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CONTROL AND STOPPING OF A DIFFUSION PROCESS ON AN INTERVAL

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Consider a process $X(\cdot) = \{X(t),\ 0 \le t < \infty\}$ which takes values in the interval $I = (0, 1)$, satisfies a stochastic differential equation

$$dX(t) = \beta(t)\,dt + \sigma(t)\,dW(t), \qquad X(0) = x \in I$$

and, when it reaches an endpoint of the interval I, it is absorbed there. Suppose that the parameters β and σ are selected by a controller at each instant $t \in [0, \infty)$ from a set depending on the current position. Assume also that the controller selects a stopping time τ for the process and seeks to maximize $\mathbf{E}u(X(\tau))$, where $u\colon [0, 1] \to \Re$ is a continuous "reward" function. If $\lambda := \inf\{x \in I\colon u(x) = \max u\}$ and $\rho := \sup\{x \in I\colon u(x) = \max u\}$, then, to the left of λ, it is best to *maximize* the mean-variance ratio (β/σ^2) or to stop, and to the right of ρ, it is best to *minimize* the ratio (β/σ^2) or to stop. Between λ and ρ, it is optimal to follow any policy that will bring the process $X(\cdot)$ to a point of maximum for the function $u(\cdot)$ with probability 1, and then stop.

1. Formulation of the problem. Suppose that for every x in the interval $I = (l, r)$ with $-\infty < l < r < \infty$, there is a nonempty subset $\mathscr{K}(x)$ of $\Re \times (0, \infty)$ that specifies the drift–diffusion pairs (β, σ) available for controlling the stochastic process $X(\cdot)$ at any time $t \in [0, \infty)$, when the current position is $X(t) = x$. We also set $\mathscr{K}(l) = \mathscr{K}(r) = \{(0, 0)\}$, meaning that the endpoints of the interval I are absorbing barriers once they are reached.

More formally, consider a controlled diffusion process $X(\cdot)$ on a filtered probability space $(\Omega, \mathscr{F}, \mathbf{P})$, $\mathbf{F} = \{\mathscr{F}(t),\ 0 \le t < \infty\}$, and such that

$$dX(t) = \beta(t)\,dt + \sigma(t)\,dW(t), \qquad X(0) = x \in I. \tag{1}$$

Here $W(\cdot)$ is an $\mathbf{F}$-Brownian motion, and $\beta(\cdot)$, $\sigma(\cdot)$ are $\mathbf{F}$-progressively measurable processes that satisfy

$$\int_0^t [|\beta(u)| + \sigma^2(u)]\,du < \infty, \qquad (\beta(t), \sigma(t)) \in \mathscr{K}(X(t))$$

almost surely, for all $0 \le t < \infty$.

Given an initial position $X(0) = x \in I$, let us denote by $\mathscr{A}(x)$ the set of all processes $X(\cdot)$ that can be constructed this way (and are thus "available" to the controller at the initial position x). For every such process $X(\cdot) \in \mathscr{A}(x)$, let $\mathbf{F}^X := \{\mathscr{F}^X(t),\ 0 \le t < \infty\}$ be the filtration generated by the process

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$X(\cdot)$, where $\mathscr{F}^X(t) := \sigma(X(s),\ 0 \le s \le t)$ denotes the "history" of the process $X(\cdot)$ up to time t. Also, let $\mathscr{S}_X$ be the class of $\mathbf{F}^X$-stopping times, namely measurable functions τ: $\Omega \to [0, \infty]$ with the property $\{\tau \le t\} \in \mathscr{F}^X(t)$, for every $0 \le t < \infty$. Finally, let u: $[l, r] \to \Re$ be a continuous "reward" function.

The "leavable" stochastic control problem (with discretionary stopping) addressed in this paper, is to find, for each $x \in I$, a process $X^*(\cdot) \in \mathscr{A}(x)$ and an $\mathbf{F}^{X^*}$-stopping time τ_* that attain the supremum

$$V(x) := \sup_{X(\cdot)\in\mathscr{A}(x),\ \tau\in\mathscr{S}_X} \mathbf{E}u(X(\tau)). \tag{2}$$

Under mild regularity conditions, the problem of (2) admits a surprisingly simple solution. This is presented in Section 4, after some preliminary material on one-dimensional diffusions (Section 2) and on optimal stopping (Section 3). The solution incorporates features of the theory of optimal stopping for Markov processes [e.g., Dynkin and Yushkevich (1969), Fakeev (1971), Shiryaev (1978)], as well as aspects of "controlling a diffusion process to a goal" [cf. Pestien and Sudderth (1985)], in a rather unexpected way.

2. A brief review of one-dimensional diffusions. Consider a diffusion process

$$dX(t) = b(X(t))\,dt + s(X(t))\,dW(t), \qquad X(0) = x \in I, \tag{3}$$

where b: $I \to \Re$, s: $I \to \Re$ are Borel-measurable functions that satisfy

$$s^2(x) > 0, \qquad \int_{x-\varepsilon}^{x+\varepsilon} s^{-2}(y)[1 + |b(y)|]\,dy < \infty \quad \text{for some } \varepsilon > 0 \tag{4}$$

at every $x \in I$. Under these conditions, the stochastic differential equation (S.D.E.) of (3) has a weak solution which is unique in the sense of probability law, up to the "explosion time"

$$S := \inf\{t \ge 0\colon X(t) \notin I\} = \lim_{n\to\infty} \uparrow S_n. \tag{5}$$

We have set $S_n := \inf\{t \ge 0\colon X(t) \notin (l_n, r_n)\}$, where the sequence $\{l_n\}$ decreases to the left endpoint l of I, and the sequence $\{r_n\}$ increases to the right endpoint r of I [cf. Karatzas and Shreve (1991), page 341, Theorem 5.15]. On $\{S < \infty\}$ we set $X(t) := X(S)$ for $S \le t < \infty$, so that both l and r are *absorbing barriers*.

An important tool in the study of diffusion processes and for the solution to our problem is the *scale function*

$$p(x) := \int_c^x \exp\left[-2\int_c^\xi (b/s^2)(u)\,du\right] d\xi, \qquad x \in I \tag{6}$$

(with arbitrary but fixed $c \in I$). The function $p(\cdot)$ is a one-to-one mapping of the interval I onto the interval $\tilde{I} = (\tilde{l}, \tilde{r})$ with $\tilde{l} := p(l+)$, $\tilde{r} := p(r-)$; it is continuous, with derivative

$$p'(x) = \exp\left[-2\int_c^x (b/s^2)(u)\,du\right] > 0, \tag{7}$$

which is absolutely continuous, namely

$$p'(x) = 1 + \int_c^x p''(\xi)\,d\xi \quad \text{where } p''(x) := -\frac{2b(x)}{s^2(x)}\,p'(x)$$

for $x \in I$. The inverse mapping q: $\tilde{I} \to I$ satisfies $p(q(y)) = y$ and has derivative

$$q'(y) = \frac{1}{p'(q(y))} > 0, \qquad y \in \tilde{I}.$$

It is easy to see then that the process

$$Y(t) := p(X(t)), \qquad 0 \le t < \infty \tag{8}$$

satisfies the S.D.E. with zero-drift (in "natural scale")

$$dY(t) = \tilde{s}(Y(t))\,dW(t) \tag{9}$$

with $Y(0) = p(x) \in \tilde{I}$, where

$$\tilde{s}(y) := (p's)(q(y)), \qquad y \in \tilde{I}. \tag{10}$$

Observe that $\tilde{s}^2(y) > 0$ for every $y \in \tilde{I}$, by (4) and (7).

REMARK 2.1. Under the conditions (4), the process $X(\cdot)$ exits from every proper subinterval of I with probability 1 [cf. Karatzas and Shreve (1991), page 344]. Hence, the process $Y(\cdot)$ exits from every proper subinterval of $\tilde{I}$ with probability 1.

REMARK 2.2. Let us consider now the function

$$v(x) := \int_c^x p'(y) \int_c^y \frac{2\,dz}{p'(z)s^2(z)}\,dy, \qquad x \in I. \tag{11}$$

Feller's (1952) *test for explosions* states that a necessary and sufficient condition for the explosion time S of (5) to be infinite almost surely, is

$$v(l+) = v(r-) = \infty \tag{12}$$

[cf. Karatzas and Shreve (1991), page 348]. Of course, S is also the time of first exit for $Y(\cdot)$ from $\tilde{I}$; since this latter process is time-changed Brownian motion, it is clear that (12) is satisfied, if

$$p(l+) = -\infty, \qquad p(r-) = \infty.$$

Condition (12) fails if $\mathbf{P}[S < \infty]$ is positive. We have in fact $\mathbf{P}[S < \infty] = 1$, if and only if one of the following conditions

$$\begin{aligned} &v(l+) < \infty \quad \text{and} \quad v(r-) < \infty, \quad \text{or} \\ &p(l+) = -\infty \quad \text{and} \quad v(r-) < \infty, \quad \text{or} \\ &v(l+) < \infty \quad \text{and} \quad p(r-) = \infty \end{aligned} \tag{13}$$

holds; cf. Karatzas and Shreve (1991), page 350. The first of these conditions implies $\mathbf{E}(S) < \infty$, as well as

$$p(l+) > -\infty \quad \text{and} \quad p(r-) < \infty. \tag{14}$$

3. Optimal stopping of a diffusion process. Let $X(\cdot)$ be a diffusion process as in (3), and let $u\colon [l, r] \to \Re$ be a continuous function. Define the value function of the associated *optimal stopping problem*

$$Q(x) := \sup_{\tau \in \mathscr{S}_X} \mathbf{E}^x u(X(\tau)), \qquad x \in I. \tag{15}$$

Here and in the sequel, the superscript records the initial position of the process under consideration; $\mathscr{S}_X$ denotes the collection of $\mathbf{F}^X$-stopping times $\tau\colon \Omega \to [0, \infty]$, and we are using the convention $\xi(\infty) := \limsup_{t\to\infty} \xi(t)$.

PROPOSITION. *Assume the conditions* (4) *and* (14). *Then the function* $Q(\cdot)$ *is continuous on* $[l, r]$ *and can be written as the lower envelope of all affine transformations of the scale function* $p(\cdot)$ *that dominate* $u(\cdot)$, *namely*,

$$Q(x) = \inf\{\alpha + \beta p(x)\colon \alpha, \beta \in \Re,\ \alpha + \beta p(\cdot) \geq u(\cdot)\}, \qquad x \in I. \tag{16}$$

The stopping time

$$\tau_* := \inf\{t \geq 0\colon u(X(t)) = Q(X(t))\} \tag{17}$$

belongs to $\mathscr{S}_X$, *and attains the supremum in* (15).

REMARK 3.1. The optimal stopping region $\Sigma := \{x \in I\colon u(x) = Q(x)\}$ is a closed subset of the interval I, and τ_* is the time of first entry for the process $X(\cdot)$ into this region. From Remarks 2.2 and 2.1, respectively, it should be clear that τ_* is almost surely finite if condition (13) holds [since τ_* is no greater than the time it takes $X(\cdot)$ to leave the interval I], or if

(18) there exists $\varepsilon > 0$, such that the intervals $[l, l + \varepsilon]$ and $[r - \varepsilon, r]$ are included in Σ.

Under either of these two conditions, one can restrict attention to stopping times in $\mathscr{S}_X$ that are almost surely finite, without changing the value of the supremum in (15).

For the special case that $X(\cdot)$ is Brownian motion, an elegant and elementary proof of the proposition is given by Dynkin and Yushkevich (1969). More general treatments of optimal stopping problems for continuous-time processes can be found in Shiryaev (1973, 1978), Fakeev (1970, 1971), El Karoui (1981). The arguments in Dynkin and Yushkevich (1969) apply with only minor changes to the diffusion-with-zero-drift process $Y(\cdot)$ of (8) and (9), and it is easy to pass from the optimal stopping problem for $Y(\cdot)$ to that for $X(\cdot)$, as we now demonstrate.

Clearly $X(\cdot)$ and $Y(\cdot)$ generate the same filtration: $\mathbf{F}^X = \mathbf{F}^Y$. Consequently, with $\mathscr{S}_Y$ denoting the class of $\mathbf{F}^Y$-stopping times and setting

$$\tilde{u}(y) := u(q(y)), \tag{19}$$

$$\tilde{Q}(y) := \sup_{\tau \in \mathscr{S}_Y} \mathbf{E}^y \tilde{u}(Y(\tau)) \tag{20}$$

for $y \in \tilde{I}$, we have

$$Q(x) = \sup_{\tau \in \mathscr{S}_Y} \mathbf{E}^{p(x)} u(q(Y(\tau))) = \tilde{Q}(p(x)). \tag{21}$$

The proposition is thus immediate from the lemma below.

LEMMA. *Under the conditions* (4) *and* (14), *the function* $\tilde{Q}(\cdot)$ *is concave and continuous on the interval* $[\tilde{l}, \tilde{r}]$ *and is the lower envelope of all the affine functions that dominate* $\tilde{u}(\cdot)$,

$$\tilde{Q}(y) = \inf\{\alpha + \beta y \colon \alpha,\ \beta \in \Re,\ \alpha + \beta\xi \geq \tilde{u}(\xi) \text{ for all } \xi \in \tilde{I}\}, \qquad y \in \tilde{I}. \tag{22}$$

The stopping time τ_* *of* (17) *can also be written as*

$$\tau_* = \inf\{t \geq 0 \colon \tilde{u}(Y(t)) = \tilde{Q}(Y(t))\}, \tag{23}$$

and attains the supremum in (20).

SKETCH OF THE PROOF. The process $Y(\cdot)$ of (9) with $Y(\cdot) = y$ is a local martingale, and takes values in the compact interval $[\tilde{l}, \tilde{r}]$ [thanks to (9), (14)]. It is thus a bounded martingale and, if $\varphi(\cdot)$ is an affine (or even concave) function dominating $\tilde{u}(\cdot)$, we have by Jensen's inequality and the optional sampling theorem,

$$\mathbf{E}\tilde{u}(Y(\tau)) \leq \mathbf{E}\varphi(Y(\tau)) \leq \varphi(\mathbf{E}(Y(\tau)) = \varphi(y), \tag{24}$$

for every $\tau \in \mathscr{S}_Y$. It follows that $\tilde{Q}(\cdot) \leq \varphi(\cdot)$ and that the right-hand side of (22) dominates the left. On the other hand, $\tilde{Q}(\cdot) \geq \tilde{u}(\cdot)$ since we can choose $\tau \equiv 0$ in (20), and $\tilde{Q}(\cdot)$ is concave [cf. Dynkin and Yushkevich (1969), page 115]. Thus $\tilde{Q}(\cdot)$ dominates the right-hand side of (22), as this latter coincides with the smallest concave majorant of $\tilde{u}(\cdot)$, and the equality in (22) is proved.

To see that the stopping time τ_* of (17) *is optimal* for the problem (20), introduce the function

$$h(y) := \mathbf{E}^y \tilde{u}(Y(\tau_*)), \qquad y \in \tilde{I}$$

and argue [as in Dynkin and Yushkevich (1969), pages 117 and 118] that $h(\cdot)$ is a concave majorant of $\tilde{u}(\cdot)$; hence, $h(\cdot) \geq \tilde{Q}(\cdot)$. The reverse inequality is an immediate consequence of the definition (20) of $\tilde{Q}(\cdot)$. The *continuity* of $\tilde{Q}(\cdot)$ on the open interval $\tilde{I} = (\tilde{l}, \tilde{r})$ follows from concavity; its continuity at the endpoints $\tilde{l}$ and $\tilde{r}$ requires a special argument, which is provided in Dynkin and Yushkevich [(1969), pages 116 and 117]. □

From elementary properties of concave functions [e.g., Karatzas and Shreve (1991), Section 3.6], $\tilde{Q}(\cdot)$ has left- and right-derivatives $D_\pm \tilde{Q}(\cdot)$ everywhere on $\tilde{I}$; these latter functions are decreasing and left- (resp., right-) continuous, and satisfy

$$D_+ \tilde{Q}(y) \le D_- \tilde{Q}(y), \qquad y \in \tilde{I}. \tag{25}$$

The inequality is strict for at most countably many points.

From (21) and (24), we have

$$D_\pm Q(x) = p'(x) D_\pm \tilde{Q}(p(x)) \tag{26}$$

and thus

$$D_- Q(x) - D_+ Q(x) = p'(x)[D_- \tilde{Q}(p(x)) - D_+ \tilde{Q}(p(x))] \ge 0, \qquad x \in I. \tag{27}$$

Again, strict inequality occurs on a set which is at most countable.

Let us introduce now the smallest and largest locations of the maximum

$$u^* := \max_{x \in [l, r]} u(x) \tag{28}$$

of the reward function $u(\cdot)$, namely,

$$\lambda := \inf\{x \in I\colon u(x) = u^*\}, \qquad \rho := \sup\{x \in I\colon u(x) = u^*\}, \tag{29}$$

respectively, and their counterparts

$$\tilde{\lambda} := p(\lambda) = \inf\{y \in \tilde{I}\colon \tilde{u}(y) = u^*\}, \qquad \tilde{\rho} := p(\rho) = \sup\{y \in \tilde{I}\colon \tilde{u}(y) = u^*\}.$$

From the fact that $\tilde{Q}(\cdot)$ is the smallest concave majorant of $\tilde{u}(\cdot)$ (cf. the Lemma), we have

$$\begin{cases} D_+ \tilde{Q}(\cdot) \ge 0, & \text{on } (\tilde{l}, \tilde{\lambda}), \\ D_- \tilde{Q}(\cdot) \le 0, & \text{on } (\tilde{\rho}, \tilde{r}), \\ \tilde{Q}(\cdot) = u^*, & \text{on } [\tilde{\lambda}, \tilde{\rho}]. \end{cases} \tag{30a}$$

By virtue of (21), these lead to

$$\begin{cases} D_+ Q(\cdot) \ge 0, & \text{on } (l, \lambda), \\ D_- Q(\cdot) \le 0, & \text{on } (\rho, r), \\ Q(\cdot) = u^*, & \text{on } [\lambda, \rho]. \end{cases} \tag{30b}$$

4. Solution of the problem. Consider again the stochastic control problem of Section 1. We shall assume that there exist two pairs (b_r, s_r) and (b_l, s_l) of Borel-measurable, real-valued functions on I, each of which satisfies the conditions of (4) and (14), as well as

$$(b_l(x), s_l(x)) \in \mathscr{K}(x), \qquad (b_r(x), s_r(x)) \in \mathscr{K}(x), \tag{31}$$

$$\frac{b_l(x)}{s_l^2(x)} = \sup\left\{\frac{\beta}{\sigma^2}: (\beta, \sigma) \in \mathscr{K}(x)\right\}, \tag{32}$$

$$\frac{b_r(x)}{s_r^2(x)} = \inf\left\{\frac{\beta}{\sigma^2}: (\beta, \sigma) \in \mathscr{K}(x)\right\}, \tag{33}$$

for all $x \in I$. Consider the corresponding diffusion processes $X_l(\cdot)$, $X_r(\cdot)$ with these parameters, namely,

$$dX_l(t) = b_l(X_l(t))\,dt + s_l(X_l(t))\,dW(t),$$

$$dX_r(t) = b_r(X_r(t))\,dt + s_r(X_r(t))\,dW(t)$$

as in (3), and let $Q_l(\cdot) = \sup_{\tau \in \mathscr{S}_{X_l}} \mathbf{E}^{\cdot} u(X_l(\tau))$, $Q_r(\cdot) = \sup_{\tau \in \mathscr{S}_{X_r}} \mathbf{E}^{\cdot} u(X_r(\tau))$ be the value functions for the associated optimal stopping problems in (15) with the same reward function $u(\cdot)$.

THEOREM. *With the above assumptions and notation, the value function* $V(\cdot)$ *for the stochastic control problem of* (2) *satisfies*

$$V(x) = \begin{cases} Q_l(x), & x \in (l, \lambda), \\ Q_r(x), & x \in (\rho, r), \\ u^*, & x \in [\lambda, \rho]. \end{cases} \tag{34}$$

In other words, the theorem states the following.

1. To the left of λ, it is best either to *maximize* the mean-variance ratio (β/σ^2), or to *stop* [when this is optimal in the stopping problem (15) for the diffusion process $X_l(\cdot)$].
2. To the right of ρ, it is best either to *minimize* the mean-variance ratio (β/σ^2), or to *stop* [when this is optimal in the stopping problem (15) for the diffusion process $X_r(\cdot)$].
3. In the interval $[\lambda, \rho]$, it is optimal to follow any policy that will bring the state-process $X(\cdot)$ arbitrarily close to a point of maximum of the function $u(\cdot)$, with probability 1—for example, by following the dynamics of either $X_l(\cdot)$ or $X_r(\cdot)$—and then stop.

Notice that any one of the intervals (l, λ), (ρ, r) or (λ, ρ) could be empty.

REMARK 4.1. It was shown by Pestien and Sudderth (1985) that the best way to control a diffusion, so as to maximize the probability of ever reaching the right-hand endpoint of an interval ("goal"), is to *maximize* the mean-variance ratio (β/σ^2). By analogy then, the best way to maximize the chance of reaching a "goal" on the left, is to *minimize* this same ratio. Thus, the strategy described in 1–3 above seeks to maximize the chance of moving toward the location of a maximum of the reward function $u(\cdot)$, or to stop. For the problem of Section 1, it is not obvious a priori that one should follow such a strategy everywhere on I, even in the vicinity of local maxima of $u(\cdot)$.

PROOF OF THE THEOREM. Let $H(\cdot)$ be the function on the right-hand side of (34). Then $H(\cdot) \le V(\cdot)$ on I because, for every $x \in I$, $H(x)$ is the expected payoff from the policy described in 1–3 above. Thus, it remains to be shown that $V(\cdot) \le H(\cdot)$.

Clearly from (34), it suffices to show $V(\cdot) \le Q_l(\cdot)$ on the interval (l, λ) [a similar methodology will show as well that $V(\cdot) \le Q_r(\cdot)$ holds on the interval (ρ, r)]. Thus, let us fix $x \in (l, \lambda)$ and observe that the value function of (2) can also be written then as

$$V(x) = \sup_{X(\cdot)\in\mathscr{A}(x),\, \tau\in\mathscr{S}_X} \mathbf{E}u(X(\tau \wedge \tau_\lambda^X)), \qquad x \in (l, \lambda), \tag{35}$$

where we have set

$$\tau_\lambda^X := \inf\{t \ge 0\colon X(t) = \lambda\} \tag{36}$$

for any given process $X(\cdot) \in \mathscr{A}(x)$. Indeed, *once the nearest global maximum of the reward function $u(\cdot)$ has been reached, there is no point in proceeding any further*.

On the other hand, let us consider the bounded processes

$$\xi_l(t) := p_l(X(t)), \qquad 0 \le t < \infty \tag{37}$$

and

$$\eta_l(t) := \tilde{Q}_l(\xi_l(t)) = Q_l(X(t)), \qquad 0 \le t < \infty. \tag{38}$$

From Itô's rule and (6), (32) we conclude that

$$\begin{aligned}\xi_l(t) &= p_l(x) + \int_0^t p_l'(X(s))\,dX(s) + \left(\frac{1}{2}\right)\int_0^t p_l''(X(s))\sigma^2(s)\,ds\\ &= p_l(x) - \int_0^t \left[\frac{b_l(X(s))}{s_l^2(X(s))} - \frac{\beta(s)}{\sigma^2(s)}\right] p_l'(X(s))\sigma^2(s)\,ds\\ &\quad + \int_0^t p_l'(X(s))\sigma(s)\,dW(s), \qquad 0 \le t < \infty\end{aligned}$$

is a supermartingale. It is an easy consequence of Jensen's inequality that *a concave, bounded and increasing function of a supermartingale is again a supermartingale*; therefore, from (38) and (30), we deduce that

$$\text{the process } \eta_l(\cdot \wedge \tau_\lambda^X) \text{ is a supermartingale.} \tag{39}$$

It follows then, from (39) and the optional sampling theorem, that we have

$$Q_l(x) = \eta_l(0) \ge \mathbf{E}\eta_l(\tau \wedge \tau_\lambda^X) = \mathbf{E}Q_l(X(\tau \wedge \tau_\lambda^X)) \ge \mathbf{E}u(X(\tau \wedge \tau_\lambda^X)) \tag{40}$$

for arbitrary $X(\cdot) \in \mathscr{A}(x)$ and $\tau \in \mathscr{S}_X$. Taking the supremum in (40) over $X(\cdot) \in \mathscr{A}(x)$ and $\tau \in \mathscr{S}_X$, we obtain in conjunction with (35) the inequality

$$Q_l(x) \geq V(x),$$

as promised. □

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