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# BILINEAR STOCHASTIC SYSTEMS WITH FRACTIONAL BROWNIAN MOTION INPUT[1]

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The partial derivatives with respect to time and the fractional Brownian motion of a particular class of stationary processes are defined. Although the fractional Brownian motion is not semimartingale, the bilinear SDE with fractional Brownian motion input is considered and solved. The solution is explicitly given in both the frequency and time domains in the case when the coefficient of the bilinear term is pure imaginary. The stationary Stratonovich solution of the bilinear SDE with white noise input is also considered.

**1. Introduction.** The applications of long-range dependent models involve several fields of science and economics such as geophysics, hydrology, turbulence, weather and so on. Good lists of references for this field are given in [1] and [9]. The basic stochastic process of this kind is the fractional Brownian motion defined in [5]. The fractional Brownian motion is given as a particular fractional operator on the standard Brownian motion. The Gaussian parametric models of long-range dependent phenomena are usually considered as either linear stochastic differential equations (SDE) with fractional Brownian motion input or linear stochastic differential equations with Brownian motion input and with a fractional operator on the output. Actually, these two types of processes are equivalent. Several linear models with long-range dependence are discussed in [7]. Most of the observations are not Gaussian, so there is a need for nonlinear modelling for long-range dependence. One of the possibilities for keeping the Gaussian input and at the same time getting rid of Gaussianity is the application of bilinear models. The easy way to get a long-range non-Gaussian process is to put the fractional operator on a non-Gaussian process, for example, on the solution of the bilinear SDE. This procedure can be carried out without difficulties. It requires more care if one considers a bilinear SDE with fractional Brownian motion input, because several difficulties arise. This might be the reason why no one has considered any nonlinear model with fractional noise input either in discrete or in continuous time.

In this paper we start with the bilinear SDE with white noise input and list some basic ideas leading to the stationary Itô solution, given in both the time domain and chaotic frequency domain forms. The first step in studying fractional Brownian motion as an input of bilinear SDE is to understand its

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basic properties in the frequency domain and introduce the integral of some deterministic functions with respect to it. We use the Itô calculus extensively in the frequency domain for the definition of the partial derivatives of a particular class of stationary processes with respect to time and fractional Brownian motion. One of the basic problems one has to pay attention to is that fractional Brownian motion is not semimartingale. Based on these derivatives, the bilinear SDE with fractional Brownian motion input is considered and solved. In the case when the coefficient of the bilinear term is pure imaginary, the solution of the bilinear SDE is explicitly given in both the frequency and time domains. It is shown that, in spite of using the Itô calculus in the frequency domain, the result is that the solution of the SDE follows the ordinary chain rule; therefore it is a Stratonovich solution. The main reason for this is that the quadratic variation of the fractional Brownian motion is zero. The stationary Stratonovich solution of the bilinear SDE with white noise input is also considered.

## 2. Brownian motion input.

2.1. *Time domain.* Throughout this paper a single scalar-valued input process will be considered. At the same time the output process will be complex-valued because of the complex constants of the equations. Let us start with the stochastic differential equation

$$dy_t = (\mu + \alpha y_t)\,dt + (\beta + \gamma y_t)\,dw_t, \tag{1}$$

where $w_t$ is Brownian motion with variance $\sigma^2$. The $\sigma^2$ is considered to be 1, otherwise one can use the transformation $w_t/\sigma$, $\sigma\beta$, $\sigma\gamma$. Equation (1) is a linear differential equation. Nevertheless, it is called bilinear in system theory to distinguish between the situations when $\gamma$ is zero and nonzero, that is, when the solution is Gaussian and non-Gaussian, in other words, when the model is linear and nonlinear. The Itô solution of (1) is well known (see [3], page 111):

$$\begin{aligned} y_t = \exp\biggl(\Bigl(\alpha - \frac{\gamma^2}{2}\Bigr)t + \gamma w_t\biggr)\biggl(y_0 + (\mu - \beta\gamma)\int_0^t \exp\biggl(-\Bigl(\alpha - \frac{\gamma^2}{2}\Bigr)s - \gamma w_s\biggr)\,ds \\ + \beta\int_0^t \exp\biggl(-\Bigl(\alpha - \frac{\gamma^2}{2}\Bigr)s - \gamma w_s\biggr)\,dw_s\biggr). \end{aligned}$$

When $\gamma = 0$, $\operatorname{Re}\alpha < 0$ and $\beta \neq 0$, this provides the stationary Gaussian Ornstein–Uhlenbeck process. We are interested in the stationary non-Gaussian solution of (1); therefore, it is necessary to assume that $|\mu|^2 + |\beta|^2 > 0$, $\gamma \neq 0$ and not only $\operatorname{Re}\alpha < 0$ but $2\operatorname{Re}\alpha + |\gamma|^2 < 0$ as well. The starting value $y_0$ is also well defined and then the stationary physically realizable solution of (1) is

$$\begin{aligned} y_t = (\mu - \beta\gamma)\int_{-\infty}^t \exp\biggl(\Bigl(\alpha - \frac{\gamma^2}{2}\Bigr)(t-s) + \gamma(w_t - w_s)\biggr)\,ds \\ + \beta\int_{-\infty}^t \exp\biggl(\Bigl(\alpha - \frac{\gamma^2}{2}\Bigr)(t-s) + \gamma(w_t - w_s)\biggr)\,dw_s. \end{aligned} \tag{2}$$

Note that $\mu\gamma \neq \alpha\beta$ must hold; otherwise (1) has only the degenerate solution $y_t = -\beta/\gamma = -\mu/\alpha$. From now on we shall assume that $\beta = 0$ and $\mu \neq 0$; that is, there is no second term in (2). This assumption can be fulfilled by the following transformation:

$$\tilde{y}_t = \frac{\gamma\mu}{\gamma\mu - \alpha\beta} y_t + \frac{\beta}{\gamma}. \tag{3}$$

The stochastic differential equation (1) without any restriction of generality becomes

$$dy_t = (\mu + \alpha y_t)\,dt + \gamma y_t\,dw_t, \tag{4}$$

and this equation will be the subject of our investigation. The expectation of $y_t$ can be calculated from (4) as

$$\mathsf{E}\,y_t = -\frac{\mu}{\alpha}.$$

One can easily get the autocovariance function $R(t) = \mathsf{E}\,(y_t - \mathsf{E}\,y_t)\overline{(y_0 - \mathsf{E}\,y_0)}$ of $y_t$ directly from (4) if $t \neq 0$ and for the variance $R(0)$ from (2),

$$\begin{aligned} R(t) &= R(0)e^{\alpha t} = \frac{-|\mu|^2|\gamma|^2}{|\alpha|^2(2\operatorname{Re}\alpha + |\gamma|^2)} e^{\alpha t}, \qquad t > 0, \\ R(t) &= \overline{R(-t)}, \qquad t \le 0. \end{aligned} \tag{5}$$

2.2. *Frequency domain.* Suppose there exists a stationary physically realizable solution of (4) which is subordinated to the integrator process $w_t$. The frequency domain representation theorem says (see [2]) that all such solutions can be changed into the so-called chaotic spectral representation form

$$y_t = \sum_{k=0}^{\infty} \int_{\mathbb{R}^k} \exp(it\Sigma\omega_{(k)}) f_k(\omega_{(k)}) W(d\omega_{(k)}), \tag{6}$$

where $\omega_{(k)} = (\omega_1, \omega_2, \dots, \omega_k)$, $\Sigma\omega_{(k)} = \sum_{j=1}^{k} \omega_j$ and $W(d\omega_{(k)})$ is the $k$-dimensional multiple Wiener–Itô spectral measure according to the Wiener process $w_t$. The representation (6) is unique up to the permutation of the variables of the transfer functions $f_k$. One of the main tools of the multiple Wiener–Itô integral technique is the diagram formula, which in general expresses the product of multiple integrals in terms of a linear combination of multiple integrals; see [4]. In this paper the diagram formula is used for the following particular case. Let $g \in L^2(\mathbb{R})$ and $f \in L^2(\mathbb{R}^k)$ be Fourier transforms of some real-valued functions and let $f$ be symmetric, then

$$\begin{aligned} &\int_{\mathbb{R}} g(\omega) W(d\omega) \int_{\mathbb{R}^k} f(\omega_{(k)}) W(d\omega_{(k)}) \\ &\quad = \int_{\mathbb{R}^{k+1}} f(\omega_{(k)}) g(\omega_{k+1}) W(d\omega_{(k+1)}) \\ &\qquad + k \int_{\mathbb{R}^{k-1}} \int_{\mathbb{R}} \bar{g}(\omega_k) f(\omega_{(k)})\, d\omega_k\, W(d\omega_{(k-1)}). \end{aligned} \tag{7}$$

Now under the assumption that we are looking for the stationary Itô solution (2) of (4), that is, by putting by definition the principal value for the integral

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{1-\exp(-it\omega)}{i\omega}\, d\omega \doteq 0,$$

it follows from the diagram formula that the following recursion is valid for the transfer functions $f_k$ (see [10]):

$$f_0 = -\frac{\mu}{\alpha}, \qquad f_k(\omega_{(k)}) = \frac{\gamma f_{k-1}(\omega_{(k-1)})}{i\Sigma\omega_{(k)} - \alpha}, \qquad k \geq 1. \tag{8}$$

To make it easier to understand (32), which concerns the transfer functions according to the fractional Brownian motion input, we remark that the symmetrized version $\tilde{f}_k = \operatorname{sym} f_k$ of these transfer functions can be written in the form

$$\tilde{f}_k(\omega_{(k)}) = \mu \frac{\gamma^k}{k!}\int_0^\infty \exp(\alpha u) \prod_{j=1}^{k} \frac{1-\exp(-iu\omega_j)}{i\omega_j}\, du. \tag{9}$$

We recall here that the symmetrized version $\tilde{f}$ of $f$ by the vector $\omega_{(k)}$ is the average of those values of $f$ which are taken by all possible permutations of entries of $\omega_{(k)}$. The spectrum of $y_t$ can be calculated from the autocovariance function (5); that is,

$$\varphi(\omega) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-it\omega} R(t)\, dt = -\frac{R(0)\operatorname{Re}\alpha}{\pi|i\omega-\alpha|^2}, \qquad \omega \in \mathbb{R}.$$

It should be noted that there is no difference between the spectrum of an Ornstein–Uhlenbeck process and $\varphi(\omega)$; therefore, it is necessary to consider higher order spectra for bilinear processes.

## 3. Stochastic differential equations with fractional Brownian motion integrator process.

3.1. *Fractional Brownian motion.* The definition of the fractional Brownian motion with parameter $h \in (-\frac{1}{2}, \frac{1}{2})$ due to [5] is the following:

$$w_t^{(h)} \doteq \frac{1}{\Gamma(1+h)}\left\{\int_{-\infty}^{0} [(t-s)^h - (-s)^h]\, dw_s + \int_0^t (t-s)^h\, dw_s\right\},$$

$t \in \mathbb{R}$. Formally $w_t^{(h)}$ can be expressed with the help of the fractional integral operator $I_x^{(h)}(f)$ defined as

$$I_x^{(h)}(f) = \frac{1}{\Gamma(h)}\int_{-\infty}^{x} (x-y)^{h-1} f(y)\, dy,$$

namely, the $h$th fractional integral process of the Brownian motion, adjusted to zero at zero; that is,

$$\begin{aligned} w_t^{(h)} &= \frac{1}{\Gamma(1+h)} \int_{-\infty}^{t} (t-s)^{(1+h)-1} w_s' \, ds - \frac{1}{\Gamma(1+h)} \int_{-\infty}^{0} (-s)^{(1+h)-1} w_s' \, ds \\ &= I_t^{(1+h)}(w') - I_0^{(1+h)}(w') \\ &= I_t^{(h)}(w) - I_0^{(h)}(w). \end{aligned}$$

Clearly, $w_t^{(0)} = w_t$.

We shall consider only the case $0 < h < \frac{1}{2}$ because of the following motivations. First, the long memory or long-range dependence property that we are interested in appears only for a positive $h$. It is widely known from experience that fractional Brownian motion with a negative $h$ hardly occurs in practice. Furthermore, the latter case would need very different considerations and even the object of this paper would reach a deadlock at some point. Again, we shall assume in the sequel that $0 < h < \frac{1}{2}$.

The most important properties of $w_t^{(h)}$ (see [5]) are the following:

1. $w_0^{(h)} = 0$;
2. $w_t^{(h)}$ is mean square continuous and continuous with probability 1;
3. it has stationary increment processes;
4. in any $t \in \mathbb{R}$, $w_t^{(h)}$ is not differentiable with probability 1;
5. it is self-similar with self-similarity parameter $h + \frac{1}{2}$; that is, the vectors $(w_{ct_1}^{(h)}, \dots, w_{ct_k}^{(h)})$ and $(|c|^{h+1/2} w_{t_1}^{(h)}, \dots, |c|^{h+1/2} w_{t_k}^{(h)})$ have the same distribution;
6. its first and second order moments are

$$\mathsf{E}\, w_t^{(h)} = 0,$$

$$\operatorname{Cov}\big(w_t^{(h)}, w_s^{(h)}\big) = \frac{\kappa(h)}{2} \big(|t|^{2h+1} + |s|^{2h+1} - |t-s|^{2h+1}\big), \tag{10}$$

$$\operatorname{Var} w_t^{(h)} = \kappa(h) |t|^{2h+1}, \tag{11}$$

where

$$\kappa(h) \doteq \frac{1}{2\pi} \int_{\mathbb{R}} \left| \frac{e^{i\omega} - 1}{i\omega} \right|^2 |\omega|^{-2h} \, d\omega. \tag{12}$$

In the rest of the paper, the constant $\kappa(h)$ will frequently appear. In the Appendix, Lemma A.2, it is expressed in terms of the gamma function as well.

It is evident that we have at least as many difficulties integrating with respect to $w_t^{(h)}$ as integrating by $w_t$ in the simple case. What is more, $w_t^{(h)}$ is not semimartingale since its quadratic variation process is zero; see Lemma A.1

in the Appendix. That fact makes it necessary to create a new stochastic integration concept.

The spectral domain representation of $w_t^{(h)}$ has already been mentioned in [5] and [8], but not exactly in the form that follows.

THEOREM 3.1. *Let the spectral domain representation of the Brownian motion $w_t$ be*

$$w_t = \int_{\mathbb{R}} \frac{e^{it\omega} - 1}{i\omega} W(d\omega),$$

*where $W(d\omega)$ is a complex Gaussian white noise spectral measure and* $\mathsf{E}\,|W(d\omega)|^2 = (1/2\pi)\,d\omega$. *Then the spectral domain representation of $w_t^{(h)}$ is*

$$w_t^{(h)} = \int_{\mathbb{R}} \frac{e^{it\omega} - 1}{i\omega} (i\omega)^{-h} W(d\omega). \tag{13}$$

From (13) we get the formal representation of the "derivative process," which exists only in the sense of Schwarz distribution (see [5]):

$$\frac{dw_t^{(h)}}{dt} = (w_t^{(h)})' = \int_{\mathbb{R}} e^{it\omega} (i\omega)^{-h} W(d\omega).$$

The next result is the invertibility of $w_t^{(h)}$. For that reason we define the integration of a nonrandom function with respect to $w_t^{(h)}$. Formally,

$$\begin{aligned}\int_{\mathbb{R}} f(t)\,dw_t^{(h)} &= \int_{\mathbb{R}} f(t)(w_t^{(h)})'\,dt = \int_{\mathbb{R}} f(t) \int_{\mathbb{R}} e^{it\omega}(i\omega)^{-h} W(d\omega)\,dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{it\omega} f(t)\,dt (i\omega)^{-h} W(d\omega).\end{aligned}$$

This idea is given a precise meaning by the definition.

DEFINITION 3.1. Let $f\colon \mathbb{R} \to \mathbb{R}$, $f \in L^2(\mathbb{R})$ and

$$\int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{it\omega} f(t)\,dt \right|^2 |\omega|^{-2h}\,d\omega < \infty.$$

Then

$$\int_{\mathbb{R}} f(t)\,dw_t^{(h)} \doteq \int_{\mathbb{R}} \int_{\mathbb{R}} e^{it\omega} f(t)\,dt (i\omega)^{-h} W(d\omega).$$

The use of this definition is seen in the following theorem.

THEOREM 3.2.

$$w_t = \frac{1}{\Gamma(1-h)} \int_{-\infty}^{t} \left\{ [(t-s)^{-h} - (-s)^{-h}] \chi_{(-\infty,0]}(s) + (t-s)^{-h} \chi_{(0,t]}(s) \right\} dw_s^{(h)}.$$

DEFINITION 3.2. We shall denote the $\sigma$-algebras generated by the pasts of the processes $w_t$ and $w_t^{(h)}$ by $\mathscr{F}^{w}_{(-\infty,t]}$ and $\mathscr{F}^{w^{(h)}}_{(-\infty,t]}$, respectively.

It is not difficult to show that in Theorem 3.2 the integrand is not only in $L^2(\mathbb{R})$, but in $L^1(\mathbb{R})$ as well. So one can apply Lemma A.3 in the Appendix to obtain from Theorem 3.2 the following theorem

THEOREM 3.3. $\mathscr{F}^{w}_{(-\infty,t]} = \mathscr{F}^{w^{(h)}}_{(-\infty,t]}$.

The spectral domain approach, the idea of which arises from [10], is especially useful in our situation. It has the advantage that we can exploit the simplicity with which the spectral representation form of the processes under consideration depends on time. It is based on the technique of spectral domain representation of square integrable stationary functionals of the fractional Brownian motion; see [2].

DEFINITION 3.3. Let us denote the space of all complex-valued stationary processes $y_t$ which are measurable with respect to the $\sigma$-algebra $\mathscr{F}(w^{(h)}) \doteq \sigma\{w_t^{(h)}, t \in \mathbb{R}\}$ by $S_2(w^{(h)})$, that is,

$$S_2(w^{(h)}) \doteq \{y_t \in L^2(\mathscr{F}(w^{(h)})) \colon y_t \text{ is second-order stationary}\}.$$

It is clear that by Theorem 3.3, $S_2(w^{(h)}) = S_2(w)$. Thus, every $y_t \in S_2(w^{(h)})$ can be transformed into the spectral domain chaotic representation form

$$y_t = \sum_{k=0}^{\infty} \int_{\mathbb{R}^k} \exp(it\Sigma\omega_{(k)}) f_k(\omega_{(k)}) \prod_1^k (i\omega_j)^{-h} W(d\omega_{(k)}), \tag{14}$$

where

$$\mathsf{E}\,|y_t|^2 = \sum_{k=0}^{\infty} \frac{k!}{(2\pi)^k} \int_{\mathbb{R}^k} |f_k(\omega_{(k)})|^2 \prod_1^k |\omega_j|^{-2h}\, d\omega_{(k)} < \infty,$$

and every $y_t$ of that form is in $S_2(w^{(h)})$. Since $S_2(w^{(h)})$ does not depend on $h$, from now on we shall use the notation $S_2 \doteq S_2(w^{(h)})$.

By transfer functions we shall mean the functions $f_k$. An essential property of the representation (14) is that the transfer functions $f_k$ are unique, at least up to permutation of their variables. To reach more symmetrical formulas we assume in advance that the transfer functions are symmetric functions, that is,

$$f_k(\omega_{(k)}) = \operatorname*{sym}_{\omega_{(k)}} f_k(\omega_{(k)}).$$

3.2. *Stochastic integration and differential equations with respect to fractional Brownian motion.* We would like to give meaning to the stochastic linear differential form

$$dy_t = \xi_t\, dt + \eta_t\, dw_t^{(h)}. \tag{15}$$

Equation (15) is for the shortened form of the integral equation

$$y_T = y_0 + \int_0^T (\xi_t\,dt + \eta_t\,dw_t^{(h)}), \tag{16}$$

where the processes $\xi_t$, $\eta_t \in S_2$. For the form (16) to be well defined on the one hand and unique on the other, one needs to make some assumptions for $\xi_t$ and $\eta_t$. Assume that the representations of $\xi_t$ and $\eta_t$ are as (14) with transfer functions $f_k^{(\xi)}(\omega_{(k)})$ and $f_k^{(\eta)}(\omega_{(k)})$, respectively.

The assumptions for $\xi_t$ and $\eta_t$ are

$$\int_{\mathbb{R}} \big|f_{k+1}^{(\eta)}(\omega_{(k)},\lambda)\big| |\lambda|^{-2h} d\lambda < \infty, \qquad k = 0, 1, 2, \dots, \omega_{(k)} \in \mathbb{R}^k, \tag{A1}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{k!}{(2\pi)^k} \int_{\mathbb{R}^k} \frac{1}{(\Sigma\omega_{(k)})^2} \bigg| f_k^{(\xi)}(\omega_{(k)}) + \operatorname*{sym}_{\omega_{(k)}} f_{k-1}^{(\eta)}(\omega_{(k-1)}) \\ + \frac{(k+1)}{2\pi} \int_{\mathbb{R}} f_{k+1}^{(\eta)}(\omega_{(k)},\omega)|\omega|^{-2h}\,d\omega \bigg|^2 \\ \times \prod_1^k |\omega_j|^{-2h}\,d\omega_{(k)} < \infty.\end{aligned} \tag{A2}$$

An intuitive definition of the stochastic integral

$$\int_0^T (\xi_t\,dt + \eta_t\,dw_t^{(h)}) \tag{17}$$

is as follows. Apply the diagram formula [see (7)] for multiplying $\eta_t$ by $(w_t^{(h)})'$, add it to $\xi_t$ and integrate from 0 to $T$. If one carries out this algebra, one has the spectral domain chaotic representation for (17),

$$\begin{aligned}\int_0^T (\xi_t\,dt + \eta_t\,dw_t^{(h)}) = T\bigg(f_0^{(\xi)} + \frac{1}{2\pi}\int_{\mathbb{R}} f_1^{(\eta)}(\omega)|\omega|^{-2h}\,d\omega\bigg) \\ + \sum_{k=1}^{\infty} \int_{\mathbb{R}^k} \big(\exp(iT\Sigma\omega_{(k)}) - 1\big) g_k(\omega_{(k)}) \\ \times \prod_1^k (i\omega_j)^{-h} W(d\omega_{(k)}),\end{aligned} \tag{18}$$

where

$$\begin{aligned}g_k(\omega_{(k)}) = \frac{1}{i\Sigma\omega_{(k)}}\bigg(f_k^{(\xi)}(\omega_{(k)}) + \operatorname*{sym}_{\omega_{(k)}} f_{k-1}^{(\eta)}(\omega_{(k-1)}) \\ + \frac{(k+1)}{2\pi}\int_{\mathbb{R}} f_{k+1}^{(\eta)}(\omega_{(k)},\omega)|\omega|^{-2h}\,d\omega\bigg), \qquad k \ge 1.\end{aligned} \tag{19}$$

Since $(w_t^{(h)})'$ does not exist in $L^2(\Omega)$ and because of the change of order of the various types of integrations,

$$\sum_k, \int_{\mathbb{R}^k} dW(\omega_{(k)}), \int_0^T dt,$$

that method would not be a priori justified without assumptions (A1) and (A2).

DEFINITION 3.4. Let us suppose that the processes $\xi_t$, $\eta_t \in S_2$ fulfill assumptions (A1) and (A2). Then the stochastic integral (17) is defined by (18) and (19).

REMARK 3.1. For every fixed $T \in \mathbb{R}$, the stochastic integral (18) is a random variable with a finite second moment.

The following theorem is also an obvious consequence of (18) and (19).

THEOREM 3.4. *Let us suppose that the processes* $\xi_t$, $\eta_t \in S_2$ *fulfill assumptions* (A1), (A2) *and*

$$f_0^{(\xi)} = -\frac{1}{2\pi}\int_{\mathbb{R}} f_1^{(\eta)}(\omega)|\omega|^{-2h} d\omega \tag{A3}$$

*holds. Then there exists a process* $y_t \in S_2$ *which satisfies* (16) *such that* $y_t$ *is unique in* $S_2$ *apart from its expectation. The kth* ($k > 0$) *order transfer functions of* $y_t$ *are those in* (19).

We have given meaning to the stochastic differential and integral forms (15) and (16). But if we also require the uniqueness of the representation (16) of $y_T - y_0$, we need to make another assumption, namely, that

$$\int_{\mathbb{R}} \left| \int_{\mathbb{R}} f_{k+1}^{(\eta)}(\omega_{(k-1)}, \lambda, \mu)|\mu|^{-2h}\, d\mu \right|^2 |\lambda|^{-2h}\, d\lambda < \infty, \tag{A4}$$
$$k \in \mathbb{N},\ \ \omega_{(k-1)} \in \mathbb{R}^{k-1}.$$

Let us define the space of the integral processes $y_t$. By Theorem 3.4, the integral processes $y_t$ are functionals of proper processes $\xi_t$, $\eta_t \in S_2$ and of complex constants, that is, $y_t = \Phi(\xi, \eta, f_0^{(y)})$, where $\mathsf{E}\, y_t = f_0^{(y)}$.

DEFINITION 3.5.

$$S_{2\mathbb{I}}(w^{(h)}) \doteq \big\{ y_t\colon y_t \text{ satisfies (16)},\ \mathsf{E}\, y_t \in \mathbb{C},\ \xi_t,\ \eta_t \in S_2,$$
$$\text{(A1), (A2), (A3), (A4) hold} \big\}.$$

It is clear that $S_{2\mathbb{I}}(w^{(h)})$ is a linear subspace of $S_2$, since the representation (16) of complex linear combinations of the elements of $S_{2\mathbb{I}}(w^{(h)})$ inherits assumptions (A1), (A2), (A3), (A4).

THEOREM 3.5. *The representation* (16) *of any* $y_t \in S_{2\mathbb{I}}(w^{(h)})$, *is unique.*

Because of the uniqueness of (16), we can define the partial derivatives of any $y_t \in S_{2\mathbb{I}}(w^{(h)})$.

DEFINITION 3.6. If $y_t \in S_{2\mathbb{I}}(w^{(h)})$ has the representation

$$dy_t = \xi_t \, dt + \eta_t \, dw_t^{(h)},$$

then the partial derivatives are defined as

$$\partial_t y_t \doteq \xi_t, \qquad \partial_{w_t^{(h)}} y_t \doteq \eta_t.$$

REMARK 3.2. Let us define for any $t_1, t_2 \in \mathbb{R}$ the operator $\Delta_{t_1, t_2}$ on $S_{2\mathbb{I}}(w^{(h)})$ by

$$\begin{aligned}\Delta_{t_1, t_2} y &\doteq \int_{t_1}^{t_2} ((\partial_s y_s)\, ds + (\partial_{w_s^{(h)}} y_s)\, dw_s^{(h)}) \\ &\doteq \int_0^{t_2} ((\partial_s y_s)\, ds + (\partial_{w_s^{(h)}} y_s)\, dw_s^{(h)}) - \int_0^{t_1} ((\partial_s y_s)\, ds + (\partial_{w_s^{(h)}} y_s)\, dw_s^{(h)}).\end{aligned}$$

Then $\Delta_{t_1, t_2}$ coincides with the ordinary difference operator, that is, $\Delta_{t_1, t_2} y = y_{t_2} - y_{t_1}$.

REMARK 3.3. Because of (A2) and (A3), $\xi_t$ and $\eta_t$ are connected with one another. $\int_0^T \xi_t \, dt$ and $\int_0^T \eta_t \, dw_t^{(h)}$ are not defined one-by-one. Though each of them could be defined without the other one, it could happen that neither of them would give the difference of a stationary process; only the sum of them would. We are constrained to work in the space $S_2$ to avoid the time dependence of the transfer functions. That is the reason why we have defined only the "common" integral $\int_0^T (\xi_t \, dt + \eta_t \, dw_t^{(h)})$. By the way, one might introduce

$$\int_0^T \eta_t \, dw_t^{(h)} \doteq \int_0^T (\xi_t \, dt + \eta_t \, dw_t^{(h)}) - \int_0^T \xi_t \, dt,$$

but this is not of primary importance; the common integral comes before.

EXAMPLE 3.1 (Stationary Ornstein–Uhlenbeck process with fractional Brownian motion input). Let us consider the stochastic differential equation

$$dy_t = \alpha y_t \, dt + \beta \, dw_t^{(h)}. \tag{20}$$

It is shown (see the Appendix) that the partial derivatives of $y_t$ are

$$\begin{aligned}\partial_t y_t &= \int_{\mathbb{R}} e^{it\omega} f_1^{(\xi)}(\omega)(i\omega)^{-h} W(d\omega), \\ \partial_{w_t^{(h)}} y_t &= \beta,\end{aligned}$$

with the transfer function

$$f_1^{(\xi)}(\omega) = \alpha\beta \frac{1}{i\omega - \alpha}.$$

Therefore the unique solution of (20) is

$$y_t = \int_{\mathbb{R}} e^{it\omega} \frac{\beta}{i\omega - \alpha} (i\omega)^{-h} W(d\omega).$$

Remember that we have defined the integral of proper nonrandom functions with respect to $w_t^{(h)}$; see Definition 3.1. Now, we are able to represent $y_t$ in the time domain as well. Assume that $\operatorname{Re}\alpha < 0$. Then

$$\begin{aligned} y_t &= \int_{\mathbb{R}} e^{it\omega} \frac{\beta}{i\omega - \alpha} (i\omega)^{-h} W(d\omega) \\ &= \int_{\mathbb{R}} e^{it\omega} \beta \int_0^{\infty} e^{\alpha u - iu\omega}\, du (i\omega)^{-h} W(d\omega) \qquad (21) \\ &= \int_{\mathbb{R}} \int_{-\infty}^{t} e^{is\omega} \beta e^{\alpha(t-s)}\, ds (i\omega)^{-h} W(d\omega) = \beta \int_{-\infty}^{t} e^{\alpha(t-s)}\, dw_s^{(h)}. \end{aligned}$$

Another type of the time domain integral representation form of $y_t$ is

$$\begin{aligned} y_t &= \beta \int_{\mathbb{R}} \int_{-\infty}^{t} e^{is\omega} e^{\alpha(t-s)}\, ds (i\omega)^{-h} W(d\omega) \\ &= -\alpha\beta \int_{\mathbb{R}} \int_{-\infty}^{t} e^{\alpha(t-s)} \frac{e^{it\omega} - e^{is\omega}}{i\omega}\, ds (i\omega)^{-h} W(d\omega) \qquad (22) \\ &= -\alpha\beta \int_{-\infty}^{t} e^{\alpha(t-s)} (w_t^{(h)} - w_s^{(h)})\, ds, \end{aligned}$$

where the last integral is an $L^2$-integral.

If $\operatorname{Re}\alpha > 0$, then $y_t$ has time domain integral representation forms similar to (21) and (22) depending on the future of $w_t^{(h)}$, that is, the domain of integration is $(t, \infty)$. The spectrum of $y_t$ can be easily derived from the spectral representation of the process and shows its long-range dependence. It is worth noting here that because of the linearity of (20) there is no difference between taking the fractional integral operator $I_t^{(h)}$ on an ordinary Ornstein–Uhlenbeck process and the solution of equation (20) with fractional Brownian motion input.

EXAMPLE 3.2 (Stationary Hermite degree 2 bilinear process with fractional Brownian motion input). Consider the pair of stochastic differential equations

$$\begin{aligned} dx_t &= \alpha_1 x_t\, dt + \beta\, dw_t^{(h)}, \\ dy_t &= \alpha_2 y_t\, dt + \gamma x_t\, dw_t^{(h)}, \end{aligned} \qquad (23)$$

where $\alpha_1, \alpha_2, \beta, \gamma \in \mathbb{C}$, $\operatorname{Re}\alpha_1 \neq 0$, $\operatorname{Re}\alpha_2 \neq 0$. The unique solution $x_t$ of the first equation is the stationary Ornstein–Uhlenbeck process with fractional

Brownian motion input, given in Example 3.1. Now we give the $y_t$ component of the unique solution (see the Appendix for the proof):

$$y_t = \sum_{k=0}^{2} \int_{\mathbb{R}^k} \exp(it\Sigma\omega_{(k)}) f_k^{(y)}(\omega_{(k)}) \prod_{j=1}^{2} (i\omega_j)^{-h} W(d\omega_{(k)}),$$

with $f_0^{(y)} = f_0^{(\xi)}/\alpha_2$ and

$$f_1^{(y)}(\omega) = 0, \qquad f_2^{(y)}(\omega_{(2)}) = \beta\gamma \operatorname*{sym}_{\omega_{(2)}} \frac{1}{i\omega_1 - \alpha_1} \frac{1}{i(\omega_1 + \omega_2) - \alpha_2}.$$

The partial derivatives of $y_t$ are

$$\partial_t y_t = f_0^{(\xi)} + \int_{\mathbb{R}^2} \exp(it\Sigma\omega_{(2)}) f_2^{(\xi)}(\omega_{(2)})(i\omega_1)^{-h}(i\omega_2)^{-h} W(d\omega_{(2)}),$$

with

$$\begin{aligned} f_0^{(\xi)} &= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{-\beta\gamma}{i\omega - \alpha_1} |\omega|^{-2h}\, d\omega \\ &= \frac{-\beta\gamma(-\alpha_1)^{-2h}}{\pi} \Gamma(2h)\Gamma(1-2h)\cos\left(\frac{\pi}{2}(1-2h)\right) \end{aligned}$$

and

$$f_2^{(\xi)}(\omega_{(2)}) = \frac{\alpha_2\beta\gamma}{i\Sigma\omega_{(2)} - \alpha_2} \operatorname*{sym}_{\omega_{(2)}} \frac{1}{i\omega_1 - \alpha_1},$$

$$\partial_{w_t^{(h)}} y_t = \gamma x_t.$$

In case $\alpha_2 = \alpha_1 = \alpha$, $\operatorname{Re}\alpha < 0$, the time domain integral representation form of the solution $y_t$ is

$$y_t = -\alpha\beta\gamma \int_{-\infty}^{t} e^{\alpha(t-s)} \frac{(w_t^{(h)} - w_s^{(h)})^2}{2}\, ds.$$

Now there is a major difference between considering (23) with fractional Brownian motion input and taking the fractional integral operator $I_t^{(h)}$ on the solution of (23) with standard Brownian motion input. The latter has the following spectral representation:

$$\int_{\mathbb{R}^2} \exp(it\Sigma\omega_{(2)}) \frac{\gamma\beta}{(i\omega_1 - \alpha_1)(i(\omega_1 + \omega_2) - \alpha_2)} (i(\omega_1 + \omega_2))^{-h} W(d\omega_{(2)}).$$

It is seen here that the fractional weight is $(i(\omega_1 + \omega_2))^{-h}$ instead of $(i\omega_1)^{-h}(i\omega_2)^{-h}$ and the expectation of this process is zero.

3.3. *Bilinear stochastic differential equation with fractional Brownian motion integrator process.* The bilinear stochastic differential equation with fractional Brownian motion is analogous to (1):

$$dy_t = (\alpha y_t + \mu)\, dt + (\gamma y_t + \beta)\, dw_t^{(h)}. \tag{24}$$

This equation is interpreted in accordance with the previous subsection. If $\operatorname{Re}\gamma \neq 0$, then we suspect that (24) has no stationary solution, but it does if $\gamma$ is purely imaginary. Therefore consider the stochastic differential equation

$$dy_t = (\alpha y_t + \mu)\, dt + (i\gamma y_t + \beta)\, dw_t^{(h)}, \tag{25}$$

where $\alpha, \mu, \beta \in \mathbb{C}$, $\operatorname{Re}\alpha < 0$, $0 \neq \gamma \in \mathbb{R}$. From now on we shall assume that $\beta = 0$ and $\mu \neq 0$; otherwise apply the linear transformation (3), as is done in the nonfractional case. So our problem is to solve the stochastic differential equation

$$dy_t = (\alpha y_t + \mu)\, dt + i\gamma y_t\, dw_t^{(h)}, \tag{26}$$

in the set of processes

$$\Big\{ y_t \in S_{2\mathbb{I}}(w^{(h)}) \colon \text{(A1), (A2), (A3), (A4) hold for } \partial_t y_t = \alpha y_t + \mu \text{ and } \partial_{w_t^{(h)}} y_t = i\gamma y_t \Big\}.$$

Applying (26) we get from assumption (A3) and from (19) the following infinite system of equations for the transfer functions of $y_t$ having the form (14):

$$0 = \alpha f_0 + \mu + i\gamma \frac{1}{2\pi} \int_{\mathbb{R}} f_1(\omega)|\omega|^{-2h}\, d\omega,$$

$$f_k(\omega_{(k)}) = \frac{1}{i\Sigma\omega_{(k)}} \bigg[ \alpha f_k(\omega_{(k)}) + i\gamma \bigg( \operatorname*{sym}_{\omega_{(k)}} f_{k-1}(\omega_{(k-1)}) + \frac{(k+1)}{2\pi} \int_{\mathbb{R}} f_{k+1}(\omega_{(k)}, \omega)|\omega|^{-2h}\, d\omega \bigg) \bigg], \qquad k \in \mathbb{N}. \tag{27}$$

In that case, assumptions (A1), (A2) and (A4) are

$$\int_{\mathbb{R}} \big|f_{k+1}(\omega_{(k)}, \lambda)\big| |\lambda|^{-2h} d\lambda < \infty, \qquad k = 0, 1, 2, \ldots, \ \omega_{(k)} \in \mathbb{R}^k, \tag{28}$$

$$\sum_{k=1}^{\infty} \frac{k!}{(2\pi)^k} \int_{\mathbb{R}^k} \frac{1}{(\Sigma\omega_{(k)})^2} \bigg| \alpha f_k(\omega_{(k)}) + i\gamma \bigg( \operatorname*{sym}_{\omega_{(k)}} f_{k-1}(\omega_{(k-1)}) + \frac{k+1}{2\pi} \int_{\mathbb{R}} f_{k+1}(\omega_{(k)}, \omega)|\omega|^{-2h}\, d\omega \bigg) \bigg|^2 \times \prod_1^k |\omega_j|^{-2h}\, d\omega_{(k)} < \infty, \tag{29}$$

$$\int_{\mathbb{R}} \bigg| \int_{\mathbb{R}} f_{k+1}(\omega_{(k-1)}, \lambda, \mu)|\mu|^{-2h}\, d\mu \bigg|^2 |\lambda|^{-2h} d\lambda < \infty, \qquad k \in \mathbb{N}, \ \omega_{(k-1)} \in \mathbb{R}^{k-1}. \tag{30}$$

We shall frequently use the following function so we define the notation as

$$K(u) \doteq \exp\Big( \alpha u - \frac{\gamma^2}{2} \kappa(h) u^{2h+1} \Big), \qquad u \geq 0. \tag{31}$$

THEOREM 3.6. *The following system of transfer functions*

$$f_k(\omega_{(k)}) = \mu \frac{(i\gamma)^k}{k!} \int_0^\infty K(u) \prod_1^k \frac{1-\exp(-iu\omega_j)}{i\omega_j}\, du, \qquad k = 0, 1, 2, \dots, \tag{32}$$

*where we used the conventional notations* $f_0(\omega_{(0)}) = f_0$ *and* $\prod_1^0 = 1$, *fulfills equations* (27) *and assumptions* (28), (29) *and* (30). *Thus the stationary process*

$$y_t = \sum_{k=0}^{\infty} \int_{\mathbb{R}^k} \exp(it\Sigma\omega_{(k)}) f_k(\omega_{(k)}) \prod_1^k (i\omega_j)^{-h} W(d\omega_{(k)}), \qquad t \in \mathbb{R} \tag{33}$$

*is a solution of the stochastic differential equation* (26). *Moreover*, $y_t$ *has the time domain representation*

$$y_t = \mu \int_{-\infty}^{t} \exp(\alpha(t-s) + i\gamma(w_t^{(h)} - w_s^{(h)}))ds, \qquad t \in \mathbb{R}, \tag{34}$$

*where the integral is an* $L^2$*-integral.*

REMARK 3.4. From a certain point of view, the following model can motivate the previous theorem. Consider the system of stochastic differential equations

$$\begin{aligned} dy_{0,t} &= (\mu + \alpha y_{0,t})\, dt, \\ dy_{1,t} &= \alpha y_{1,t}\, dt + \gamma y_{0,t}\, dw_t^{(h)}, \\ dy_{2,t} &= \alpha y_{2,t}\, dt + \gamma y_{1,t}\, dw_t^{(h)}, \\ &\ \vdots \\ dy_{N,t} &= \alpha y_{N,t}\, dt + \gamma y_{N-1,t}\, dw_t^{(h)}, \end{aligned} \tag{35}$$

$N \geq 1$, where $\mu, \alpha, \gamma \in \mathbb{C}$, $\operatorname{Re}\alpha < 0$. Then $y_{1,t}$ is a stationary Ornstein–Uhlenbeck process with fractional Brownian motion input and $y_{2,t}$ is a stationary Hermite degree 2 process with fractional Brownian motion input. See Examples 3.1 and 3.2 in the previous subsection. One can prove that for all $N \in \mathbb{N}$ there is a unique solution $y_{N,t} \in S_{2\mathbb{I}}(w^{(h)})$ and the transfer functions in the frequency domain chaotic representation of $y_{N,t}$ are the following:

$$\begin{aligned} f_k^{(N)}(\omega_{(k)}) = \mu \frac{\gamma^k}{k!} \int_0^\infty &\exp(\alpha u) \frac{((\gamma^2/2)\kappa(h)u^{2h+1})^{(N-k)/2}}{((N-k)/2)!} \\ &\times \prod_1^k \frac{1-\exp(-iu\omega_j)}{i\omega_j}\, du, \end{aligned} \tag{36}$$

if $0 \leq k \leq N$ and $k$ has the same evenness as $N$; otherwise $f_k^{(N)}(\omega_{(k)}) = 0$.

Using the inversion formula for the Hermite polynomials,

$$\frac{x^N}{N!} = \sum_{k=0}^{N}{}' \frac{(t/2)^{(N-k)/2}}{((N-k)/2)!} H_k(x, t),$$

where $\sum'$ means summing for all $k$ having the same evenness as $N$, and by Lemma A.6 in the Appendix, one can get the time domain $L^2$-integral representation form

$$y_{N,t} = \frac{\gamma^N}{N!} \int_{-\infty}^{t} e^{\alpha(t-s)} \big(w_t^{(h)} - w_s^{(h)}\big)^N \, ds. \tag{37}$$

In this remark, $\gamma$ was an arbitrary complex number. From now on, let $\gamma$ be purely imaginary or, as we do from (25), denote it with $i\gamma$, where $\gamma \in \mathbb{R}$. So let us replace $\gamma$ with $i\gamma$ in each of the previous formulas in this remark.

Now, first summing the equations of (35) and letting $N \to \infty$ or, equivalently, summing the $N$th equations for $N = 0, 1, 2, \dots,$ we get the bilinear stochastic differential equation (26). Second, for a fixed $k$, by summing the $k$th order transfer functions in (36) for $N = 0, 1, 2, \dots,$ we obtain (32). Third, by summing the equations in (37) for $N = 0, 1, 2, \dots,$ we get (34).

We do not follow this idea in the proof of Theorem 3.6 as a more direct argument can be found.

REMARK 3.5. It is an open problem whether the solution of the bilinear stochastic differential equation (26) given in Theorem 3.6 is unique. In other words, do the parameters $\mu$, $\alpha$, $\gamma$ in (26) determine the solution uniquely or not? Obviously, the question is whether the homogeneous equation, that is, equation (26) with $\mu = 0$, has the only solution $y_t = 0$.

REMARK 3.6. The solution $y_t$, given in Theorem 3.6, of the bilinear stochastic differential equation (26) has an almost surely bounded and continuous modification. The reason is that $|\exp(i\gamma(w_t^{(h)} - w_s^{(h)}))| = 1$ in (34).

In the following theorem we present the expectation, the autocovariance function and the spectral density function of the solution $y_t$, given in Theorem 3.6, in the form of simple and twofold integrals.

THEOREM 3.7. *The expectation, the autocovariance function and the spectrum of the solution* (32)-(33)-(34) *are*

$$\mathsf{E}\, y_t = \mu \int_0^\infty K(u)\, du,$$

$$\begin{aligned} R(t) &= \mathsf{E}\,(y_t - \mathsf{E}\, y_t)\overline{(y_0 - \mathsf{E}\, y_0)} \\ &= |\mu|^2 \int_0^\infty \int_0^\infty K(u_1)\overline{K(u_2)} \\ &\qquad \times \bigg(\exp\Big(-\frac{\gamma^2}{2}\kappa(h)\Big(|t|^{2h+1} - |t-u_1|^{2h+1} - |t+u_2|^{2h+1} \\ &\qquad\qquad + |t-u_1+u_2|^{2h+1}\Big)\Big) - 1\bigg)\, du_1\, du_2, \end{aligned}$$

$$\varphi(\lambda)=|\mu|^2\int_0^\infty\int_0^\infty K(u_1)\overline{K(u_2)}\big(\exp(*(\gamma^2 A^{(u_1,u_2)}))(\lambda)-1\big)\,du_1\,du_2, \tag{38}$$

*respectively, where*

$$A^{(u_1,u_2)}(\omega)=\frac{1}{2\pi}\frac{1-\exp(-iu_1\omega)}{i\omega}\overline{\left(\frac{1-\exp(-iu_2\omega)}{i\omega}\right)}|\omega|^{-2h}$$

*and*

$$e^{*g}(\omega)\doteq\sum_{j=0}^{\infty}\frac{1}{j!}g^{*j}(\omega)\doteq\sum_{j=0}^{\infty}\frac{1}{j!}\underbrace{g}_{1}*\underbrace{g}_{2}*\cdots*\underbrace{g}_{j}(\omega)$$

*is the convolution exponential of a function* $g\in L^1(\mathbb{R})$.

REMARK 3.7. The solution $y_t$, given in Theorem 3.6, of the bilinear stochastic differential equation (26) is a long-range dependent or long memory process. We can justify this as follows. From (38),

$$\varphi(\lambda)=\sum_{j=1}^{\infty}\frac{|\mu|^2\gamma^{2j}}{j!}\int_0^\infty\int_0^\infty K(u_1)\overline{K(u_2)}(A^{(u_1,u_2)})^{*j}(\lambda)\,du_1\,du_2. \tag{39}$$

In the infinite sum in (39) each term is nonnegative because for each $j$ the $j$th term is the spectrum of the $j$th order chaotic component in (33). The first term is

$$\begin{aligned}|\mu|^2\gamma^2\int_0^\infty\int_0^\infty K(u_1)\overline{K(u_2)}A^{(u_1,u_2)}(\lambda)\,du_1\,du_2\\ =\frac{|\mu|^2\gamma^2}{2\pi}\left|\int_0^\infty K(u)\frac{1-e^{-iu\lambda}}{i\lambda}\,du\right|^2|\lambda|^{-2h}.\end{aligned}$$

It follows from Lebesgue's dominated convergence theorem that

$$\lim_{\lambda\to 0}\int_0^\infty K(u)\frac{1-e^{-iu\lambda}}{i\lambda}\,du=\int_0^\infty K(u)u\,du\neq 0.$$

Thus the limit of the first term in (39) is infinite as $\lambda\to 0$. Hence, $\varphi(\lambda)$ is not bounded. This fact implies that the autocovariance function $R(t)$, of $y_t$, is not integrable, and this is just the definition of long-range dependence.

Finally we mention a fact regarding a special feature of the stochastic integral with respect to $w_t^{(h)}$.

REMARK 3.8. It is not surprising that, for each of the solution processes we have mentioned in the examples and in Theorem 3.6, it is not the Itô differential rule but the usual deterministic chain rule that holds. That is, the ordinary differential operators coincide on the above processes with the differential operators $\partial_t$ and $\partial_{w_t^{(h)}}$ described in Definition 3.6. This is in line with the fact that the quadratic variation process, which causes the extra term in the Itô formula in the time domain, is zero for our integrator process $w_t^{(h)}$.

**4. Stratonovich solution of bilinear stochastic differential equations with Brownian motion input.** The simplest example showing the difference between the Itô and Stratonovich calculus is the integral

$$\int_0^T w_t \, dw_t.$$

Let us turn it into frequency domain; by using the diagram formula (7) and Lemma A.6 in the Appendix, we get

$$\begin{aligned}\int_0^T w_t \, dw_t &= \frac{1}{2}\int_{\mathbb{R}^2} \frac{\exp(iT\omega_1)-1}{i\omega_1}\frac{\exp(iT\omega_2)-1}{i\omega_2} W(d\omega_{(2)}) \\ &\quad + \int_0^T \left(\frac{1}{2\pi}\int_{\mathbb{R}} \frac{1-\exp(-it\omega)}{i\omega} d\omega\right) dt \\ &= \frac{w_T^2 - T}{2} + \int_0^T \left(\frac{1}{2\pi}\int_{\mathbb{R}} \frac{1-\exp(-it\omega)}{i\omega}\, d\omega\right) dt.\end{aligned}$$

Now the problem is that the integral

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{1-e^{-it\omega}}{i\omega}\, d\omega,$$

is not well defined. If one takes the principal value, that is, 0, then the Itô calculus follows, and taking either Abelian or Gaussian mean, that is, 1/2, the Stratonovich rule is implied. There was no such type of decision in the case of fractional noise input since all the integrals were well defined. Let us denote the solution (32)-(33)-(34) by $y_t^{(h)}$; that is, we are going to point to the dependence on $h$. We have not admitted the case $h = 0$ but we may ask whether the solution $y_t^{(h)}$ converges if $h \to 0$. It follows easily, for example, by (34) and by Theorem 3.1, that $\operatorname{l.i.m.}_{h\to 0} y_t^{(h)} = y_t^{(0)}$ in $L^2$, where $y_t^{(0)}$ is (34) with 0 in place of $h$ in it. It is not surprising, again because of the validity of the deterministic chain rule, that $y_t^{(0)}$ is the solution of the bilinear stochastic differential equation in the Stratonovich sense.

We shall now consider the stationary Stratonovich solution for the general bilinear equation (4):

$$dy_t = (\alpha y_t + \mu)\, dt + \gamma y_t \, dw_t,$$

instead of considering only the pure imaginary coefficient $\gamma$. It is easy to see that

$$y_t = \mu \int_{-\infty}^t \exp(\alpha(t-s) + \gamma(w_t - w_s))\, ds, \qquad t \in \mathbb{R}, \tag{40}$$

is the stationary Stratonovich solution given in the time domain. In the frequency domain we suppose that the solution is given by the multiple Wiener–Itô spectral representation

$$y_t = \sum_{k=0}^{\infty} \int_{\mathbb{R}^k} \exp(it\Sigma\omega_{(k)}) f_k(\omega_{(k)}) W(d\omega_{(k)}), \qquad t \in \mathbb{R},$$

where the transfer functions are given by the following system of equations:

$$0 = \alpha f_0 + \mu + \gamma \frac{1}{2\pi} \int_{\mathbb{R}} f_1(\omega)\, d\omega,$$

$$f_k(\omega_{(k)}) = \frac{1}{i\Sigma\omega_{(k)}} \bigg[ \alpha f_k(\omega_{(k)}) + \gamma \bigg( \operatorname*{sym}_{\omega_{(k)}} f_{k-1}(\omega_{(k-1)}) + \frac{(k+1)}{2\pi} \int_{\mathbb{R}} f_{k+1}(\omega_{(k)}, \omega)\, d\omega \bigg) \bigg], \qquad k \in \mathbb{N}. \tag{41}$$

If we define by either the Abelian or Gaussian mean the integral

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{1 - e^{-it\omega}}{i\omega}\, d\omega \doteq \frac{1}{2},$$

instead of taking its principal value, then the solution of the equations (41) gives the transfer functions according to the Stratonovich solution. It is easy to see that the solution of the equations (41) is given by the following system of functions:

$$f_k(\omega_{(k)}) = \mu \frac{\gamma^k}{k!} \int_0^\infty \exp\bigg( \bigg( \alpha + \frac{\gamma^2}{2} \bigg) u \bigg) \times \prod_1^k \frac{1 - \exp(-iu\omega_j)}{i\omega_j}\, du, \qquad k = 0, 1, 2, \ldots. \tag{42}$$

The proof of the following theorem comes by a straightforward easy calculation.

THEOREM 4.1. *The bilinear stochastic equation* (4) *has a stationary Stratonovich solution if and only if* $\operatorname{Re}\alpha + (\operatorname{Re}\gamma)^2 < 0$. *In that case, the solution is given by* (40) *in the time domain and by the transfer functions* (42) *in the frequency domain. Then* $\mathsf{E}\, y_t = -\mu/(\alpha + \gamma^2/2)$, *the covariance function of* $y_t$ *is*

$$R(t) = R(0) \exp\bigg( \bigg( \alpha + \frac{\gamma^2}{2} \bigg) t \bigg) = \frac{-|\mu|^2 |\gamma|^2}{2|\alpha + \gamma^2/2|^2 [\operatorname{Re}\alpha + (\operatorname{Re}\gamma)^2]} \exp\bigg( \bigg( \alpha + \frac{\gamma^2}{2} \bigg) t \bigg), \qquad t > 0,$$

*and the spectrum is*

$$\varphi(\omega) = -\frac{R(0) \operatorname{Re}(\alpha + \gamma^2/2)}{\pi |i\omega - \alpha - \gamma^2/2|^2}, \qquad \omega \in \mathbb{R}.$$

## APPENDIX

LEMMA A.1. *The quadratic variation process of* $w_t^{(h)}$ *is zero and what is more,*

$$\lim_{n\to\infty} \mathsf{E} \sum_{k=0}^{n-1} \big( w^{(h)}_{t((k+1)/n)} - w^{(h)}_{t(k/n)} \big)^2 = 0, \qquad t \in \mathbb{R}.$$

PROOF. The process $w_t^{(h)}$ has stationary increments, so by (11),

$$\lim_{n\to\infty} \mathsf{E} \sum_{k=0}^{n-1} \big(w_{t((k+1)/n)}^{(h)} - w_{t(k/n)}^{(h)}\big)^2 = \lim_{n\to\infty} \left( n\kappa(h) \left( \frac{|t|}{n} \right)^{2h+1} \right) = 0. \qquad \square$$

LEMMA A.2.

$$\kappa(h) = \frac{\Gamma(1-2h)}{h(2h+1)\pi} \cos\left( \frac{\pi}{2}(1-2h) \right),$$

*where* $\kappa(h)$ *was defined in* (12).

PROOF.

$$\begin{aligned} &\frac{1}{2\pi} \int_{\mathbb{R}} \left| \frac{\exp(i\omega) - 1}{i\omega} \right|^2 |\omega|^{-2h} \, d\omega \\ &\quad = \frac{1}{2\pi} \int_{\mathbb{R}} \left| \int_0^1 \exp(iv\omega) \, dv \right|^2 |\omega|^{-2h} \, d\omega \\ &\quad = \frac{1}{\pi} \operatorname{Re} \int_{\mathbb{R}} \int_0^1 \int_0^{v_1} \exp(i(v_1 - v_2)\omega) \, dv_2 \, dv_1 |\omega|^{-2h} \, d\omega \\ &\quad = \frac{1}{\pi} \lim_{N\to\infty} \int_0^1 \int_0^{u_1} \int_{-u_2 N}^{u_2 N} \exp(i\lambda) |\lambda|^{-2h} \, d\lambda {u_2}^{2h-1} \, du_2 \, du_1. \end{aligned} \tag{43}$$

By [6], formula 2.3.3.1.,

$$\lim_{M\to\infty} \int_{-M}^{M} e^{i\lambda} |\lambda|^{-2h} \, d\lambda = \int_{-\infty}^{\infty} e^{i\lambda} |\lambda|^{-2h} \, d\lambda = 2\Gamma(1-2h) \cos\left( \frac{\pi}{2}(1-2h) \right),$$

so

$$\left| \int_{-u_2 N}^{u_2 N} e^{-i\lambda} |\lambda|^{-2h} \, d\lambda {u_2}^{2h-1} \right| \le c {u_2}^{2h-1}$$

with some constant $c > 0$. Thus we can apply Lebesgue's dominated convergence theorem in (43) to change the order of $\lim_{N\to\infty}$ and $\int_0^1 \int_0^{u_1}$. In this manner we have

$$\frac{1}{2\pi} \int_{\mathbb{R}} \left| \frac{e^{i\omega} - 1}{i\omega} \right|^2 |\omega|^{-2h} \, d\omega = \frac{\Gamma(1-2h)}{h(2h+1)\pi} \cos\left( \frac{\pi}{2}(1-2h) \right). \qquad \square$$

PROOF OF THEOREM 3.1. By the well-known theorem about the spectral representation of linear $L^2$-functionals of the Brownian motion,

$$\begin{aligned} w_t^{(h)} &= \frac{1}{\Gamma(h+1)} \int_{-\infty}^{\infty} \big[ \chi_{(-\infty,0)}(s)((t-s)^h - (-s)^h) \\ &\qquad\qquad + \chi_{(0,t)}(s)(t-s)^h \big] \, dw_s \\ &= \int_{\mathbb{R}} \frac{1}{\Gamma(h+1)} \left[ \int_{-\infty}^0 e^{is\omega}((t-s)^h - (-s)^h) \, ds \right. \\ &\qquad\qquad \left. + \int_0^t e^{is\omega}(t-s)^h \, ds \right] W(d\omega). \end{aligned} \tag{44}$$

Let us introduce the notation

$$\varphi(t,\omega) \doteq \frac{1}{\Gamma(h+1)}\left[\int_{-\infty}^{0} e^{is\omega}((t-s)^h-(-s)^h)\,ds + \int_0^t e^{is\omega}(t-s)^h\,ds\right].$$

For a fixed $t \in \mathbb{R}$, $\varphi(t,\omega)$ is an inverse Fourier transform. We assume that $\omega > 0$ as one may conjugate for $\omega < 0$.

It causes us some trouble that the function which is the subject of the inverse Fourier transform is not in $L^1(\mathbb{R})$, though it is in $L^2(\mathbb{R})$. To ensure integrability, we shall calculate $\varphi(t,z)$ for $z = \omega + i\lambda$, $\lambda < 0$, then take the limit as $\lambda \to 0$. Hence,

$$\begin{aligned}\varphi(t,z) &= \frac{1}{\Gamma(h+1)}\left[\int_{-\infty}^{0} e^{izs}((t-s)^h-(-s)^h)\,ds + \int_0^t e^{izs}(t-s)^h\,ds\right]\\ &= \frac{1}{\Gamma(h)}\frac{1}{iz}(e^{izt}-1)\int_0^{\infty} e^{-izu}u^{h-1}\,du,\end{aligned} \tag{45}$$

where we have used integration by parts and transformed the integration intervals to $(0,\infty)$. By [6], formula 2.3.3.1.,

$$\int_0^{\infty} e^{-izu}u^{h-1}\,du = \Gamma(h)(iz)^{-h},$$

since $\operatorname{Re} z = \lambda < 0$. Thus we have

$$\varphi(t,z) = \frac{e^{itz}-1}{iz}(iz)^{-h}.$$

Now,

$$\lim_{\lambda\to 0}\varphi(t,\omega+i\lambda) = \frac{e^{it\omega}-1}{i\omega}(i\omega)^{-h}.$$

On the other hand, taking the limit in the $L^2(\mathbb{R})$-sense in the first row of (45), we have

$$\begin{aligned}&\operatorname*{l.i.m.}_{\lambda\to 0}\ \varphi(t,\omega+i\lambda)\\ &\quad= \frac{1}{\Gamma(h+1)}\left[\int_{-\infty}^{0} e^{i\omega s}((t-s)^h-(-s)^h)\,ds + \int_0^t e^{i\omega s}(t-s)^h\,ds\right].\end{aligned}$$

This can be justified easily by applying Lebesgue's dominated convergence theorem for the $L^2$-norm of the differences. So, we have

$$\varphi(t,\omega) = \frac{e^{it\omega}-1}{i\omega}(i\omega)^{-h}, \qquad \omega \neq 0,$$

and taking into account (44), the proof is complete. □

PROOF OF THEOREM 3.2. First of all it is not difficult to see that the function that we have to integrate is in $L^2(\mathbb{R})$. For $t > 0$, by the definition,

$$\begin{aligned}
&\int_{-\infty}^{t} \{[(t-s)^{-h} - (-s)^{-h}]\chi_{(-\infty,0]}(s) + (t-s)^{-h}\chi_{(0,t]}(s)\}\,dw_s^{(h)} \\
&\quad = \int_{\mathbb{R}} \left\{ \int_{-\infty}^{0} [(t-s)^{-h} - (-s)^{-h}]e^{is\omega}\,ds + \int_0^t (t-s)^{-h} e^{is\omega}\,ds \right\} (i\omega)^{-h} W(d\omega) \\
&\quad = \int_{\mathbb{R}} \left\{ \int_0^{\infty} u^{-h} e^{-iu\omega}\,du e^{it\omega} - \int_0^{\infty} u^{-h} e^{-iu\omega}\,du \right\} (i\omega)^{-h} W(d\omega) \\
&\quad = \Gamma(1-h) \int_{\mathbb{R}} \frac{e^{it\omega} - 1}{i\omega} W(d\omega) = \Gamma(1-h) w_t.
\end{aligned}$$

The case $t = 0$ is trivial and the case $t < 0$ requires similar treatment. □

LEMMA A.3. *Let* $f\colon \mathbb{R} \to \mathbb{R}$, $f \in L^1(\mathbb{R}) \bigcap L^2(\mathbb{R})$. *Then* $\int_{-\infty}^{t} f(s)\,dw_s^{(h)}$ *is* $\mathscr{F}_{(-\infty,t]}^{w^{(h)}}$*-measurable.*

PROOF. One can apply the usual technique, that is, approximation by step functions. It is simple to verify the fact that for a step function the integral is $\mathscr{F}_{(-\infty,t]}^{w^{(h)}}$-measurable.

On the other hand, any $f \in L^1(-\infty,t] \bigcap L^2(-\infty,t]$ can be approximated, in the norm $\|f\|_{1,2} = \|f\|_1 + \|f\|_{L^2}$, by step functions. So, $\int_{-\infty}^{t} f(s)\,dw_s^{(h)}$ can be approximated by $\mathscr{F}_{(-\infty,t]}^{w^{(h)}}$-measurable random variables of the form $\sum_j c_j^{(n)}(w_{s_{j+1}}^{(h)} - w_{s_j}^{(h)})$.

Namely, if

$$c^{(n)}(s) = \sum_j c_j^{(n)} \chi_{[s_j^{(n)}, s_{j+1}^{(n)}]}(s),$$

then

$$\sum_j c_j^{(n)}(w_{s_{j+1}}^{(h)} - w_{s_j}^{(h)}) = \int_{-\infty}^{t} c^{(n)}(s)\,dw_s^{(h)}$$

and

$$\begin{aligned}
&\mathsf{E}\left| \int_{-\infty}^{t} f(s)\,dw_s^{(h)} - \int_{-\infty}^{t} c^{(n)}(s)\,dw_s^{(h)} \right|^2 \\
&\quad = \frac{1}{2\pi} \int_{-1}^{1} \left| \int_{-\infty}^{t} (f(s) - c^{(n)}(s)) e^{is\omega}\,ds \right|^2 |\omega|^{-2h}\,d\omega \\
&\qquad + \frac{1}{2\pi} \int_{\mathbb{R}\setminus[-1,1]} \left| \int_{-\infty}^{t} (f(s) - c^{(n)}(s)) e^{is\omega}\,ds \right|^2 |\omega|^{-2h}\,d\omega \\
&\quad = I_1 + I_2,
\end{aligned}$$

where

$$I_1 \le \frac{1}{2\pi}\left(\int_{-\infty}^{t}\left|f(s)-c^{(n)}(s)\right| ds\right)^2 \int_{-1}^{1}|\omega|^{-2h}\,d\omega \to 0 \qquad \text{as } n\to\infty$$

since $c^{(n)} \to f$ in $L^1(-\infty, t]$ as $n\to\infty$. Moreover,

$$\begin{aligned} I_2 &\le \frac{1}{2\pi}\int_{\mathbb{R}\setminus[-1,1]}\left|\int_{-\infty}^{t}(f(s)-c^{(n)}(s))e^{is\omega}\,ds\right|^2 d\omega \\ &\le \frac{1}{2\pi}\int_{\mathbb{R}}\left|\int_{-\infty}^{t}(f(s)-c^{(n)}(s))e^{is\omega}\,ds\right|^2 d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{t}|f(s)-c^{(n)}(s)|^2\,ds \to 0 \qquad \text{as } n\to\infty \end{aligned}$$

since $c^{(n)} \to f$ in $L^2(-\infty, t]$ as $n\to\infty$. □

PROOF OF THEOREM 3.4. Because of the linearity of the stochastic integral, it is enough to prove that $\xi_t = \eta_t = 0$ follows from

$$0 = \int_0^T (\xi_t dt + \eta_t\, dw_t^{(h)}).$$

For the transfer functions this means that

$$0 = f_k^{(\xi)}(\omega_{(k)}) + \operatorname*{sym}_{\omega_{(k)}} f_{(k-1)}^{(\eta)}(\omega_{(k-1)}) + \frac{(k+1)}{2\pi}\int_{\mathbb{R}} f_{(k+1)}^{(\eta)}(\omega_{(k)},\omega)|\omega|^{-2h}\,d\omega$$

for $k \ge 1$. The first and third terms on the right-hand side are in space $L^2(\mathbb{R}, |\omega_k|^{-2h})$, the third one because of assumption (A4). So the second term must be in $L^2(\mathbb{R}, |\omega_k|^{-2h})$, too. But such an "oversymmetrized" function can only be in $L^2(\mathbb{R}, |\omega_k|^{-2h})$ if it is zero. Thus all the transfer functions $f_k^{(\eta)}(\omega_{(k)})$, $k = 0, 1, 2, \dots$ are zero. Therefore all the $f_k^{(\xi)}(\omega_{(k)})$ must be zero, too. This means that $\xi_t = \eta_t = 0$. □

LEMMA A.4. *For $u > 0$,*

$$\begin{aligned} \frac{1}{2\pi}\int_{\mathbb{R}}\frac{1-e^{-iu\omega}}{i\omega}|\omega|^{-2h}\,d\omega &= \frac{\kappa(h)}{2}(2h+1)u^{2h} \\ &= \frac{\kappa(h)}{2}\frac{d}{du}u^{2h+1}, \end{aligned}$$

*where the constant $\kappa(h)$ was defined in* (12).

PROOF.

$$\begin{aligned} \frac{1}{2\pi}\int_{\mathbb{R}}\frac{1-e^{-iu\omega}}{i\omega}|\omega|^{-2h}\,d\omega &= \lim_{N\to\infty}\frac{1}{2\pi}\int_{-N}^{N}\frac{1-e^{-iu\omega}}{i\omega}|\omega|^{-2h}\,d\omega \\ &= \lim_{N\to\infty}\frac{1}{2\pi}\int_0^u\int_{-vN}^{vN}e^{-i\lambda}|\lambda|^{-2h}\,d\lambda\, v^{2h-1}\,dv. \end{aligned} \tag{46}$$

By [6], formula 2.3.3.1 (see Lemma A.2),

$$\begin{aligned}\lim_{M\to\infty}\int_{-M}^{M} e^{-i\lambda}|\lambda|^{-2h}\,d\lambda &= \int_{-\infty}^{\infty} e^{-i\lambda}|\lambda|^{-2h}\,d\lambda\\ &= 2\Gamma(1-2h)\cos\left(\frac{\pi}{2}(1-2h)\right)\\ &= 2\pi h(2h+1)\kappa(h),\end{aligned}$$

hence

$$\left|\int_{-vN}^{vN} e^{-i\lambda}|\lambda|^{-2h}\,d\lambda v^{2h-1}\right| \le cv^{2h-1}$$

with some constant $c > 0$. Thus we can apply Lebesgue's dominated convergence theorem to change the order of $\lim_{N\to\infty}$ and $\int_0^u$ in (46). Therefore

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{R}}\frac{1-e^{-iu\omega}}{i\omega}|\omega|^{-2h}\,d\omega &= \frac{1}{2\pi}\int_0^u\int_{\mathbb{R}} e^{-i\lambda}|\lambda|^{-2h}\,d\lambda v^{2h-1}\,dv\\ &= \frac{h(2h+1)\kappa(h)}{2h}u^{2h},\end{aligned}$$

from which we obtain the statement of the lemma. □

LEMMA A.5.

$$k\,\underset{\omega_{(k)}}{\operatorname{sym}}\prod_1^{k-1}\frac{1-e^{-iu\omega_j}}{i\omega_j} = i\sum_1^k\omega_j\prod_1^k\frac{1-e^{-iu\omega_j}}{i\omega_j} + \frac{d}{du}\prod_1^k\frac{1-e^{-iu\omega_j}}{i\omega_j}.$$

PROOF.

$$\begin{aligned}&\frac{d}{du}\prod_1^k\frac{1-\exp(-iu\omega_j)}{i\omega_j}\\ &\quad= k\,\underset{\omega_{(k)}}{\operatorname{sym}}\left(\exp(-iu\omega_k)\prod_1^{k-1}\frac{1-\exp(-iu\omega_j)}{i\omega_j}\right)\\ &\quad= k\prod_1^k\frac{1-\exp(-iu\omega_j)}{i\omega_j}\sum_1^k\frac{i\omega_j}{1-\exp(-iu\omega_j)}\exp(-iu\omega_j)\\ &\quad= -i\sum_1^k\omega_j\prod_1^k\frac{1-\exp(-iu\omega_j)}{i\omega_j} + k\,\underset{\omega_{(k)}}{\operatorname{sym}}\prod_1^{k-1}\frac{1-\exp(-iu\omega_j)}{i\omega_j}.\end{aligned}$$ □

The following lemma is known as the Itô formula; see [4] Theorem 4.2.

LEMMA A.6. *Let $H_k$ be the kth Hermite polynomial*

$$H_k(x,t) \doteq \frac{(-t)^k}{k!}\exp\left(\frac{x^2}{2t}\right)\frac{d^k}{dx^k}\exp\left(-\frac{x^2}{2t}\right), \qquad x\in\mathbb{R},\ t>0,$$

*for* $k = 0, 1, 2, \ldots$ *and*

$$\xi \doteq \int_{\mathbb{R}} \varphi(\omega) W(d\omega),$$

*where* $W(d\omega)$ *is a complex Gaussian white noise spectral measure*, $\mathsf{E}|W(d\omega)|^2 = (1/2\pi)\, d\omega$, *and* $\varphi$ *is some square integrable function. Define* $\sigma^2 \doteq \mathsf{E}\,|\xi|^2$. *Then*

$$H_k(\xi, \sigma^2) = \frac{1}{k!} \int_{\mathbb{R}^k} \prod_1^k \varphi(\omega_j) W(d\omega_{(k)}).$$

PROOF OF EXAMPLE 3.1. We are going to construct the solution of stochastic equation (20). We define the stationary processes $\xi_t$, $\eta_t$ by their spectral representations

$$\xi_t \doteq \int_{\mathbb{R}} e^{it\omega} f_1^{(\xi)}(\omega)(i\omega)^{-h} W(d\omega), \qquad f_1^{(\xi)}(\omega) \doteq \alpha\beta \frac{1}{i\omega - \alpha},$$
$$\eta_t \doteq f_0^{(\eta)} \doteq \beta,$$

where $\alpha, \beta \in \mathbb{C}$, $\operatorname{Re}\alpha \neq 0$. Then $\xi_t, \eta_t \in S_2(w^{(h)})$ and assumptions (A1), (A3) and (A4) are trivially valid. Assumption (A2) is fulfilled too, because

$$\begin{aligned}
&\frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{\omega^2} \left| f_1^{(\xi)}(\omega) + f_0^{(\eta)} \right|^2 |\omega|^{-2h}\, d\omega \\
&\quad = \frac{1}{2\pi} \int_{\mathbb{R}} \left| \alpha\beta \frac{1}{i\omega - \alpha} + \beta \right|^2 |\omega|^{-2-2h}\, d\omega \\
&\quad = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{|\beta|^2}{(\omega - \operatorname{Im}\alpha)^2 + (\operatorname{Re}\alpha)^2} |\omega|^{-2h}\, d\omega < \infty.
\end{aligned}$$

By Theorem 3.4, the integral process

$$y_t \doteq y_0 + \int_0^t (\xi_s\, ds + \eta_s\, dw_s^{(h)})$$

is well defined apart from its expectation. By Definition 3.4,

$$y_t = f_0^{(y)} + \int_{\mathbb{R}} e^{it\omega} f_1^{(y)}(\omega)(i\omega)^{-h} W(d\omega)$$

with an arbitrary $f_0^{(y)} \in \mathbb{C}$ and

$$f_1^{(y)}(\omega) = \frac{1}{i\omega}(f_1^{(\xi)}(\omega) + f_0^{(\eta)}) = \frac{1}{i\omega}\left(\frac{\alpha\beta}{i\omega - \alpha} + \beta\right) = \frac{\beta}{i\omega - \alpha}.$$

Define $f_0^{(y)} = 0$. Then $\xi_t = \alpha y_t$, so $y_t \in S_{2\mathbb{I}}(w^{(h)})$ satisfies the stochastic differential equation (20), and what is more, that is the only solution of (20) in

$S_{2\mathbb{I}}(w^{(h)})$, because the respective transfer function equation system has only one solution. □

PROOF OF EXAMPLE 3.2. Define the stationary processes $\xi_t$ and $\eta_t$ as follows:

$$\xi_t \doteq f_0^{(\xi)} + \int_{\mathbb{R}^2} \exp(it\Sigma\omega_{(2)}) f_2^{(\xi)}(\omega_{(2)})(i\omega_1)^{-h}(i\omega_2)^{-h} W(d\omega_{(2)}),$$

$$\eta_t \doteq \gamma x_t,$$

where

$$\begin{aligned} f_0^{(\xi)} &\doteq \frac{1}{2\pi}\int_{\mathbb{R}} \frac{-\gamma\beta}{i\omega-\alpha_1}|\omega|^{-2h}\,d\omega, \\ f_2^{(\xi)}(\omega_{(2)}) &\doteq \alpha_2\gamma\beta \operatorname*{sym}_{\omega_{(2)}}\left(\frac{1}{i\omega_1-\alpha_1}\right)\frac{1}{i\Sigma\omega_{(2)}-\alpha_2}. \end{aligned} \tag{47}$$

Then $\xi_t, \eta_t \in S_2$. It is obvious that assumption (A1) is fulfilled. Assumption (A3) is just the definition of $f_0^{(\xi)}$ and assumption (A4) is trivial. Let us check (A2):

$$\begin{aligned} &f_2^{(\xi)}(\omega_{(2)}) + \operatorname*{sym}_{\omega_{(2)}} f_1^{(\eta)}(\omega_1) \\ &\quad = \operatorname*{sym}_{\omega_{(2)}} \frac{1}{i\omega_1-\alpha_1}\frac{\alpha_2\gamma\beta}{i(\omega_1+\omega_2)-\alpha_2} + \operatorname*{sym}_{\omega_{(2)}} \frac{\gamma\beta}{i\omega_1-\alpha_1} \\ &\quad = \frac{i(\omega_1+\omega_2)}{i(\omega_1+\omega_2)-\alpha_2}\operatorname*{sym}_{\omega_{(2)}}\frac{\gamma\beta}{i\omega_1-\alpha_1}. \end{aligned}$$

Hence,

$$\begin{aligned} &\int_{\mathbb{R}^2} \frac{1}{(\omega_1+\omega_2)^2}\left|f_2^{(\xi)}(\omega_{(2)}) + \operatorname*{sym}_{\omega_{(2)}} f_1^{(\eta)}(\omega_1)\right|^2 |\omega_1|^{-2h}|\omega_2|^{-2h}\,d\omega_1\,d\omega_2 \\ &\qquad \le |\gamma\beta|^2 \int_{\mathbb{R}^2} \frac{1}{|i(\omega_1+\omega_2)-\alpha_2|^2}\frac{1}{|i\omega_1-\alpha_1|^2}|\omega_1|^{-2h}|\omega_2|^{-2h}\,d\omega_1\,d\omega_2. \end{aligned} \tag{48}$$

Now,

$$\begin{aligned} &\int_{\mathbb{R}} \frac{1}{|i(\omega_1+\omega_2)-\alpha_2|^2}|\omega_2|^{-2h}\,d\omega_2 \\ &\qquad \le \frac{1}{(\operatorname{Re}\alpha_2)^2}\int_{-1}^{1}|\omega_2|^{-2h}\,d\omega_2 + \int_{\mathbb{R}} \frac{1}{\omega_2^2+(\operatorname{Re}\alpha_2)^2}\,d\omega_2 < \infty. \end{aligned} \tag{49}$$

Equation (49) and the Fubini theorem yield the finiteness of (48). Thus assumption (A2) is satisfied, too. By Theorem 3.4, the integral process

$$y_t \doteq y_0 + \int_0^t (\xi_s\, ds + \eta_s\, dw_s^{(h)})$$

is well defined apart from its expectation. By Definition 3.4,

$$y_t = \sum_{k=0}^{2} \int_{\mathbb{R}^k} \exp(it\Sigma\omega_{(k)}) f_k^{(y)}(\omega_{(k)}) \prod_{j=1}^{2} (i\omega_j)^{-h} W(d\omega_{(k)})$$

with an arbitrary $f_0^{(y)} \in \mathbb{C}$ and

$$f_1^{(y)}(\omega) = \frac{1}{i\omega}\left( f_1^{(\xi)}(\omega) + f_0^{(\eta)} + \frac{2}{2\pi}\int_{\mathbb{R}} f_2^{(\eta)}(\omega, \lambda)|\lambda|^{-2h}\, d\lambda \right) = 0,$$

$$f_2^{(y)}(\omega_{(2)}) = \frac{1}{i(\omega_1+\omega_2)}\Big( f_2^{(\xi)}(\omega_{(2)}) + \operatorname*{sym}_{\omega_{(2)}} f_1^{(\eta)}(\omega_1) \Big)$$

$$= \gamma\beta \operatorname*{sym}_{\omega_{(2)}} \frac{1}{i\omega_1 - \alpha_1} \frac{1}{i(\omega_1+\omega_2) - \alpha_2}.$$

For the latter formula, see (47). Define

$$f_0^{(y)} \doteq \frac{1}{\alpha_2} f_0^{(\xi)}.$$

Then $\xi_t = \alpha_2 y_t$. Since $\eta_t = \gamma x_t$, so $y_t \in S_2$ fulfils the second equation of (23). Furthermore, it is the unique solution in $S_{2\mathbb{I}}(w^{(h)})$, for the same reason as in Example 3.1.

Now let $\alpha \doteq \alpha_1 = \alpha_2$, $\operatorname{Re}\alpha < 0$. We show the time domain integral representation form of the solution $y_t$. First, it is very simple to get by integration by parts that

$$\operatorname*{sym}_{\omega_{(2)}} \frac{1}{i\omega_1 - \alpha} \frac{1}{i\Sigma\omega_{(2)} - \alpha} = -\frac{\alpha}{2}\int_{-\infty}^{t} \exp(\alpha(t-s)) \prod_{1}^{2} \frac{1-\exp(-i(t-s)\omega_j)}{i\omega_j}\, ds. \tag{50}$$

Second, one can show, using the method applied in the proof of Lemma A.4 and by partial integration twice, that

$$\frac{1}{2\pi\alpha}\int_{\mathbb{R}} \frac{1}{i\omega - \alpha}|\omega|^{-2h}\, d\omega = \frac{\alpha}{4\pi}\int_{-\infty}^{t} e^{\alpha(t-s)} \int_{\mathbb{R}} \left| \frac{e^{it\omega} - e^{is\omega}}{i\omega} \right|^2 |\omega|^{-2h}\, d\omega\, du. \tag{51}$$

Let us substitute (50) and (51) into the frequency domain chaotic representation of $y_t$, take into account the isometrical isomorphism between the second-order chaotic space of $W$ and the $L^2$-space of its two-variable transfer functions and finally use the diagram formula to get the frequency domain chaotic representation of $(w_t^{(h)} - w_s^{(h)})^2$. Thus,

$$\begin{aligned} y_t &= f_0^{(y)} + \int_{\mathbb{R}^2} \exp(it\Sigma\omega_{(2)}) f_2^{(y)}(\omega_{(2)}) \prod_1^2 (i\omega_j)^{-h} W(d\omega_{(2)}) \\ &= -\alpha\beta\gamma \int_{-\infty}^{t} \frac{\exp(\alpha(t-s))}{2} \biggl( \frac{1}{2\pi} \int_{\mathbb{R}} \left| \frac{\exp(it\omega) - \exp(is\omega)}{i\omega} \right|^2 |\omega|^{-2h}\, d\omega \\ &\qquad + \int_{\mathbb{R}^2} \prod_1^2 \frac{\exp(it\omega_j) - \exp(is\omega_j)}{i\omega_j} \\ &\qquad \times \prod_1^2 (i\omega_j)^{-h} W(d\omega_{(2)}) \biggr)\, ds \\ &= -\alpha\beta\gamma \int_{-\infty}^{t} \exp(\alpha(t-s)) \frac{(w_t^{(h)} - w_s^{(h)})^2}{2}\, ds. \end{aligned}$$

□

PROOF OF THEOREM 3.6. Let us first verify (28). For $k = 0$,

$$\int_{\mathbb{R}} |f_1(\lambda)| |\lambda|^{-2h}\, d\lambda \le |\mu\gamma| \int_{\mathbb{R}} \int_0^\infty |K(u)| \left| \frac{1 - e^{-iu\lambda}}{i\lambda} \right| du |\lambda|^{-2h}\, d\lambda.$$

Since

$$\int_{\mathbb{R}} \left| \frac{1 - e^{-iu\lambda}}{i\lambda} \right| |\lambda|^{-2h}\, d\lambda = cu^{2h} < \infty$$

for $u > 0$, with some constant $c$, therefore

$$\int_{\mathbb{R}} |f_1(\lambda)| |\lambda|^{-2h}\, d\lambda \le c|\mu\gamma| \int_0^\infty |K(u)| u^{2h}\, du < \infty.$$

For $k \ge 1$,

$$\begin{aligned} &\int_{\mathbb{R}} \bigl| f_{k+1}(\omega_{(k)}, \lambda) \bigr| |\lambda|^{-2h}\, d\lambda \\ &\quad \le |\mu| \frac{|\gamma|^k}{k!} \int_0^\infty |K(u)| \prod_1^k \left| \frac{1 - \exp(-iu\omega_j)}{i\omega_j} \right| \\ &\qquad \times \int_{\mathbb{R}} \left| \frac{1 - \exp(-iu\lambda)}{i\lambda} \right| |\lambda|^{-2h}\, d\lambda\, du \\ &\quad = c|\mu| \frac{|\gamma|^k}{k!} \int_0^\infty |K(u)| \prod_1^k \left| \frac{1 - \exp(-iu\omega_j)}{i\omega_j} \right| u^{2h}\, du < \infty, \end{aligned} \tag{52}$$

for almost all $\omega_{(k)} \in \mathbb{R}^k$. Thus we have proved (28).

Now, for $k \geq 1$,

$$\frac{(k+1)}{2\pi}\int_{\mathbb{R}} f_{k+1}(\omega_{(k)},\omega)|\omega|^{-2h}\,d\omega$$
$$= \mu\frac{(i\gamma)^{k+1}}{k!}\kappa(h)\int_0^\infty \frac{1}{2}\frac{d}{du}u^{2h+1}K(u)\prod_1^k \frac{1-\exp(-iu\omega_j)}{i\omega_j}\,du$$

follows from Lemma A.4. Thus

$$\frac{i\gamma(k+1)}{2\pi}\int_{\mathbb{R}} f_{k+1}(\omega_{(k)},\omega)|\omega|^{-2h}\,d\omega$$
$$= -\alpha f_k(\omega_{(k)}) - \mu\frac{(i\gamma)^k}{k!}\int_0^\infty K(u)\frac{d}{du}\prod_1^k \frac{1-\exp(-iu\omega_j)}{i\omega_j}\,du$$
$$= -\alpha f_k(\omega_{(k)}) - i\gamma \operatorname*{sym}_{\omega_{(k)}} f_{k-1}(\omega_{(k-1)}) + i\sum_1^k \omega_j f_k(\omega_{(k)}),$$

where we used Lemma A.5. Thus,

$$\begin{aligned}&\alpha f_k(\omega_{(k)}) + i\gamma\left(\operatorname*{sym}_{\omega_{(k)}} f_{(k-1)}(\omega_{(k-1)}) + \frac{(k+1)}{2\pi}\int_{\mathbb{R}} f_{k+1}(\omega_{(k)},\omega)|\omega|^{-2h}\,d\omega\right)\\ &\quad = i\sum_1^k \omega_j f_k(\omega_{(k)}).\end{aligned} \tag{53}$$

Hence, the infinite sum in (29) is

$$\sum_{k=1}^\infty \frac{k!}{(2\pi)^k}\int_{\mathbb{R}^k} |f_k(\omega_{(k)})|^2 \prod_1^k |\omega_j|^{-2h}\,d\omega_{(k)}.$$

We have to prove the finiteness of that infinite sum:

$$\begin{aligned}&\sum_{k=1}^\infty \frac{k!}{(2\pi)^k}\int_{\mathbb{R}^k} |f_k(\omega_{(k)})|^2 \prod_1^k |\omega_j|^{-2h}\,d\omega_{(k)}\\
&\quad = |\mu|^2\sum_{k=1}^\infty \frac{(\gamma^2)^k}{k!}(2\pi)^{-k}\int_{\mathbb{R}^k}\int_0^\infty\int_0^\infty K(u_1)\overline{K(u_2)}\\
&\qquad \times\prod_1^k\left[\frac{1-\exp(-iu_1\omega_j)}{i\omega_j}\overline{\left(\frac{1-\exp(-iu_2\omega_j)}{i\omega_j}\right)}\right]du_1\,du_2\prod_1^k|\omega_j|^{-2h}\,d\omega_{(k)}\\
&\quad = |\mu|^2\sum_{k=1}^\infty \frac{(\gamma^2)^k}{k!}\int_0^\infty\int_0^\infty K(u_1)\overline{K(u_2)}\\
&\qquad \times\left(\frac{1}{2\pi}\int_{\mathbb{R}}\frac{1-\exp(-iu_1\omega)}{i\omega}\overline{\left(\frac{1-\exp(-iu_2\omega)}{i\omega}\right)}|\omega|^{-2h}d\omega\right)^k du_1\,du_2,\end{aligned} \tag{54}$$

where the fact that we may change the order of $\int_{\mathbb{R}^k}$ and $\int_0^\infty \int_0^\infty$ can be easily justified because

$$\int_0^\infty |K(u)| \int_{\mathbb{R}} \left| \frac{1-e^{-iu\omega}}{i\omega} \right| |\omega|^{-2h} \, d\omega \, du \tag{55}$$

$$= \int_{\mathbb{R}} \left| \frac{1-e^{-i\lambda}}{i\lambda} \right| |\lambda|^{-2h} \, d\lambda \int_0^\infty |K(u)| u^{2h} \, du < \infty.$$

Continuing (54), let us notice that we may use (13) and then (10) to get

$$\sum_{k=1}^\infty \frac{k!}{(2\pi)^k} \int_{\mathbb{R}^k} |f_k(\omega_{(k)})|^2 \prod_1^k |\omega_j|^{-2h} \, d\omega_{(k)}$$

$$\le |\mu|^2 \int_0^\infty \int_0^\infty |K(u_1)||K(u_2)|$$

$$\times \sum_{k=1}^\infty \frac{(\kappa(h)(\gamma^2/2))^k}{k!} \left(u_1^{2h+1} + u_2^{2h+1} - |u_1 - u_2|^{2h+1}\right)^k du_1 \, du_2$$

$$\le |\mu|^2 \left( \int_0^\infty \exp(\operatorname{Re} \alpha u) \, du \right)^2 - |\mu|^2 \left( \int_0^\infty |K(u)| \, du \right)^2 < \infty,$$

since $\operatorname{Re}\alpha < 0$. We have finished proving that the transfer functions (32) satisfy assumption (29).

Let us verify (30). For fixed $\omega_{(k-1)} \in \mathbb{R}^{k-1}$,

$$\int_{\mathbb{R}} \left| \int_{\mathbb{R}} f_{k+1}(\omega_{(k-1)}, \lambda, \mu) |\mu|^{-2h} \, d\mu \right|^2 |\lambda|^{-2h} \, d\lambda$$

$$\le c_3 \int_{\mathbb{R}} \int_0^\infty |K(u)| u^{4h+1} \prod_1^{k-1} \left| \frac{1-\exp(-iu\omega_j)}{i\omega_j} \right| du < \infty$$

with some constant $c_3$, where the inequality is based on (52) and the Jensen inequality. So we have justified (30) too.

Now let us see the system of equations (27). For $k > 0$ we have already obtained (27); see (53). So we have to prove only the first equation, that is, for $k = 0$. However, that can be easily carried out taking into account that $\prod_1^k$ means 1 for $k = 0$.

Now we are going to prove the time domain form of the solution. By the isometrical isometry between $L^2(W)$ and the Fock space of $W$, a consequence of Lemma A.6 and Theorem 3.1 and the Hermite series expansion of a square integrable function of a Gaussian random variable, we can see that

$$y_t = \mu \sum_{k=0}^\infty \int_{\mathbb{R}^k} \frac{(i\gamma)^k}{k!} \int_{-\infty}^t K(t-s) \prod_1^k \frac{\exp(it\omega_j) - \exp(is\omega_j)}{i\omega_j} \, ds$$

$$\times \prod_1^k (i\omega_j)^{-h} W(d\omega_{(k)})$$

$$= \mu \int_{-\infty}^{t} K(t-s) \sum_{k=0}^{\infty} (i\gamma)^k H_k(w_t^{(h)} - w_s^{(h)}, \kappa(h)(t-s)^{2h+1})\, ds$$

$$= \mu \int_{-\infty}^{t} \exp(\alpha(t-s)) \exp(i\gamma(w_t^{(h)} - w_s^{(h)}))\, ds. \qquad \square$$

PROOF OF THEOREM 3.7. From (34),

$$\mathsf{E}\, y_t = \mu \int_{-\infty}^{t} \exp(\alpha(t-s)) \mathsf{E} \exp(i\gamma(w_t^{(h)} - w_s^{(h)}))\, ds = \mu \int_0^\infty K(u)\, du.$$

To calculate $R(t)$, let us use (32) and (33):

$$\begin{aligned} R(t) &= \mathsf{E}\,(y_t - \mathsf{E}\, y_t)\overline{(y_0 - \mathsf{E}\, y_0)} \\ &= |\mu|^2 \int_0^\infty \int_0^\infty K(u_1)\overline{K(u_2)} \\ &\quad \times \sum_{k=1}^{\infty} \frac{\gamma^{2k}}{k!} \left( \frac{1}{2\pi} \int_{\mathbb{R}} \exp(it\omega) \frac{1-\exp(-iu_1\omega)}{i\omega} \right. \\ &\qquad \left. \times \overline{\left( \frac{1-\exp(-iu_2\omega)}{i\omega} \right)} |\omega|^{-2h} d\omega \right)^k du_1 du_1 \end{aligned} \tag{56}$$

$$= |\mu|^2 \int_0^\infty \int_0^\infty K(u_1)\overline{K(u_2)} \sum_{k=1}^{\infty} \frac{\gamma^{2k}}{k!} \left(\mathsf{E}\left((w_t^{(h)} - w_{t-u_1}^{(h)})(-w_{-u_2}^{(h)})\right)\right)^k du_1 du_2 \tag{57}$$

$$\begin{aligned} &= |\mu|^2 \int_0^\infty \int_0^\infty K(u_1)\overline{K(u_2)} \\ &\quad \times \left( \exp\left( -\frac{\gamma^2}{2} \kappa(h)\big(|t|^{2h+1} - |t-u_1|^{2h+1} - |t+u_2|^{2h+1} \right.\right. \\ &\qquad \left.\left. + |t-u_1+u_2|^{2h+1}\big) \right) - 1 \right) du_1\, du_2. \end{aligned} \tag{58}$$

Regarding the change of order of the various types of integrations, we refer to the Fubini theorem and to (55).

Let us now consider the spectrum. First we prove that the left-hand side of (38), which we shall denote by $\psi(\lambda)$, exists in $L^1(\mathbb{R})$. For any function $g \in L^1(\mathbb{R})$, $\|g * g\|_1 \le (\|g\|_1)^2$, thus $\|g^{*k}\|_1 \le (\|g\|_1)^k$. Hence $e^{*g}(\omega) \doteq \sum_{k=0}^{\infty}(1/k!)g^{*k}(\omega)$ really exists in $L^1(\mathbb{R})$ since

$$\left\| \sum_{k=0}^{\infty} \frac{1}{k!} g^{*k} \right\|_1 \le \sum_{k=0}^{\infty} \frac{1}{k!} \|g^{*k}\|_1 \le \sum_{k=0}^{\infty} \frac{1}{k!} (\|g\|_1)^k = \exp(\|g\|_1). \tag{59}$$

Now, using the Cauchy–Schwarz inequality, for fixed $0 < u_1, u_2 < \infty$,

$$\begin{aligned} \|A^{(u_1,u_2)}\|_1 &= \frac{1}{2\pi} \int_{\mathbb{R}} \left| \frac{1-\exp(-iu_1\omega)}{i\omega} \right| \left| \frac{1-\exp(-iu_2\omega)}{i\omega} \right| |\omega|^{-2h}\, d\omega \\ &= \kappa(h) u_1^{h+1/2} u_2^{h+1/2}. \end{aligned} \tag{60}$$

From (59) and (60),

$$\left\|\exp(*(\gamma^2 A^{(u_1,u_2)}))(\cdot)-1\right\|_1 \le \exp(\gamma^2\kappa(h)u_1^{h+1/2}u_2^{h+1/2})-1.$$

Thus,

$$\begin{aligned}
\|\psi\|_1 &\le |\mu|^2\int_0^\infty\int_0^\infty |K(u_1)||K(u_2)|\left\|\exp(*(\gamma^2 A^{(u_1,u_2)}))(\cdot)-1\right\|_1 du_1\,du_2\\
&\le |\mu|^2\int_0^\infty\int_0^\infty |K(u_1)||K(u_2)|\big(\exp(\gamma^2\kappa(h)u_1^{h+1/2}u_2^{h+1/2})-1\big)\,du_1\,du_2\\
&= |\mu|^2\int_0^\infty\int_0^\infty \exp\Big(\operatorname{Re}\alpha(u_1+u_2)-\frac{\gamma^2}{2}\kappa(h)\big(u_1^{2h+1}+u_2^{2h+1}\big)\\
&\qquad\qquad + \gamma^2\kappa(h)u_1^{h+1/2}u_2^{h+1/2}\Big)\,du_1\,du_2\\
&\quad - |\mu|^2\int_0^\infty\int_0^\infty \exp\Big(\operatorname{Re}\alpha(u_1+u_2)-\frac{\gamma^2}{2}\kappa(h)(u_1^{2h+1}+u_2^{2h+1})\Big)\,du_1\,du_2\\
&= |\mu|^2\int_0^\infty\int_0^\infty \exp\Big(\operatorname{Re}\alpha(u_1+u_2)-\frac{\gamma^2}{2}\kappa(h)(u_1^{h+1/2}-u_2^{h+1/2})^2\Big)\,du_1\,du_2\\
&\quad - |\mu|^2\int_0^\infty\int_0^\infty \exp\Big(\operatorname{Re}\alpha(u_1+u_2)-\frac{\gamma^2}{2}\kappa(h)(u_1^{2h+1}+u_2^{2h+1})\Big)\,du_1\,du_2\\
&\le |\mu|^2\int_0^\infty\int_0^\infty \exp(\operatorname{Re}\alpha(u_1+u_2))\,du_1 d\,u_2 < \infty,
\end{aligned}$$

that is, $\psi \in L^1(\mathbb{R})$. Now we can apply the Fubini theorem to change the order of the integrals $\int_0^\infty\int_0^\infty(\cdot)\,du_1\,du_2$ and $\int_{\mathbb{R}}(\cdot)\,d\lambda$ in the calculation of the inverse Fourier transform of $\psi$:

$$\begin{aligned}
&\int_{\mathbb{R}}\exp(it\lambda)\psi(\lambda)\,d\lambda\\
&\quad= |\mu|^2\int_0^\infty\int_0^\infty K(u_1)\overline{K(u_2)}\\
&\qquad\qquad \times\int_{\mathbb{R}}\exp(it\lambda)(\exp(\gamma^2 A^{(u_1,u_2)})(\lambda)-1)\,d\lambda\,du_1\,du_2 \qquad (61)\\
&\quad= |\mu|^2\int_0^\infty\int_0^\infty K(u_1)\overline{K(u_2)}\\
&\qquad\qquad \times\left(\exp\left(\int_{\mathbb{R}}\exp(it\lambda)\gamma^2 A^{(u_1,u_2)}(\lambda)\,d\lambda\right)-1\right)du_1\,du_2.
\end{aligned}$$

Here the second equality is a consequence of the fact that

$$\sum_{k=1}^{N}\big(\gamma^2 A^{(u_1,u_2)}\big)^{*k}(\lambda)\to\exp(*(\gamma^2 A^{(u_1,u_2)}))(\lambda)-1 \quad \text{as } N\to\infty$$

in $L^1$-norm and that the inverse Fourier transform converts a convolution into the product of the inverse Fourier transforms. We have already obtained the argument of the exponential function in the third row of (61),

$$\begin{aligned}
&\int_{\mathbb{R}} \exp(it\lambda)\gamma^2 A^{(u_1,u_2)}(\lambda)\,d\lambda \\
&\quad = \gamma^2 \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\exp(it\lambda)-\exp(i(t-u_1)\lambda)}{i\lambda}\overline{\left(\frac{1-\exp(-iu_2\lambda)}{i\lambda}\right)}|\lambda|^{-2h}\,d\lambda \\
&\quad = \gamma^2 \mathsf{E}\left(\left(w_t^{(h)} - w_{t-u_1}^{(h)}\right)\left(-w_{-u_2}^{(h)}\right)\right) \\
&\quad = -\frac{\gamma 2}{2}\kappa(h)\left(|t|^{2h+1} - |t-u_1|^{2h+1} - |t+u_2|^{2h+1} + |t-u_1+u_2|^{2h+1}\right);
\end{aligned}$$

see (56), (57) and (58). So, the right side of (61) is equal to (58), that is,

$$\int_{\mathbb{R}} e^{it\lambda}\psi(\lambda)\,d\lambda = R(t),$$

hence $\psi$ must be the spectrum $\varphi$. □

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