left[u + \frac{1}{2M}(e^{-2Mu} - 1)\right],$$

with the Legendre transforms

$$g^*_+(x) = \frac{1}{2M^2}x^2,$$
$$g^*_-(x) = \frac{\gamma^2}{4M^2}f\left(\frac{2M}{\gamma^2}x\right),$$

where $f(u) = \log(1 - u)(1 - u) + u$ *for* $u < 1$ *and* $f(u) = \infty$ *otherwise.*

To prove this result we need the following lemma.

LEMMA 2.4. *Suppose* $\{U^{\varepsilon}_n\}$ *is of type* g_+ *and suppose furthermore that the eigenvalues of the covariance matrices* $\mathbf{E}((U^{\varepsilon}_{n+1})(U^{\varepsilon}_{n+1})^T|\mathscr{F}_n)$ *are bounded below by some positive constant* γ^2 *or in other words,*

$$\mathbf{E}(\langle\theta, U^{\varepsilon}_{n+1}\rangle^2|\mathscr{F}_n) \ge \gamma^2||\theta||^2$$

for all $\theta \in \mathbf{R}^m$. *Then there exists a* C^2 *strictly convex function* $g_-\colon \mathbf{R} \to \mathbf{R}_+$ *with* $g_-(0) = g'_-(0) = 0$ *and* $g''_-(0) > 0$, *such that*

$$\mathbf{E}(\exp\langle\theta, U^{\varepsilon}_{n+1}\rangle)|\mathscr{F}_n) \ge \exp(g_-(||\theta||)).$$

In case $\sup_{n,\varepsilon} ||U_n|| \le M$ *we can choose for* g_+ *and* g_- *the functions given in Proposition* 2.3(iii).

PROOF. To shorten notation, let $U = U^{\varepsilon}_{n+1}$, $X = \langle\theta, U^{\varepsilon}_{n+1}\rangle$ and write $E(\cdot)$ for $\mathbf{E}(\cdot|\mathscr{F}_n)$ and $P(\cdot)$ for $\mathbf{P}(\cdot|\mathscr{F}_n)$.

Let $k(t) = \log(E(e^{tX}))$ for $t \ge 0$. It is well known (and easy to verify) that $k(0) = k'(0) = 0$ and

$$k''(t) = E_t[(X - E_t(X))^2],$$

where E_t is the operator defined by $E_t(Y) = E(Y(e^{tX}/E(e^{tX})))$. Thus

$$k''(t) \geq E[(X - E_t(X))^2 \exp(tX)] \exp(-g_+(t||\theta||))$$
$$\geq E\big[(X - E_t(X))^2 \mathbf{1}_{\{t|X| \leq M\}}\big] \exp(-(M + g_+(t||\theta||))).$$

We claim that there exists a continuous function M: $\mathbf{R}_+ \to \mathbf{R}_+$ such that

$$E\big[(X - E_t(X))^2 \mathbf{1}_{\{t|X| > M(t||\theta||)\}}\big] \leq 1/2\gamma^2 ||\theta||^2.$$

Suppose for the moment that the claim is true. Set $\psi(s) = M(s) + g_+(s)$. Because $E(X) = 0$ we have

$$E[(X - E_t(X))^2] = E(X^2) + E_t(X)^2 \geq E(X^2) \geq \gamma^2 ||\theta||^2.$$

Therefore

$$k''(t) \geq 1/2\gamma^2 ||\theta||^2 \exp(-\psi(t||\theta||)).$$

Let $\rho(s) = 1/2\gamma^2 e^{-\psi(s)}$. By successive integrations we find that

$$k(1) \geq ||\theta||^2 \int_0^1 \int_0^t \rho(s||\theta||)\, ds\, dt = g_-(||\theta||),$$

where

$$g_-(t) = \int_0^t \int_0^r \rho(s)\, ds\, dr.$$

It is clear that g_- is a C^2 nonnegative strictly convex function vanishing at the origin. This proves the result.

We now prove the claim. Let

$$A(M) = E\big[(X - E_t(X))^2 \mathbf{1}_{\{t|X| > M\}}\big].$$

We have

$$(X - E_t(X))^2 \leq 2(X^2 + E_t(X)^2) \leq 2||\theta||^2\big[||U||^2 + (E_t(||U||))^2\big]$$
$$\leq 2||\theta||^2 \left[||U||^2 + E(||U||^2)\frac{E(e^{2tX})}{(E(e^{tX}))^2}\right]$$

by the Hölder inequality. Let $r = g_+(t||\theta||) + \log(4)$;

$$E(e^{tX}) \geq E\big(\exp(tX)\mathbf{1}_{t|X| \leq r}\big) \geq \exp(-r)(1 - P(t|X| > r))$$
$$\geq \exp(-r)(1 - 2\exp(-r)\exp(g_+(t||\theta||))) = 1/8 \exp(-g_+(t||\theta||)).$$

Also

$$E(||U||^k) \leq k! E(\exp(||U||)) \leq k! E\big(\exp(\sqrt{m} \sup_i |U_i|)\big)$$
$$\leq k! 2m \exp(g_+(\sqrt{m})).$$

Thus $E(||U||^k) \leq e^{a_k}$ for some constant $a_k > 0$. It follows that

$$E(||U||^2)\frac{E(e^{2tX})}{(E(e^{tX}))^2} \leq e^{\phi(t||\theta||)}$$

with $\phi(s) = a_2 + g_+(2s) + 2g_+(s) + 6\log(2)$. Therefore

$$A(M) \le 2\|\theta\|^2\big[E(\|U\|^4) + \exp(\phi(t\|\theta\|))\big]P(t|X| > M)$$
$$\le 4\|\theta\|^2 \exp[a_4 + \phi(t\|\theta\|) + g_+(t\|\theta\|) - M].$$

It now suffices to choose $M(s) = a_4 + \phi(s) + g_+(s) + c$ with c large enough to prove the claim.

In case $\|U\| \le M$ we have $k''(t) = E_t(X^2) - E_t(X)^2 \le E_t(X^2) \le M^2\|\theta\|^2$ and $k''(t) \ge E[(X - E_t(X))^2]e^{-2Mt} \ge \gamma^2\|\theta\|^2 e^{-2Mt}$. Then we get

$$\frac{\gamma^2}{2M}\left[\|\theta\| + \frac{1}{2M}(e^{-2M\|\theta\|} - 1)\right] \le k(1) \le \frac{1}{2}M^2\|\theta\|^2$$

by integration. □

PROOF OF PROPOSITION 2.3. The proof of Proposition 2.3 is now straightforward. By definitions $H(x, \alpha)$ and U_n^ε we have

$$L(x, v) = \sup_{\alpha \in \mathbf{R}^m} \langle \alpha, v\rangle - H(x, \alpha) \le \sup_{\alpha \in \mathbf{R}^m} \langle \alpha, v - F(x)\rangle - g_+(\|\alpha\|)$$
$$= \sup_{t \in \mathbf{R}_+} t\|v - F(x)\| - g_+(t).$$

Thus $L(x, v) \le g_-^*(\|v - F(x)\|)$, and similarly $L(x, v) \ge g_+^*(\|v - F(x)\|)$.

3. Limiting measures and attractor-free sets.

Limiting measures. The following result is essential to our analysis. It is similar to Theorem 1.1 of Kifer (1988).

PROPOSITION 3.1. *Assume Hypotheses* 1.1 *and* 1.4(iii)(b). *Let* $\mathscr{I}^\varepsilon$ *denote the set of invariant probability measures of* $\{Z_n^\varepsilon\}_{n \in \mathbf{N}}$. *Suppose that the family*

$$\mathscr{J} = \{\mu^\varepsilon = \nu^\varepsilon \circ \Pi^{-1}\colon \nu^\varepsilon \in \mathscr{I}^\varepsilon,\ \varepsilon > 0\}$$

is tight. Then any limit point $\mu = \lim_{\varepsilon_i \to 0} \mu^{\varepsilon_i}$ *of* $\mathscr{J}$ *is an invariant measure of the flow* Φ.

PROOF. First remark that Hypothesis 1.4(iii)(b) implies the following *averaging property*: Let $K \subset \mathbf{R}^m$ be a compact set, $T \ge 0$ and $\delta > 0$, then

$$\lim_{\varepsilon \to 0} \sup_{x \in K} \mathbf{P}\Big(\sup_{0 \le t \le T} \|\overline{X}^{\varepsilon, x}(t) - \Phi_t(x)\| \ge \delta\Big) = 0. \tag{6}$$

Indeed, it suffices to apply Hypothesis 1.4(iii)(b) to the set

$$\mathscr{A} = \Big\{h \in C_T(\mathbf{R}^m)\colon \sup_{0 \le t \le T} \|h(t) - \Phi_t(x)\| \ge \delta\Big\}.$$

Here $\mathscr{A}$ is a closed set and for any $b \in K$ the infimum of $\mathscr{L}_{b,T}^+$ on $\mathscr{A}$ is positive. It easily follows that μ is an invariant measure of the flow Φ [see, e.g., Corollary 3.2 of Benaïm (1998) for more details]. □

A set $A \subset \mathbf{R}^m$ is an *attractor* provided A is a nonempty compact invariant set that has a neighborhood N such that $\lim_{t\to\infty} \operatorname{dist}(\Phi_t(x), A) = 0$ uniformly in $x \in N$. The basin of attraction of A is the open set

$$B(A) = \Big\{x \in \mathbf{R}^m\colon \lim_{t\to\infty} \operatorname{dist}(\Phi_t(x), A) = 0\Big\}.$$

A compact invariant set $K \subset \mathbf{R}^m$ is called *attractor-free* if the restricted flow $\Phi|K$ admits no attractor other than K itself. By a result of Conley (1978), K is attractor-free if and only if K is connected and every point of K is chain-recurrent for $\Phi|K$.

The following proposition is an immediate consequence of the Poincaré recurrence theorem.

PROPOSITION 3.2. *Let μ be a an invariant Borel probability measure of the flow Φ generated by a dissipative vector field. Then each component of the support of μ is attractor-free.*

PROOF. Let $X \subset \mathbf{R}^m$ denote the support of μ. By the Poincaré recurrence theorem, $\mu\{x \in \mathbf{R}^m\colon x \in \omega(x)\} = 1$ where $\omega(x)$ denotes the omega limit set of x. Therefore $X = \operatorname{clos}\{x \in X\colon x \in \omega(x)\}$. Hence every point of X is chain recurrent for $\Phi|X$. $\square$

Properties of attractor-free sets. The *vector order* in $\mathbf{R}^m$ is written $x \geq y$, with the meaning that $x_i \geq y_i$ for all i. If $x \geq y$ and $x \neq y$ then we write $x > y$. If $x_i > y_i$ for all i then we write $x \gg y$. For a cooperative and irreducible vector field, the flow Φ enjoys the fundamental property of being *strongly monotone* [Hirsch (1985), Kunze and Siegel (1994), Smith (1995)]. That is, $x > y$ implies

$$\Phi_t(x) \gg \Phi_t(y)$$

for all $t > 0$.

Given two subsets $A, B \subset \mathbf{R}^m$ we write $A \geq B$ ($A > B,\ A \gg B$) if $a \geq b$ ($a > b,\ a \gg b$) for all $a \in A,\ b \in B$. A set A is called *unordered* if no two of its points are related by $>$.

The following theorem is proved in Hirsch (1996).

THEOREM 3.3. *Let $K \subset \mathbf{R}^m$ be an attractor-free set for the flow Φ generated by a the vector field F as in Hypothesis* 1.2. *Then either K is unordered, or K is a simply ordered C^1 arc of equilibria whose relative interior points are stable.*

Let $K \subset \mathbf{R}^m$ be a compact unordered invariant set. Define

$$H^+(K) = \big\{x \in \mathbf{R}^m\colon \exists\ y \in K,\ s \geq 0\colon \Phi_s(x) \gg y\big\}$$

and

$$H^-(K) = \big\{x \in \mathbf{R}^m\colon \exists\ y \in K,\ s \geq 0\colon \Phi_s(x) \ll y\big\}.$$

The sets $H^+(K)$ and $H^-(K)$ are open positively invariant sets whose boundaries $V^+(K) = \partial H^+(K)$ and $V^-(K) = \partial H^-(K)$ are closed invariant unordered hypersurfaces homeomorphic to $\mathbf{R}^{m-1}$ which contain K. [See, e.g., Hirsch (1988a) and Takáč (1992)]. Recently Tereščák (1994) proved that these hypersurfaces are smooth (C^1).

COROLLARY 3.4. *Let $K \subset \mathbf{R}^m$ be an unordered attractor-free set that contains more than one point. Then there exists an attractor $A^+(K) \gg K$ whose basin of attraction contains $H^+(K)$. Similarily there exists an attractor $A^-(K) \ll K$ whose basin contains $H^-(K)$.*

PROOF. Let $[K, \infty[= \{x \in \mathbf{R}^m\colon x \geq K\}$ and let $p \in \mathbf{R}^m$ be the minimal element of $[K, \infty[$ for the vector ordering $\geq$. Clearly, $[K, \infty[= [p, \infty[$.

Since K is unordered, p is not in K. Therefore $p > K$, and by strong monotonicity and invariance of K we have $\Phi_t(p) \gg p$ for $t > 0$. It follows that the set $[K, \infty[= [p, \infty[$ is mapped into its interior by Φ_t for all $t > 0$. This last property together with the fact that the flow is dissipative imply that the set

$$A^+(K) = \bigcap_{t \geq 0} \Phi_t([K, \infty[)$$

is an attractor whose basin of attraction $B(A^+(K))$ contains $[K, \infty[$.

To show that $H^+(K) \subset B(A^+(K))$, it suffices to show that if $x \gg y$ for some $y \in K$, then $\omega(x) \geq K$.

Let $y \in K$ and $x \gg y$. By the limit set dichotomy [see Smith (1996)] either $\omega(x) \gg \omega(y)$ or else $\omega(x) = \omega(y) = \{e\}$ for some equilibrium e.

Suppose $\omega(x) = \omega(y) = e$. Let $\lambda_1(e)$ denote the largest real part of the eigenvalues of $DF(e)$. Since $DF(e)$ is irreducible, the Perron–Frobenius theorem applied to $DF(e) + aI$ for large $a > 0$ implies that $\lambda_1(e)$ is a simple eigenvalue whose corresponding eigenspace is spanned by a positive vector $u \gg 0$. Now by monotonicity, the open set $\{z\colon x \gg z \gg y\}$ is attracted by e. Therefore $\lambda_1(e) \leq 0$ and all other eigenvalues have negative real parts. This implies that e is an attractor for $\Phi | V^+(K)$. This is contradictory because K is attractor-free and $K \neq \{e\}$.

Now suppose $\omega(x) \gg \omega(y)$. Set $W = \{w \in K\colon w \ll \omega(x)\}$. Then W is a nonempty open subset of K positively invariant. Also $\Phi_t(\operatorname{clos}(W)) \subset W$ for $t > 0$ by strong monotonicity. Thus W contains an attractor for $\Phi | K$. Therefore $W = K$. □

The next proposition gives a result similar to Corollary 3.4 for unstable equilibria.

PROPOSITION 3.5. *Let $p \in \mathbf{R}^m$ be an unstable equilibrium for Φ. Then at least one of the two following conditions hold.*

(i) *There exists an attractor $A^+(p) \gg p$ whose basin contains $H^+(p)$; or*
(ii) *There exists an attractor $A^-(p) \ll p$ whose basin contains $H^-(p)$.*

If p is linearly unstable, both conditions (i) *and* (ii) *hold. If p is an endpoint of a simply ordered arc of equilibria $J \subset \mathcal{E}$, then condition* (i) *holds if $p = \inf J$ and* (ii) *holds if $p = \sup J$.*

PROOF. By a theorem of Mierczyński (1994), there exists a C^1 connected one-dimensional manifold $W_1 = W_1(p)$ through p defined in a neighborhood of p with the following properties: W_1 is tangent to the line L_p at p, W_1 is simply ordered by $\gg$ and W_1 is locally invariant in the sense that for every neighborhood N of p small enough, there exists $t_0 > 0$ such that that $\Phi_t(W_1 \cap N) \subset W_1$ for $|t| \leq t_0$. Since W_1 is one-dimensional and simply ordered, for every $x \in W_1 \cap N$ either $\Phi_t(x) \leq x$ for all $0 < t \leq t_0$ or $\Phi_t(x) \geq x$ for all $0 < t \leq t_0$.

Let $x, y \in W_1 \cap N$ with $x \gg p$ and $y \ll p$. Suppose that $\Phi_t(x) \leq x$ and $\Phi_t(y) \geq y$ for $0 < t \leq t_0$. Then $\Phi_t(x) \leq x$ and $\Phi_t(y) \geq y$ for all $t \geq 0$ by local invariance, and by monotonicity the set $[y, x] = \{z\colon y \leq z \leq x\}$ defines a positively invariant neighborhood of p.

Therefore, the assumption that p is unstable implies the existence of a neighborhood N of p and a positive number t_0 such that for all $x, y \in W_1 \cap N$ with $x \gg p$ and $y \ll p$ $\Phi_t(x) > x$ for all $0 < t \leq t_0$ or $\Phi_t(y) < y$ for all $0 < t \leq t_0$. Suppose, for example, that the first condition holds. Then there exists an equilibrium $e \gg p$ such that for all $x \in W_1 \cap N$ with $x \gg p$, $\lim_{t \to \infty} \Phi_t(x) = e$. Choose $x_0 \in W_1 \cap N$ with $x_0 \gg p$. Strong monotonicity implies that $\Phi_t[x_0, \infty[\subset \operatorname{int}([x_0, \infty[)$ for all $t > 0$. Thus by an argument similar to the one used in the proof of Corollary 3.4, we deduce the existence of an attractor $A^+(p)$ whose basin contains $[x_0, \infty[$. Let now $x \in H^+(p)$. We have $\Phi_t(x) \gg p$ for some $t \geq 0$. Thus $\Phi_t(x) \gg y$ for some $y \in W_1 \cap N$. Thus $\omega(x) \geq \lim_{t \to \infty} \Phi_t(y) = e \gg x_0$. Therefore x is in the basin of $A^+(p)$. □

The arguments used in the proof of Proposition 3.5 can be easily adapted to prove the following.

PROPOSITION 3.6. *Let p be a stable equilibrium of F. Suppose that there exists a neighborhood N of p such that $\mathcal{E} \cap N$ is unordered. Then p is asymptotically stable.*

In addition, we have the following proposition.

PROPOSITION 3.7. *Suppose F is real analytic in an open set $U \subset \mathbf{R}^m$. Let $p \in U$ be a stable equilibrium for F. Then p is asymptotically stable.*

The proof is a consequence of a result in Jiang (1991), implying that a real analytic dissipative vector field F is cooperative and irreducible, then it cannot have a nondegenerate, compact, totally ordered arc of equilibria; see also Lemma 3.3 and Theorem 2 in Jiang and Yu (1995), and Chow and Hale [(1982), page 321].

4. Proof of main results. In this section we will prove the following result (Theorem 4.1), and from it deduce Theorem 1.5. Throughout this section, Hypotheses 1.1, 1.2 and 1.4(ii) and (iii) are implicitly assumed.

THEOREM 4.1. *Let ν^ε be an invariant measure of $\{Z_n^\varepsilon\}_{n\in\mathbf{N}}$ and let $\mu^\varepsilon = \nu^\varepsilon \circ \Pi^{-1}$. Let $K \subset \mathbf{R}^m$ be an attractor-free set for Φ which contains either a nonequilibrium point or an unstable equilibrium point. Suppose that L^- is nondegenerate at K. Then there exists a neighborhood U of K such that $\lim_{\varepsilon\to 0} \mu^\varepsilon(U) = 0$.*

The key point needed to prove Theorem 4.1 is given by Lemma 4.4 below. In order to prove Lemma 4.4 we need some preliminary technical results given by the next two lemmas.

LEMMA 4.2. *Let $u \gg 0$ be a positive vector. For $r \in \mathbf{R}$, let $\Phi^{r,u}$ denote the flow induced by the ordinary differential equation*

$$\frac{dx}{dt} = F(x) + ru.$$

For all $t > 0$, $r_0 > 0$ the set

$$\Gamma(x, r_0, t, u) = \{\Phi^{r,u}(t, x)\colon -r_0 \le r \le r_0\}$$

is a C^1 simply ordered arc.

PROOF. The fact that $\Gamma(x, r_0, t, u)$ is a C^1 arc follows from standard results on the smoothness of solutions of differential equations depending on a parameter. Let $x, y \in \mathbf{R}^m$ with $x \ge y$. We claim that for $r > r'$ and $t > 0$, $\Phi^{r,u}(t, x) \gg \Phi^{r',u}(t, y)$.

Let $K = \{\Phi^{r,u}(t, x)\colon 0 \le t \le 1,\ |r| \le r_0\} \cup \{\Phi^{r,u}(t, y)\colon 0 \le t \le 1,\ |r| \le r_0\}$. Continuity of the map $t, r, z \to \Phi^{r,u}(t, z)$ makes K a compact set. Set $h(t) = \Phi^{r,u}(t, x) - \Phi^{r',u}(t, y)$. For $0 \le t \le 1$, a Taylor formula gives

$$\begin{aligned} h(t) &= h(0) + th'(0) + O(t^2) = x - y + t(F(x) - F(y)) + t(r - r')u + O(t^2) \\ &= \int_0^1 (x - y + tDF(x + sy)(x - y))\, ds + t(r - r')u + O(t^2). \end{aligned}$$

Since nondiagonal entries of $DF(z)$ are nonnegative, there exists $\alpha > 0$ such that $Id + tDF(x + sy) \ge 0$ for $0 \le t \le \alpha$ and $0 \le s \le 1$. Also, there exists $0 < \beta \le 1$ such that $t(r - r')u + O(t^2) \gg 0$ for $0 < t \le \beta$. Then $\Phi^{r,u}(t, x) - \Phi^{r',u}(t, y) \gg 0$ for $0 < t \le \eta = \min(\alpha, \beta)$. If now $t > \eta$, it suffices to write $t = p\eta + t'$, $p \in \mathbf{N}$, $0 \le t' < \eta$ and to use the flow property to show that $\Phi^{r,u}(t, x) - \Phi^{r',u}(t, y) \gg 0$. This proves the claim and Lemma 4.2 follows by choosing $y = x$. □

Let $K \subset \mathbf{R}^m$ be as in Theorem 4.1. By Theorem 3.3 one of the three following conditions holds:

(i) K is unordered and is not an equilibrium, or
(ii) $K = \{p\}$ is an unstable equilibrium, or
(iii) K is a subarc of a simply ordered arc of equilibria with an unstable endpoint p and $p \in K$.

Now, define the set $K^+ \gg K$ as follows. If K is as (i) or (ii) then $K^+ = A^+(K)$ where $A^+(K)$ is given by Corollary 3.4 in case (i) and Proposition 3.5 in case (ii). If K is as in (iii) we may assume that p is the upper endpoint of K (the other case being similar) and we set $K^+ = A^+(p)$ with $A^+(p)$ given by Proposition 3.5.

Finally, we let U_1 and U_2 denote disjoint compact neighborhoods of K and K^+ and we choose U_1 small enough to be contained in the neighborhood given by Definition 1.3.

LEMMA 4.3. *Given $\eta > 0$ and $V_2 \subset U_2$ a neighborhood of K^+. There exist a neighborhood of K, $V_1 \subset U_1$, times $T_1 \geq T_0 \geq 0$ and a continuous map*

$$\psi\colon V_1 \times \mathbf{R}_+ \to \mathbf{R}^m,$$
$$(x, t) \to \psi_x(t)$$

such that for all $x \in V_1$:

(a) *$\psi_x(0) = x$ and $\psi_x(t) \in V_2$ for all $t \geq T_1$;*
(b) *$\mathscr{L}^-_{x,t}(\psi_x) \leq \eta$ for all $t \geq 0$;*
(c) *The Lebesgue measure of set $\{t \geq 0\colon \psi_x(t) \in V_1\}$ is bounded by T_0.*

PROOF. We will consider two cases.

Case 1. K is as in (i) or (ii) above. By the nondegeneracy condition and the choice of the neighborhood U_1 there exist numbers $r_0 > 0$, $C > 0$ and a unit positive vector $u \gg 0$ such that $L^-(x, F(x) + ru) \leq C$ for all $x \in U_1$, $|r| \leq r_0$. Let $t_0 = \eta/C$. Using the notations of Lemma 4.2, for each $x \in U_1$, let J_x denote the ordered arc $J_x = \Gamma(x, r_0, t_0, u)$ and let $e(x) = \Phi^{r_0, u}(t_0, x)$ the right endpoint of $J(x)$. By Lemma 4.2, $e(x) \gg x$. Thus, for every $x \in K$, $e(x) \in H^+(K)$. Also, since K is compact and unordered there exists $\alpha > 0$ such that for every $x \in K$, $\operatorname{dist}(e(x), V^+(K)) > \alpha$. Set

$$V_1 = U_1 \cap e^{-1}\big(\{e \in H^+(K)\colon \operatorname{dist}(e, V^+(K)) > \alpha\}\big).$$

Since $H^+(K)$ is open and $x \to e(x)$ is continuous, V_1 defines a neighborhood of K.

Given $x \in V_1$ we now define a continuous function $\Psi_x\colon \mathbf{R}_+ \to \mathbf{R}^m$ by $\Psi_x(t) = \Phi_t^{r_0, u}(x)$ for $0 \leq t \leq t_0$ and $\Psi_x(t) = \Phi_t(e(x))$ for $t > t_0$. With this choice for Ψ_x, assertion (a) of Lemma 4.3 is satisfied because, according to Propositions 3.4 and 3.5, $t \to \Phi_t(e(x))$ converges toward K^+, and $\operatorname{clos}(e(V_1))$ is a compact subset of the basin of attraction of K^+.

To verify assertion (b) write

$$\mathscr{L}^{-}_{x,t}(\Psi_x) = \int_0^{t_0} L^{-}(\Phi_s^{r_0,u}(x), F(\Phi_s^{r_0,u}(x)) + r_0 u)\, ds + \int_{t_0}^{t} L^{-}(\Phi_s(x), F(\Phi_s(x)))\, ds.$$

The second integral in the right-hand side of this equality is zero and the first integral is bounded by $\int_0^{t_0} C\, ds = \eta$.

To prove (c) it suffices to show that the amount of time spent by the forward trajectory $\{\Phi_t(e(x)),\ t \geq 0\}$ in V_1, is bounded by some constant independent of the choice of $x \in V_1$. This is obvious because $\operatorname{clos}(e(V_1))$ is a compact subset of the basin of attraction of K^+ and V_1 is disjoint from U_2. This proves the lemma in this case.

Case 2. K is as in (iii). Let p be the upper endpoint of K and let $q \in W_1(p)$ with $q \gg p$ where $W_1(p)$ is a C^1 connected invariant manifold through p as used in the proof of Theorem 3.5. Set $J = K \cup (W_1(p) \cap [p, q])$. Since K and $W_1(p)$ are C^1 and ordered, there exists a piecewise C^1 map Θ: $[0,1] \to \mathbf{R}^m$ such that $\Theta([0,1]) = J$, $\Theta(1) = q$ and $\Theta(s) \ll \Theta(t)$ for $s < t$. We let S: $J \to [0,1]$ denote the inverse of Θ and we define the constant $A = 2(1 + \sup_{0 \leq t \leq 1} \|\Theta'(t)\|)$.

Let $\xi > 0$. Because K consists of equilibria, there exists $0 < \delta < 1$ such that for all $x \in U_\delta(K) = \{x \in \mathbf{R}^m\colon \operatorname{dist}(x, K) \leq \delta\}$, $\|F(x)\| \leq \xi$. Choose $m \in J \cap U_\delta(K)$ with $m \gg p$ and set $V_1 = \{x \in U_\delta(K)\colon x \leq m\}$. Now for each $x \in V_1$ let $c_x \in K$ be such that $\operatorname{dist}(x, K) = \|x - c_x\|$ and define a function h_x: $[0,1] \to \mathbf{R}^m$ by $h_x(t) = x + 2t(x - c_x)$ for $0 \leq t \leq 1/2$ and $h_x(t) = \Theta(S(x) + (2t-1))(S(m) - S(x))$ for $t \geq \frac{1}{2}$. Hence $h_x(0) = x$, $h_x(1) = m$ and $\|h_x'(t)\| \leq \max(2\delta, 2\sup_{0 \leq t \leq 1} \|\Theta'(t)\|) \leq A$.

Now, set $\Psi_x(t) = h_x(\xi t)$ for $t \leq 1/a$ and $\Psi_x(t) = \Phi_t(x)$ for $t \geq 1/\xi$.

Since m belongs to the basin of K^+, assertion (a) of Lemma 4.3 is satisfied, and since $m \in W_1(p)$ we must have $\Phi_t(m) \gg m \geq V_1$ for all $t > 0$. Thus assertion (c) of Lemma 4.3 is also satisfied. For $\xi > 0$ small enough $\|F(\Psi_x(t)) - \Psi_x'(t)\| \leq \xi + A\xi \leq r_0$. Thus by the nondegeneracy condition (Definition 1.3) we get that $L(\Psi_x(t), \Psi_x'(t)) = O(\xi^\alpha)$.

Then for all $t > 0$,

$$\mathscr{L}_{x,t}(\Psi_x) \leq \mathscr{L}_{x,1/\xi}(\Psi_x) = O(\xi^{\alpha-1}).$$

Since $\alpha > 1$ we can obviously choose ξ such that $\mathscr{L}_{x,t}(\Psi_x) \leq \eta$. $\square$

Next we follow Freidlin and Wentzell (1984) by introducing a convenient induced chain.

For any Borel set $V \subset \mathbf{R}^m$ set $\tilde{V} = \Pi^{-1}(V) \subset M$. Then define the *induced chain* on $\tilde{V}$ as the Markov chain $\{Z_n^{\varepsilon,\tilde{V}}\}_{n \in \mathbf{N}}$ living in $\tilde{V}$ whose transition probabilities $\tilde{P}_z^\varepsilon(\cdot)$ are given by

$$\tilde{P}_z^\varepsilon(B) = \mathbf{P}(Z^\varepsilon_{T_{\tilde{V}}} \in B \mid Z_0^\varepsilon = z),$$

where $T_{\tilde{V}} = \inf\{n \geq 1\colon Z_n^\varepsilon \in \tilde{V}\}$ and B is any Borel subset of $\tilde{V}$.

LEMMA 4.4. *There exists a neighborhood of K, $V_1 \subset U_1$, a neighborhood of K^+, $V_2 \subset U_2$, positive constants $\delta > \eta > 0$ and an integer-valued function $\varepsilon \to n(\varepsilon) \in \mathbf{N}$ such that for $V = V_1 \cup V_2$, the induced chain on $\tilde{V}$ satisfies*:

(i) $\liminf_{\varepsilon\to 0} \varepsilon \log(\tilde{P}_z^\varepsilon(n(\varepsilon), V_2)) \geq -\eta$ *uniformly in* $z \in \tilde{V}_1$.
(ii) $\limsup_{\varepsilon\to 0} \varepsilon \log(\tilde{P}_z^\varepsilon(n(\varepsilon), V_1)) \leq -\delta$ *uniformly in* $z \in \tilde{V}_2$.

Here $\tilde{P}_z^\varepsilon(n, \cdot)$ denotes the transition kernel of the induced chain in n steps.

PROOF. since K^+ is an attractor, we can choose a compact neighborhood of K^+, $V_2 \subset U_2$ with the property that

$$\inf\{\operatorname{dist}(\Phi_t(x), \partial U_2) \colon x \in V_2,\ t \geq 0\} > 0.$$

Fix an open neighborhood of K^+, $V_2' \subset V_2$ such that

$$\operatorname{dist}(V_2', {V_2}^c) > 0.$$

Let $\tau > 1$ be such that the time spent in $U_2 \setminus V_2'$ by any forward trajectory of Φ is bounded by $(\tau - 1)$ and $0 < \varepsilon_0 < 1$ such that

$$\varepsilon_0 \sup\{||F(x)|| \colon x \in U_2\} < \operatorname{dist}(V_2', {V_2}^c).$$

Given $x \in V_2$ let $\Gamma^{\varepsilon, x}$ denote the event that $\{X_n^{\varepsilon, x}\}$ leaves U_2 before reentering V_2. That is,

$$\Gamma^{\varepsilon, x} = \big\{\exists\ k \geq 1 \colon \forall 1 \leq i \leq k\ X_i^{\varepsilon, x} \notin V_2 \text{ and } X_k^{\varepsilon, x} \notin U_2\big\}.$$

Our first goal is to estimate the probability of $\Gamma^{\varepsilon, x}$. Define closed sets

$$\mathscr{A}_1 = \big\{h \in C_\tau(\mathbf{R}^m) \colon h(t) \in \partial U_2 \text{ for some } 0 \leq t \leq \tau\big\},$$

$$\mathscr{A}_2 = \big\{h \in C_\tau(\mathbf{R}^m) \colon \forall \varepsilon_0 \leq t \leq \tau\ h(t) \in U_2 \setminus V_2'\big\}$$

and

$$\mathscr{A}_3 = \big\{h \in C_\tau(\mathbf{R}^m) \colon V(h, \varepsilon_0, \tau) \geq \operatorname{dist}(V_2', V_2^c)\big\},$$

where

$$V(h, \varepsilon_0, \tau) = \sup\big\{||h(t+s) - h(t)|| \colon 0 \leq t \leq t+s \leq \tau,\ 0 \leq s \leq \varepsilon_0\big\}.$$

Clearly for $\varepsilon < \varepsilon_0$,

$$\begin{aligned}\Gamma^{\varepsilon, x} &\subset \big\{\overline{X}^{\varepsilon, x} \in \mathscr{A}_1\big\} \cup \big\{\overline{X}^{\varepsilon, x} \notin \mathscr{A}_1 \text{ and } \forall\ j = 1, \ldots, [\tau/\varepsilon]\ \overline{X}^{\varepsilon, x}(j\varepsilon) \notin V_2\big\}\\ &\subset \big\{\overline{X}^{\varepsilon, x} \in \mathscr{A}_1\big\} \cup \big\{\overline{X}^{\varepsilon, x} \in \mathscr{A}_2\big\} \cup \big\{\overline{X}^{\varepsilon, x} \in \mathscr{A}_3\big\}.\end{aligned}$$

Thus by the upper large deviation principle Hypothesis 1.4(iii)(b), there exist $a_i \in K$ such that

$$\limsup_{\varepsilon\to 0}\Big[\sup_{x\in V_2} \varepsilon \log(\mathbf{P}(\Gamma^{\varepsilon, x})\Big] \leq -\inf_{i=1,2,3} \mathscr{L}^+_{a_i, \tau}(\mathscr{A}_i) = -\delta, \tag{7}$$

where $\delta > 0$ because for all $i = 1, 2, 3$ the forward trajectory $t \to \Phi_t(a_i)$ is not in $\mathscr{A}_i$.

Fix $0 < \eta < \delta/2$. Now that V_2 and η have been chosen, Lemma 4.3 provides us a neighborhood V_1 of K and we define V as $V = V_1 \cup V_2$.

Using the notation of Lemma 4.3, set $T = T_0 + T_1$ and $n(\varepsilon) = [T/\varepsilon]$ and let $\mathscr{O} = \{h \in C_T(\mathbf{R}^m)\colon \|h - \psi_{h(0)}\| < \alpha\}$. Here $\mathscr{O}$ is an open set of paths and it is easy to see that

$$\{\overline{X}^{\varepsilon, x} \in \mathscr{O}\} \subset \{Z^{\varepsilon, \tilde{V}}_{n(\varepsilon)} \in \tilde{V}_2\}$$

for all $x \in V_1$ provided $\alpha > 0$ is small enough. Thus, by the large deviation principle [Hypothesis 1.4(iii)(a)] there exists $a \in V_1$ such that

$$\liminf_{\varepsilon \to 0} \varepsilon \log\big(\tilde{P}^{\varepsilon}_z(n(\varepsilon), V_2)\big) \geq -\mathscr{L}^{-}_{a, T}(\Psi_a) \geq -\eta$$

uniformly in $z \in \tilde{V}_1$. This proves inequality (i) of the lemma.

To prove inequality (ii) observe that inequality (7) implies

$$\limsup_{\varepsilon \to 0} \varepsilon \log\big(\tilde{P}^{\varepsilon}_z(\tilde{V}_1)\big) \leq -\delta \tag{8}$$

uniformly in $z \in \tilde{V}_2$. Now, by the Chapman–Kolmogorov formula we have

$$\begin{aligned} \tilde{P}^{\varepsilon}_z(n, \tilde{V}_1) &= \int_{\tilde{V}_2} \tilde{P}^{\varepsilon}_z(dy) \tilde{P}^{\varepsilon}_y(n-1, \tilde{V}_1) + \int_{\tilde{V}_1} \tilde{P}^{\varepsilon}_z(dy) \tilde{P}^{\varepsilon}_y(n-1, \tilde{V}_1) \\ &\leq \sup_{y \in \tilde{V}_2} \tilde{P}^{\varepsilon}_y(n-1, \tilde{V}_1) + \tilde{P}^{\varepsilon}_z(\tilde{V}_1) \leq n \sup_{y \in \tilde{V}_2} \tilde{P}^{\varepsilon}_y(\tilde{V}_1). \end{aligned} \tag{9}$$

Since $n(\varepsilon) \leq T/\varepsilon$, $\limsup_{\varepsilon \to 0} \varepsilon \log(n(\varepsilon)) = 0$. Therefore inequalities (8) and (9) yields the desired inequality. $\square$

The end of the proof is now a straightforward application of an argument used in Freidlin and Wentzell (1984). Let ν^{ε} be an invariant measure of $\{Z^{\varepsilon}_n\}$. Let V be as in Lemma 4.4. If $\nu^{\varepsilon}(\tilde{V}) \neq 0$, set $\tilde{\nu}^{\varepsilon} = \nu^{\varepsilon}/\nu^{\varepsilon}(\tilde{V})$. By Proposition 5.3 of Kifer (1988), $\tilde{\nu}^{\varepsilon}$ is an invariant measure of the induced chain. Now, for $i, j \in \{1, 2\}$ and $i \neq j$, define

$$m^{\varepsilon}_{i, j} = \frac{1}{\tilde{\nu}^{\varepsilon}(\tilde{V}_i)} \int_{\tilde{V}_i} \tilde{P}^{\varepsilon}_z(n(\varepsilon), \tilde{V}_j) \tilde{\nu}^{\varepsilon}(dz)$$

if $\tilde{\nu}^{\varepsilon}(\tilde{V}_i) \neq 0$ and $m^{\varepsilon}_{i, j} = 0$ otherwise. Set $m^{\varepsilon}_{1, 1} = 1 - m^{\varepsilon}_{1, 2}$ and $m^{\varepsilon}_{2, 2} = 1 - m^{\varepsilon}_{2, 1}$. Since $\tilde{\nu}^{\varepsilon}$ is an invariant measure of the induced chain, the probability vector $\pi^{\varepsilon} = (\pi^{\varepsilon}_1, \pi^{\varepsilon}_2)$ defined by $\pi^{\varepsilon}_i = \tilde{\nu}^{\varepsilon}(\tilde{V}_i)$, $i = 1, 2$, is an invariant probability vector of the 2×2 Markov chain defined by the transition matrix $(m^{\varepsilon}_{i, j})_{i, j = 1, 2}$. Thus

$$\pi^{\varepsilon}_2 m^{\varepsilon}_{2, 1} = \pi^{\varepsilon}_1 m^{\varepsilon}_{1, 2}.$$

This equality combined with Lemma 4.4 shows that

$$\frac{\pi^{\varepsilon}_1}{\pi^{\varepsilon}_2} = \frac{m^{\varepsilon}_{2, 1}}{m^{\varepsilon}_{1, 2}} \leq C \exp\left(\frac{\eta - \delta}{\varepsilon}\right)$$

for some constant $C > 0$. Therefore $\tilde{\nu}^\varepsilon(\tilde{V}_1)$ [and consequently $\nu^\varepsilon(\tilde{V}_1)$] goes to zero as $\varepsilon \to 0$. This concludes the proof of Theorem 4.1. □

If one looks carefully at the proof of Theorem 4.1, one can see that we use the full strength of the nondegeneracy assumption only in case K is a subarc of simply ordered arc of equilibria (Case 2 of Lemma 4.3).

Under a milder nondegeneracy assumption we have the following.

THEOREM 4.5. *Let ν^ε be an invariant measure of $\{Z_n^\varepsilon\}_{n\in\mathbf{N}}$ and let $\mu^\varepsilon = \nu^\varepsilon \circ \Pi^{-1}$. Let $K \subset \mathbf{R}^m$ be an attractor-free set for Φ. Assume K is an unstable point or K contains a nonequilibrium point. Assume furthermore that K satisfies the following mild nondegeneracy condition: there exists a neighborhood U of K, a unit positive vector $u \gg 0$ and a real number $r_0 > 0$ such that*

$$\sup_{x\in U,\,|r|<r_0} L(x, F(x) + ru) < \infty.$$

Then there exists a neighborhood U of K such that

$$\lim_{\varepsilon\to 0} \mu^\varepsilon(U) = 0.$$

PROOF OF THEOREM 1.5. Let $\mu = \lim_{\varepsilon_i\to 0} \mu^{\varepsilon_i}$ be a limit point of $\{\mu^\varepsilon\}$ and H a component of μ. By Proposition 3.2, H is attractor-free. Suppose H contains a nonequilibrium point or an unstable point. Then by Theorem 4.1, there exists a neighborhood U of H such that $\lim_{\varepsilon\to 0} \mu^\varepsilon(U) = 0$. We can always suppose that $\mu(\partial U) = 0$. Thus tightness of $\{\mu^\varepsilon\}_{\varepsilon>0}$ implies that $\mu(U) = \lim_{\varepsilon_i\to 0} \mu_{\varepsilon_i}(U) = 0$. This is contradictory with the fact that H is contained in the support of μ. Now Theorem 3.3 implies that H is either a stable equilibrium or a C^1 subarc of a maximal arc of stable equilibria. In case $\mathscr{E}$ is finite or F is real analytic, Propositions 3.6 and 3.7 show that H is an asymptotically stable equilibrium. □

5. Application: learning in coordination games. In this section we apply our results to study a class of learning or evolutionary processes which arise in game theory [Fudenberg and Levine (1998), Weibull (1995)].

We first set up the notation. Consider a *strategic game* in normal form in which a group of m players $i = 1, \dots, m$ play a stage game against one another. We assume that each player has two *pure strategies* or *actions* denoted by 0 and 1. In a one-shot game each player i chooses an action $s_i \in \{0, 1\}$ independently of the other players. As a result of these choices player i receives a *payoff* $U_i(s) \in \mathbf{R}$ where $s = (s_1, \dots, s_m)$ denotes the pure *strategy profile* of the players.

As usual in game theory we allow the possibility that players randomize their choices. The set of *mixed strategies* for player i is the unit interval $[0, 1]$ which is identified with the space of probability distributions over $\{0, 1\}$. Thus a mixed strategy $0 \le \sigma_i \le 1$ can be interpreted as the probability of playing action 1 by player i.

The payoff to player i corresponding to the mixed strategy profile $\sigma = (\sigma_1, \dots, \sigma_m)$ is the average payoff

$$U_i(\sigma) = \sum_{s \in \{0,1\}^m} \mathbf{P}_\sigma(s) U_i(s),$$

where

$$\mathbf{P}_\sigma(s) = \prod_{j=1}^{m} [\sigma_j s_j + (1-\sigma_j)(1-s_j)]$$

denotes the probability of the pure strategy profile $s = (s_1, \dots, s_m)$ corresponding to σ.

Given a mixed strategy profile σ for all the players, we denote by σ^{-i} the corresponding strategy profile for the players other than i. We let $U_i^0(\sigma^{-i})$ [respectively, $U_i^1(\sigma^{-i})$] denote the payoff obtained by player i when she plays 0 (respectively, 1) and her opponents play the mixed strategy profile σ^{-i}.

We assume players act on beliefs as follows. If player i believes her opponents will play the mixed strategy profile σ^{-i}, then the probability that she takes action 1 is given by a function $BR_i\colon [0,1]^{m-1} \to [0,1]$ of the form

$$BR_i(\sigma^{-i}) = \Psi_i\big[U_i^1(\sigma^{-i}) - U_i^0(\sigma^{-i})\big],$$

where $\Psi_i\colon \mathbf{R} \to [0,1]$ is some probability distribution function: an increasing function with $\lim_{t\to\infty} \Psi_i(t) = 1$ and $\lim_{t\to-\infty} \Psi_i(t) = 0$. We assume BR_i is smooth and call it *smooth best response function* for player i.

There are several interpretations for the use of smooth best response functions. One is that players attempt to choose actions that optimize expected payoffs, but make random mistakes. Another is that players decide to randomize their choice to avoid the possibility that their opponents exploit their choices [see Section 4.7 of Fudenberg and Levine (1998)]. A third interpretation, proposed by Fudenberg and Kreps in the spirit of Harsanyi's theory, is to assume that payoffs are subjected to small random perturbations. For more details and further game theoretic explanations we refer the interested reader to Chapter 4 of Fudenberg and Levine (1998).

DEFINITION 5.1. A mixed strategy profile σ is called a Nash distribution equilibrium if $\sigma_i = BR_i(\sigma^{-i})$ for all $i = 1 \dots m$.

We shall now consider a classical model of learning or evolution whose idea goes back to Nash in his Ph.D. dissertation. Consider a finite population of size N divided in m groups of players. Let $N_i = p_i N$ denote the size of group i where $p_i > 0$ and $\sum_i p_i = 1$. A *population state* is a vector $x = (x_1, \dots, x_m)$ where $0 \le x_i \le 1$ is the fraction of players in group i adopting strategy 1.

At each time $k = 1, 2, \dots,$ exactly m players, one in each group, are randomly chosen to play the strategic game. Let $\varepsilon = 1/N$ and let $X_k^\varepsilon = (x_1^\varepsilon(k), \dots, x_m^\varepsilon(k))$ be the population state at time k. Since player i (the player chosen in group i) does not know her opponents she sees $(X_k^\varepsilon)^{-i}$ as the mixed strategy profile of her opponents at time k. Therefore, at time $k+1$ she plays action $s_i(k+1) \in \{0, 1\}$ according to the probabilities

$$\mathbf{P}(s_i(k+1) = 1) = 1 - \mathbf{P}(s_i(k+1) = 0) = BR_i((X_k^\varepsilon)^{-i}).$$

At the end of the round $k+1$, all the players observe the strategies which have been played.

The process X^ε is thus a discrete-time Markov chain taking value in a finite lattice $L^\varepsilon \subset [0, 1]^m$. It satisfies the recursion

$$X_{k+1}^\varepsilon - X_k^\varepsilon = \varepsilon f(k, X_k^\varepsilon),$$

where $f(k, x) = (f_1(k, x), \ldots, f_m(k, x))$ and $\{f_i(k, x)\colon i = 1, \ldots, m;\ k = 1, 2, \ldots\}$ are independent random variables which verify

$$\mathbf{P}(f_i(k, x) = 1/p_i) = (1 - x_i) BR_i(x^{-i}),$$

$$\mathbf{P}(f_i(k, x) = -1/p_i) = x_i(1 - BR_i(x^{-i}))$$

and

$$\mathbf{P}(f_i(k, x) = 0) = (1 - x_i)(1 - BR_i(x^{-i})) + x_i BR_i(x^{-i}).$$

An important question about this kind of learning process is to investigate the long-term behavior of $\{X_k^\varepsilon\}$ and to verify whether or not players learn to play a Nash equilibrium. For two players and two strategies games, the process is always (in a sense to be made precise) a convergent process [see Benaïm and Hirsch (1997)]. However, for games with more that three players and without further assumption on the game to be played, there is no reason to expect such a "convergent" behavior. It is possible to construct examples where fictitious play leads to cyclic behavior and players do not learn to play Nash equilibria.

We shall now apply the mathematical results obtained in this paper to analyze the long term behavior of the process for a broad class of game that we now define.

DEFINITION 5.2. Given two distinct players i, j and a pair of actions $a, b \in \{0, 1\}$, let $T_{a,b}^{i,j}\colon \{0, 1\}^m \to \{0, 1\}^m$ be the map defined by

$$T_{a,b}^{i,j}(s) = \big(s_1, \ldots, s_{i-1}, a, s_{i+1}, \ldots, s_{j-1}, b, s_{j+1}, \ldots, s_m\big).$$

We say that player i coordinates with j if the function

$$C^{i,j} = \big[U_i \circ T_{1,1}^{i,j} + U_i \circ T_{0,0}^{i,j}\big] - \big[U_i \circ T_{0,1}^{i,j} + U_i \circ T_{1,0}^{i,j}\big]$$

is nonnegative. We say that i strictly coordinates with j if $C^{i,j}$ is nonnegative and is not identically zero.

The game will be called a *coordination game* if all players coordinates. To a coordination game we can associate a directed graph with vertices $\{1, \dots, m\}$ and oriented edges (i, j) if i strictly coordinates with j. We call a coordination game irreducible if this graph is irreducible.

LEMMA 5.3. *Assume*:

(a) *The game is an irreducible game of coordination.*
(b) *The maps* Ψ_i *are smooth* (C^1) *and verify* $0 < \Psi_i < 1$ *and* $\Psi_i' > 0$.

Then the dynamical system defined on $[0, 1]^m$ *by*

$$\frac{dx_i}{dt} = \frac{1}{p_i}(-x_i + BR_i(x^{-i})) \tag{10}$$

is a C^1 *cooperative and irreducible vector field with a compact attractor* $\Lambda \subset]0, 1[^m$.

PROOF. The fact that this vector field is cooperative and irreducible easily follows from Definition 5.2 and the conditions on Ψ_i. Since $0 < \Psi_i < 1$, the vector field points inward $[0, 1]^m$ at each point of the boundary of $[0, 1]^m$. This implies the existence of a global attractor $\Lambda \subset]0, 1[^m$. $\square$

Observe that the equilibria of (10) are the Nash distribution equilibria of the game.

THEOREM 5.4. *Suppose that the assumptions* (a) *and* (b) *of Lemma* 5.3 *hold. Let* $\mathscr{E} \subset [0, 1]^m$ *denote the set of stable equilibria of* (10). *Then for every neighborhood* U *of* $\mathscr{E}$,

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \mathbf{P}(X_n^\varepsilon \in U) = 1.$$

If, furthermore, the Ψ_i *are real analytic we can choose for* $\mathscr{E}$ *the set of asymptotically stable equilibria of* (10).

PROOF. For $x \in [0, 1]^m$, $\mathbf{E}(f(k, x)) = F(x)$ where F is the vector field given by (10) and the covariance matrix of $f(k, x)$ is the diagonal matrix whose ith entry is

$$\gamma_i^2(x) = (1 - x_i)x_i + (1 - BR_i(x^{-i}))BR_i(x^{-i}) \geq (1 - BR_i(x^{-i}))BR_i(x^{-i}).$$

By assumption (b) of Lemma 5.3, there exists $\gamma^2 > 0$ such that $\gamma_i^2(x) \geq \gamma^2$. Therefore, statements (i), (ii) and (iii) of Proposition 2.3 apply. Also, assumption (b) of Lemma 5.3 again, makes X^ε a finite aperiodic irreducible Markov chain. Thus it has a unique invariant measure μ^ε and $\lim_{n\to\infty} \mathbf{P}(X_n^\varepsilon \in U) = \mu^\varepsilon(U)$. The result then follows from Proposition 3.2 and Theorems 3.3 and 4.1. In case $\mathscr{E}$ is finite or Ψ_i real analytic, we use Propositions 3.6 and 3.7. $\square$

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