The Annals of Applied Probability
1999, Vol. 9, No. 1, 110–145

ASYMPTOTICS OF FIRST PASSAGE TIMES FOR RANDOM WALK IN AN ORTHANT[1]

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We wish to describe how a chosen node in a network of queues overloads. The overloaded node may also drive other nodes into overload, but the remaining "super" stable nodes are only driven into a new steady state with stochastically larger queues. We model this network of queues as a Markov additive chain with a boundary. The customers at the "super" stable nodes are described by a Markov chain, while the other nodes are described by an additive chain. We use the existence of a harmonic function h for a Markov additive chain provided by Ney and Nummelin and the asymptotic theory for Markov additive processes to prove asymptotic results on the mean time for a specified additive component to hit a high level l. We give the limiting distribution of the "super" stable nodes at this hitting time. We also give the steady-state distribution of the "super" stable nodes when the specified component equals l. The emphasis here is on *sharp* asymptotics, not rough asymptotics as in large deviation theory. Moreover, the limiting distributions are for the unscaled process, not for the fluid limit as in large deviation theory.

1. Introduction. The following introduction describes the model and states the main theorems. It is highly recommended that the Flatto–Hahn–Wright example in Section 3.1 be read in parallel with this introduction. Unless a proof immediately follows the statement of a result, it is deferred to Section 2.3. The symbol for a Markov chain W may be twisted, in which case it will be written in script letters as $\mathscr{W}$. The chain W may be transformed into a free process, in which case it will be noted by W^∞. The time reversal will be denoted by W^*. We will use this convention for a variety of symbols throughout the paper.

1.1. *Definitions.* Many large stochastic systems are modeled as a Harris recurrent (irreducible) Markov chain W with a state space $(S, \mathscr{S})$ with a stationary probability distribution π and kernel K. Let $L := K - I$ be the discrete generator. We are often interested in the measure $\pi(F)$ of some rare, forbidden set $F \in \mathscr{S}$, where $\pi(F) > 0$. Alternatively, we may consider the expected value of the time T_F until the chain reaches F. We may also be interested in the hitting distribution on F.

The problem of finding the mean time $m(x)$ for the chain to hit F starting from any initial point x means solving the linear system

$$Lm(x) = -1 \quad \text{for } x \in B := F^c \qquad \text{and} \qquad m(x) = 0 \quad \text{for } x \in F. \tag{1.1}$$

Received July 1995; revised February 1998.
[1]Supported in part by NSERC Grant A4551.
AMS 1991 *subject classifications*. Primary 60K25; secondary 60K20.
Key words and phrases. Random walk, rare events, change of measure.

The problem of finding $e_A(x)$, the probability of hitting F in $A \subseteq F$ (where $A \in \mathscr{S}$) starting from x, means solving the problem

$$\begin{aligned} Le_A(x) = 0 \quad \text{for } x \in B, \qquad e_A(x) = 1 \quad \text{for } x \in A \\ \text{and} \quad e_A(x) = 0 \quad \text{for } x \in F \setminus A. \end{aligned} \tag{1.2}$$

If the space is high-dimensional, then both of these problems may become numerically intractable. Worse, if $\pi(F)$ is small, then they cannot even be solved by simulation because it would take so long to simulate the event of interest; that is, an entrance into F!

Here we consider a sequence of forbidden sets $F_l \equiv F$ such that $\pi(F) \to 0$. We address the problem of how to give the exact asymptotics of the expected value of the time T_F to reach the forbidden set F. Our second problem is to find the limiting hitting distribution on F, and the third problem is to give an asymptotic expression for π on F. The main results in this paper, Theorems 1.10, 1.13 and 1.6, give answers to these three problems for Markov chains which may be viewed as a Markov additive chain with a boundary.

Given the kernel K, we can always construct the kernel $K' := (K + I)/2$. Note that this new kernel has null transitions with probability 1/2. Both K and K' have stationary distribution π. Moreover, let L' be the discrete generator associated with K' and let m' and e'_A be the solutions of the systems (1.1) and (1.2) with L' replacing L. Clearly, $m'(x) = 2m(x)$ and $e'_A(x) = e_A(x)$. Consequently, by adjusting the time scale, we may as well assume our Markov chain W has null transitions with probability 1/2.

We note that our results apply to continuous-time Markov jump processes on $(S, \mathscr{S})$ with bounded generators G [see Iscoe and McDonald (1994a) for definitions]. We may construct a Markov chain W on $(S, \mathscr{S})$ with discrete generator $L := G/q$, where q is the event rate of the generator G. W has the same stationary probability distribution as the jump process. Moreover, the hitting distribution on F by W is the same as that of the original jump process. Finally, the time until the jump process hits F is the same as the time for the uniformization of the Markov chain W to hit F. By calculation, the expected time for the uniformized chain to hit F is $m(x)/q$. If we assume the time scale is such that $q = 1$, then the results for Markov chains immediately imply corresponding results for Markov jump processes.

Consider a Markov additive chain $W^\infty \equiv (\widetilde{W}^\infty, \widehat{W}^\infty)$ defined on a probability space $(\Omega, \mathscr{F}, P)$. W^∞ is a Markov chain with kernel K^∞ on the measurable space $(S^\infty, \mathscr{S}^\infty) := (R^r \times \widehat{S}, \mathscr{B}^r \otimes \widehat{\mathscr{S}})$, where R and $\mathscr{B}$ respectively denote the integers and subsets of the integers in the discrete-time case (in the discrete case, the integers are also denoted by $\mathbb{Z}$) or the reals and the Borel sets in the continuous time case (in the continuous case, the reals are also denoted by $\mathbb{R}$). The Markovian component $\widehat{W}^\infty$ is a ψ-recurrent [see Meyn and Tweedie (1993) for the definition], aperiodic Markov chain with state space $(\widehat{S}, \widehat{\mathscr{S}})$. The additive component $\widetilde{W}^\infty$ can be viewed as a multidimensional random walk taking values in $(R^r, \mathscr{B}^r)$.

Decompose a point $\mathbf{x} \in S^\infty$ as $\mathbf{x} = (\tilde{x}, \hat{x}) \equiv (x_1, x_2, \ldots, x_r, \hat{x})$, where $\tilde{x} \in R^r$, $\hat{x} \in \widehat{S}$. If $\Gamma \in \mathscr{B}^r$ and $\widehat{A} \in \widehat{\mathscr{S}}$, then the kernel of W^∞ satisfies

$$K^\infty(\mathbf{x}, \Gamma \times \widehat{A}) = P\big(\widehat{W}^\infty[1] \in \widehat{A}, \tilde{x} + \widetilde{W}^\infty[1] - \widetilde{W}^\infty[0] \in \Gamma | \widehat{W}^\infty[0] = \hat{x}\big).$$

Let L^∞ denote the generator of W^∞. As above, W^∞ decomposes into components $(\widetilde{W}^\infty, \widehat{W}^\infty)$. Let m denote the counting measure in the discrete case and the Lebesgue measure in the continuous case.

Define the additive increment associated with the jump from state $\widehat{W}^\infty[n-1]$ to $\widehat{W}^\infty[n]$ to be $X^\infty[n] := \widetilde{W}^\infty[n] - \widetilde{W}^\infty[n-1]$. Hence, if $\widetilde{W}^\infty[0] = X^\infty[0]$, $\widetilde{W}^\infty[n] = \sum_{i=0}^n X^\infty[i]$ describes the sum of the increments associated with getting to state $\widehat{W}^\infty[n]$ in n steps. The components of the increment X^∞ may, of course, be negative or zero.

Now consider a Markov chain W with probability transition kernel K, stationary probability measure π and state space $S \subset S^\infty$. We assume the existence of a boundary $\blacktriangle \subset S^\infty$ such that the probability transition kernel K of the Markov chain W in S has transitions which agree with those of K^∞ within the interior of S; that is, $K(\mathbf{x}, C) = K^\infty(\mathbf{x}, C)$ if $\mathbf{x} \in S^\infty \setminus \blacktriangle$ and $C \subset S^\infty \setminus \blacktriangle$. Let the edge of the boundary $\blacktriangle$ be denoted by $\triangle \equiv \blacktriangle \cap S$. In many cases, S will be the measurable product space of the positive orthant times $\widehat{S}$ given by $(S, \mathscr{S}) := ((R^+)^r \times \widehat{S}, (\mathscr{B}^+)^r \otimes \widehat{\mathscr{S}})$, where R^+, respectively $\mathscr{B}^+$, denotes the nonnegative integers (also denoted by $\mathbb{N}_0$), respectively subsets of nonnegative integers in the discrete case and the nonnegative reals (also denoted by $\mathbb{R}^+$), respectively the Borel sets on the nonnegative reals, in the continuous case.

Decompose W into components $(\widetilde{W}, \widehat{W})$, where $\widehat{W} \in \widehat{S}$. We shall systematically use an upper index of ∞ to indicate kernels (like K^∞) or chains (like W^∞) which are *free* in the sense that the additive components are unconstrained in R^r. Let the origin D be such that $\pi(D) > 0$ and such that $D \subseteq \triangle$. For any set $C \in \mathscr{S}$, let $\pi_C(\cdot) := \pi(\cdot \cap C)/\pi(C)$ and let $E_C := \int_C \pi_C(d\mathbf{z}) E_{\mathbf{z}}$. Of course, $E_{\mathbf{z}}$ will always represent the expectation operator for a chain started at state $\mathbf{z}$.

In some examples, there is a natural extension of this theory. It is not necessary that $\widetilde{W}^\infty$ be an additive process. Instead, the kernel K^∞ could have the following decomposition:

$$\begin{aligned} &K^\infty\big((z_1, z_2, \ldots, z_r, \hat{z});\ (z_1 + dx_1, dx_2, \ldots, dx_r, d\hat{x})\big) \\ &\qquad = \widehat{K}^\infty(\hat{z}, d\hat{x}) k_1(dx_1 | \hat{z}, \hat{x}) \tilde{k}\big((z_2, \ldots, z_r);\ (dx_2, \ldots, dx_r) | \hat{z}, \hat{x}, z_1, x_1\big), \end{aligned}$$

where $\tilde{k}$ is a probability transition kernel conditioned on the transitions of $\widehat{W}^\infty$. Hence, given the transition of $\widehat{W}^\infty$, the components $W_2^\infty, \ldots, W_r^\infty$ make a Markovian transition dependent on the additive first component. In this extension, there is no general methodology for finding the harmonic function required in Section 1.2; however, if Conditions 1–7 given below hold, then we can redo the proofs of Theorems 1.6, 1.13 and 1.10. Only Theorem 2.10 requires an extra hypothesis.

1.2. *Hypotheses.* Suppose there exists a positive function h such that $h(\mathbf{x}) = \int_{\mathbf{y} \in S^\infty} K^\infty(\mathbf{x}, d\mathbf{y}) h(\mathbf{y})$. Hence h is harmonic for K^∞. Perform the h-

transform to construct the twisted kernel $\mathscr{K}^\infty(\mathbf{x}, d\mathbf{y}) := K^\infty(\mathbf{x}, d\mathbf{y})h(\mathbf{y})/h(\mathbf{x})$. Let the generator of the associated twisted chain $\mathscr{W}^\infty$ be $\mathscr{L}^\infty$. In Section 2.1, we use the theory in Ney and Nummelin (1987a) to construct a harmonic function for K^∞ of the form $h(\mathbf{x}) = \exp(\alpha x_1)\hat{a}(\hat{x})$. We shall systematically denote any object which has been twisted with script letters.

Decompose $\mathscr{W}^\infty$ as a Markov additive chain $(\widetilde{\mathscr{W}}^\infty, \widehat{\mathscr{W}}^\infty)$. The Markov chain $\widehat{\mathscr{W}}^\infty$ has state space $\widehat{S}$. The vector $\widetilde{\mathscr{W}}^\infty[n]$ can be viewed as a multidimensional additive component taking values in R^r. Define the increment associated with the jump from state $\widetilde{\mathscr{W}}^\infty[n-1]$ to $\widetilde{\mathscr{W}}^\infty[n]$ to be $\mathscr{X}^\infty[n] := \widetilde{\mathscr{W}}^\infty[n] - \widetilde{\mathscr{W}}^\infty[n-1]$. Hence, if $\widetilde{\mathscr{W}}^\infty[0] = \mathscr{X}^\infty[0]$, then $\widetilde{\mathscr{W}}^\infty[n] = \sum_{i=0}^n \mathscr{X}^\infty[i]$ describes the sum of the increments to get to state $\widetilde{\mathscr{W}}^\infty[n]$ in n steps. The pairs $(\widetilde{\mathscr{W}}^\infty[n], \widehat{\mathscr{W}}^\infty[n])_{n=0}^\infty$ define a Markov chain on $S^\infty := R^r \times \widehat{S}$. If $\Gamma \in \mathscr{B}^r$ and $\widehat{A} \in \widehat{\mathscr{S}}$, then the kernel of $(\widetilde{\mathscr{W}}^\infty[n], \widehat{\mathscr{W}}^\infty[n])$ satisfies

$$\mathscr{K}^\infty(\mathbf{x}, \Gamma \times \widehat{A}) = P(\widehat{\mathscr{W}}^\infty[1] \in \widehat{A}, \tilde{x} + \widetilde{\mathscr{W}}^\infty[1] - \widetilde{\mathscr{W}}^\infty[0] \in \Gamma | \widehat{\mathscr{W}}^\infty[0] = \hat{x}).$$

The associated twisted generator applied to a bounded measurable function g satisfies

$$\begin{aligned} \mathscr{L}^\infty g(\mathbf{x}) &:= \int_{\mathbf{y} \in \mathscr{S}^\infty} [g(\mathbf{y}) - g(\mathbf{x})] K^\infty(\mathbf{x}, d\mathbf{y}) \frac{h(\mathbf{y})}{h(\mathbf{x})} \\ &= \int_{\mathbf{y} \in \mathscr{S}^\infty} \left[\frac{h(\mathbf{y})}{h(\mathbf{x})} g(\mathbf{y}) - g(\mathbf{x}) \right] K^\infty(\mathbf{x}, d\mathbf{y}) \\ &:= \frac{1}{h(\mathbf{x})} L^\infty(h \cdot g)(\mathbf{x}) \end{aligned} \tag{1.3}$$

for all $\mathbf{x} \in S^\infty$.

We recall that, without loss of generality, we may assume K has a null transition with probability 1/2. Hence $\widehat{K}^\infty$ and $\widehat{\mathscr{K}}^\infty$, the kernels of the chains $\widehat{W}^\infty$ and $\widehat{\mathscr{W}}^\infty$, are aperiodic. We impose the following conditions.

CONDITION 1. $\widehat{\mathscr{W}}^\infty$ is a ψ-recurrent Markov chain with a stationary probability measure φ [see Meyn and Tweedie (1993) for definitions].

CONDITION 2. The Markov additive chain $(\widetilde{\mathscr{W}}^\infty[n], \widehat{\mathscr{W}}^\infty[n])$ satisfies Conditions M1 and N1 in Section 2.1. In the discrete case, $\widetilde{\mathscr{W}}^\infty$ is aperiodic as defined in Condition P1, while in the continuous case, $\widetilde{\mathscr{W}}^\infty$ is spread out as defined in Condition P2.

CONDITION 3. $\tilde{d}_1 > 0$, where

$$\begin{aligned} \tilde{d} &\equiv \int_{\hat{z}} \varphi(d\hat{z}) E_{(\tilde{0}, \hat{z})} \mathscr{X}^\infty[1] \\ &= \int_{\hat{z}} \varphi(d\hat{z}) \int_{\hat{w}} \int_{\tilde{u}} \tilde{u} \mathscr{K}^\infty((\tilde{0}, \hat{z}), d\tilde{u} \times d\hat{w}). \end{aligned}$$

(If $S = (R^+)^r \times \widehat{S}$, then $\tilde{d} > 0$ componentwise to satisfy Condition 4.)

CONDITION 4. There exists a set $C \subseteq \triangle$ such that $\pi(C) > 0$ and such that, for $\mathbf{z} \in C$, $P_{\mathbf{z}}(\mathscr{T}_{\blacktriangle}^{\infty} = \infty) > 0$, where $\mathscr{T}_{\blacktriangle}^{\infty}$ is the first return time to $\blacktriangle$ by $\mathscr{W}^{\infty}$.

CONDITION 5. $\int_{\widehat{S}} \hat{a}^{-1}(\hat{z})\varphi(d\hat{z}) < \infty$.

CONDITION 6. The measure $\lambda(d\mathbf{x}) := \int_{\mathbf{z}\in\triangle} \pi(d\mathbf{z})K(\mathbf{z}, d\mathbf{x})\chi\{\mathbf{x} \in \triangle^c\}h(\mathbf{x})$ is finite.

CONDITION 7. If $\hat{a}^{-1}$ is unbounded, we require that the measure $\hat{\lambda}(d\hat{x}) := \int_{\tilde{x}} \lambda(d\tilde{x}, d\hat{x})$ is $\hat{a}^{-1}$-regular for the chain $\widehat{\mathscr{W}}^{\infty}$ as defined in Meyn and Tweedie [(1993), Chapter 14.1].

Let $f(\mathbf{z}) := \chi_{\triangle}(\mathbf{z}) \cdot \int_{\mathbf{x}\in\triangle^c} K(\mathbf{z}, d\mathbf{x})h(\mathbf{x})$. To verify Condition 6, we must verify $\int_{\mathbf{z}} f(\mathbf{z})\pi(d\mathbf{z}) < \infty$. This can be done using Lyapounov functions [see Meyn and Tweedie (1993), Theorem 14.3.7].

LEMMA 1.1. *Suppose there exist positive functions V and s on S such that*

$$LV(\mathbf{y}) \equiv KV(\mathbf{y}) - V(\mathbf{y}) \leq -f(\mathbf{z}) + s(\mathbf{y}). \tag{1.4}$$

Then $\int_{\mathbf{z}} f(\mathbf{z})\pi(d\mathbf{z}) \leq \int_{\mathbf{x}} s(\mathbf{x})\pi(d\mathbf{x})$.

Typically, $s(\mathbf{x}) = b \cdot \chi_C(\mathbf{x})$, where b is a positive constant and $C \in \mathscr{S}$.

In many examples, $\chi\{\mathbf{z}\in\triangle\}K(\mathbf{z}, d\mathbf{x})\chi\{\mathbf{x}\in\triangle^c\} = \chi\{\mathbf{z}\in\triangle\}K^{\infty}(\mathbf{z}, d\mathbf{x})\chi\{\mathbf{x}\in\triangle^c\}$. In this case, the measure λ becomes $\lambda(d\mathbf{x}) = \int_{\triangle} \pi(d\mathbf{z})h(\mathbf{z})\mathscr{K}^{\infty}(\mathbf{z}, d\mathbf{x})\chi\{\mathbf{x} \in \triangle^c\}$, so to check Condition 6, it suffices to verify that $\int_{\triangle} \pi(d\mathbf{z})h(\mathbf{z}) < \infty$ (using Lemma 1.1).

To check $\hat{\lambda}(d\hat{x})$ is $\hat{a}^{-1}$-regular for the chain $\widehat{\mathscr{W}}^{\infty}$, we need to find a petite set $\widehat{C} \in \widehat{\mathscr{S}}$ such that

$$E_{\hat{\lambda}}\left[\sum_{n=1}^{\tau_{\widehat{C}}} \hat{a}^{-1}(\widehat{\mathscr{W}}^{\infty}[n])\right] < \infty. \tag{1.5}$$

The most practical method to verify that the sum (1.5) is finite is to check condition (V3) given in Meyn and Tweedie (1993). It is sufficient to find a constant b and an extended real-valued function $\widehat{V}\colon \widehat{S} \to [0, \infty]$ such that

$$\int \widehat{\mathscr{K}}^{\infty}(\hat{y}, d\hat{z})\widehat{V}(\hat{z}) - \widehat{V}(\hat{y}) \leq -\hat{a}^{-1}(\hat{y}) + b\chi_{\widehat{C}}(\hat{y}),$$

and such that $\int \widehat{V}(\hat{x})\hat{\lambda}(d\hat{x}) < \infty$.

A natural candidate for $\widehat{V}$ is a multiple of $\hat{a}^{-1}$. Just note that

$$\begin{aligned}\int_{\hat{z}} \widehat{\mathscr{K}}^{\infty}(\hat{y}, d\hat{z})\hat{a}^{-1}(\hat{z}) &= \int_{\mathbf{z}} K^{\infty}((\tilde{0}, \hat{y}), (d\tilde{z}, d\hat{z}))\exp(\alpha z_1)\frac{\hat{a}(\hat{z})}{\hat{a}(\hat{y})}\hat{a}^{-1}(\hat{z}) \\ &= \left(\int_{\mathbf{z}} K^{\infty}((\tilde{0}, \hat{y}), (d\tilde{z}, d\hat{z}))\exp(\alpha z_1)\right)\hat{a}^{-1}(\hat{y}).\end{aligned}$$

If it can be shown that $\int_{\mathbf{z}} K^{\infty}((\tilde{0},\hat{y}),(d\tilde{z},d\hat{z}))\exp(\alpha z_1) < 1$ uniformly outside a petite set $\widehat{C}$ and uniformly bounded in $\hat{y}$ on $\widehat{C}$, then we can take $\widehat{V}$ to be some multiple of $\hat{a}^{-1}$.

1.3. *Stochastic networks.* The main application of our work is the estimation of rare event probabilities for stochastic networks. The Flatto–Hahn–Wright example in Section 3.1 is a special case of the networks considered below. Consider a network with $r+m$ nodes which is modeled as a Markov jump process with a state space $S \equiv (\mathbb{N}_0^r, \mathbb{N}_0^m)$, where $\mathbb{N}_0$ denotes the nonnegative integers. If $\beta = \{i_1, i_2, \dots, i_b\}$, we say $\mathbf{x}$ is on the boundary S_β if $x_i = 0$ for $i \in \beta$ but $x_i > 0$ for $i \notin \beta$. We assume the jump process modeling the stochastic network is the uniformization of a (homogeneous) nearest-neighbor random walk W in the orthant $\mathbb{N}_0^{r+m}$ having transition kernel K:

$$K(\mathbf{x},\mathbf{y}) = \begin{cases} J(\mathbf{y}-\mathbf{x}), & \text{if } \mathbf{x} \notin S_\beta \text{ for any } \beta \neq \varnothing, \\ J_\beta(\mathbf{y}-\mathbf{x}), & \text{if } \mathbf{x} \in S_\beta. \end{cases}$$

Here $J(\mathbf{x})$ gives the nearest-neighbor jump probability in the direction $\mathbf{x}$ in $\mathscr{N} = \{(x_1, x_2, \dots, x_{r+m})\colon x_i \in \{-1,0,1\}\}$. On a boundary S_β, we can only allow transitions in the directions $\mathscr{N}_\beta = \{(z_1, z_2, \dots, z_{r+m})\colon z_i \in \{-1,0,1\},\ i \in \beta^c;\ z_i \in \{0,1\},\ i \in \beta\}$. So J_β is of the form

$$J_\beta(\mathbf{z}) = \begin{cases} J(\mathbf{z}), & \text{if } \mathbf{z} \in \mathscr{N}_\beta \text{ and } \mathbf{z} \neq \mathbf{0}, \\ 1 - \sum_{\mathbf{s} \in \mathscr{N}_\beta \setminus \{\mathbf{0}\}} J(\mathbf{s}), & \text{if } \mathbf{z} = \mathbf{0}. \end{cases}$$

We assume the kernel K is associated with an irreducible Markov chain W with a stationary probability distribution π. Let $D = \{(0,0,\dots,0)\}$. We are interested in the rare event when the first node overloads; that is, when the first coordinate of W exceeds a level l. When the first node overloads, other nodes may remain stable even though they are subject to higher loads. The coordinates corresponding to these "super" stable nodes are assumed to be coordinates $r+1$ through $r+m$. Unfortunately, when the first overloads, it may drive other nodes into overload. We assume these nodes correspond to coordinates 2 through r. We look for a harmonic function of the form $h(\mathbf{x}) := \exp(\alpha x_1)\hat{a}(\hat{x})$ since, in addition to twisting the first component to become transient, we must judiciously twist only those other components which remain recurrent after twisting. Furthermore, the twist must make nodes 1 through r transient to plus infinity.

To find h, we take $\blacktriangle = \{\mathbf{x}\colon x_i \le 0,\ i = 1,\dots,r;\ x_i \ge 0,\ i = r+1,\dots,m\}$, thus removing the boundaries for the first r coordinates. Hence $S^\infty = \mathbb{Z}^r \times \mathbb{N}_0^m$, where $\mathbb{Z}$ represents the integers. If $\beta \subseteq \{r+1,\dots,r+m\}$ and $x_i = 0$ for $i \in \beta$ but $x_i > 0$ for $i \in \{r+1,\dots,r+m\} \setminus \beta$, then $\mathbf{x} \in S_\beta^\infty$. Extend K to K^∞ on S^∞ by defining

$$K^\infty(\mathbf{x},\mathbf{y}) = \begin{cases} J(\mathbf{y}-\mathbf{x}), & \text{if } \mathbf{x} \notin S_\beta^\infty \text{ for any } \beta \subseteq \{r+1,\dots,r+m\}, \\ J_{\beta\cap\{r+1,\dots,r+m\}}(\mathbf{y}-\mathbf{x}), & \text{if } \mathbf{x} \in S_{\beta\cap\{r+1,\dots,r+m\}}^\infty. \end{cases}$$

Finding a harmonic function h for K^∞ of the form $\exp(\alpha x_1)\hat{a}(\hat{x})$ is possible in great generality, as is seen in Section 2.1. However, since K^∞ is the transition kernel of a nearest-neighbor walk with steps outside the boundary $\triangle$ replaced with null transitions, it is often possible to find h in exponential form. For $\mathbf{x} = (x_1, \dots, x_{r+m}) \in S^\infty$, define $h(\mathbf{x}) := \mathbf{a}^\mathbf{x} \equiv a_1^{x_1} a_2^{x_2} \cdots a_{r+m}^{x_{r+m}}$, where $\mathbf{a} = (a_1, a_2, \dots, a_m)$ is a vector of positive constants such that $a_2 = 1, \dots, a_r = 1$.

If h is harmonic at $\mathbf{x} \in \operatorname{int}(S^\infty)$, then

$$\sum_{\mathbf{z}\in\mathscr{N}} \mathbf{a}^\mathbf{z} J(\mathbf{z}) = 1, \text{ where } \mathscr{N} \text{ is the set of nearest neighbors to the origin.} \tag{1.6}$$

Similarly, for $\mathbf{y} \in S^\infty_\beta$, harmonicity yields the constraint

$$\sum_{\mathbf{z}\in\mathscr{N}_\beta} \mathbf{a}^\mathbf{z} J^\infty_\beta(\mathbf{z}) = 1. \tag{1.7}$$

In general, this means 2^m constraints! Of course, there is always the solution $a_i = 1$ for all i, but in general, another positive solution may not exist.

Fortunately, solutions do exist in many interesting cases! For example, we may further assume that $J(\mathbf{z}) = 0$ when at least two coordinates z_i and z_j are -1 among $i \ge r+1$, $j \ge r+1$. In this case, satisfying the constraint (1.7) gives a condition on each boundary $S_{\{k\}}$; $k \ge r+1$:

$$\sum_{\mathbf{z}\in\mathscr{N}_{\{k\}}} \mathbf{a}^\mathbf{z} J(\mathbf{z}) = \sum_{\mathbf{z}\in\mathscr{N}_{\{k\}}} J(\mathbf{z}). \tag{1.8}$$

Subtracting this from (1.6) gives

$$\sum_{\mathbf{z}\in\mathscr{D}_{\{k\}}} a_k^{-1} \prod_{i\ne k} a_i^{z_i} J(\mathbf{z}) = \sum_{\mathbf{z}\in\mathscr{D}_{\{k\}}} J(\mathbf{z}), \text{ where } \mathscr{D}_{\{k\}} = \mathscr{N}^c_{\{k\}}. \tag{1.9}$$

The m constraints given by (1.9) plus the constraint (1.6) are equal to the m constraints given by (1.8) plus the constraint (1.6).

Now subtract the general constraint given in (1.7) from (1.6). Hence,

$$\sum_{\mathbf{z}\in\mathscr{D}_\beta} \mathbf{a}^\mathbf{z} J(\mathbf{z}) = \sum_{\mathbf{z}\in\mathscr{D}_\beta} J(\mathbf{z}), \tag{1.10}$$

where $\mathscr{D}_\beta = \mathscr{N}^c_\beta$. However, since we are assuming at most one negative jump among the coordinates $\{r+1, \dots, r+m\}$, it follows that

$$\sum_{\mathbf{z}\in\mathscr{D}_\beta} \mathbf{a}^\mathbf{z} J(\mathbf{z}) = \sum_{k\in\beta} \sum_{\mathbf{z}\in\mathscr{D}_{\{k\}}} \mathbf{a}^\mathbf{z} J(\mathbf{z}),$$

$$\sum_{\mathbf{z}\in\mathscr{D}_\beta} J(\mathbf{z}) = \sum_{k\in\beta} \sum_{\mathbf{z}\in\mathscr{D}_{\{k\}}} J(\mathbf{z}).$$

This means the constraint (1.10) can be obtained by summing the constraints (1.9) over $k \in \beta$. Consequently, all the constraints are equivalent to the m constraints given in (1.9) plus (1.6); that is, m constraints in all. Consequently, there is at least one solution.

Of course, the solution h must produce a twisted process such that $\widetilde{\mathscr{W}}^\infty$ drifts to plus infinity while $\widehat{\mathscr{W}}^\infty$ must be a stable Markov chain! If this fails, then we must try again by twisting another set of coordinates; that is, we must redefine the super stable nodes.

1.4. *Steady state.* Recall that $\mathscr{T}_l^\infty = \min\{n \geq 1: \widetilde{\mathscr{W}}_1^\infty[n] \geq l\}$ is $\mathscr{W}^\infty$'s first hitting time at $F := \{\mathbf{x} \in S^\infty,\ x_1 \geq l\}$ and we denote the hitting time at $\blacktriangle$ by $\mathscr{T}_\blacktriangle^\infty$. Let $\mathscr{R}^\infty[l] = \widetilde{\mathscr{W}}_1^\infty[\mathscr{T}_l^\infty] - l$ denote the excess beyond l of the ladder height. The following is shown in Section 2.2.

LEMMA 1.2. *Under Conditions* 1–6,

$$P_{\mathbf{z}}\big(\widehat{\mathscr{W}}^\infty[\mathscr{T}_l^\infty] \in d\hat{y},\ \mathscr{R}^\infty[l] \in du,\ \mathscr{T}_l^\infty < \mathscr{T}_\blacktriangle^\infty\big) \to H(\mathbf{z})\mu(du, d\hat{y})$$

in total variation,

where $H(\mathbf{z}) \equiv P_{\mathbf{z}}(\mathscr{T}_\blacktriangle^\infty = \infty)$. *This means that, given* $\mathscr{W}^\infty$ *hits* F, μ *is the limiting hitting distribution of* $(\mathscr{R}^\infty, \widehat{\mathscr{W}}^\infty)$.

It is not necessary to impose the spread-out condition in Condition 2.

LEMMA 1.3. *Let* $f\colon\ : [0,\infty) \times \widehat{S} \to R$ *be a bounded measurable function such that* $f(u, \hat{y})$ *is continuous* (*or piecewise continuous*) *in* u. *Then, in the continuous case under Conditions* 1–6 *but without assuming the spread-out condition*,

$$E_{\mathbf{z}}\big(f(\mathscr{R}^\infty[l], \widehat{\mathscr{W}}^\infty[\mathscr{T}_l^\infty])\chi\{\mathscr{T}_l^\infty < \mathscr{T}_\blacktriangle^\infty\}\big) \to H(\mathbf{z})\int\!\!\int f(u, \hat{y})\mu(du, d\hat{y}).$$

The proof follows by Athreya, McDonald and Ney [(1978), Theorem 3.1]. In Theorems 1.5, 1.6 and 1.7, we will need the above convergence for functions which are continuous or piecewise continuous in the (first) additive component. Consequently, these theorems could be proved without the spread-out condition. On the other hand, the convergence in total variation in Theorem 1.13 will fail without the mixing condition, Condition P2. We assume the stronger mixing condition for simplicity.

The uniform integrability afforded by Condition 7 gives the following.

LEMMA 1.4. *If Conditions* 1–7 *hold, then*

$$\begin{aligned}\lim_{l\to\infty}\int_{\mathbf{x}} \lambda(d\mathbf{x})E_{\mathbf{x}}\big(\chi\{\mathscr{T}_l^\infty < \mathscr{T}_\blacktriangle^\infty\}\exp(-\alpha\mathscr{R}^\infty[l])\hat{a}^{-1}(\widehat{\mathscr{W}}^\infty[\mathscr{T}_l^\infty])\big)\\ = \int_{\mathbf{x}} \lambda(d\mathbf{x})H(\mathbf{x}) \cdot \int_{\hat{y}}\int_{u\geq 0} \exp(-\alpha u)\hat{a}^{-1}(\hat{y})\mu(du, d\hat{y}).\end{aligned}$$

It is well known [see Orey (1971), Theorem 7.2, (7.2), or Meyn and Tweedie (1993), Theorem 10.0.1] that we can represent the probability of $A \subseteq F$ as the expected number of visits to A before returning to $\triangle$:

$$\pi(A) = \int_\triangle \pi(d\mathbf{z})E_{\mathbf{z}}\bigg(\sum_{n=1}^{T_\triangle} \chi_A(W[n])\bigg),\ \text{where } T_\triangle \text{ is the first time } W \text{ hits } \triangle.$$

We use this representation of $\pi(A)$ to prove the following theorem.

THEOREM 1.5. *Consider a set A in $\mathscr{B} \otimes R^{r-1} \otimes \widehat{\mathscr{S}}$ (hence measurable with respect to x_1 and $\hat{x}$) such that $m_A^\infty(u, \hat{y}) := E_{(u,0,\dots,0,\hat{y})}(\sum_{n=0}^\infty \chi_A(W^\infty[n]))$ is uniformly bounded in u and $\hat{y}$. Assuming Conditions* 1–7, *we have*

$$\int_A \pi((l + dx_1) \times R^{r-1} \times d\hat{x}) \sim e^{-\alpha l} f \int_{\hat{w}} \int_{u \geq 0} \hat{a}^{-1}(\hat{w}) e^{-\alpha u} m_A^\infty(u, \hat{w}) \mu(du, d\hat{w}),$$

where

$$f := \int_{\mathbf{z} \in \triangle} \pi(d\mathbf{z}) \int_{\mathbf{x} \notin \triangle} K(\mathbf{z}, d\mathbf{x}) h(\mathbf{x}) H(\mathbf{x}).$$

We see the asymptotics of $\pi((l + \Gamma) \times R^{r-1} \times \widehat{A})$ are given by $\exp(-\alpha l)$ as long as the right-hand side of the above expression is strictly positive. This is the case as long as $H(\mathbf{x})$ is strictly positive and this follows from Condition 4. Remark that if $\widehat{S}$ is countable and $A = \{0 \leq x_1 \leq L, \hat{x} = \hat{p}\}$, then $m_A^\infty(u, \hat{y})$ is maximal at $u = L$ and $\hat{y} = \hat{p}$. For general state spaces, if $A \subseteq [0, L] \times R^{r-1} \times C'_{k_0}$ for some k_0 and $L < \infty$, where

$$C'_k = \{\mathbf{x} = (\tilde{0}, \hat{x}) \in S^\infty \colon P_{\mathbf{x}}(\widetilde{W}_1^\infty[m] \geq mk^{-1} \text{ for all } m \geq k) \geq 1/4\},$$

then A is *directly Riemann integrable* as defined in Kesten (1974), and by Kesten [(1974), Lemma 6], $m_A^\infty(u, \hat{y})$ is uniformly bounded above.

The asymptotic expression in Theorem 1.5 can be reevaluated to give the following.

THEOREM 1.6 (Steady state). *Assume Conditions* 1–7 *and assume that A is as in Theorem* 1.5. *Then*

$$\int_A \pi((l + dx_1) \times R^{r-1} \times d\hat{x}) \sim e^{-\alpha l} f \int_{\hat{x}} \int_{x_1 \geq 0} \chi_A(x_1, \hat{x}) \hat{a}^{-1}(\hat{x}) \varphi(d\hat{x}) \times \frac{1}{\tilde{d}_1} \exp(-\alpha x_1) m(dx_1),$$

where φ is the stationary distribution of $\widehat{\mathscr{W}}^\infty$ given by Condition 1.

This means that, for large l, the stationary measure $\pi((l+dx_1) \times R^{r-1} \times d\hat{x})$ is a constant times $\exp(-\alpha l)$ times a product of the measures $\hat{a}^{-1}(\hat{x})\varphi(d\hat{x})$ and $\exp(-\alpha x_1) m(dx_1)$.

1.5. *Hitting Times.* We wish to describe the hitting time when W_1, the first coordinate of W, reaches some high level l. Let $T_l \equiv T_F$ and let T_D denote the hitting time at D. Let $f(x)$ denote the probability that, starting from $x \in B \setminus (D \cup F)$, W hits D before F; that is,

$$f(x) = P_x(W \text{ hits } D \text{ before } F) = P_x(T_D < T_F). \tag{1.11}$$

f can be extended to all S to satisfy the following Dirichlet problem:

$$\begin{cases} Lf(\mathbf{x}) = 0, & \text{in } \mathbf{x} \in B \setminus D; \\ f(\mathbf{x}) = 0, & \text{in } \mathbf{x} \in F; \\ f(\mathbf{x}) = 1, & \text{in } x \in D. \end{cases} \tag{1.12}$$

Using the fact that f satisfies (1.12), we have that

$$\begin{aligned} \Lambda &:= \int_{\mathbf{y}\in F} \pi(d\mathbf{y}) \int_{\mathbf{x}\in B} K(\mathbf{y}, d\mathbf{x}) f(\mathbf{x}) \\ &= \int_D \pi(d\mathbf{z}) \int_{\mathbf{x}} K(\mathbf{z}, d\mathbf{x})(1 - f(\mathbf{x})). \end{aligned} \tag{1.13}$$

Using the above expression, we can interpret $\Lambda/\pi(D)$ as the probability starting in steady state in D that the chain leaves D and does not return before hitting F. Call this probability p_D.

Define the Palm measure $P_0^{F(\to D)}$ associated with the stationary point process of returns to D after passing through F. The associated expectation $E_0^{F(\to D)} T_l$ is a natural measure of the time until overload starting from the point in D reached after the last recovery from overload. By the Hitting Time Theorem in Baccelli and McDonald (1996), we have the following.

COROLLARY 1.7. *As* $\pi(F) \to 0$, $(E_0^{F(\to D)} T_F)^{-1} \sim \Lambda$.

This result is an extension of Lemma 1 in Meyn and Frater (1993). With additional hypotheses, Baccelli and McDonald (1996) show when $E_0^{F(\to D)} T_F$ is asymptotic to $E_D T_F$.

It seems using Corollary 1.7 and (1.13) to estimate $E_0^{F(\to D)} T_F$ is useless without an expression for π, and if visits to the forbidden set are rare events, then even simulating π on F is practically impossible. On the other hand, if f is bounded away from 0 and 1 on F, then $E_0^{F(\to D)} T_F$ clearly has the same rough asymptotics as $\pi(F)$. In fact, we can say much more because we can perform and approximate h transformation on W. Next, we replace $f(x)$ by a good guess! Let $\rho(x)$ denote the probability that the random walk W, starting at x, hits $\triangle$ before hitting F. Like f, the function ρ satisfies a Dirichlet problem:

$$\begin{cases} L\rho(\mathbf{x}) = 0, & \text{in } \mathbf{x} \in B \setminus \triangle; \\ \rho(\mathbf{x}) = 0, & \text{in } \mathbf{x} \in F; \\ \rho(\mathbf{x}) = 1, & \text{in } \mathbf{x} \in \triangle. \end{cases} \tag{1.14}$$

On $B \setminus \triangle$, the equation $L\rho(\mathbf{x}) = 0$ is equivalent to the equation $L^\infty \rho(\mathbf{x}) = 0$ because the jump kernels K and K^∞ agree inside $S \setminus \triangle$ and ρ is constant on the edge $\triangle$.

Assuming for the moment that $\rho(\mathbf{x})$ is sufficiently close to $f(\mathbf{x})$, we may approximate

$$\Lambda = \int_{\mathbf{y}\in F} \pi(d\mathbf{y}) \int_{\mathbf{x}\in B} K(\mathbf{y}, d\mathbf{x}) f(\mathbf{x})$$

by

$$
\begin{aligned}
b &:= \int_{\mathbf{y}\in F} \pi(d\mathbf{y}) \int_{\mathbf{x}\in B} K(\mathbf{y}, d\mathbf{x})\rho(\mathbf{x}) \\
&= \int_{\mathbf{z}\in\triangle} \pi(d\mathbf{z}) L\rho(\mathbf{z}) \\
&= \int_{\mathbf{z}\in\triangle} \pi(d\mathbf{z}) \int_{\mathbf{x}} K(\mathbf{z}, d\mathbf{x})(1-\rho(\mathbf{x})).
\end{aligned} \tag{1.15}
$$

The above equivalence follows from the Hitting Time Theorem in Baccelli and McDonald (1996) by identifying $\triangle$ and D.

The fact that Λ and b are close is shown in the following key lemma.

LEMMA 1.8 (Comparison lemma). *If Conditions* 1–7 *hold, then* $\lim_{l\to\infty} |\Lambda - b|/b = 0$.

Using Corollary 1.7 and Lemma 1.8, we have the following.

COROLLARY 1.9. *If Conditions* 1–7 *hold, then* $\Lambda = \pi(D)p_D$, $b = \pi(\triangle)p_\triangle$ *and*

$$
E_0^{F(\to D)} T_F \sim \Lambda^{-1} \sim b^{-1} \sim E_0^{F(\to \Delta)} T_F,
$$

where $p_\triangle$ *is the probability starting in equilibrium on* $\triangle$ *of hitting* F *before* $\triangle$.

Using the fact that h is harmonic for L^∞, define $\Psi \equiv \Psi_l$ by

$$
\rho(\mathbf{x}) = 1 - \exp(-\alpha l) h(\mathbf{x}) \Psi(\mathbf{x}). \tag{1.16}
$$

ρ is harmonic for L^∞ on $B \setminus \blacktriangle$, so Ψ is harmonic with respect to $\mathscr{L}^\infty$ on $B \setminus \blacktriangle$. Moreover, we then have the following boundary-value problem for Ψ, defined through (1.16):

$$
\begin{cases}
\mathscr{L}^\infty \Psi(\mathbf{x}) = 0, & \mathbf{x} \in B \setminus \blacktriangle; \\
\Psi(\mathbf{y}) = \exp(\alpha l) h(\mathbf{y})^{-1}, & \mathbf{y} \in F; \\
\Psi(\mathbf{z}) = 0, & \mathbf{z} \in \blacktriangle.
\end{cases} \tag{1.17}
$$

The boundary-value problem (1.17) has a probabilistic solution. Hence,

$$
\Psi(\mathbf{x}) = E_{\mathbf{x}}\big[\hat{a}^{-1}(\widehat{\mathscr{W}}^\infty[\mathscr{T}_l^\infty]) \exp(-\alpha \mathscr{R}^\infty[l]) \chi\{\mathscr{T}_l^\infty < \mathscr{T}_\blacktriangle^\infty\}\big] \quad \text{for } \mathbf{x} \in S \setminus (\triangle \cup F). \tag{1.18}
$$

Evaluating (1.15), we see

$$
\begin{aligned}
b &= \int_{\mathbf{y}\in\triangle} \pi(d\mathbf{y}) L\rho(\mathbf{y}) \\
&= \int_{\mathbf{z}\in\triangle} \pi(d\mathbf{z}) e^{-\alpha l} \int_{\mathbf{x}\in B\setminus\blacktriangle} K(\mathbf{z}, d\mathbf{x}) h(\mathbf{x}) \Psi(\mathbf{x}) \\
&= \int_{\mathbf{z}\in\triangle} \pi(d\mathbf{z}) e^{-\alpha l} \int_{\mathbf{x}\in B\setminus\triangle} K(\mathbf{z}, d\mathbf{x}) h(\mathbf{x}) \\
&\qquad \times E_{\mathbf{x}}\big[\hat{a}^{-1}(\widehat{\mathscr{W}}^\infty[\mathscr{T}_l^\infty]) \exp(-\alpha \mathscr{R}^\infty[l]) \chi\{\mathscr{T}_l^\infty < \mathscr{T}_\blacktriangle^\infty\}\big].
\end{aligned} \tag{1.19}
$$

The above expression has no closed-form expression, although it can be estimated by simulation. The expression for Ψ near the origin is obtained by simulating $\mathscr{W}^\infty$ with absorption on $\blacktriangle$.

Using Theorem 2.5, expression (1.19) and Corollary 1.7, we shall prove the following.

THEOREM 1.10 (Mean Hitting Time). *Under Conditions* 1–7,

$$E_0^{F(\to D)} T_l \sim \exp(\alpha l) g^{-1} \quad \text{as } l \to \infty,$$

where

$$g \equiv \int_{\mathbf{z}\in\triangle} \pi(d\mathbf{z}) \int_{\mathbf{x}\notin\triangle} K(\mathbf{z}, d\mathbf{x}) h(\mathbf{x}) H(\mathbf{x}) \int_{\hat{y}} \int_{u\geq 0} \hat{a}^{-1}(\hat{y}) e^{-\alpha\cdot u} \mu(du, d\hat{y}), \tag{1.20}$$

where $H(\mathbf{x})$ is the probability $\mathscr{W}^\infty$, starting at $\mathbf{x}$, never hits $\blacktriangle$.

The rough asymptotics of $E_0^{F(\to D)} T_l$ are given by $\exp(\alpha l)$. The constant g can be obtained by simulation. This is not too onerous because we only need π on $\triangle$ and $\mathscr{W}^\infty$ is transient toward F.

1.6. *Hitting Distribution.* Let the time reversal of the Markov chain W with respect to π be denoted by W^* and let the kernel of the time reversal be K^*. Let $f^*(\mathbf{x})$ denote the probability that W^*, starting from $\mathbf{x} \in B$, hits D before F. We denote the first entrance time by W^* into F by T_F^*. T_D^* denotes the first entrance time at D. By time reversal, we have the following.

LEMMA 1.11. *If $A \in \mathscr{S}$ and $A \subseteq F$, then*

$$P_D\big(W[T_F] \in A | T_F < T_D\big) = \frac{1}{\pi(D) p_D} \int_{\mathbf{y}\in A} \pi(d\mathbf{y}) \int_{\mathbf{x}\in B} K^*(\mathbf{y}, d\mathbf{x}) f^*(\mathbf{x}),$$

where $p_D = \int_D \pi(d\mathbf{z}) P_{\mathbf{z}}(T_F < T_D)/\pi(D)$.

To apply Lemma 1.11, we again make an approximation. This time, substitute $\rho^*(\mathbf{x})$ for $f^*(\mathbf{x})$, where $\rho^*(x)$ is the probability W^* hits $\triangle$ before returning to F. The error introduced in the hitting distribution on F is shown to be asymptotically small.

PROPOSITION 1.12. *If $A \in \mathscr{S}$ and $A \subseteq F$, then*

$$\int_{\mathbf{y}\in A} \pi(d\mathbf{y}) \int_{\mathbf{x}\in B} K^*(\mathbf{y}, d\mathbf{x}) f^*(\mathbf{x}) \sim \int_{\mathbf{y}\in A} \pi(d\mathbf{y}) \int_{\mathbf{x}\in B} K^*(\mathbf{y}, d\mathbf{x}) \rho^*(\mathbf{x}).$$

We therefore investigate the expression

$$\begin{aligned} &\frac{1}{\pi(D) p_D} \int_A \pi(d\mathbf{y}) \int_{\mathbf{x}\in B} K^*(\mathbf{y}, d\mathbf{x}) \rho^*(\mathbf{x}) \\ &\quad = \frac{\pi(\triangle) p_\triangle}{\pi(D) p_D} P_\triangle\big(W[T_F] \in A | T_F < T_\triangle\big), \end{aligned} \tag{1.21}$$

using Corollary 1.11 with D replaced by $\triangle$. By Corollary 1.9,

$$\pi(\triangle)p_{\triangle}/\pi(D)p_D \to 1,$$

so asymptotically, $P_D(W[T_F] \in A|T_F < T_D)$ and $P_{\triangle}(W[T_F] \in A|T_F < T_{\triangle})$ are the same.

Now take $A = (l + \Gamma) \times R^{r-1} \times \hat{A}$ in $\mathscr{S}$. By a change of measure, we see

$$\begin{aligned}
&P_{\triangle}(W[T_F] \in A,\ T_F < T_{\triangle}) \\
&\quad = \frac{1}{\pi(\triangle)} \int_{\mathbf{z}\in\triangle} \pi(d\mathbf{z}) \int_{\mathbf{x}\notin\triangle} K(\mathbf{z}, d\mathbf{x}) \int_A P_{\mathbf{x}}(W^\infty[T_l^\infty] \in d\mathbf{y},\ T_F^\infty < T_{\blacktriangle}^\infty) \\
&\quad = \frac{1}{\pi(\triangle)} \int_{\mathbf{z}\in\triangle} \pi(d\mathbf{z}) \int_{\mathbf{x}\notin\triangle} K(\mathbf{z}, d\mathbf{x}) h(\mathbf{x}) \\
&\qquad\qquad \times \int_A \hat{a}^{-1}(\hat{y}) \exp(-\alpha y_1) P_{\mathbf{x}}(\mathscr{W}^\infty[\mathscr{T}_l^\infty] \in d\mathbf{y},\ \mathscr{T}_F^\infty < \mathscr{T}_{\blacktriangle}^\infty) \\
&\quad = \frac{e^{-\alpha l}}{\pi(\triangle)} \int_{\mathbf{z}\in\triangle} \pi(d\mathbf{z}) \int_{\mathbf{x}\notin\triangle} K(\mathbf{z}, d\mathbf{x}) h(\mathbf{x}) \\
&\qquad\qquad \times \int_{\hat{A}} \int_{u\in\Gamma} \hat{a}^{-1}(\hat{y}) e^{-\alpha u} P_{\mathbf{x}}(\widehat{\mathscr{W}}^\infty[\mathscr{T}_l^\infty] \in d\hat{y},\ \mathscr{R}^\infty[l] \in du,\ \mathscr{T}_F^\infty < \mathscr{T}_{\blacktriangle}^\infty).
\end{aligned}$$

Similarly,

$$\begin{aligned}
P_{\triangle}(T_F < T_{\triangle}) &= \frac{1}{\pi(\triangle)} \int_{\mathbf{z}\in\triangle} \pi(d\mathbf{z}) \int_{\mathbf{x}\notin\triangle} K(\mathbf{z}, d\mathbf{x}) h(\mathbf{x}) \\
&\qquad \times \int_A h^{-1}(\mathbf{y}) P_{\mathbf{x}}(\widehat{\mathscr{W}}^\infty[\mathscr{T}_l^\infty] \in d\hat{y},\ \mathscr{T}_F^\infty < \mathscr{T}_{\blacktriangle}^\infty) \\
&= e^{-\alpha l} \frac{1}{\pi(\triangle)} \int_{\mathbf{x}} \lambda(d\mathbf{x}) \int_{\hat{y}} \int_{u\geq 0} \hat{a}^{-1}(\hat{y}) \exp(-\alpha u) \\
&\qquad \times P_{\mathbf{x}}(\widehat{\mathscr{W}}^\infty[\mathscr{T}_l^\infty] \in d\hat{y},\ \mathscr{R}^\infty[l] \in du,\ \mathscr{T}_F^\infty < \mathscr{T}_{\blacktriangle}^\infty).
\end{aligned}$$

Cancelling out $\exp(-\alpha l)$, we see $P_D(\widehat{W}[T_l] \in A|T_l < T_{\triangle})$ is estimated by

$$\begin{aligned}
&\Bigg[\int_{\mathbf{x}\in\triangle^c} \lambda(d\mathbf{x}) \int_{\hat{A}} \int_{u\in\Gamma} h^{-1}(\hat{y}) \exp(-\alpha u) \\
&\quad \times P_{\mathbf{x}}\Big(\widehat{\mathscr{W}}^\infty[\mathscr{T}_l^\infty] \in d\hat{y},\ \mathscr{R}^\infty[l] \in du,\ \mathscr{T}_F^\infty < \mathscr{T}_{\blacktriangle}^\infty\Big)\Bigg] \\
&\quad \times \left(\int_{\mathbf{x}\in\triangle^c} \lambda(d\mathbf{x}) \Psi(\mathbf{x})\right)^{-1}.
\end{aligned} \tag{1.22}$$

Again this provides a practical solution to the hitting problem since the above expression can be simulated quickly.

Lemma 1.12 gives the following theorem.

THEOREM 1.13 (Hitting distribution). *Under Conditions* 1–7 *above, as* $l \to \infty$,

$$\frac{1}{\pi(D)} \int_{\mathbf{z} \in D} \pi(d\mathbf{z}) P_{\mathbf{z}}(\widehat{W}[T_l] \in d\hat{y},\ R[T_l] \in du \mid T_l < T_D)$$
$$\to \hat{a}^{-1}(\hat{y}) e^{-\alpha \cdot u} \mu(du, d\hat{y}) \Big/ \left(\int_{\hat{y}} \int_{u \geq 0} \hat{a}^{-1}(\hat{y}) e^{-\alpha \cdot u} \mu(du, d\hat{y}) \right),$$

where $\mu(du, d\hat{y})$ *is the stationary distribution of* $(\mathscr{R}^\infty[l], \widehat{\mathscr{W}}^\infty[T_l^\infty])$.

1.7. *Literature review.* The main technique used here is to find a function h which is harmonic outside some boundary ▲ in order to perform an h-transformation or twist of the Markov chain. The h-transform, introduced by Doob (1959), has a long history. In the case of random walk on the line, the twist is the associated random walk discussed in Feller [(1971), Section XII.4]. Kesten [(1974), Section 4] studied the asymptotics of a Markov additive process using the twist. Ney and Nummelin (1987a, b) used the twist to study large deviations of the additive component of a Markov additive process.

The representation of the steady-state measure of a rare event $A \subseteq F$ employed in Section 1.4 is the basis of the $\triangle$-cycle (called A-cycle there) technique in importance sampling exploited by Nicola, Shahabuddin, Heidelberger and Glynn (1992). $\triangle$-cycles consist of trajectories which start in $\triangle$, the edge of ▲ in S, and end just before a return to $\triangle$. The equilibrium measure π on $\triangle$ is unknown (unless $\triangle$ is a single point), but by generating N (untwisted) $\triangle$-cycles, it can be estimated. Just sample W whenever it returns to $\triangle$. Let $T_\triangle(k)$, $k = 1, \ldots, N$, denote the return times to $\triangle$ in the N untwisted $\triangle$-cycles.

Each return by an untwisted $\triangle$-cycle determines the start of a twisted $\triangle$-cycle and we alternate between untwisted and twisted $\triangle$-cycles. At the start of the kth twisted $\triangle$-cycle, first generate a step K and next generate steps using the twisted kernel $\mathscr{K}^\infty$ until $\mathscr{W}^\infty$ either hits ▲ or it hits F at time $\mathscr{T}_l^\infty$. If it hits ▲ before F, set $V_k = 0$. If $\mathscr{W}^\infty$ hits F before ▲, then turn off the twist and count N_k, the number of visits to A before W^∞ returns to $\triangle$. In this case, let

$$V_k := \hat{a}^{-1}(\widehat{\mathscr{W}}^\infty[\mathscr{T}_l^\infty]) \exp(-\alpha \mathscr{R}^\infty[l]) N_k.$$

The expected value $\exp(-\alpha l)\pi(\triangle) E_\triangle V_k$ is precisely expression (2.31). Hence,

$$\sum_{k=1}^N V_k \Big/ \sum_{k=1}^N T_\triangle(k)$$

is an asymptotically unbiased estimator of $\exp(\alpha l)\pi(A)$. This application of the twist and $\triangle$-cycles to produce an importance sampling estimator is described in detail in Bonneau (1996). Also see the survey paper by Heidelberger (1995). Note also that the $\triangle$-cycle technique may be used to estimate constants such as

$$f \equiv \left(\int_{\mathbf{z}\in\triangle} \pi(d\mathbf{z}) \int_{\mathbf{x}\notin\triangle} K(\mathbf{z}, d\mathbf{x}) h(\mathbf{x}) H(\mathbf{x}) \right),$$

$$g \equiv \int_{\mathbf{z}\in\triangle} \pi(d\mathbf{z}) \int_{\mathbf{x}\notin\triangle} K(\mathbf{z}, d\mathbf{x}) h(\mathbf{x}) H(\mathbf{x}) \int_{\hat{y}} \int_{u\geq 0} \hat{a}^{-1}(\hat{y}) e^{-\alpha\cdot u} \mu(du, d\hat{y}),$$

in Theorem 1.6 and Theorem 1.10.

Note that if Conditions 1–7 hold, the importance sampling estimator above will have a standard deviation of order $O(1/\sqrt{N})$. If the conditions fail, the asymptotics of $\pi(A)$ are *not* given by $\exp(-\alpha l)$. In practice, the importance sampling technique will fail because $\triangle$-cycles will abort by hitting $\triangle$ before F.

Glasserman and Kou (1995) described problems with importance sampling when estimating the probability of overloading a network. In the two-node example discussed in Glasserman and Kou [(1995), Section 4.1], these problems arise because cycles end when the network empties. The likelihood ratio is poorly controlled on the boundary when the second queue empties. We would take $\triangle = \{(x_1, x_2)\colon x_2 = 0\}$ and the problem would be eliminated (but of course we have to estimate π on $\triangle$).

The mean time until a rare event occurs is another useful descriptor. By Lemma 1.7 and Corollary 1.9, the inverse of the mean time for the chain in equilibrium to hit F, starting from the first return point in D after visiting F, is asymptotic to $b = \pi(\triangle)p_{\triangle}$, where $p_{\triangle}$ is the probability starting in equilibrium on $\triangle$ of hitting F before $\triangle$. We can estimate $\pi(\triangle)$ by the average number of steps $T_{\triangle}$ in an untwisted $\triangle$-cycle. The importance sampling estimator of b is obtained by remarking that

$$\begin{aligned} b &= \pi(\triangle)p_{\triangle} = \pi(\triangle)P_{\triangle}(T_l < T_{\triangle}) \\ &= \int_{\mathbf{z}\in\triangle} \pi(d\mathbf{z}) \int_{\mathbf{x}\notin\triangle} K(\mathbf{z}, d\mathbf{x}) h(\mathbf{x}) E_{\mathbf{x}}(h^{-1}(\mathscr{W}^{\infty}[\mathscr{T}_l^{\infty}]) \chi\{\mathscr{T}_F^{\infty} < \mathscr{T}_{\blacktriangle}^{\infty}\}) \\ &= \exp(-\alpha l) \int_{\mathbf{z}\in\triangle} \pi(d\mathbf{z}) \int_{\mathbf{x}\notin\triangle} K(\mathbf{z}, d\mathbf{x}) h(\mathbf{x}) \\ &\qquad \times E_{\mathbf{x}}\big(\hat{a}^{-1}(\widehat{\mathscr{W}}^{\infty}[\mathscr{T}_l^{\infty}]) \exp(-\alpha \mathscr{R}^{\infty}[l]) \chi\{\mathscr{T}_F^{\infty} < \mathscr{T}_{\blacktriangle}^{\infty}\}\big). \end{aligned} \tag{1.23}$$

This can be estimated as above. At the start of the kth $\triangle$-cycle, generate first a step K and then generate steps using the twisted kernel $\mathscr{K}^{\infty}$ until $\mathscr{W}^{\infty}$ either hits $\blacktriangle$ or it hits F at time $\mathscr{T}_l^{\infty}$. If it hits $\blacktriangle$ before F, set $V_k = 0$. If $\mathscr{W}^{\infty}$ hits F before $\blacktriangle$, then set $V_k = \hat{a}^{-1}(\widehat{\mathscr{W}}[\mathscr{T}_l]) \exp(-\alpha \mathscr{R}[l])$. The expected value $\exp(-\alpha l)\pi(\triangle)E_{\triangle}V_k$ is precisely expression (1.23). Consequently,

$$\sum_{k=1}^{N} V_k \Big/ \sum_{k=1}^{N} T_{\triangle}(k)$$

is an asymptotically unbiased estimator of $\exp(\alpha l)b$. Again, this importance sampling estimator will have a standard deviation of order $O(1/\sqrt{N})$ if Conditions 1–7 hold.

Parekh and Walrand (1989) and Frater, Lennon and Anderson (1991) calculated the change of measure or *twist* associated with a large deviation of the total number of customers in a Jackson network. They employed this twist to implement the above $\triangle$-cycle technique to find importance sampling estimates of the probability of the rare event when the total number of customers exceeds some level l as well as an estimate of the mean time until the network overloads. They showed that in some cases these estimates have optimally small standard deviation. Labrèche (1995) and Labrèche and McDonald (1995) found a harmonic function for a Jackson network in order to study overloads. The associated twist is precisely the one found by Frater, Lennon and Anderson (1991).

The main thrust of our work is exact asymptotics. Exact asymptotics for the steady-state probabilities of rare events have been studied previously by Höglund (1991) for ruin problems and by Shwartz and Weiss (1993) in the special case of the FHW queueing network originally proposed by Flatto and Hahn (1985). Recently, Sadowsky and Szpankowski (1995) gave the exact asymptotics of a fast teller queueing system using methods similar to those employed in Section 1.4. Here we analyze the FHW example.

Our method is to cut out the boundary $\blacktriangle$ where W fails to be a Markov additive chain. This represents a broader point of view than that taken in Sadowsky and Szpankowski (1995) (or in previous works) since the boundary there is simply those states where a (one-dimensional) queue is empty. Using the theory of Ney and Nummelin (1987a, b), one can then find a harmonic function on the complement of $\blacktriangle$. The associated h-transform produces the twisted process. The main novelty is Theorem 1.10, giving the exact asymptotics of the mean time until a rare event occurs and Theorem 1.13, giving the limiting hitting distribution. The main ingredient in the proof is the *Comparison* Lemma, which shows we may replace an arbitrary starting set with an initial steady-state distribution on $\triangle$. The generality of the boundary makes Conditions 4–7 necessary. The representation of the limiting steady-state probability of a rare event given in Theorem 1.6 is also new, as is the *Joint Bottlenecks* Theorem 2.10.

The range of applications is quite large. In Section 3, we find the harmonic function associated with a simple example, the Flatto–Hahn–Wright model. The fast teller model was studied in Beck, Dabrowski and McDonald (1997). The join-the-shortest-queue model was partially analyzed in McDonald (1996). The harmonic function was given explicitly and Conditions 1–5 were checked. Conditions 6 and 7 were verified in Foley and McDonald (1997). In Huang and McDonald (1996), the representation in Theorem 1.6 was used to give the asymptotics of the queue length distribution of an $M|D|1$ queue. Also, the key heuristic of guessing ρ for f can be used in conjunction with the theory in Iscoe and McDonald (1994a, b) to give error bounds for the estimated mean exit time.

In Section 2.1, we review the conditions necessary for determining a twist. In Section 2.2, we recall the asymptotic theory of Markov additive chains. Section 2.3 gives all the proofs which were deferred in earlier sections. In

Section 2.4, we determine the behavior of the queues at those nodes which become transient when the first is overloaded. In Section 3, we apply the above results to the FHW example.

2. Tools and proofs.

2.1. *The twist.* We now show why we may expect that there exists a harmonic function h for K^∞ of the form $h(\mathbf{x}) = \exp(\alpha x_1)\hat{a}(\hat{x})$. To make the connection to the work of Ney and Nummelin (1987a, b) easier, we make the identification $(\widetilde{W}^\infty, \widehat{W}^\infty) \equiv (V, Z)$ throughout this section [the superscript ∞ would be redundant since (V, Z) is always free]. The generating function $\widehat{K}^\infty_\gamma$ of the transition kernel of the Markov additive chain (V, Z) is given by

$$\widehat{K}^\infty_\gamma(\hat{x}, \widehat{A}) = E(\exp(\gamma \cdot X[1])\chi\{Z[1] \in \widehat{A}\} | Z[0] = \hat{x}),$$

where $X[1] := (V[1] - V[0])$. The existence of eigenvalues and eigenvectors for this "Feynman–Kac" operator was studied in Ney and Nummelin (1987a, b) under Condition M1 below.

CONDITION M1. There exists a probability measure ν on $(\widehat{S}, \widehat{\mathscr{S}})$ and a family of (positive) measures $\{h(\hat{x}, \cdot)\}$ on $(\widehat{S}, \widehat{\mathscr{S}})$ such that $\int \psi(d\hat{x}) h(\hat{x}, R) > 0$ and

$$h(\hat{x}, \Gamma)\nu(\widehat{A}) \leq K^\infty((\tilde{0}, \hat{x}), \Gamma \times \widehat{A})$$

for all $\hat{x} \in \widehat{S}$, $\Gamma \in \mathscr{B}^r$ and $\widehat{A} \in \widehat{\mathscr{S}}$.

Under this hypothesis, Ney and Nummelin [(1984), Lemma 3.1] constructed a regenerative structure for the Markov additive chains.

LEMMA 2.1. *Under Condition* M1, *there exist random variables* $0 < T_0 < T_1 < \cdots$ *with the following properties*:

(i) $(T_{i+1} - T_i;\ i = 0, 1, \ldots)$ *are i.i.d. random variables*;
(ii) *the random blocks*

$$(Z[T_i], \ldots, Z[T_{i+1} - 1], X[T_i + 1], \ldots X[T_{i+1}]), \qquad i = 0, 1, \ldots,$$

are independent;
(iii)

$$P_{\mathbf{x}}(Z[T_i] \in \widehat{A} | \mathscr{F}_{T_i - 1}, X[T_i]) = \nu(\widehat{A}),$$

for $\widehat{A} \in \widehat{\mathscr{S}}$, *where* $\mathscr{F}_n$ *is the* σ*-algebra generated by* $\{Z[0], \ldots, Z[n], X[1], \ldots, X[n]\}$.

At times $\{T_i\}$, the process Z behaves as if it has returned to an atom in the state space. This generates a succession of cycles from $[T_{i-1}, T_i)$. The increments of V between successive returns to this atom are therefore independent. Define

$$\tau_\nu =_D T_{i+1} - T_i, \quad \text{and} \quad S[\tau_\nu] =_D V[T_{i+1}] - V[T_i], \qquad i = 1, 2, \dots .$$

The variables τ_ν and $S[\tau_\nu]$ have a joint distribution which can be obtained as above from any of the identical regenerative cycles.

Define $\varsigma := \mathrm{Supp}_\nu(S[\tau_\nu]/\tau_\nu)$, where $\mathrm{Supp}_\nu(Y)$ is the convex hull of the support of the measure $P_\nu(Y \in \cdot)$. To ensure that T_i is genuinely r-dimensional, we assume the following.

CONDITION N1. The interior of ς is nonempty.

We now impose mixing conditions so that two independent copies of (V_1, Z) with different initial distributions may be coupled together from some T_i onward.

CONDITION P1. In the discrete case, we assume the distribution of $S_1[\tau_\nu]$, the (marginal) distribution of the first component of $S[\tau_\nu]$, is aperiodic.

CONDITION P2. In the continuous case, we suppose that the distribution of $S_1[\tau_\nu]$ is spread out so it has a nonsingular component relative to Lebesgue measure.

In the continuous case, we could impose the sufficient condition such as (iii) given in Sadowsky and Szpankowski [(1995), Appendix A]. It would suffice that the measure h in Condition M1 satisfy

$$h(x, d\mathbf{y}) \geq c \prod_{k=1}^{r} (b_k - a_k)^{-1} \chi_{[a_k, b_k]}(y_k) m(d\mathbf{y}) \text{ for some constant } c.$$

In fact, we could dispense with a mixing condition, Condition P2, altogether and the Theorems 1.5, 1.6 and 1.7 will still hold using Theorem 3.1 in Athreya, McDonald and Ney (1978). On the other hand, the convergence in total variation in Theorem 1.13 will fail without the mixing condition, Condition P2. Lemma 1.3 indicates how the theory goes under these weaker conditions.

Under Condition M1, we may define the generating function

$$\psi(\gamma, \zeta) = E_\nu \exp(\gamma \cdot S[\tau_\nu] - \zeta \tau_\nu).$$

Let $\mathscr{U} := \{(\gamma, \zeta) \in \mathbb{R}^{r+1}: \psi(\gamma, \zeta) < \infty\}$ and define $\Lambda(\gamma) = \inf\{\zeta: \psi(\gamma, \zeta) \leq 1\}$. We impose the following conditions.

CONDITION M2. $\mathscr{U}$ is an open set.

CONDITION M3. $E_\nu V_1[\tau_\nu] < 0$.

If Conditions M1 and M2 hold, then by Ney and Nummelin [(1987a), Theorem 4.1], $\psi(\gamma, \Lambda(\gamma)) = 1$ and $\exp(\Lambda(\gamma))$ is an eigenvalue of $\widehat{K}_\gamma^\infty$ associated with the right eigenvector $r(\hat{x};\gamma)$. We study the case where $\gamma = (\gamma_1, 0, \dots, 0)$ and we wish to find γ_1 other than 0 such that $\Lambda(\gamma) = 0$.

By Ney and Nummelin [(1987a), Lemma 3.3], we have that

$$\nabla\Lambda(\mathbf{0}) = (E_\nu \tau_\nu)^{-1} E_\nu V[\tau_\nu].$$

Hence, if Condition M3 holds, then the derivative of Λ with respect to γ_1 is negative at $\gamma_1 = 0$. Naturally, $\Lambda(\mathbf{0}) = 0$. Moreover, the function $\Lambda(\alpha)$ is strictly convex by Ney and Nummelin [(1987), Corollary 3.3]. Hence, $\Lambda(\gamma_1, 0, \dots, 0)$ passes through 0 at most at two points, 0 and $\alpha > 0$. If we impose the condition that $P_\nu(S_1[\tau_\nu] > 0) > 0$, then by the definition of ψ and the fact that $\psi(\gamma, \Lambda(\gamma)) = 1$, we have $\Lambda(\gamma_1, 0, \dots, 0) \to \infty$ as $\gamma_1 \to \infty$. Finally, by Condition M2, $\Lambda(\gamma_1, 0, \dots, 0)$ increases continuously to infinity, so α exists.

We note the following in passing.

LEMMA 2.2. *If Condition* M3 *holds, then* $\nabla\Lambda(\alpha, 0, \dots, 0) \cdot (1, 0, \dots, 0) > 0$.

PROOF. By strict convexity,

$$\Lambda(\mathbf{0}) > \Lambda(\alpha, 0, \dots, 0) + \nabla\Lambda(\alpha, 0, \dots, 0) \cdot (-\alpha, 0, \dots, 0).$$

The result follows since $\Lambda(\mathbf{0}) = \Lambda(\alpha, 0, \dots, 0) = 0$. □

α is the uniquely chosen value for γ_1 ensuring that the associated Perron–Frobenius eigenvalue of $\widehat{K}_\gamma^\infty$ is 1! Let $\boldsymbol{\alpha} := (\alpha, 0, \dots, 0)$. Hence there exists a positive Perron–Frobenius eigenvector $\hat{a}(\hat{x}) \equiv r(\hat{x};\boldsymbol{\alpha})$ satisfying $\hat{a}(\hat{x}) = \int \widehat{K}_\gamma^\infty(\hat{x}, d\hat{y})\hat{a}(\hat{y})$. We may therefore define a positive harmonic function for the additive chain (V, Z) by $h(\mathbf{x}) := \exp(\mathbf{a} \cdot \tilde{x})\hat{a}(\hat{x}) \equiv \exp(\alpha x_1)\hat{a}(\hat{x})$. To show it is harmonic for K^∞, pick $(\tilde{x}, \hat{x}) \in S^\infty$ such that $x_1 = s$, so

$$\begin{aligned}\int K^\infty(\mathbf{x}, d\mathbf{y})h(\mathbf{y}) &= \int_{(\tilde{v},\hat{y})\in S^\infty} K^\infty((\tilde{x}, \hat{x}); (\tilde{x} + d\tilde{v}, d\hat{y}))\hat{a}(\hat{y}) \exp(\alpha \cdot (s + \tilde{v}_1)) \\ &= \exp(\alpha s) \int_{\hat{y}} \widehat{K}_{\boldsymbol{\alpha}}^\infty(\hat{x}, d\hat{y})\hat{a}(\hat{y}) \\ &= \exp(\alpha s)\hat{a}(\hat{x}) = h(\mathbf{x}),\end{aligned}$$

since $\hat{a}$ is a right eigenvector for $\widehat{K}_{\boldsymbol{\alpha}}^\infty$ associated with the eigenvalue 1.

The kernel $\widehat{\mathscr{K}}^\infty$ and the stationary distribution φ are denoted by $Q(\alpha)$ and $\pi(\alpha)$ in Ney and Nummelin (1987a). The increment of the additive chain is denoted by S_1 in Ney and Nummelin (1987a), so the expression

$$\tilde{d}_1 \equiv E_\varphi^{\widehat{\mathscr{W}}^\infty} X_1[1] = E_{\pi(\alpha)}^{Q(\alpha)} S_1 = \nabla\Lambda(\alpha, 0, \dots, 0) \tag{2.24}$$

follows by Lemma 5.3 there.

Let the stationary distribution of $\widehat{K}^\infty$ be denoted by $\hat{\pi}^\infty$.

LEMMA 2.3 (Drift). *If Conditions* M1 *and* M2 *hold and if* $E_{\hat{\pi}^\infty} X_1[1] < 0$, *then* $\tilde{d}_1 > 0$.

PROOF. $\hat{\pi}^\infty$ is denoted by $\pi(0)$ in Ney and Nummelin (1987). By Lemma 5.3, $E_{\hat{\pi}^\infty} X_1[1] = \nabla\Lambda(\mathbf{0})$. It follows from the hypothesis that $\nabla\Lambda(\mathbf{0}) \cdot (1, 0, \dots, 0) < 0$. Hence Condition M3 holds. Therefore, by Lemma 2.2, $\nabla\Lambda(\alpha, 0, \dots, 0) \cdot (1, 0, \dots, 0) > 0$. The result follows from the above expression for $\tilde{d}_1$. $\square$

Condition M2 is too strong for many applications. By direct computation, it may be possible to find the left and right Perron–Frobenius eigenmeasure and eigenfunction associated with the eigenvalue $\exp(\Lambda(\gamma))$ of $\widehat{K}^\infty_\gamma$. The left and right eigenmeasure and eigenfunction are denoted by $\hat{\pi}_\gamma$ and $r(\cdot, \gamma)$, respectively, and we suppose that $\int r(\hat{x}, \gamma)\hat{\pi}_\gamma(d\hat{x}) = 1$. It may be that $\hat{\pi}_\gamma$ is not even a probability measure! The conditions of Lemma 5.3 in Ney and Nummelin (1987) may not be checkable but (2.24) still holds. Take the derivate of the expression

$$\int\int \hat{\pi}_\gamma(d\hat{x})\widehat{K}^\infty_\gamma(\hat{x}, d\hat{y})r(\hat{y}, \gamma) = \exp(\Lambda(\gamma)).$$

We get

$$\begin{aligned}\Lambda'(\gamma)\exp(\Lambda(\gamma)) = {} & \int \frac{d}{d\gamma}\hat{\pi}_\gamma(d\hat{x})r(\hat{x}, \gamma)\exp(\Lambda(\gamma)) \\ & + \int \hat{\pi}_\gamma(d\hat{y})\exp(\Lambda(\gamma))\frac{d}{d\gamma}r(\hat{y}, \gamma) \\ & + \int\int \hat{\pi}_\gamma(d\hat{x})\frac{d}{d\gamma}\widehat{K}^\infty_\gamma(\hat{x}, d\hat{y})r(\hat{y}, \gamma).\end{aligned}$$

But,

$$\frac{d}{d\gamma}\left(\int \hat{\pi}_\gamma(d\hat{x})r(\hat{y}, \gamma)\right) = \frac{d}{d\gamma}1 = 0$$

and

$$\begin{aligned}&\int\int \hat{\pi}_\gamma(d\hat{x})\frac{d}{d\gamma}K^\infty((\tilde{0}, \hat{x}), (d\tilde{y}, d\hat{y}))y_1\exp(\gamma y_1)r(\hat{y}, \gamma) \\ &\quad = \int\int \hat{\pi}_\gamma(d\hat{x})K^\infty((\tilde{0}, \hat{x}), (d\tilde{y}, d\hat{y}))y_1\exp(\gamma y_1)\frac{r(\hat{y}, \gamma)}{r(\hat{x}, \gamma)}r(\hat{x}, \gamma).\end{aligned}$$

Evaluating at $\boldsymbol{\alpha}$, where $\Lambda(\boldsymbol{\alpha}) = 0$, gives

$$\begin{aligned}&\int r(\hat{x}, \alpha)\hat{\pi}_\alpha(d\hat{x})\int \mathscr{K}^\infty((\tilde{0}, \hat{x}), (d\tilde{y}, d\hat{y}))y_1 \\ &\quad = \int \varphi(d\hat{x})\int \mathscr{K}^\infty((\tilde{0}, \hat{x}), (d\tilde{y}, d\hat{y}))y_1 \\ &\quad = \tilde{d}_1.\end{aligned}$$

Therefore, $\tilde{d}_1 = \Lambda'(\boldsymbol{\alpha})$, so we can establish $\tilde{d}_1 > 0$ from the convexity of Λ.

2.2. *Asymptotics of Markov additive processes.* In this section, we apply the results in Kesten (1974) to obtain the asymptotic behavior of $\mathscr{W}^\infty$. To make the connection easier, we identify $(\widetilde{\mathscr{W}}^\infty, \widehat{\mathscr{W}}^\infty) \equiv (\mathscr{V}, \mathscr{Z})$ [again the superscript ∞ is redundant since $(\mathscr{V}, \mathscr{Z})$ is always free]. Following Kesten (1974), consider a two-sided process $\{\mathscr{X}[n]^\#, \mathscr{Z}[n]^\#;\ -\infty < n < \infty\}$ with probability measure $P^\#$ determined by

$$
\begin{aligned}
&P^\#(\mathscr{Z}[k+i]^\# \in d\hat{z}_i,\ 0 \le i \le n,\ \mathscr{X}^\#[k+i] \in d\lambda_i,\ 1 \le i \le n)\\
&\quad = \varphi(d\hat{z}_0)\mathscr{K}^\infty((\tilde{0}, \hat{z}_0), d\lambda_1 \times d\hat{z}_1)\\
&\qquad \cdot \mathscr{K}^\infty((\tilde{0}, \hat{z}_1), d\lambda_2 \times d\hat{z}_2) \cdots \mathscr{K}^\infty((\tilde{0}, \hat{z}_{n-1}), d\lambda_n \times d\hat{z}_n),
\end{aligned}
$$

for any sequence of states $\{\hat{z}_i,\ 0 \le i \le n\}$ in $\widehat{S}$ and any sojourn times $\{\lambda_i,\ 0 \le i < n\}$ and any integer k. The pairs $\{(\mathscr{X}[n]^\#, \mathscr{Z}[n]^\#)\}_{-\infty<n<\infty}$ form a stationary Markov chain and, in particular,

$$P^\#(\mathscr{Z}[k]^\# \in d\hat{z}) = \varphi(d\hat{z}).$$

Moreover, given $\mathscr{Z}[k]^\# = \hat{z}$, it follows that $\{\mathscr{X}[k+n]^\#,\ \mathscr{Z}[k+n]^\#\}_{1\le n<\infty}$ has the same law as $\{\mathscr{V}[n] - \mathscr{V}[n-1],\ \mathscr{Z}[n]\}_{1\le n<\infty}$ given $\mathscr{Z}[0] = \hat{z}$.

Define

$$
\mathscr{V}[n]^\# = \begin{cases} \displaystyle\sum_{i=1}^{n} \mathscr{X}[i]^\#, & \text{if } n > 0,\\ 0, & \text{if } n = 0,\\ \displaystyle -\sum_{i=n+1}^{0} \mathscr{X}[i]^\#, & \text{if } n < 0, \end{cases}
$$

and ladder indices for the sequence $\{\mathscr{V}[n]^\#\}_{-\infty<n<\infty}$:

$$
\begin{aligned}
\nu_0^\# &= \max\Bigl\{n \le 0\colon \mathscr{V}_1[n]^\# > \sup_{j<n} \mathscr{V}_1[j]^\#\Bigr\},\\
\nu_{i+1}^\# &= \min\Bigl\{n > \nu_i^\#\colon \mathscr{V}_1[n]^\# > \mathscr{V}_1[\nu_i^\#]^\#\Bigr\}.
\end{aligned}
\tag{2.25}
$$

The index $\nu_0^\#$ represents the time, $n \le 0$, when the last strict maximum of $\mathscr{V}_1[n]^\#$ occurred.

We now construct the Markov chain $Y[n]^\# = \mathscr{Z}[\nu_n^\#]^\#$ which records the position of the chain $\mathscr{Z}[n]^\#$ after each ascending ladder height. Kesten [(1974), Lemma 2] showed that, for $\hat{z} \in \widehat{S}$,

$$
\begin{aligned}
\psi(d\hat{z}) &= P^\#(\nu_0^\# = 0,\ \mathscr{Z}[0]^\# \in d\hat{z})\\
&= P^\#\Bigl(\sup_{n<0} \mathscr{V}_1[n]^\# < 0,\ \mathscr{Z}[0]^\# \in d\hat{z}\Bigr)
\end{aligned}
\tag{2.26}
$$

is an invariant measure for the chain $Y[n]^\#$. He also showed $q := P^\#(\nu_0^\# = 0) > 0$, so $\psi(\hat{z})/q$ is the stationary probability of $Y[n]^\#$. Moreover,

$$\int_{\hat{z}} \psi(d\hat{z}) E_{\hat{z}}\{\nu_1^\#\} = 1 \quad \text{and} \quad \int_{\hat{z}} \psi(d\mathbf{z}) E_{\hat{z}} \mathscr{V}_1[\nu_1^\#] = \tilde{d}_1 > 0. \tag{2.27}$$

Also, as in Kesten [(1974), Lemma 4], define

$$\mu(du, d\hat{y}) := \tilde{d}_1^{-1} \int_{\hat{z}} \psi(d\hat{z}) P_{\hat{z}}(Y[1]^{\#} \in d\hat{y},\ \mathscr{V}_1[\nu_1^{\#}]^{\#} \geq u) m(du).$$

This is a probability distribution by (2.27). By the stationarity of $Y^{\#}$, this is equivalent to

$$\mu(du, d\hat{y}) = \tilde{d}_1^{-1} P^{\#}(\nu_0^{\#} = 0,\ \mathscr{Y}[0]^{\#} \in d\hat{y},\ \mathscr{V}_1[\nu_1^{\#}]^{\#} \geq u). \tag{2.28}$$

This representation will prove useful later.

PROPOSITION 2.4.

$$\begin{aligned}\mu(du, d\hat{y}) &= \tilde{d}_1^{-1} P^{\#}(\nu_0^{\#} = 0,\ \mathscr{Y}[0]^{\#} \in d\hat{y},\ \mathscr{V}_1[\nu_1^{\#}]^{\#} \geq u) m(du)\\ &\leq \frac{1}{\tilde{d}_1} \varphi(d\hat{y}) m(du) P^{\#}(\mathscr{V}_1[\nu_1^{\#}]^{\#} \geq u | \mathscr{Y}[0]^{\#} = \hat{y})\\ &\leq \frac{1}{\tilde{d}_1} \varphi(d\hat{y}) m(du).\end{aligned}$$

We can couple $(\mathscr{V}_1[n], \mathscr{Y}[n])$ to $(\mathscr{V}_1[n]^{\#}, \mathscr{Y}[n]^{\#})$ [or alternatively, apply Kesten (1974), Theorem 1, and in particular (1.19)] to show the following.

THEOREM 2.5. *Under Conditions* 1–3, $(\mathscr{R}[l], \mathscr{Y}[\mathscr{T}_l])$ *converges in total variation to* μ *as* $l \to \infty$; *that is, the hitting distribution on* F *converges in total variation to* μ ($\mathscr{R}[l] = \mathscr{V}_1\mathscr{T}_l - l$).

By Condition 3, the mean increment $\tilde{d}$ of $\mathscr{V}$ is positive. Moreover, by Condition 1, $\mathscr{Y}$ has a stationary probability measure φ. Next, by Condition 2, we may construct a sequence of stopping times $\mathscr{T}_i$ such that, relative to a starting measure ν, the increments

$$\tau_\nu =_D \mathscr{T}_{i+1} - \mathscr{T}_i \quad \text{and} \quad \mathscr{S}[\tau_\nu] =_D \mathscr{V}[\mathscr{T}_{i+1}] - \mathscr{V}[\mathscr{T}_i], \qquad i = 1, 2, \ldots,$$

are i.i.d. By Ney and Nummelin [(1987), Lemma 5.2], $E_\nu[\mathscr{S}[\tau_\nu]]/E_\nu[\tau_\nu] = \tilde{d}$. By the law of large numbers, it follows that $\mathscr{V}[\mathscr{T}_i]/\mathscr{T}_i \to \tilde{d}$ as $i \to \infty$. It follows that $\mathscr{V}[n]/n \to \tilde{d}$ almost surely. The average behavior of $\mathscr{V}[n]$ is summarized as follows.

LEMMA 2.6. *If Conditions* 1–3 *are satisfied, then* $\mathscr{V}[n]/n \to \tilde{d}$ *almost surely in each component as* $n \to \infty$.

Next, recall $T_l^\infty = \inf\{n\colon \mathscr{V}_1[n] \geq l\}$. By the renewal theorem, $T_l^\infty/l \to \tilde{d}_1$. Consequently,

$$\begin{aligned}\lim_{l\to\infty} \frac{1}{l}\mathscr{V}[T_l^\infty] &= \left(\lim_{l\to\infty} \frac{1}{T_l^\infty}\mathscr{V}[T_l^\infty]\right)\left(\lim_{l\to\infty} \frac{1}{l} T_l^\infty\right)\\ &= \tilde{d} \cdot \frac{1}{\tilde{d}_1},\end{aligned}$$

since $\mathscr{V}[n]/n \to \tilde{d}/\tilde{d}_1$. This gives the following.

LEMMA 2.7. *If Conditions* 1–3 *are satisfied, then almost surely* $\mathscr{V}[\mathscr{T}_l^\infty]/l \to \tilde{d}/\tilde{d}_1$ *as* $n \to \infty$.

2.3. *Proofs.* In Theorem 2.5, we showed the distribution of $\widehat{\mathscr{W}}^\infty[\mathscr{T}_l^\infty]$ converges in total variation as $l \to \infty$. We will need to show the following.

LEMMA 2.8. *If Conditions* 1–7 *hold, then* $\hat{a}^{-1}(\widehat{\mathscr{W}}^\infty[\mathscr{T}_l^\infty])$ *converges in* $L^1(\lambda)$, *where* $\mathscr{T}_l^\infty = \min\{n\colon \mathscr{W}_1^\infty[n] \geq l\}$.

PROOF. This is automatic if $\hat{a}^{-1}$ is bounded, and fortunately the tools in Meyn and Tweedie [(1993), Chapter 14.1] apply if $\hat{a}^{-1}$ is unbounded. Using Theorem 2.5, $\hat{a}^{-1}(\widehat{\mathscr{W}}^\infty[\mathscr{T}_l^\infty])$ converges in distribution. We will use the Regenerative Decomposition (14.6) in Meyn and Tweedie (1993) to show convergence in $L^1(\lambda)$. Extending (14.8) in Meyn and Tweedie (1993), we need to show

$$E_\lambda[\hat{a}^{-1}(\widehat{\mathscr{W}}^\infty[\mathscr{T}_l^\infty])\chi\{\tau_{\widehat{C}} \geq \mathscr{T}_l^\infty\}] \to 0 \tag{2.29}$$

as $l \to \infty$.

By Meyn and Tweedie [(1993), Theorem 14.0.1], using Condition 5, there exists a petite set $\widehat{C} \in \widehat{\mathscr{S}}$ such that

$$E_{\mathbf{x}}\left[\sum_{n=0}^{\tau_{\widehat{C}}-1} \hat{a}^{-1}(\widehat{\mathscr{W}}^\infty[n])\right] < \infty$$

for all $\mathbf{x} = (\tilde{0}, \hat{x})$ and such that the above is uniformly bounded for $\hat{x} \in \widehat{C}$. Next, $\hat{\lambda}$ is $\hat{a}^{-1}$-regular by Condition 7, so by definition, (1.5) holds. If $n = \mathscr{T}_l^\infty \leq \tau_{\widehat{C}}$, then the nth term in (1.5) is $\hat{a}^{-1}(\widehat{\mathscr{W}}^\infty[\mathscr{T}_l^\infty])$, so the sum in (1.5) bounds the one term in (2.29). This gives (2.29) and the result. □

PROOF OF LEMMA 1.2. Pick $\widehat{A} \in \widehat{\mathscr{S}}$ and $\Gamma \in \mathscr{B}$. Clearly, $\chi\{\mathscr{T}_l^\infty < \mathscr{T}_\blacktriangle^\infty\} \to \chi\{\mathscr{T}_\blacktriangle^\infty = \infty\}$, so pick an l sufficiently large that the difference in probability is less than ε. Now consider the limit, as $l \to \infty$, of

$$P_{\mathbf{z}}\big(\widehat{\mathscr{W}}^\infty[\mathscr{T}_l^\infty] \in \widehat{A},\ \mathscr{R}^\infty[l] \in \Gamma,\ \mathscr{T}_\blacktriangle^\infty = \infty\big).$$

The Markov chain $(\widehat{\mathscr{W}}^\infty[\mathscr{T}_l^\infty],\ \mathscr{R}^\infty[l])$ converges in total variation to its stationary measure regardless of the initial distribution. It therefore has a trivial tail field, and by Orey [(1971), Theorem 4.1], the above limit is $H(\mathbf{z})\mu(\Gamma \times \widehat{A})$. This means the limit, as $l \to \infty$, of

$$P_{\mathbf{z}}\big(\widehat{\mathscr{W}}^\infty[\mathscr{T}_l^\infty] \in \widehat{A},\ \mathscr{R}^\infty[l] \in \Gamma,\ \mathscr{T}_l^\infty < \mathscr{T}_\blacktriangle^\infty\big)$$

is within ε of $H(\mathbf{z})\mu(\Gamma \times \widehat{A})$. The proof follows. □

PROPOSITION 2.9. *Under Conditions* 1–7,

$$\Psi_l(\mathbf{x}) \equiv \Psi(\mathbf{x}) \to H(\mathbf{x})\int_{\hat{y},\,u\geq 0} \hat{a}^{-1}(\hat{y})\exp(-\alpha\cdot u)\mu(du, d\hat{y}) \quad \textit{for } \mathbf{x} \in S \setminus (\triangle \cup F)$$

as $l \to \infty$, *where* $H(\mathbf{x}) := P_{\mathbf{x}}(\mathscr{T}_\blacktriangle^\infty = \infty)$.

PROOF. By Lemma 1.2, the chain $(\mathscr{R}^\infty[l], \widehat{\mathscr{W}}^\infty[\mathscr{T}_l])$ converges to $\hat{a}^{-1}(\hat{y})\exp(-\alpha\cdot u)\mu(du, d\hat{y})$ in distribution. Next, by Proposition 2.4,

$$\mu(dy_1, d\hat{y}) \le m(dy_1)\varphi(d\hat{y})/\tilde{d}_1.$$

Consequently,

$$\int_{\hat{y}}\int_{y_1\ge 0} \hat{a}^{-1}(\hat{y})\exp(-\alpha y_1)\mu(dy_1, d\hat{y}) \le \frac{\alpha}{\tilde{d}_1}\int_{\hat{y}} \hat{a}^{-1}(\hat{y})\varphi(d\hat{y}) < \infty$$

by Condition 5. By Meyn and Tweedie [(1993), Theorem 14.0.1], we have that

$$\begin{aligned} &E_{\mathbf{x}}\big(\chi\{\mathscr{T}_l^\infty < \mathscr{T}_\blacktriangle^\infty\}\hat{a}^{-1}(\widehat{\mathscr{W}}^\infty[\mathscr{T}_l])\exp(-\alpha\mathscr{R}^\infty[l])\big) \\ &\qquad \to H(\mathbf{x})\int_{\hat{y}}\int_{y_1\ge 0} \hat{a}^{-1}(\hat{y})\exp(-\alpha y_1)\mu(dy_1, d\hat{y}) \end{aligned}$$

for any starting point $\mathbf{x}$. $\square$

PROOF OF LEMMA 1.4. By Proposition 2.9, we have that

$$\Psi_l(\mathbf{x}) \to H(\mathbf{x})\int_{\mathbf{y}}\int_{y_1\ge 0} \hat{a}^{-1}(\hat{y})\exp(-\alpha y_1)\mu(dy_1, d\hat{y})$$

for any starting point $\mathbf{x}$. The sequence $\Psi_l(\mathbf{x})$ is uniformly integrable with respect to λ if $E_{\mathbf{x}}\hat{a}^{-1}(\widehat{\mathscr{W}}^\infty[\mathscr{T}_l^\infty])$ is, and this follows from Lemma 2.8. $\square$

PROOF OF THEOREM 1.5. Since W agrees with W^∞ on $S^\infty \setminus \blacktriangle$, we have

$$\begin{aligned} &\pi(\triangle)E_\triangle\bigg(\sum_{n=1}^{T_\triangle}\chi_A(W[n])\bigg) \\ &\quad = \int_{\mathbf{z}\in\triangle}\pi(d\mathbf{z})\int_{\mathbf{x}\notin\triangle}K(\mathbf{z}, d\mathbf{x})E_{\mathbf{x}}\bigg(\sum_{n=1}^{T_\triangle^\infty}\chi_A\big(W^\infty[n]\big)\bigg) \qquad (2.30)\\ &\quad = \int_{\mathbf{z}\in\triangle}\pi(d\mathbf{z})\int_{\mathbf{x}\notin\triangle}K(\mathbf{z}, d\mathbf{x})\int_{\mathbf{y}\in F}m_A(l, u, \hat{y}) \\ &\qquad\qquad \times P_{\mathbf{x}}(\widehat{W}^\infty[T_l^\infty]\in d\hat{y},\ R^\infty[T_l^\infty]\in du,\ T_l^\infty < T_\triangle^\infty), \end{aligned}$$

where we have conditioned on the point where W^∞ overshoots l, that is, at $\mathbf{y} = (\tilde{y}, \hat{y})$, where $u = y_1 - l$, and we have defined $m_A(l, u, \hat{y}) = E_{\mathbf{y}}(\sum_{n=1}^{T_\triangle}\chi_A(W^\infty[n]))$ to be the expected rewards for hitting A obtained by W^∞ (or W) after hitting F before returning to $\blacktriangle$.

By a change of measure,

$$\begin{aligned} &P_{\mathbf{x}}(\widehat{W}^\infty[T_l^\infty]\in d\hat{y},\ R^\infty[l]\in du,\ T_l^\infty < T_\triangle^\infty) \\ &\quad = h(\mathbf{x})E_{\mathbf{x}}\big(h^{-1}(\mathscr{W}^\infty[\mathscr{T}_l^\infty]) \\ &\qquad\qquad \times \chi\{\widehat{\mathscr{W}}^\infty[\mathscr{T}_l^\infty]\in d\hat{y},\ \mathscr{R}^\infty[l]\in du,\ \mathscr{T}_l^\infty < \mathscr{T}_\triangle^\infty\}\big) \\ &\quad = h(\mathbf{x})\exp(-\alpha l)\hat{a}^{-1}(\hat{y})\exp(-\alpha u) \\ &\qquad \times P_{\mathbf{x}}\big(\widehat{\mathscr{W}}^\infty[\mathscr{T}_l^\infty]\in d\hat{y},\ \mathscr{R}^\infty[\mathscr{T}_l^\infty]\in du,\ \mathscr{T}_l^\infty < \mathscr{T}_\triangle^\infty\big). \end{aligned}$$

Substituting the above expression into (2.30), we get

$$\int_A \pi((l+dx_1)\times R^{r-1}\times d\hat{x}) \tag{2.31}$$
$$= e^{-\alpha l}\int_{\mathbf{x}}\lambda(d\mathbf{x})\int_{\mathbf{y}\in F} m_A(l,u,\hat{y})\hat{a}^{-1}(\hat{y})e^{-\alpha u}$$
$$\times P_{\mathbf{x}}(\widehat{\mathscr{W}}^{\infty}[\mathscr{T}_l^{\infty}]\in d\hat{y},\ \mathscr{R}^{\infty}[l]\in du,\ \mathscr{T}_l^{\infty}<\mathscr{T}_{\triangle}^{\infty}).$$

Now, for any $\mathbf{y}=(\tilde{y},\hat{y})$, $y_1>l$,

$$m_A(l,u,\hat{y})\to m_A^{\infty}(u,\hat{y}):=E_{(u,0,\ldots,0,\hat{y})}\Big(\sum_{n=0}^{\infty}\chi_A\big(W^{\infty}[n]\big)\Big)\quad\text{as } l\to\infty,$$

by monotone convergence. By hypothesis, m_A^{∞} is uniformly bounded above so, using Lemma 1.4, we see that

$$e^{\alpha l}\int_A \pi((l+dx_1)\times R^{r-1}\times d\hat{x})$$
$$\to \int_{\mathbf{x}}\lambda(d\mathbf{x})H(\mathbf{x})\cdot\int_{\hat{y}\in\widehat{S}}\int_{u\geq 0}\hat{a}^{-1}(\hat{y})e^{-\alpha u}m_A^{\infty}(u,\hat{y})\mu(du,d\hat{y}). \qquad \square$$

PROOF OF THEOREM 1.6. By Theorem 1.5,

$$\int \chi_A(x_1,\hat{x})\pi((l+dx_1)\times R^{r-1}\times d\hat{x}) \tag{2.32}$$
$$\sim e^{-\alpha l}\int_{\mathbf{z}\in\triangle}\pi(d\mathbf{z})\int_{\mathbf{x}\notin\triangle}K(\mathbf{z},d\mathbf{x})h(\mathbf{x})H(\mathbf{x})$$
$$\times\int_{\hat{y}}\int_{u\geq 0}\hat{a}^{-1}(\hat{y})e^{-\alpha u}m_A^{\infty}(u,\hat{y})\mu(du,d\hat{y}),$$

where $m_A^{\infty}(u,\hat{y}):=E_{(u,0,\ldots,0,\hat{y})}(\sum_{n=0}^{\infty}\chi_A(W^{\infty}[n]))$.

For $\mathbf{y}=(u,y_2,\ldots,y_r,\hat{y})$,

$$P_{\mathbf{y}}\big(W^{\infty}[n]\in d\mathbf{w}\big)=(K^{\infty})^n(\mathbf{y},dw_1\times R^{r-1}\times d\hat{w})$$
$$=\exp(\alpha(u-v))\frac{\hat{a}(\hat{y})}{\hat{a}(\hat{w})}(\mathscr{K}^{\infty})^n(\mathbf{y},dv\times R^{r-1}\times\hat{w}).$$

Hence,

$$m_A^{\infty}(u,\hat{y})=E_{(u,0,\ldots,0,\hat{y})}\Big(\sum_{n=0}^{\infty}\chi_A\big(W^{\infty}[n]\big)\Big)$$
$$=\int_{v\geq 0}\int_{\hat{w}}\exp(\alpha(u-v))\frac{\hat{a}(\hat{y})}{\hat{a}(\hat{w})}\chi_A(v,\hat{w})E_{(u,0,\ldots,0,\hat{y})}$$
$$\times\Big(\sum_{n=0}^{\infty}\chi\{\mathscr{W}^{\infty}[n]\in dv\times R^{r-1}\times d\hat{w}\}\Big).$$

Substituting this into (2.32) gives

$$\exp(\alpha l)\pi(A) \to f\int_{\hat{y}}\int_{u\geq 0}\mu(du,d\hat{y})\int_{v\geq 0}\int_{\hat{w}}\frac{1}{\hat{a}(\mathbf{w})}\exp(-\alpha v)\chi_A(v,\hat{w})E_{(u,0,\dots,0,\hat{y})}$$
$$\times\left(\sum_{n=0}^{\infty}\chi\{\mathscr{W}^{\infty}[n]=dv\times R^{r-1}\times d\hat{w}\}\right)$$
$$= f\int_{v\geq 0}\int_{\hat{w}}\frac{1}{\hat{a}(\mathbf{w})}\exp(-\alpha v)\chi_A(v,\hat{w})\int_{\hat{y}}\int_{u\geq 0}\mu(du,d\hat{y})E_{(u,0,\dots,0,\hat{y})}$$
$$\times\left(\sum_{n=0}^{\infty}\chi\{\mathscr{W}^{\infty}[n]\in dv\times R^{r-1}\times d\hat{w}\}\right)$$
$$= f\int_{\hat{w}}\int_{v\geq 0}\chi_A(v,\hat{w})\hat{a}^{-1}(\hat{w})\varphi(d\hat{w})\frac{1}{\tilde{d}_1}\exp(-\alpha v)m(dv),$$

since φ is the steady state of $\widehat{\mathscr{K}}^{\infty}$, so

$$\frac{m(dv)\varphi(d\hat{w})}{\tilde{d}_1}=\int_{\hat{y},u}\mu(du,d\hat{y})E_{(u,0,\dots,0,\hat{y})}\left(\sum_{n=0}^{\infty}\chi\{\mathscr{W}^{\infty}[n]\in dv\times R^{r-1}\times d\hat{w}\}\right)$$

is the mean number of visits of $\mathscr{W}^{\infty}$ to $dv\times R^{r-1}\times d\hat{w}$. The asymptotics of $\pi(d\mathbf{y})$ conditioned on $y_1\in l+\Gamma$ follow from the above by summing. □

Note that $\Pi(d\mathbf{x})K^*(\mathbf{x},d\mathbf{y})=\Pi(d\mathbf{x})(K^{\infty})^*(\mathbf{x},d\mathbf{y})=\Pi(d\mathbf{y})\mathscr{K}^{\infty}(\mathbf{y},d\mathbf{x})$ for $\mathbf{x},\mathbf{y}\in S^{\infty}\setminus\blacktriangle$, where we denote the measure $h(\mathbf{y})\pi(d\mathbf{y})$ by $\Pi(d\mathbf{y})$. From Theorem 1.6, we have that, as l tends to infinity,

$$\Pi(du\times R^{r-1}\times d\hat{x})\to g\left(\int_{\hat{y},v}\hat{a}^{-1}(\hat{y})e^{-\alpha\cdot v}\mu(dv,d\hat{y})\right)^{-1}\varphi(d\hat{x})\times\frac{m(du)}{\tilde{d}_1},$$

where we recall m denotes counting measure in the discrete case and Lebesgue measure in the continuous case. Consequently, $K^*(\mathbf{x},dy_1\times R^{r-1}\times d\hat{y})$ is given asymptotically by the time reversal of $\mathscr{K}^{\infty}$ with respect to the measure $m(dx_1)\times\varphi(d\hat{x})$.

This connects the results in Section 2.2 and Corollary 1.11. By Corollary 1.11, the probability that W hits the set $du\times R^{r-1}\times d\hat{y}$ is asymptotically

$$\frac{1}{\pi(D)p_D}\int_{y_2,\dots,y_r}\pi(du\times dy_2\times\cdots\times dy_r\times d\hat{y})\int_{x_1<l}K^*(\mathbf{y},d\mathbf{x})f^*(\mathbf{x})$$

where $f^*(\mathbf{x})$ is probability the time reversal of W hits D before F

$$\sim e^{\alpha l}g^{-1}\int_{y_2,\dots,y_r}\pi(du\times dy_2\times\cdots\times dy_r\times d\hat{y})P_{\mathbf{y}}(W^*\text{ never returns to }F).$$

Using the asymptotic expression for $\Pi(du\times R^{r-1}\times d\hat{x})$, the above expression is asymptotic to

$$\left(\int_{\hat{y},v}\hat{a}^{-1}(\hat{y})e^{-\alpha\cdot v}\mu(dv,d\hat{y})\right)^{-1}$$
$$\times\hat{a}^{-1}(\hat{y})e^{-\alpha u}P_{\mathbf{y}}(\,(\mathscr{W}^{\infty})^*\text{ never returns to }F\,)\varphi(d\hat{y})\times\frac{m(du)}{\tilde{d}_1},$$

since the time reversal of $\mathscr{W}^\infty$ with respect to $\varphi(d\hat{y}) \times [m(du)/\tilde{d}_1]$ is asymptotically the same as W^* and since the probability that the time reversal of $\mathscr{W}^\infty$ leaves $\mathbf{y}$ and never returns to F is independent of $y_2, \dots, y_r$. Finally,

$$P_{\mathbf{y}}(\text{the time reversal of } \mathscr{W}^\infty \text{ never returns to } F)\varphi(d\hat{y}) \times \frac{m(du)}{\tilde{d}_1}$$

is equal to (2.28), so the above expression collapses to

$$\hat{a}^{-1}(\hat{y})e^{-\alpha u}\mu(du, d\hat{y})\left(\int_{\hat{y}, v} \hat{a}^{-1}(\hat{y})e^{-\alpha \cdot v}\mu(dv, d\hat{y})\right)^{-1},$$

which we already know is asymptotic to the probability W hits $\mathbf{y} \in F$.

PROOF OF LEMMA 1.8. Since $D \subseteq \triangle$, it follows that $\rho \geq f$. In fact, for any $\mathbf{x}$, the difference $\rho(\mathbf{x}) - f(\mathbf{x})$ is the probability of those trajectories of W which start at $\mathbf{x}$ and hit $\triangle \setminus D$ and then climb back to F before finally returning to D. This is because the chain W is Harris recurrent so that it will eventually hit either D or F.

These trajectories must hit $\triangle \setminus D$ for the last time before returning to F. Decomposing over this last exit time (say m) and the return time to F (say $n + m$), we can represent the probability

$$\int_{\mathbf{y}\in F} \pi(d\mathbf{y}) \int_{x \in B} K(\mathbf{y}, d\mathbf{x})(\rho(\mathbf{x}) - f(\mathbf{x}))$$

by

$$\begin{aligned}\sum_{m,n} \int_{\mathbf{y}\in F} \pi(d\mathbf{y}) \int_{\mathbf{z}\in \triangle\setminus D} P_{\mathbf{y}}\Big(&W[1] \notin F \cup D, \dots, W[m-1] \notin F \cup D,\ W[m] \in d\mathbf{z},\\ &W[m+1] \notin \triangle \cup F, \dots, W[m+n-1] \notin \triangle \cup F,\\ &W[m+n] \in F\Big).\end{aligned}$$

For m, n and $\mathbf{z}$ fixed, the above sum decomposes as the past and future around the time point m. Use time reversal on the past before time m and then sum over all m and n to get

$$\begin{aligned}&\int_{\mathbf{y}\in F} \pi(d\mathbf{y}) \int_{x\in B} K(\mathbf{y}, d\mathbf{x})(\rho(\mathbf{x}) - f(\mathbf{x}))\\ &\quad = \int_{\mathbf{z}\in\triangle,\, \mathbf{z}\notin D} \pi(d\mathbf{z})P_{\mathbf{z}}\Big(W[n] \in B \setminus D, -1 \leq n \leq -T_l^*;\\ &\qquad\qquad W[n] \notin \triangle,\ 1 \leq n \leq T_l\Big),\end{aligned}$$

where T_l^* is the first time the time-reversed process W^* reaches F.

Therefore, since the future of a Markov chain is independent of the past when we condition on the present, we have

$$P_{\mathbf{z}}(W[n] \in B \setminus D,\ -1 \leq n \leq -T_l^*;\ W[n] \notin \triangle,\ 1 \leq n \leq T_l) = U^*(\mathbf{z}) \cdot V(\mathbf{z}),$$

where

$$U^*(\mathbf{z}) = P_{\mathbf{z}}\big(W^* \text{ hits } F \text{ before } D\big) \tag{2.33}$$

and

$$V(\mathbf{z}) = P_{\mathbf{z}}\big(W[n] \in B \setminus \triangle,\ 1 \le n < T_l\big) \tag{2.34}$$

$$= \int_{\mathbf{x}} K(\mathbf{z}, d\mathbf{x})(1 - \rho(\mathbf{x})). \tag{2.35}$$

Hence,

$$\begin{aligned} b - \Lambda &= \int_{\mathbf{z}\in\triangle,\, \mathbf{z}\notin D} \pi(d\mathbf{z}) U^*(\mathbf{z}) \cdot V(\mathbf{z}) \\ &= \int_{\mathbf{z}\in\triangle,\, \mathbf{z}\notin D} \pi(d\mathbf{z}) U^*(\mathbf{z}) \cdot e^{-\alpha l} \int_{\mathbf{x}} K(\mathbf{z}, d\mathbf{x}) h(\mathbf{x}) \Psi(\mathbf{x}) \end{aligned}$$

by (1.19) and the definition of ρ.

By (1.15),

$$b = \int_{\mathbf{z}\in\triangle} \pi(d\mathbf{z}) \int_{\mathbf{x}} K(\mathbf{z}, d\mathbf{x})(1 - \rho(\mathbf{x}))$$

$$= \int_{\mathbf{z}\in\triangle} \pi(d\mathbf{z}) V(\mathbf{z}) \tag{2.36}$$

$$= \int_{\mathbf{z}\in\triangle} \pi(d\mathbf{z}) e^{-\alpha l} \int_{\mathbf{x}} K(\mathbf{z}, d\mathbf{x}) h(\mathbf{x}) \Psi(\mathbf{x}). \tag{2.37}$$

We conclude that

$$(b - \Lambda)/b = \frac{\int_{\mathbf{z}\in\triangle,\, \mathbf{z}\notin D} \pi(d\mathbf{z}) U^*(\mathbf{z}) \cdot \int_{\mathbf{x}} K(\mathbf{z}, d\mathbf{x}) h(\mathbf{x}) \Psi(\mathbf{x})}{\int_{\mathbf{z}\in\triangle} \pi(d\mathbf{z}) \int_{\mathbf{x}} K(\mathbf{z}, d\mathbf{x}) h(\mathbf{x}) \Psi(\mathbf{x})}.$$

By Lemma 1.4,

$$\begin{aligned} &\int_{\mathbf{z}\in\triangle} \int_{\mathbf{x}} \pi(d\mathbf{z}) K(\mathbf{z}, d\mathbf{x}) h(\mathbf{x}) \Psi(\mathbf{x}) \\ &\quad \equiv \int_{\mathbf{x}} \lambda(d\mathbf{x}) \Psi(\mathbf{x}) \to \int_{\mathbf{x}} \lambda(d\mathbf{x}) H(\mathbf{x}) \int_{\hat{y}} \int_{y_1 \ge 0} \hat{a}^{-1}(\hat{y}) \exp(-\alpha y_1) \mu(dy_1, d\hat{y}). \end{aligned}$$

By Condition 4, the set of $\mathbf{x}$ such that $H(\mathbf{x}) > 0$ has positive measure, so the above limit is strictly positive. Hence, the denominator of the above expression tends to a positive constant. On the other hand, $U^*(\mathbf{z}) \to 0$ for any $\mathbf{z}$ fixed as $l \to \infty$. Using the fact that Ψ_l converges in $L^1(\lambda)$, it follows that the numerator of the above expression tends to 0. This gives the result. □

PROOF OF THEOREM 1.10.

$$\lim_{l\to\infty} e^{\alpha l} b = \lim_{l\to\infty} e^{\alpha l} \int_{\mathbf{y}\in\triangle} \pi(\mathbf{y}) L\rho(\mathbf{y})$$

$$\begin{aligned} &= g \equiv \int_{\mathbf{z}\in\triangle} \pi(d\mathbf{z}) \int_{\mathbf{x}\notin\triangle} K(\mathbf{z}, d\mathbf{x}) h(\mathbf{x}) H(\mathbf{x}) \\ &\qquad \times \int_{\hat{y}} \int_{y_1\ge 0} \exp(-\alpha y_1) \hat{a}^{-1}(\hat{y}) \mu(dy_1, d\hat{y}). \end{aligned} \tag{2.38}$$

By Lemma 1.8 and Lemma 1.7, $E_0^{F(\to D)} T_l \sim b^{-1} \sim e^{\alpha l} g^{-1}$. The result now follows from Corollary 1.9. □

PROOF OF PROPOSITION 1.12. We made a substitution $\rho^*(\mathbf{x})$ for $f^*(\mathbf{x})$, where $\rho^*(\mathbf{x})$ is the first entrance time of the walk W^* into $\triangle$. The error introduced in calculating the probability of hitting F in A is

$$\int_A \pi(d\mathbf{y}) \int_{\mathbf{x}\in B} K^*(\mathbf{y}, d\mathbf{x})(\rho^*(\mathbf{x}) - f^*(\mathbf{x})) = \int_A \pi(d\mathbf{y}) P_{\mathbf{y}}(T^*_\triangle < T^*_l < T^*_D),$$

since $\rho^*(\mathbf{x}) > f^*(\mathbf{x})$ because of trajectories of W^* which hit $\triangle \setminus D$ and then climb back up to F before finally hitting D. Conditioning on the first entrance time in $\triangle \setminus D$ and using time reversal, the above expression is equal to

$$\int_{\mathbf{z}\in\triangle\setminus D} \pi(d\mathbf{z}) V_A(\mathbf{z}) U^*(\mathbf{z}),$$

where U^* was defined at (2.33) and $V_A(\mathbf{z}) := P_{\mathbf{z}}(W[T_F] \in A, T_F < T_\triangle)$.

Again by time reversal,

$$\int_A \pi(d\mathbf{y}) \int_{\mathbf{x}\in B} K^*(\mathbf{y}, d\mathbf{x}) \rho^*(\mathbf{x}) = \int_{\mathbf{z}\in\triangle} \pi(d\mathbf{z}) V_A(\mathbf{z}).$$

By the same argument as in the proof of Lemma 1.8, we have

$$\frac{\int_{\mathbf{z}\in\triangle\setminus D} \pi(d\mathbf{z}) V_A(\mathbf{z}) U^*(\mathbf{z})}{\int_{\mathbf{z}\in\triangle} \pi(d\mathbf{z}) V_A(\mathbf{z})} \to 0.$$

This gives the result. □

PROOF OF THEOREM 1.13. The conditional hitting distribution

$$P_D(R[l] \in \Gamma,\ \widehat{W}[T_l] \in \widehat{A} | T_l < T_D)$$

is asymptotically the same as (1.22). Using Lemma 1.4, the limit as $l \to \infty$ of (1.22) is

$$\frac{f}{g} \int_{\widehat{A}} \int_\Gamma \exp(-\alpha u) \hat{a}^{-1}(\hat{y}) \mu(du, d\hat{y}). \tag{2.39}$$

Since the sum over u and $\hat{y}$ of this limit must be 1, it follows that the above expression is just a constant times the density $\exp(-\alpha y_1)\hat{a}^{-1}(\hat{y})\mu(dy_1, d\hat{y})$. This gives the result. □

2.4. *Asymptotics of the twisted process.* If Conditions 1–3 hold, then by Lemma 2.7, $\mathscr{V}^\infty[\mathscr{T}_l^\infty] \to \tilde{d}/\tilde{d}_1$ as $l \to \infty$. Using this, we can also give some information about the nodes which are not "super" stable.

THEOREM 2.10 (Joint Bottlenecks). *If Conditions* 1–7 *hold, then the conditional distribution of* $\widetilde{W}[T_l]/l$, *given* $T_l < T_D$, *converges to a unit point measure at* $\tilde{d}/\tilde{d}_1$.

This result means the nodes driven into overload by the first grow linearly with the length of the queue at the first node. Of course, much more could be said about the conditional limiting distribution of $\widetilde{W}[T_l]$, but we will not pursue that enquiry here.

PROOF OF THEOREM 2.10. By definition, $\tilde{d}=(\tilde{d}_1,\dots,\tilde{d}_r)$. Let $\tilde{\varepsilon}=(\varepsilon,\dots,\varepsilon)$, where $0<\varepsilon<\min\{\tilde{d}_1,\dots\tilde{d}_r\}$. Let A denote the set $\{\mathbf{z}\in F\colon \tilde{z}/l\in[\tilde{d}-\tilde{\varepsilon},\tilde{d}+\tilde{\varepsilon}]\}$. The conditional distribution of $\widetilde{W}[T_l]/l$ given $T_l<T_D$ is close to $\tilde{d}$ can be expressed as

$$P_D\left(\frac{\widetilde{W}[T_l]}{l}\in[\tilde{d}-\tilde{\varepsilon},\tilde{d}+\tilde{\varepsilon}],T_l<T_D\right)\Big/P_D(T_l<T_D)$$
$$=\frac{P_D(\widetilde{W}[T_l]\in A\cap\{T_l<T_D\})}{P_D(T_l<T_D)}.$$

Now by Lemma 1.9,

$$\Lambda=\pi(D)P_D(T_l<T_D)=\pi(D)p_D\sim\pi(\triangle)p_\triangle=\pi(\triangle)P_\triangle(T_l<T_\triangle)=b.$$

By time reversal,

$$\pi(D)P_D\left(\widetilde{W}[T_l]\in A\cap\{T_l<T_D\}\right)=\int_A\pi(d\mathbf{z})\int_B K^*(\mathbf{z},d\mathbf{x})f^*(\mathbf{x})$$

and

$$\pi(\triangle)P_\triangle(\widetilde{W}[T_l]\in A\cap\{T_l<T_\triangle\})=\int_A\pi(d\mathbf{z})\int_B K^*(\mathbf{z},d\mathbf{x})\rho^*(\mathbf{x}),$$

where f^* and ρ^* are defined in Section 1.5. The difference between the two expressions above is bounded by

$$\int_F\pi(d\mathbf{z})\int_B K^*(\mathbf{z},d\mathbf{x})\,(\rho^*(\mathbf{x})-f^*(\mathbf{x}))\,.$$

The above expression was shown to be $o(b)$ in the proof of Theorem 1.10. Consequently,

$$\frac{P_D(\widetilde{W}[T_l]\in A\cap\{T_l<T_D\})}{P_D(T_l<T_D)}$$
$$\sim\frac{P_\triangle(\widetilde{W}[T_l]\in A\cap\{T_l<T_\triangle\})}{P_\triangle(T_l<T_\triangle)}$$
$$=\frac{P_\triangle(\widetilde{W}^\infty[T_l^\infty]\in A\cap\{T_l^\infty<T_\blacktriangle^\infty\})}{P_\triangle(T_l^\infty<T_\blacktriangle^\infty)}$$
$$=\frac{\int\lambda(d\mathbf{x})E_\mathbf{x}[\chi\{\widetilde{W}^\infty[T_l^\infty]\in A\cap\{\mathcal{T}_l^\infty<\mathcal{T}_\blacktriangle^\infty\}\}\cdot(h^{-1}(\mathscr{W}^\infty[\mathcal{T}_l^\infty]))]}{\int\lambda(d\mathbf{x})E_\mathbf{x}[\chi\{\mathcal{T}_l^\infty<\mathcal{T}_\blacktriangle^\infty\}\cdot(h^{-1}(\mathscr{W}^\infty[\mathcal{T}_l^\infty]))]}.$$

By Lemma 2.7, $\lim_{l\to\infty}\widetilde{\mathscr{W}}^\infty[\mathcal{T}_l^\infty]/l=\tilde{d}/\tilde{d}_1$. Hence,

$$P_\mathbf{x}(\mathscr{W}^\infty[\mathcal{T}_l^\infty]\in A)\to 1\quad\text{as } l\to\infty.$$

If we cancel out the factor $\exp(-\alpha l)$ from the numerator and denominator of the above ratio, we see that both the numerator and denominator converge in $L^1(\lambda)$ to the same limit using Conditions 4–7. Consequently, the conditional distribution of $\widetilde{W}[T_l]/l$ given $T_l < T_D$ is concentrated arbitrarily close to $\tilde{d}$ as $l \to \infty$. This is what we wanted to prove. □

If the kernel $\mathscr{K}^\infty$ has decomposition given at the end of Section 1 and if, in addition to Conditions 1–7, we can also show that $\widetilde{\mathscr{W}}^\infty[\mathscr{T}_l^\infty]/l \to \tilde{d}/\tilde{d}_1$ as $l \to \infty$ for some mean drift $\tilde{d}$, then the above proof of Theorem 2.10 still works.

3. Examples.

3.1. *Flatto–Hahn–Wright model.* In the Flatto–Hahn–Wright model, customers arrive at nodes 1 and 2 according to two independent Poisson processes with rates λ and η, respectively. There is also a third independent Poisson stream with rate ν which feeds both nodes simultaneously. The service rates at node 1 and node 2 are exponential with rates α and β, respectively, and the customers queue until they are served. Let (x, y) denote the number of customers waiting or being served at queue 1 and queue 2. The above Markov jump process has jump rates given by Figure 1.

Using standard methods, Wright (1992) showed the stationary distribution π of this jump process exists precisely when $\max\{(\lambda+\nu)/\alpha, (\eta+\nu)/\beta\} < 1$. The generator L of this jump process is given as an operator on a bounded function g on S.

$$\begin{aligned}Lg(x, y) = {}& \lambda(g(x+1) - g(x, y)) + \nu(g(x+1, y+1) - g(x, y))\\ &+ \eta(g(x, y+1) - g(x, y)) + \alpha(x)(g(x-1, y) - g(x, y))\\ &+ \beta(y)(g(x, y-1) - g(x, y)),\end{aligned}$$

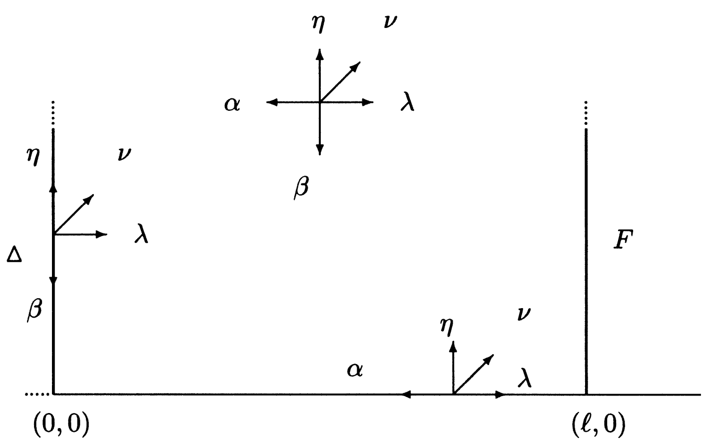

FIG. 1. *Rates for the FHW model.*

where $\beta(x) = \beta$ if $x \geq 1$ and 0 otherwise and where $\alpha(y)$ is defined analogously. The event rate of this jump process is $q = \lambda + \nu + \eta + \beta + \alpha$. If we regard L as the discrete generator of a Markov chain W on S, then the FHW jump process is precisely the homogenization of this chain. In other words, the transition kernel at time t of the FHW jump process is given by $\exp(qtL)$. Consequently, π is also the stationary distribution of W. W is a nearest-neighbor random walk in S. Without loss of generality, we shall assume $q = 1$, so we can simply confound the generators of the two processes.

In this model, $S = \mathbb{N}_0 \times \mathbb{N}_0$. Take $S^\infty = \mathbb{Z} \times \mathbb{N}_0$ and $\blacktriangle = \{(x, y)\colon x \leq 0,\ x \in \mathbb{Z},\ y \in \mathbb{N}_0\}$. To calculate the twist constants, remark that the constraint in the interior, $\operatorname{int}(S)$, is

$$\lambda a_1 + \nu a_1 a_2 + \eta a_2 + \alpha a_1^{-1} + \beta a_2^{-1} = 1.$$

The constraint on the x-axis, $S_{\{2\}}$, is

$$\lambda a_1 + \nu a_1 a_2 + \eta a_2 + \alpha a_1^{-1} = \lambda + \nu + \eta + \alpha. \tag{3.40}$$

Subtracting the latter constraint from the first yields $\beta a_2^{-1} = \beta$, which corresponds to (1.9). Consequently, $a_2 = 1$. Substituting into the first constraint gives $a_1 = \alpha/(\nu + \lambda)$. (Of course, the other solution is $a_1 = 1$.) If the chain is stable, $a_1 > 1$.

We can now check the conditions for Theorem 1.10. The twisted kernel $\mathscr{K}^\infty$ is given by

$$J(1,0) = \lambda a_1 = \lambda\alpha/(\nu + \lambda), \qquad J(1,1) = \nu a_1 = \nu\alpha/(\nu + \lambda),$$
$$J(0,1) = \eta, \qquad J(-1,0) = \alpha a_1^{-1} = \nu + \lambda, \qquad J(0,-1) = \beta.$$

The kernel $\widehat{\mathscr{K}}^\infty$ of the y-coordinate process $\widehat{\mathscr{W}}^\infty$ is as follows: for $y \geq 1$,

$$\widehat{\mathscr{K}}^\infty(y, y+1) = \eta + \nu\frac{\alpha}{\nu + \lambda},$$
$$\widehat{\mathscr{K}}^\infty(y, y) = \lambda\frac{\alpha}{\nu + \lambda} + (\nu + \lambda),$$
$$\widehat{\mathscr{K}}^\infty(y, y-1) = \beta.$$

For $y = 0$,

$$\widehat{\mathscr{K}}^\infty(0, 1) = \eta + \nu\frac{\alpha}{\nu + \lambda},$$
$$\widehat{\mathscr{K}}^\infty(0, 0) = \lambda\frac{\alpha}{\nu + \lambda} + \nu + \lambda + \beta.$$

The latter is a transition kernel because of constraint (3.40). $\widehat{\mathscr{K}}^\infty$ is the kernel of a positively recurrent aperiodic chain on $\mathbb{N}_0$ as long as

$$\gamma := \eta + \nu\frac{\alpha}{\nu + \lambda} < \beta. \tag{3.41}$$

Take $\psi = m$, where m counts the points in $\mathbb{N}_0$. It is trivial to check Condition M2 in Condition 2 since $\hat{\mathscr{W}}^{\infty}$ is ψ-recurrent. It suffices to take points $(\tilde{0}, \hat{a})$ and $(\tilde{b}, \hat{b})$ such that $c := \mathscr{K}^{\infty}((\tilde{0}, \hat{a}), (\tilde{b}, \hat{b})) > 0$ and define

$$\nu(d\hat{y}) := \triangle_{\hat{b}}(d\hat{y}) \quad \text{and} \quad h(\hat{x}, d\tilde{y}) := c\chi_{\hat{a}}(\hat{x})\triangle_{\tilde{b}}(d\tilde{y}).$$

Condition 3 follows because the mean drift is

$$\tilde{d}_1 = (\lambda + \nu)a_1 - \alpha a_1^{-1} = \alpha - (\lambda + \nu),$$

and this is strictly positive when the chain is stable. Condition 4 follows by the law of large numbers since the increments $\mathscr{X}^{\infty}[n]$ have positive expectation by (3.41). Condition 5 holds since $a_2 = 1$. Condition 6 holds for the same reason, as does Condition 7.

We can now apply Theorem 1.10 to conclude that as long as (3.41) holds, the mean time to reach the forbidden set F is asymptotically

$$a_1^l g^{-1} = g^{-1}\left(\frac{\alpha}{\nu + \lambda}\right)^l,$$

where the constant f can be obtained by simulating the twisted process $\mathscr{W}^{\infty}$ having kernel $\mathscr{K}^{\infty}$. This is not much of a surprise because the first node behaves like an $M|M|1$ queue with load $(\lambda + \nu)/\alpha$.

We may also apply Theorem 1.13 to show the hitting distribution on F converges to a measure proportional to μ (since $a_2 = 1$), where μ is the hitting distribution of the twisted process on F. This can be obtained quickly by simulation.

Finally, applying Theorem 1.6, we see that, as $l \to \infty$, $\pi(l, y)/\sum_y \pi(l, y)$ converges to

$$a_2^{-y}\varphi(y)\Big/\left(\int_z a_2^{-z}\varphi(z)\right) = \left(1 - \frac{\gamma}{\beta}\right)\left(\frac{\gamma}{\beta}\right)^y.$$

Now suppose the rates are such that $\gamma > \beta$, so when the first node is overloaded the second also overloads. To treat this case, create a fictitious node which neither receives nor serves customers. In other words, consider a state space $S = \mathbb{N}_0 \times \mathbb{N}_0 \times \{0\}$. The chain $W = (W_1, W_2, \widehat{W})$ is such that $\widehat{W}[t] \equiv 0$, while $(W_1[t], W_2[t])$ represents the number of customers queued at the first and second nodes, respectively. To construct W^{∞}, simply extend the transition probabilities of W to $S^{\infty} := \mathbb{Z} \times \mathbb{Z} \times \{0\}$ by taking $\blacktriangle = \{(x, y)\colon x \le 0 \text{ or } y \le 0;\ x, y \in \mathbb{Z}\}$.

Now twist W^{∞} with twist coordinates $(a_1, 1, a_3)$. Since $\widehat{W}^{\infty}$ has only one state, $a_3 = 1$, and the only equation to be satisfied is $(\lambda+\nu)a_1+(\eta+\beta)+\alpha a_1^{-1} = \lambda + \nu + \eta + \beta + \alpha$. The only solution other than $a_1 = 1$ is $a_1 = \alpha/(\nu + \lambda)$. This is the same twist as was obtained above. The stationary distribution of $\widehat{W}^{\infty}$ is a unit mass at 0 so $(\tilde{d}_1, \tilde{d}_2) = (\alpha - (\nu + \lambda), \nu + \nu\alpha/(\nu + \lambda) - \beta)$ by adding the components of $(\mathscr{W}_1^{\infty}, \mathscr{W}_2^{\infty})$. Hence $\tilde{d}_1$ and $\tilde{d}_2$ are positive, since we are assuming $\gamma \ge \beta$, so Conditions 1–5, 7 are automatic.

To check Condition 6, we must show that

$$\sum_{x=0}^{\infty} h(x,0)\pi(x,0) < \infty \quad \text{and} \quad \sum_{y=0}^{\infty} h(0,y)\pi(0,y).$$

The second sum is surely finite, since $h(0, y) = 1$ for $y \geq 0$. Next,

$$\sum_{x=0}^{\infty} h(x,0)\pi(x,0) = \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} f(x,y)\pi(x,y) \quad \text{where } f(x,y) := a_1^x \cdot \chi\{y=0\}.$$

By Lemma 1.1, it suffices to find a positive function $V(x, y)$ such that $\triangle V(x, y) \leq -f(x, y) + s(x, y)$, where s is a positive function such that $\sum_{x,y} s(x, y)\pi(x, y) < \infty$.

Define a function $V(x, y) := (a_1^x(\beta/\gamma)^y)/(\gamma-\beta)$. It is easy to check that $\triangle V(x, y) = 0$ if $x > 0$ and $y > 0$. Next, if $x > 0$, then $\triangle V(x, 0) = (\beta-\gamma)V(x, 0) = -a_1^x = -f(x, 0)$. Finally, if $y > 0$, then

$$\triangle V(0,y) = (\alpha-(\nu+\lambda))V(0,y) = (\alpha-(\nu+\lambda))(\beta/\gamma)^y/(\gamma-\beta)$$

and

$$\triangle V(0,0) = (\beta-\gamma+\alpha-(\nu+\lambda))V(0,0) = (\beta-\gamma+\alpha-(\nu+\lambda))/(\gamma-\beta).$$

Define

$$\begin{aligned} s(x,y) = {} & \frac{(\alpha-(\nu+\lambda))}{(\gamma-\beta)}(\beta/\gamma)^y \cdot \chi\{x=0\} \\ & + \left[1 + \frac{(\beta-\gamma+\alpha-(\nu+\lambda))}{(\gamma-\beta)}\right] \cdot \chi\{x=0,\ y=0\}. \end{aligned}$$

Note that s is positive because $\alpha > \lambda+\nu$ and $\beta > \nu+\eta$ by hypothesis. It follows that $\triangle V(x, y) \leq -f(x, y) + s(x, y)$. The last step is to verify that

$$\begin{aligned} \sum_{x,y} s(x,y)\pi(x,y) = {} & \left[1 + \frac{(\beta-\gamma+\alpha-(\nu+\lambda))}{(\gamma-\beta)}\right]\pi(0,0) \\ & + \sum_{y=1}^{\infty} \frac{(\alpha-(\nu+\lambda))}{(\gamma-\beta)}(\beta/\gamma)^y \pi(0,y) < \infty, \end{aligned}$$

but this holds because $\beta < \gamma$.

Applying Theorem 1.10, we get that $E_{\triangle} T_l$ is of order a_1^l. Again, this is not very surprising because the first node behaves like an $M|M|1$ queue with input rate $\nu + \lambda$ and service rate α. Applying Theorem 2.10, we get that $W_2[T_l]/l$ grows linearly in l with a rate of $\tilde{d}_2/\tilde{d}_1$.

Acknowledgments. I wish to thank Professors Arie Hordijk, Ian Iscoe and Bob Foley as well as Benoît Beck and Nadine Labrèche for their help. Thanks also to the rather harsh but very professional referees who obliged me to clearly set out what is novel. Finally, I should like to express my appreciation to François Baccelli at the INRIA and Dick Serfozo at Georgia Tech for their hospitality during the completion of this work.

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