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SINGULAR OPTIMAL STRATEGIES FOR INVESTMENT WITH TRANSACTION COSTS

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We study an investment decision problem for an investor who has available a risk-free asset (such as a bank account) and a chosen risky asset. It is assumed that the interest rate for the risk-free asset is zero. The amount invested in the risky asset is given by an Itô process with infinitesimal parameters $\mu(\cdot)$ and $\sigma(\cdot)$, which come from a control set. This control set depends on the investor's wealth in the risky asset. The wealth can be transferred between the two assets and there are charges on all transactions equal to a fixed percentage of the amount transacted. The investor's financial goal is to achieve a total wealth of $a > 0$. The objective is to find an optimal strategy to maximize the probability of reaching a total wealth a before bankruptcy. Under certain conditions on the control sets, an optimal strategy is found that consists of an optimal choice of a risky asset and an optimal choice for the allocation of wealth (buying and selling policies) between the two assets.

1. Introduction. Consider a stochastic process (X, Y) satisfying the stochastic differential equation

$$\begin{aligned} dX(t) &= \mu(t)\,dt + \sigma(t)\,dW(t) + dL(t) - dU(t),\\ dY(t) &= -(1+\alpha)\,dL(t) + (1-\lambda)\,dU(t) \end{aligned} \tag{1.1}$$

and the initial condition $(X(0-), Y(0-)) = (x, y)$. Here $\{W(t)\colon t \geq 0\}$ is a standard Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$, adapted to a right-continuous filtration $\{\mathcal{F}_t, t \geq 0\}$. Each $\mathcal{F}_t$ is contained in $\mathcal{F}$, is independent of $\{W(t+s) - W(t);\ s \geq 0\}$ and contains all P-null sets. The processes $\mu(t)$ and $\sigma(t)$ are assumed to be real valued, $\{\mathcal{F}_t\}$ adapted and to satisfy

$$\int_0^t (|\mu(s)| + \sigma^2(s))\,ds < \infty \quad \text{a.s. for every } t > 0. \tag{1.2}$$

The processes $L(t)$ and $U(t)$ are assumed to be right continuous with left limits (RCLL), nonnegative, nondecreasing and adapted to $\{\mathcal{F}_t\}$. The quantities α and λ are two positive constants and $0 < \lambda < 1$.

In our optimal control problem, the processes $\mu(\cdot)$, $\sigma(\cdot)$, $L(\cdot)$ and $U(\cdot)$ are considered as control processes. We restrict our attention to those control processes that yield nonnegative processes $X(t)$ and $Y(t)$ in (1.1). Furthermore, if $(X(t), Y(t)) = (0, 0)$ for some $t > 0$, then $(X(t), Y(t))$ is absorbed at $(0, 0)$.

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In our motivating examples, the process $X(t)$ represents the amount invested in a risky asset at time t, and $Y(t)$ represents the amount in a risk-free bank account (with zero interest rate). We allow transactions between X and Y with proportional transaction costs. The real numbers α and λ account for proportional transaction costs incurred whenever wealth is transferred from one asset to the other. At a given time t, the nominal value of the risky investment is $X(t)$, but due to the transaction costs, the amount received by the investor by selling the risky asset is $(1-\lambda)X(t)$. Hence, the effective total wealth of the investor at time t is given by $(1-\lambda)X(t)+Y(t)$. The state $(0, 0)$ for the (X, Y) process is considered to be the bankruptcy state. The processes $L(t)$ and $U(t)$ represent the cumulative wealth transferred to and from the risky asset during $[0, t]$, respectively. Note that

$$\begin{aligned} X(0) &= x + L(0) - U(0), \\ Y(0) &= y - (1+\alpha)L(0) + (1-\lambda)U(0) \end{aligned}$$

and $(X(0), Y(0))$ may differ from $(X(0-), Y(0-))$ because of a transaction at time zero.

Let $a > 0$ be the finanical goal of the investor, where $a > 0$ is a positive constant. The optimal control problem is to find processes $\mu(\cdot)$, $\sigma(\cdot)$, $L(\cdot)$ and $U(\cdot)$ to maximize the probability that the total wealth $(1-\lambda)X(t) + Y(t)$ reaches goal a before $(X(t), Y(t))$ reaches the origin (i.e., to maximize the probability that the total wealth reaches the financial goal $a > 0$ before bankruptcy).

There are no further restrictions on the processes L and U, but the possible choices of μ and σ are determined by a collection $\{C(r)\colon 0 \le r \le a/(1-\lambda)\}$ of nonempty subsets of $R \times R^+$. The controller is required to choose the pair (μ, σ) at time t from the control set $C(r)$ whenever $X(t-) = r$. More precisely, we assume that $(\mu(t), \sigma(t))$ lies in $C(X(t-))$ for all $t > 0$.

Let $\Sigma(x, y)$ be the collection of all nonnegative processes $\{(X(t), Y(t))\colon t \ge 0\}$ described above and which are available to a controller with the initial condition $(X(0-), Y(0-)) = (x, y)$. Assume that $\Sigma(x, y)$ is nonempty for each (x, y) with $0 \le (1-\lambda)x + y \le a$ and define the value function $V(x, y)$ by

$$V(x, y) = \sup\big\{P[(1-\lambda)X + Y \text{ reaches } a \text{ before } 0]\colon (X, Y) \in \Sigma(x, y)\big\}, \tag{1.3}$$

where $[(1-\lambda)X + Y$ reaches a before $0]$ represents the event [there exists a $t > 0$ such that $(1-\lambda)X(t) + Y(t) \ge a$ and $\min_{[0, t]}(1-\lambda)X(s) + Y(s) > 0]$. The key to the definition of an optimal choice for (μ, σ) is the function

$$\rho(x) = \sup\left\{\frac{\mu}{\sigma^2}\colon (\mu, \sigma) \in C(x)\right\} \quad \text{for } 0 \le x \le \frac{a}{1-\lambda}. \tag{1.4}$$

In a number of related optimal control problems [8, 10, 14], it has been proved optimal to choose the controls (μ, σ) at each x so that μ/σ^2 attains the supremum $\rho(x)$. We conjecture that this choice of μ and σ remains optimal for a more general class of problems than considered in this article. We will verify this under the assumption that ρ is a continuous decreasing function on

$[0, a/(1-\lambda)]$. This conjecture is also true when there are no transaction costs, [i.e., $\lambda = \alpha = 0$ in (1.1)] and this problem is treated in [8] under a more general assumption that ρ is a continuous function.

Throughout this article we assume that the function ρ is continuous, decreasing and can be written in the form

$$\rho(x) = \frac{\mu_0(x)}{\sigma_0(x)^2} \quad \text{for } 0 \le x \le \frac{a}{1-\lambda}. \tag{1.5}$$

Furthermore, μ_0 and σ_0 are continuous functions on $[0, a/(1-\lambda)]$, $\sigma_0(x) > 0$ for all x, and $(\mu_0(x), \sigma_0(x))$ belongs to $C(x)$ for each x in $[0, a/(1-\lambda)]$. These [i.e., (1.4), (1.5) and that ρ is continuous decreasing] are the only assumptions we impose on the control sets $\{C(x)\colon 0 \le x \le a/(1-\lambda)\}$. These control sets can be unbounded in general. The functions μ_0 and σ_0 will select the optimal μ and σ.

To describe our optimal strategy, let us define the function $\varphi(x) = S'(x) + 2\rho(x)S(x) - ((1-\lambda)/(1+\alpha))$ on the interval $[0, a/(1-\lambda)]$. Here S is the scale function and is given by $S(x) = \int_0^x \exp(-2\int_0^r \rho(u)\,du)\,dr$ and S' is its derivative.

Case 1. If $\varphi(a/(1-\lambda)) \ge 0$, then our optimal strategy is quite simple: choose μ_0 and σ_0 to satisfy (1.5). Control the process X as a reflecting diffusion process with coefficients (μ_0, σ_0) and with reflection at zero by choosing U to be identically zero and L to be the local time process of X at origin.

Case 2. If $\varphi(a/(1-\lambda)) < 0$, then since φ is continuous, we can find a value c on $(0, a/(1-\lambda))$ such that $\varphi(c) = 0$. Again choose μ_0 and σ_0 to satisfy (1.5). In this case, it is optimal to keep the process X in the interval $[0, c]$ as a reflecting diffusion process with coefficients (μ_0, σ_0) and with reflections at 0 and c (with a possible initial jump). More precisely, let $(X(0-), Y(0-)) = (x, y)$. If $0 \le x \le c$, then X process is a reflecting diffusion process with reflections at 0 and c and hence L and U are given by the local time processes of X at 0 and c, respectively. If $x > c$, then there is an initial jump to $(X(0), Y(0)) = (c, y + (1-\lambda)(x-c))$ and then the process follows the strategy defined for the case $0 \le x \le c$. In this case, L is the local time process of X at zero, $U(0) = x - c$ and $U(t) - U(0)$ is the local time of X at c for $t > 0$. The details are in Section 4.

There are only a few solvable optimal control problems with transaction costs available in the literature [2, 3, 4, 13, 16]. Most of them are related to utility maximization problems and use the standard financial model $\mu(x) = \mu x$ and $\sigma(x) = \sigma x$. Our objective function is quite different from theirs, but resembles those studied in [8], [14] and [15]. In [3], Davis and Norman constructed an optimal investment and consumption policy for a utility maximization problem with proportional transaction costs. They showed that the optimal transaction policy is to stay in a cone with a "local time" type transaction policy at each boundary. Our optimal strategy is similar to that of [3]. But instead of a cone, our optimal policy is to stay in a trapezoid. For a detailed account on the available literature on the utility maximization problems with transaction costs, we refer the reader to Section 1 of [13] and to [2].

In the next section we describe two examples which motivate the results of this article.

Similar to many optimal control problems in the literature, first we formulate an appropriate verification lemma in Section 3. The Hamilton–Jacobi–Bellman equation related to our problem reduces to the form

$$\max\{Q_{xx}+2\rho(x)Q_x,\ Q_x-(1+\alpha)Q_y,\ (1-\lambda)Q_y-Q_x\}=0 \tag{1.6}$$

on the region $E=\{(x,y): x\geq 0,\ a>y\geq 0,\ (1-\lambda)x+y\leq a\}$, together with the boundary conditions $Q(x,y)=1$ if $(1-\lambda)x+y=a$ and $Q(0,0)=0$. With our closed form expressions for the value function V, it is easy to see that V is a C^2-solution of the above Hamilton–Jacobi–Bellman equation. In Theorem 4.5, we observe that a free boundary for (1.6) is given by a vertical line $x=c$ where the constant c depends only on ρ, λ and α, but it does not depend on the goal a. Furthermore, it is easy to verify the "principle of smooth fit" across the line $x=c$ by using the explicit closed form expressions for the value function.

2. Two applications.

EXAMPLE 1. Consider an investment model with one risky asset, which we call a stock, and a risk-free bank account. The interest rate for the bank account is zero. Let $X_1(t)$ be an investor's holding in the stock at time $t\geq 0$ and $Y_1(t)$ be the amount in the investor's bank account at time $t\geq 0$. The investor begins with an initial endowment $X_1(0-)=x$, $Y_1(0-)=y$. The funds can be transferred between two assets for a transaction fee. This transaction cost is proportional to the amount transferred and the constants of proportionality are different for the two assets. Similarly to the problems studied in [3], [13], we also assume that the cumulative transaction cost during $[0,t]$ is a function of bounded variation in t.

It is assumed that there is a constant $\delta>0$ so that $X_1(t)\geq\delta$ and $Y_1(t)\geq 0$ (no loans are allowed) for all $t\geq 0$. [In particular, $X_1(0-)=x\geq\delta$.] If $X_1(t)=\delta$ for some t, it is necessary to transfer funds from Y_1 to keep X_1 above the value δ. Therefore $(X_1(t),Y_1(t))=(\delta,0)$ is considered as the bankruptcy situation and hence $(\delta,0)$ is an absorbing state for the process (X_1,Y_1). We use the standard financial model [i.e., $\mu(t)=\mu_0X_1(t)$ and $\sigma(t)=\sigma_0X(t)$ in (1.1)] to represent $X_1(t)$ and $Y_1(t)$. More precisely,

$$\begin{aligned} dX_1(t)&=X_1(t)(\mu_0\,dt+\sigma_0\,dW(t))+dL(t)-dU(t),\\ dY_1(t)&=-(1+\alpha)\,dL(t)+(1-\lambda)\,dU(t), \end{aligned} \tag{2.1}$$

and $(X_1(0-),Y_1(0-))=(x,y)$, where $\{W(t): t\geq 0\}$ is a standard Brownian motion adapted to a filtration $\{\mathscr{F}_t\}$ on a probability space $(\Omega,\mathscr{F},P)$ and L and U are nonnegative, adapted, right-continuous, nondecreasing control processes. The constants μ_0, σ_0, α and λ are all known to the investor.

Let g be the financial goal of the investor. The investor's total wealth at time t is given by $Z_1(t) = (1-\lambda)X_1(t)+Y_1(t)$. [It is assumed $g > (1-\lambda)x+y > (1-\lambda)\delta > 0$.] The only available controls are the two increasing processes L and U, and the investor's objective is to maximize the probability that the total wealth $Z(t)$ reaches the goal of g dollars before bankruptcy.

Our results give an explicit solution to this problem when $\mu_0 > 0$. To make use of Theorems 4.3, and 4.5, introduce the processes $X(t) = X_1(t) - \delta$ and $Y(t) = Y_1(t)$.

Hence

$$\begin{aligned} dX(t) &= (X(t)+\delta)(\mu_0\,dt + \sigma_0\,dW(t)) + dL(t) - dU(t) \\ dY(t) &= -(1+\alpha)\,dL(t) + (1-\lambda)\,dU(t) \quad \text{and} \\ (X(0-), Y(0-)) &= (x-\delta, y). \end{aligned} \tag{2.2}$$

Introduce $Z(t) = (1-\lambda)X(t) + Y(t) = Z_1(t) - (1-\lambda)\delta$, and let $a = g - (1-\lambda)\delta > 0$. Now (X, Y) agree with (1.1) and the problem described before. In this case, the collection of control sets $\{C(r)\colon 0 \le r \le a/(1-\lambda)\}$ is given by $C(r) = \{((r+\delta)\mu_0, (r+\delta)\sigma_0)\}$ for each $0 \le r \le a/(1-\lambda)$. The control problem here is to find an optimal choice for L and U to maximize the probability that the $Z(t)$ process reaches a before 0, that is, to maximize the probability $P[Z$ reaches a before $0|(X(0-), Y(0-)) = (x-\delta, y)]$, where [$Z$ reaches a before 0] represents the event $[Z(t) \ge a$ and $\min_{[0,t]} Z(s) > 0$ for some $t \ge 0]$. Also notice that the event $Z(t) = 0$ is same as $(X(t), Y(t)) = (0, 0)$. To describe the optimal choices which follow from our results, we introduce $\rho_0 = \mu_0/\sigma_0^2$. Hence the function $\rho(x)$ defined in (1.4) and (1.5) is given by [using (1.2)],

$$\rho(x) = \frac{\rho_0}{(x+\delta)}. \tag{2.3}$$

Since $\mu_0 > 0$, the function $\rho(x)$ is decreasing. The scale function $S(x)$ for the X-process in (2.2) is therefore defined by $S(x) = \int_0^x \exp(-2\int_0^r \rho(u)\,du)\,dr$. We introduce the function $\varphi(x)$ by

$$\varphi(x) = S'(x) + 2\rho(x)S(x) - \left(\frac{1-\lambda}{1+\alpha}\right) \quad \text{where } S' \text{ is the derivative of } S. \tag{2.4}$$

Since $\rho(x)$ is strictly decreasing, it is easy to check that φ is also strictly decreasing.

One can easily check that

$$\varphi(x) = \begin{cases} \dfrac{1}{(1-2\rho_0)}\left(\dfrac{\delta}{x+\delta}\right)^{2\rho_0} - \dfrac{2\rho_0}{(1-2\rho_0)}\left(\dfrac{\delta}{x+\delta}\right) - \left(\dfrac{1-\lambda}{1+\alpha}\right), & \text{if } \rho_0 \ne \dfrac{1}{2}, \\ \left(\dfrac{\delta}{x+\delta}\right)\left(1 - \log\left(\dfrac{\delta}{x+\delta}\right)\right) - \left(\dfrac{1-\lambda}{1+\alpha}\right), & \text{if } \rho_0 = \dfrac{1}{2}. \end{cases} \tag{2.5}$$

Hence $\varphi(a/(1-\lambda)) = \varphi(g/(1-\lambda)-\delta)$ and

$$\varphi\left(\frac{g}{(1-\lambda)}-\delta\right) = \begin{cases} \dfrac{1}{(1-2\rho_0)}\left(\dfrac{(1-\lambda)\delta}{g}\right)^{2\rho_0} - 2\rho_0\left(\dfrac{(1-\lambda)\delta}{g}\right) - \left(\dfrac{1-\lambda}{1+\alpha}\right), \\ \hfill \text{if } \rho_0 \neq \dfrac{1}{2}, \\ \left(\dfrac{(1-\lambda)\delta}{g}\right)\left[1-\log\left(\dfrac{(1-\lambda)\delta}{g}\right)\right] - \left(\dfrac{1-\lambda}{1+\alpha}\right), \quad \text{if } \rho_0 = \dfrac{1}{2}. \end{cases}$$

If $\varphi(a/(1-\lambda)) \geq 0$, it follows from Theorem 4.3, that it is optimal to choose $U(t) \equiv 0$ and $L(t)$ to be the local time process for $X(t)$ reflecting at the origin.

If $\varphi(a/(1-\lambda)) < 0$, since $\varphi(0) = (\lambda+\alpha)/(1+\alpha) > 0$ and φ is strictly decreasing, there is a unique point c such that $0 < c < a/(1-\lambda)$ and $\varphi(c) = 0$. Theorem 4.5 implies that if $X(0-) = (x-\delta) \leq c$, then it is optimal to run the X process as a reflecting diffusion process with reflecting barriers at $x = 0$ and $x = c$. Hence, an optimal choice for L and U processes is to take $L(\cdot)$ to be the local time process of X at the origin and $U(\cdot)$ to be the local time process of X at the point c. If $X(0-) = (x-\delta) > c$ and $Y(0-) = y$, initially jump to $X(0) = c$, $Y(0) = y + (1-\lambda)[(x-\delta)-c]$ and then follow the optimal strategy described above.

The constraint $X_1(t) \geq \delta > 0$ is essential for two reasons. First, if $\delta = 0$, then $\rho(x)$ defined in (1.4) and (2.3) is not continuous on $[0, a/(1-\lambda)]$ (in fact, it is unbounded near the origin), and hence it will not satisfy our assumptions in (1.5). Second, if $\delta = 0$ and $\mu_0 \geq \sigma_0^2/2$, then the choice $L(t) \equiv 0$, $U(t) \equiv 0$ is an optimal choice, since it yields $X(t) > 0$ for all t, $\sup_{t>0} X(t) = +\infty$, and the value function $V(x, y) \equiv 1$ for all (x, y).

Our next example is closely related to an example encountered by Browne [1]. This leads to an "incomplete market" due to the correlation of two Brownian motions involved here. For a more detailed description of the model, we refer to [1].

EXAMPLE 2. Consider a firm, such as a property liability insurance company or a pension management company, having a risk obligation and wishing to achieve a certain financial goal before going bankrupt. The firm is obligated to a risk process $R(t)$, which is the revenue from the obligation minus claims up to time t. The firm can store its cash reserves in an interest-free bank account or invest in a risky asset, which we call a stock. The price process of the stock at time t is given by $S(t)$. We assume it is possible to transfer funds between the stock and the bank account with proportional transaction costs.

This model without transaction costs and with a different objective was studied by Browne [1]. Also [1] gives details and related references about the diffusion approximation theory for the net claims process and the wealth process of such an insurance company.

The processes $R(t)$ and $S(t)$ satisfy

$$dR(t) = \delta\,dt + \beta\,dW_1(t) \tag{2.6}$$

and

$$dS(t) = S(t)(\theta\,dt + b\,dW_2(t)), \tag{2.7}$$

where δ, β, θ and b are constants. Here $W_1(t)$ and $W_2(t)$ are two correlated standard Brownian motions on some probability space $(\Omega, \mathscr{F}, P)$ adapted to a right continuous filtration $\{\mathscr{F}_t: t \geq 0\}$. Their correlation coefficient is r and $E[W_1(t)W_2(t)] = rt$. Each $\mathscr{F}_t$ is contained in $\mathscr{F}$, and is independent of the increments $\{(W_1(t+s) - W_1(t),\ W_2(t+s) - W_2(t));\ s \geq 0\}$ and contains all P-null sets.

Let $\underline{f}$ be an investment policy and consider $X^f(t)$ to be the amount invested in the stock at time t under the policy $\underline{f}$. For any policy $\underline{f}$, we assume that $X^f(t) \geq 0$, so that the company does not borrow money to invest in the stock. Hence

$$dX^f(t) = X^f(t)(\theta\,dt + b\,dW_2(t)) + dR(t) + dL^f(t) - dU^f(t), \tag{2.8}$$

where $L^f(\cdot)$ and $U^f(\cdot)$ are adapted, right-continuous, nondecreasing control processes and represents the transactions from and to the bank account, respectively. The amount $Y^f(t)$ in the bank account at time t is nonnegative and satisfies

$$dY^f(t) = -(1+\alpha)\,dL^f(t) + (1-\lambda)\,dU^f(t), \tag{2.9}$$

where α and λ are constants. Equation (2.8) can be rewritten in the form

$$\begin{aligned} dX^f(t) &= (\delta + \theta X^f(t))\,dt + \sqrt{\beta^2 + 2rb\beta X^f(t) + (bX^f(t))^2}\,dB(t) \\ &\quad + dL^f(t) - dU^f(t), \end{aligned}$$

where $\{B(t): t \geq 0\}$ is a Brownian motion adapted to the filtration $(\mathscr{F}_t)$ and the increment process $\{B(t+s) - B(t): s \geq 0\}$ is independent of $\mathscr{F}_t$ (Protter [12], page 80). Under a policy $\underline{f}$, the total wealth of the company at time t is given by

$$Z^f(t) = (1-\lambda)X^f(t) + Y^t(t), \tag{2.10}$$

subjected to the available controlled transaction processes $L^f(\cdot)$ and $U^f(\cdot)$. The objective of the company is to reach a goal of financial level a before reaching a bankruptcy state $Z^f(t) = 0$ for some $t > 0$. Hence it would like to maximize the probability $P[Z^f$ reaches a before $0 | X^f(0-) = x,\ Y^f(0-) = y]$.

With our setting, the control sets in this example are described by $C(z) = \{(\delta + \theta z, \sqrt{\beta^2 + 2rb\beta z + b^2z^2})\}$ for each $0 \leq z \leq a/(1-\lambda)$. In this situation, the function ρ defined in (1.4) can be written as $\rho(z) = (\delta + \theta z)/(\beta^2 + 2rb\beta z + b^2z^2)$ for $0 \leq z \leq a/(1-\lambda)$. Using our results in Section 4, we are able to derive optimal strategies for L^f and U^f, when ρ is decreasing.

3. A verification lemma. It is convenient to formulate the problem in two dimensions with the state space $E = \{(x, y)\colon x \geq 0,\ a > y \geq 0,\ (1-\lambda)x + y \leq a\}$ where the x- and y-coordinates represents the processes $X(t)$ and $Y(t)$, respectively.

Let (X, Y) be an available process in $\Sigma(x, y)$, τ be any stopping time and $\varphi(x, y)$ be a function which is twice continuously differentiable on an open set in $\mathbb{R}^2$ which contains the set E. Then Itô's rule for RCLL semi-martingales (e.g., [9], [11]; see also equation (4.7) of [13]) applied to $\varphi(X(t), Y(t))$ yields,

$$\begin{aligned}
&\varphi(X(t\wedge\tau), Y(t\wedge\tau)) \\
&\quad = \varphi(X(0-), Y(0-)) + \int_0^{t\wedge\tau} \frac{\partial\varphi}{\partial x}(X(s-), Y(s-))\sigma(s)\,dW(s) \\
&\qquad + \int_0^{t\wedge\tau} \left(\frac{1}{2}\sigma^2(s)\frac{\partial^2\varphi}{\partial x^2} + \mu(s)\frac{\partial\varphi}{\partial x}\right)(X(s-), Y(s-))\,ds \\
&\qquad + \int_0^{t\wedge\tau} \left(\frac{\partial\varphi}{\partial x} - (1+\alpha)\frac{\partial\varphi}{\partial y}\right)(X(s-), Y(s-))\,dL^c(s) \qquad (3.1)\\
&\qquad + \int_0^{t\wedge\tau} \left((1-\lambda)\frac{\partial\varphi}{\partial y} - \frac{\partial\varphi}{\partial x}\right)(X(s-), Y(s-))\,dU^c(s) \\
&\qquad + \sum_{0\leq s\leq t\wedge\tau} [\varphi(X(s), Y(s)) - \varphi(X(s-), Y(s-))],
\end{aligned}$$

where

$$L^c(t) \equiv L(t) - \sum_{0\leq s\leq t} (L(s) - L(s-)),$$

$$U^c(t) \equiv U(t) - \sum_{0\leq s\leq t} (U(s) - U(s-))$$

and

$$t\wedge\tau = \min\{t, \tau\}.$$

Notice that the processes $L^c(\cdot)$ and $U^c(\cdot)$ denote the continuous parts of $L(\cdot)$ and $U(\cdot)$ respectively.

Now we formulate a verification lemma, which will be adequate for our purposes.

LEMMA 3.1. *Let $Q(x, y)$ be a function defined on an open set G which contains the set E. Assume that*:

(i) *Q is a nonnegative twice-differentiable function on the set E*;
(ii) *$Q(0, 0) = 0$, $Q(x, y) \geq 1$ if $(1-\lambda)x + y = a$, and (x, y) is in E*;
(iii) *$\max\{(\partial^2 Q/\partial x^2) + 2\rho(x)(\partial Q/\partial x), (\partial Q/\partial x) - (1+\alpha)(\partial Q/\partial y), (\partial Q/\partial y) - (1-\lambda)(\partial Q/\partial x)\} \leq 0$ and $\partial Q/\partial x \geq 0$ on E, where ρ is given by* (1.4).

Then $Q(x, y) \geq V(x, y)$ for every (x, y) in E, where V is the value function defined in (1.3).

REMARK 3.2. (i) Proof of this verification lemma is similar to that of [14].

(ii) Since $\alpha > 0$, $0 < \lambda < 1$, condition (iii) of the lemma implies that $\partial Q/\partial x$ and $\partial Q/\partial y$ are both nonnegative.

PROOF. Let (x, y) be in E and (X, Y) belong to $\Sigma(x, y)$. Then (X, Y) satisfies (1.1). Introduce the stopping time τ by

$$\begin{aligned} \tau &= \inf\{t \geq 0: (1-\lambda)X(t) + Y(t) \geq a \text{ or } (X(t), Y(t)) = (0,0)\} \\ &= +\infty \quad \text{otherwise.} \end{aligned} \tag{3.2}$$

Let any (a, b) in E. If $l > 0$ such that $(a + l,\ b - (1+\alpha)l)$ also in E, then from condition (iii) of the lemma, we have

$$Q(a + l, b - (1+\alpha)l) - Q(a, b) \leq 0. \tag{3.3}$$

Similarly, if $l > 0$ such that $(a - l, b + (1-\lambda)l)$ also in E then

$$Q(a - l, b + (1-\lambda)l) - Q(a, b) \leq 0. \tag{3.4}$$

Thus (3.3) and (3.4) yield

$$Q(X(s), Y(s)) - Q(X(s-), Y(s-)) \leq 0 \quad \text{for all } s \geq 0. \tag{3.5}$$

We apply Itô's lemma to $Q(X(t), Y(t))$ as in (3.1). We use condition (iii) of the verification lemma and (3.5) to obtain

$$\begin{aligned} &E[Q(X(t\wedge\tau), Y(t\wedge\tau))] \\ &\quad \leq Q(x, y) + E\left[\int_0^{t\wedge\tau} \frac{1}{2}\sigma^2(s)\left(Q_{xx} + 2\frac{\mu(s)}{\sigma^2(s)}Q_x\right)(X(s-), Y(s-))\,ds\right]. \end{aligned} \tag{3.6}$$

By assumption (iii), we know $\partial Q/\partial x \geq 0$ on E. Hence, using (1.4), we get

$$\begin{aligned} &\left(Q_{xx} + 2\frac{\mu(s)}{\sigma^2(s)}Q_x\right)(X(s-), Y(s-)) \\ &\quad \leq (Q_{xx} + 2\rho(X(s-))Q_x)(X(s-), Y(s-)). \end{aligned} \tag{3.7}$$

However, the expression in the right-hand side is less than or equal to zero, from condition (iii) of the verification lemma. This together with (3.6) yields

$$E[Q(X(t\wedge\tau), Y(t\wedge\tau))] \leq Q(x, y).$$

Now employing conditions (i) and (ii) of the verification lemma and applying Fatou's lemma, it follows that

$$P[(1-\lambda)X(\tau) + Y(\tau) \geq a | (X(0-), Y(0-)) = (x, y)] \leq Q(x, y). \tag{3.8}$$

Taking the supremum over all available processes in $\Sigma(x, y)$ in the left-hand side of (3.8), we derive

$$V(x, y) \leq Q(x, y).$$

This complete the proof. □

In the next section we employ this verification lemma to prove the optimality of our chosen candidate for an optimal strategy.

4. Optimal strategies. Consider the function $\rho(\cdot)$ given in (1.4). We define the corresponding scale function $S(\cdot)$ by

$$S(x) = \int_0^x \exp\left(-2\int_0^r \rho(u)\,du\right) dr \quad \text{for } 0 \le x \le \frac{a}{(1-\lambda)}. \tag{4.1}$$

Let us introduce a candidate for an optimal strategy which we call "timid play." In this strategy the controller always uses $\mu_0(\cdot)$ and $\sigma_0(\cdot)$ given in (1.5) for the infinitesimial drift and diffusion coefficients, respectively. The process $U(t)$ defined in (1.1) is chosen to be identically zero, and the process $L(\cdot)$ in (1.1) is chosen to be the local time process of $X(t)$ process at the origin. More precisely, timid play is represented by the process (X^*, Y^*) which satisfies

$$\begin{aligned} &dX^*(t) = \mu_0(X^*(t))\,dt + \sigma_0(X^*(t))\,dW(t) + dL(t) \quad \text{and} \\ &dY^*(t) = -(1+\alpha)\,dL(t), \end{aligned} \tag{4.2}$$

where $W(t)$ is a one-dimensional Brownian motion, and $L(\cdot)$ is the local time process of X^* at the origin. This strategy is introduced in [14] and is optimal for the problems considered in [8] and [14]. Next, we introduce the total wealth process Z^* by

$$Z^*(t) = (1-\lambda)X^*(t) + Y^*(t). \tag{4.3}$$

Our next lemma gives the pay-off probability from the timid-play strategy.

LEMMA 4.1. *Let (X^*, Y^*) be the timid-play strategy defined above and Z^* be as given in* (4.3). *Define $Q(x, y)$ by*

$$Q(x, y) = P\big[Z^* \text{ reaches } a \text{ before } 0 \,|\, (X^*(0-), Y^*(0-)) = (x, y)\big], \tag{4.4}$$

then

$$\begin{aligned} Q(x, y) = 1 - &\left(1 - \frac{S(x)}{S((a-y)/(1-\lambda))}\right) \\ &\times \exp\left(-\frac{1}{1+\alpha}\int_0^y \frac{1}{S((a-r)/(1-\lambda))}\,dr\right), \end{aligned} \tag{4.5}$$

where S is the scale function defined in (4.1).

PROOF. Let (X^*, Y^*) be the solution to (4.2). Introduce the stopping time τ by

$$\begin{aligned} \tau &= \inf\{t \ge 0\colon Z^*(t) \ge a \text{ or } (X^*(t), Y^*(t)) = (0,0)\} \\ &= +\infty \quad \text{otherwise.} \end{aligned}$$

Then τ is finite a.s., since $L(\cdot)$, the local time for X^* approaches infinity as t goes to infinity.

Consider the differential equation

$$
(4.6)\qquad
\begin{aligned}
&\frac{\partial^2 Q}{\partial x^2}(x, y) + 2\rho(x)\frac{\partial Q}{\partial x}(x, y) = 0 \qquad \text{for all } (x, y) \text{ in } E;\\
&\frac{\partial Q}{\partial x}(0, y) - (1+\alpha)\frac{\partial Q}{\partial y}(0, y) = 0 \quad \text{for } 0 < y < a;\\
&Q(x, y) = 1 \quad \text{if } (1-\lambda)x + y = a \text{ and } (x, y) \text{ is in } E \text{ and}\\
&\qquad\qquad Q(0, 0) = 0.
\end{aligned}
$$

Equation (4.6) can be solved explicitly; the solution is given by (4.5). Next we apply Itô's lemma (3.1) to $Q(X^*(t \wedge \tau), Y^*(t \wedge \tau))$ to obtain

$$E\big[Q(X^*(t \wedge \tau), Y^*(t \wedge \tau))|(X^*(0-), Y^*(0-)) = (x, y)\big] = Q(x, y).$$

Since $0 \le Q \le 1$, by letting t tend to infinity and using the bounded convergence theorem, we obtain

$$E\big[Q(X^*(\tau), Y^*(\tau))|(X^*(0-), Y^*(0-)) = (x, y)\big] = Q(x, y).$$

Hence using the boundary data for Q, (4.4) and (4.5) follow. □

In our next lemma, we discuss some properties of a function φ which are very important in establishing our optimal strategies. Let

$$(4.7)\qquad \varphi(x) = S'(x) + 2\rho(x)S(x) - \left(\frac{1-\lambda}{1+\alpha}\right) \quad \text{for } 0 \le x \le \frac{a}{1-\lambda},$$

where S is given by (4.1) and S' is the derivative of S.

LEMMA 4.2. *Consider the function φ defined by* (4.7) *on* $[0, a/(1-\lambda)]$. *Then*

(i) $\varphi(0) > 0$;
(ii) φ *is decreasing on* $[0, a/(1-\lambda)]$.

PROOF. (i) $\varphi(0) = (\lambda + \alpha)/(1+\alpha) > 0$ since $S(0) = 0$ and $S'(0) = 1$, where S is given by (4.1).

(ii) First we consider the case where ρ is differentiable. Since ρ is decreasing, it yields $\rho'(x) \le 0$ on $(0, a/(1-\lambda))$. Using the fact that $S''(x)+2\rho(x)S'(x) = 0$, by direct computation it yields $\varphi'(x) = 2\rho'(x)S(x) \le 0$. Hence φ is decreasing on $[0, a/(1-\lambda)]$.

For the general case, first we approximate ρ by a sequence of functions (ρ_n) such that (a) each ρ_n is continuous and decreasing on $[0, a/(1-\lambda)]$, differentiable on $(0, a/(1-\lambda))$ and (b) $\lim_{n\to\infty} \rho_n(x) = \rho(x)$ for $0 \le x \le a/(1-\lambda)$. Now we define $S_n(x)$ for ρ_n analogous to (4.1). It is evident that $\lim_{n\to\infty} S_n(x) = S(x)$ and $\lim_{n\to\infty} S'_n(x) = S(x)$. Hence we introduce (φ_n) by

$$\varphi_n(x) = S'_n(x) + 2\rho_n(x)S_n(x) - \left(\frac{1-\lambda}{1+\alpha}\right) \quad \text{for } 0 \le x \le \frac{a}{1-\lambda}.$$

Then $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$ on $[0, a/(1-\lambda)]$, and from the above argument it follows that $\varphi_n'(x) = 2\rho_n'(x)S_n(x) \le 0$, and hence each φ_n is decreasing. Therefore $\varphi(x)$ is also decreasing on $[0, a/(1-\lambda)]$. □

Next, in Theorem 4.3 we prove that the "timid-play" strategy is indeed optimal under the assumption $\varphi(a/(1-\lambda)) \ge 0$. Theorem 4.5 gives an optimal strategy if $\varphi(a/(1-\lambda)) < 0$.

THEOREM 4.3. *Assume that*

(i) *the function* ρ *defined in* (1.4) *is decreasing on* $[0, a/(1-\lambda)]$;
(ii) $\varphi(a/(1-\lambda)) \ge 0$ *where* φ *is given by* (4.7).

Then the "timid-play" strategy is optimal.

PROOF. With the explicit formula for Q given in (4.5), Q can be easily extended to an open set G containing E. Therefore, to prove timid play is optimal, it is enough to verify conditions (i), (ii) and (iii) of the verification lemma. Conditions (i) and (ii) are evident from the formula (4.5) for Q. It remains to verify condition (iii).

The requirement $\partial Q/\partial x \ge 0$ on E is obvious from (4.5).

Since the scale function S satisfies the differential equation $S''(x) + 2\rho(x)S'(x) = 0$ on $(0, a/(1-\lambda))$, using (4.5), one can directly verify that

$$\frac{\partial^2 Q}{\partial x^2}(x, y) + 2\rho(x)\frac{\partial Q}{\partial x}(x, y) = 0 \quad \text{for } (x, y) \text{ in the interior of } E. \tag{4.8}$$

Next, let us verify that $((\partial Q/\partial x) - (1+\alpha)(\partial Q/\partial y)) \le 0$ on E.

First we assume that $\rho(\cdot)$ is differentiable on $(0, a/(1-\lambda))$ and $\rho'(x) \le 0$, since ρ is decreasing. Notice that (4.5) can be written in the form

$$Q(x, y) = A(y)S(x) + B(y), \tag{4.9}$$

where

$$A(y) = \frac{1}{S((a-y)/(1-\lambda))}\exp\left(-\frac{1}{1+\alpha}\int_0^y \frac{1}{S((a-r)/(1-\lambda))}\,dr\right) \tag{4.10}$$

and

$$B(y) = 1 - S\left(\frac{a-y}{1-\lambda}\right)A(y). \tag{4.11}$$

Hence

$$\begin{aligned}
&\left(\frac{\partial Q}{\partial x} - (1+\alpha)\frac{\partial Q}{\partial y}\right)(x, y) \\
&\quad = A(y)S(x)\left[\left(\frac{S'(x)-1}{S(x)}\right)\right. \\
&\qquad \left. - \frac{(((1+\alpha)/(1-\lambda))S'((a-y)/(1-\lambda)) - 1)}{S((a-y)/(1-\lambda))}\right].
\end{aligned} \tag{4.12}$$

Introduce $g(x) = (S'(x) - 1)/S(x)$ on $(0, a/(1-\lambda)]$ and notice that $g'(x) = (S'(x)/S(x)^2)[1 - S'(x) - 2\rho(x)S(x)]$. Set $h(x) = S'(x) + 2\rho(x)S(x)$. Then $h(0) = S'(0) = 1$ and $h'(x) = 2\rho'(x)S(x) \le 0$ on $(0, a/(1-\lambda))$. In the above computations for g' and h', we have used $S''(x) + 2\rho(x)S'(x) = 0$ on $[0, a/(1-\lambda)]$. Now, since h is decreasing, and $h(0) = 1$, we have $h(x) \le 1$ on $(0, a/(1-\lambda)]$ and therefore $g'(x) > 0$ on $(0, a/(1-\lambda))$. This implies that g is increasing.

Hence

$$\begin{aligned}\frac{S'(x)-1}{S(x)} &\le \frac{S'((a-y)/(1-\lambda))-1}{S((a-y)/(1-\lambda))}\\ &\le \frac{((1+\alpha)/(1-\lambda))S'((a-y)/(1-\lambda))-1}{S((a-y)/(1-\lambda))} \quad \text{on } E,\end{aligned} \tag{4.13}$$

since $x \le (a-y)/(1-\lambda)$ on E and S and S' are nonnegative.

Consequently, (4.12) and (4.13) imply that

$$\left(\frac{\partial Q}{\partial x} - (1+\alpha)\frac{\partial Q}{\partial y}\right)(x, y) \le 0 \quad \text{on } E. \tag{4.14}$$

To remove the differentiability assumption, we follow the same approximation procedure as in the proof of Lemma 4.2. Hence (4.14) remains true for any decreasing ρ.

It remains to verify

$$\left((1-\lambda)\frac{\partial Q}{\partial y} - \frac{\partial Q}{\partial x}\right)(x, y) \le 0 \quad \text{on } E. \tag{4.15}$$

Using (4.9), (4.10) and (4.11), and by a direct computation, we get

$$\left((1-\lambda)\frac{\partial Q}{\partial y} - \frac{\partial Q}{\partial x}\right)(x, y) = A(y)S(x)\left[f(x) - f\left(\frac{a-y}{1-\lambda}\right)\right], \tag{4.16}$$

where $f(x) = [((1-\lambda)/(1+\alpha)) - S'(x)]/S(x)$ for $0 < x \le a/(1-\lambda)$. Therefore, to obtain (4.15) it is sufficient to show that f is increasing on $(0, a/(1-\lambda))$.

However, $f'(x) = (S'(x)/(S(x))^2)\varphi(x)$ on $(0, a/(1-\lambda))$ where φ is given in (4.7). By Lemma 4.2, φ is decreasing on $[0, a/(1-\lambda)]$; hence $\varphi(x) \ge \varphi(a/(1-\lambda))$ and $\varphi(a/(1-\lambda)) \ge 0$ from assumption (b). Consequently $f' \ge 0$, and f is increasing on $(0, a/(1-\lambda))$. This yields (4.15) on E.

The inequalities (4.8), (4.14) and (4.15) verify condition (iii) of the verification lemma. Therefore it follows that $Q(x, y) \ge V(x, y)$ on E, where Q is given by (4.5) and V is the value function given by (1.3). But (X^*, Y^*) belongs to $\Sigma(x, y)$ and hence $Q(x, y) \le V(x, y)$ on E.

This enables us to conclude that $Q(x, y) = V(x, y)$ on E and hence the timid play is optimal. □

Finally, we consider the situation $\varphi(a/(1-\lambda)) < 0$, where φ is given by (4.7). In this case, our optimal strategy is more complicated than the timid play strategy.

First, notice that since φ is decreasing, $\varphi(0) > 0$ and $\varphi(a/(1-\lambda)) < 0$, there is a point c such that $0 < c < a/(1-\lambda)$ and $\varphi(c) = 0$. By integrating the differential equation $S''(x) + 2\rho(x)S'(x) = 0$ and using the form of φ in (4.7), it can be easily shown that the point $x = c$ is unique if ρ is strictly decreasing. Furthermore, if $\varphi(c_1) = \varphi(c_2) = 0$ and $c_1 < c_2$ then ρ is constant on $[c_1, c_2]$. In this case, we may choose any point c satisfying $\varphi(c) = 0$, and the pay-off probability from our candidate for an optimal strategy remains the same for any c between c_1 and c_2. We use this point c to describe our candidate for the optimal strategy.

In this strategy, the controller always uses $\mu_0(\cdot)$ and $\sigma_0(\cdot)$ given in (1.5), for the infinitesimal drift and diffusion coefficient respectively. If the initial data is (x, y) and if $0 \le x \le c$, then the processes L and U defined in (1.1) are chosen to be the local time processes of X process with the reflection barriers at the origin and at $x = c$, respectively. If $x > c$, then at time 0, make an initial jump for the (X, Y) process from (x, y) to $(c, y + (1-\lambda)(x-c))$ and then follow the strategy described for the case $0 \le x \le c$. Therefore, for simplicity, we consider the initial position to be $(c, y + (1-\lambda)(x-c))$ if $x > c$. Hence the candidate (X^*, Y^*) for our optimal strategy satisfies

$$\begin{aligned} dX^*(t) &= \mu_0(X^*(t))\,dt + \sigma_0(X^*(t))\,dW(t) + dL(t) - dU(t), \\ dY^*(t) &= -(1+\alpha)\,dL(t) + (1-\lambda)\,dU(t) \end{aligned} \tag{4.17}$$

with the initial condition

$$(X^*(0), Y^*(0)) = (x \wedge c,\ y + (1-\lambda)(x-c)^+), \tag{4.18}$$

and $W(t)$ is a one-dimensional Brownian motion, L and U are the local time processes of the reflecting diffusion process X^* at origin and $x = c$, respectively.

We compute the pay-off probability from the (X^*, Y^*) process in our next lemma.

LEMMA 4.4. *Let (X^*, Y^*) be the strategy defined in* (4.17) *and* (4.18). *Define $Q(x, y)$ by*

$$Q(x, y) = P\big[Z^* \text{ reaches } a \text{ before } 0 \,|\, (X^*(0), Y^*(0)) = (x, y)\big], \tag{4.19}$$

where Z^* represents the total wealth process and is given by $Z^*(t) = (1-\lambda)X^*(t) + Y^*(t)$, for all $t \ge 0$. Introduce $y_0 = a - (1-\lambda)c$. Then

$$Q(x, y) = \begin{cases} \dfrac{U(x, y)}{U(c, y_0)}, & \text{if } 0 \le x \le c,\ 0 \le y \le y_0 \text{ and } (x, y) \text{ in } E, \\[2ex] 1 - \dfrac{S(c)A(y_0)}{U(c, y_0)}\left(1 - \dfrac{S(x)}{S((a-y)/(1-\lambda))}\right)g(y), & \\ & \text{if } y \ge y_0 \text{ and } (x, y) \text{ in } E, \\[2ex] \dfrac{U(c, y + (1-\lambda)(x-c))}{U(c, y_0)}, & \text{if } x \ge c \text{ and } (x, y) \text{ in } E, \end{cases} \tag{4.20}$$

where $A(y) = \exp(-2\rho(c)/(1-\lambda)y)$, $g(y) = \exp((-1/(1+\alpha))\int_{y_0}^{y} 1/S((a-r)/(1-\lambda))\,dr)$, $U(x, y) = A(y)S(x) + 1/(1+\alpha)\int_0^y A(r)\,dr$ *and* S *is the scale function defined in* (4.1).

PROOF. First we consider $0 \le x \le c$, $0 \le y \le y_0$ and (x, y) in E. With this initial condition (x, y), notice that (X^*, Y^*) can reach the line $(1-\lambda)x + y = a$ only at (c, y_0). Hence we solve the differential equation

$$\begin{aligned}
&\frac{\partial^2 Q}{\partial x^2}(x, y) + 2\rho(x)\frac{\partial Q}{\partial x}(x, y) = 0 \quad \text{for } 0 < x < c,\ 0 < y < y_0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{and } (x, y) \text{ in } E; \\
&\frac{\partial Q}{\partial x}(0, y) - (1+\alpha)\frac{\partial Q}{\partial y}(0, y) = 0, \qquad 0 < y < y_0; \\
&(1-\lambda)\frac{\partial Q}{\partial x}(c, y) - \frac{\partial Q}{\partial y}(c, y) = 0, \qquad 0 < y < y_0; \\
&\qquad\qquad Q(c, y_0) = 1 \quad \text{and} \\
&\qquad\qquad Q(0, 0) = 0.
\end{aligned} \tag{4.21}$$

This gives the formula $Q(x, y) = U(x, y)/U(c, y_0)$ as in (4.20). To verify (4.19), we introduce the quitting time τ_0 by

$$\begin{aligned}
\tau_0 &= \inf\{t \ge 0\colon Z^*(t) \ge a \text{ or } (X^*(t), Y^*(t)) = (0, 0)\} \\
&= +\infty \quad \text{otherwise.}
\end{aligned} \tag{4.22}$$

Notice that τ_0 is finite a.s., since $\sigma_0(x) > 0$ for all x and the local time process L approaches infinity as t tends to infinity. Furthermore, $(X^*(\tau_0), Y^*(\tau_0))$ is equal to $(0, 0)$ or (c, y_0). To obtain (4.19), we apply Itô's rule to $Q(X^*(t \wedge \tau_0), Y^*(t \wedge \tau_0))$ and use (4.21) similarly to the proof of Lemma 4.1. We leave the details to the reader.

Next, we consider the case $y > y_0$ and (x, y) in E. Since (X^*, Y^*) has continuous sample paths for $t > 0$, introduce the stopping time τ_1 by

$$\begin{aligned}
\tau_1 &= \inf\{t \ge 0\colon (1-\lambda)X^*(t) + Y^*(t) = a \text{ or } Y^*(t) = y_0\}. \\
&= +\infty \quad \text{otherwise.}
\end{aligned} \tag{4.23}$$

Again, τ_1 is also finite since $\sigma_0(x) > 0$ for all x and $L(t)$ approaches infinity as t tends to infinity. Next we solve

$$\begin{aligned}
&\frac{\partial^2 Q}{\partial x^2}(x, y) + 2\rho(x)\frac{\partial Q}{\partial x}(x, y) = 0 \quad \text{if } y > y_0 \text{ and } (x, y) \text{ in } E; \\
&\frac{\partial Q}{\partial x}(0, y) - (1+\alpha)\frac{\partial Q}{\partial y}(0, y) = 0 \quad \text{if } y > y_0; \\
&\qquad\qquad Q(x, y) = 1 \quad \text{if } y > y_0 \text{ and } (1-\lambda)x + y = a; \\
&\qquad\qquad Q(x, y_0) = \frac{U(x, y_0)}{U(c, y_0)} \quad \text{if } 0 \le x \le c.
\end{aligned} \tag{4.24}$$

The solution to (4.24) gives the formula in (4.20) for $y \geq y_0$ and (x, y) in E. To verify (4.19), again we apply Itô's rule (3.1) to $Q(X(t\wedge\tau_0\wedge\tau_1), Y(t\wedge\tau_0\wedge\tau_1))$, as in Lemma 4.1. By letting t go to infinity, we obtain

$$E[Q(X^*(\tau_0 \wedge \tau_1),\ Y^*(\tau_0 \wedge \tau_1))|(X^*(0-), Y^*(0-)) = (x, y)] = Q(x, y). \tag{4.25}$$

Using the boundary conditions in (4.24), the left-hand side of (4.25) is equal to

$$E\big[I_{[\tau_0\leq\tau_1,\ Z^*(\tau_0)=a]} + I_{[\tau_0>\tau_1]}Q(X^*(\tau_1), Y^*(\tau_1))\big]. \tag{4.26}$$

However, on the set $[\tau_0 > \tau_1]$, $(X(\tau_1), Y(\tau_1)) = (0, y_0)$ and it follows that

$$E\big[I_{[\tau_0>\tau_1]}Q(X^*(\tau_1), Y^*(\tau_1))\big] = E\big[I_{[\tau_0>\tau_1]}Q(0, y_0)\big]. \tag{4.27}$$

Now $Q(0, y_0)$ is known from the previous case and (4.19) holds at $(0, y_0)$. Therefore, by invoking the strong Markov property at τ_1, we obtain

$$\begin{aligned} E\big[I_{\tau_0>\tau_1]}Q(0, y_0)\big] &= E\big[I_{[\tau_0>\tau_1]}E\big[I_{[Z^*(\tau_0)=a}|\mathscr{F}_{\tau_1}\big]\big] \\ &= E\big[I_{[\tau_0>\tau_1,\ Z^*(\tau_0)=a]}\big]. \end{aligned} \tag{4.28}$$

Combining (4.26), (4.27) and (4.28), we obtain

$$Q(x, y) = E\big[I_{[Z^*(\tau_0)=a]}\big] \text{ and hence (4.19) follows.}$$

If the initial position (x, y) in E is such that $x > c$, then at time 0, the (X^*, Y^*) process jumps to $(c, y + (1-\lambda)(x-c))$.

Hence

$$\begin{aligned} &P\big[Z^* \text{ reaches } a \text{ before } 0|(X^*(0-), Y^*(0-)) = (x, y)\big] \\ &\quad = P\big[Z^* \text{ reaches } a \text{ before } 0|(X^*(0), Y^*(0)) = (c, y + (1-\lambda)(x-c))\big] \\ &\quad = Q(c, y + (1-\lambda)(x-c)). \end{aligned}$$

This completes the proof of Lemma 4.4. □

Finally, we are ready to prove the optimality of the (X^*, Y^*) process introduced in Lemma 4.4, for the case $\varphi(a/(1-\lambda)) < 0$ where φ is given in (4.7).

THEOREM 4.5. *Assume that*:

(i) ρ, *defined in* (1.4), *is decreasing on* $[0, a/(1-\lambda)]$;
(ii) $\varphi(a/(1-\lambda)) < 0$ *where* φ *is given in* (4.7).

Then the strategy described in Lemma 4.4 *is optimal and the value function* $V(x, y)$ *is equal to* $Q(x, y)$ *given in* (4.20).

REMARK 4.6. Since we have a closed form expression for $Q(x, y)$ in (4.20), by direct computation, one can verify the continuity of $\partial Q/\partial x$, $\partial Q/\partial y$ and $\partial^2 Q/\partial x^2$ across the switching curve $x = c$. Condition $\varphi(c) = 0$ is essential in this verification. Therefore, the "principle of smooth fit" holds for this problem.

PROOF. Similar to the proof of Theorem 4.3, it is enough to verify the assumptions of the verification lemma. By (4.20), Q can be smoothly extended to an open set containing E. Also, assumptions (i) and (ii) of the verification lemma can be verified directly. It remains to verify only assumption (iii). Condition $\partial Q/\partial x \geq 0$ on E follows directly from (4.19).

Step 1. To show $(\partial^2 Q/\partial x^2)+2\rho(x)(\partial Q/\partial x) \leq 0$: if $x < c$, then since $S''(x)+2\rho(x)S'(x)=0$ and hence $(\partial^2 U/\partial x^2)+2\rho(x)(\partial U/\partial x)=0$ where U is given in (4.20), it easily follows that $(\partial^2 Q/\partial x^2)+2\rho(x)(\partial Q/\partial x)=0$ for $x<c$ and (x, y) in E. If $x>c$ then

$$Q(x, y) = \frac{U(c, y+(1-\lambda)(x-c))}{U(c, y_0)} \tag{4.29}$$

from (4.20) and $U(c, y_0) > 0$. Therefore it is enough to verify $(\partial^2 R/\partial x^2)+2\rho(x)(\partial R/\partial x) \leq 0$ for $x>c$, where

$$R(x, y) = U(c, y+(1-\lambda)(x-c)). \tag{4.30}$$

By a direct computation, it follows that

$$\frac{\partial^2 R}{\partial x^2}(x, y) + 2\rho(x)\frac{\partial R}{\partial x}(x, y) = 2S'(c)(\rho(x)-\rho(c))A(y+(1-\lambda)(x-c)).$$

Since ρ is decreasing, S' and A are positive, it follows that the right hand side is less than or equal to zero. Furthermore, $\partial Q/\partial x$ and $\partial^2 Q/\partial x^2$ are continuous across the line $x=c$ and hence $(\partial^2 Q/\partial x^2)+2\rho(x)(\partial Q/\partial x) \leq 0$ on E.

Step 2. To show $(\partial Q/\partial x)(x, y)-(1+\alpha)(\partial Q/\partial y)(x, y) \leq 0$ on E:

(a) If $0 \leq x < c$ and $0 \leq y \leq y_0$, by (4.20), and direct computation we get

$$\frac{\partial Q}{\partial x}(x, y) - (1+\alpha)\frac{\partial Q}{\partial y}(x, y) = \frac{A(y)}{U(c, y_0)}\left[S'(x)+2\left(\frac{1+\alpha}{1-\lambda}\right)\rho(c)S(x)-1\right].$$

Now introduce $H(x) = S'(x)+2((1+\alpha)/(1-\lambda))\rho(c)S(x)-1$ for $0 \leq x \leq c$. Clearly $H(0)=0$ and $H(c) = -((\lambda+\alpha)/(1-\lambda))S'(c) < 0$. Now first we assume ρ is differentiable, so $\rho'(x) \leq 0$ and $H''(x)+2\rho(x)H'(x) = -2\rho'(x)S'(x) \geq 0$. Hence by a maximum principle for ordinary differential equations, ([11], page 2) it follows that $H(x) \leq 0$ for $0 \leq x \leq c$. To remove the differentiability assumption on ρ, one can approximate ρ by a sequence of differentiable, decreasing functions (ρ_n) with $\rho_n'(x) \leq 0$ on $[0, c]$. Then $H_n(x)$ can be analogously defined and $H_n(0)=0$ and $H_n(c)<0$ for large n. Therefore, one can apply the maximum principle to $H_n(x)$ and let n tend to infinity to conclude $H(x) \leq 0$ on $[0, c]$.

Hence we have $(\partial Q/\partial x)(x, y)-(1+\alpha)(\partial Q/\partial y)(x, y) \leq 0$ if $0 \leq x < c$, and $0 \leq y \leq y_0$.

(b) If $x > c$ and (x, y) in E, then by (4.29) and (4.30), we have

$$\frac{\partial Q}{\partial x}(x, y) - (1+\alpha)\frac{\partial Q}{\partial y}(x, y) = -\frac{(\lambda+\alpha)}{U(c, y_0)}\frac{\partial U}{\partial y}(c, y+(1-\lambda)(x-c)) < 0.$$

(c) If $y > y_0$ and (x, y) in E, then by (4.20) and a direct computation we get

$$\begin{aligned}&\frac{\partial Q}{\partial x}(x, y) - (1+\alpha)\frac{\partial Q}{\partial y}(x, y)\\&\qquad = \frac{S(c)A(y_0)g(y)}{U(c, y_0)S((a-y)/(1-\lambda))}\\&\qquad\quad \times \left[\frac{S'(x)-1}{S(x)} - \frac{[((1+\alpha)/(1-\lambda))S'((a-y)/(1-\lambda))-1]}{S((a-y)/(1-\lambda))}\right],\end{aligned}$$

where $A(\cdot)$, $g(\cdot)$ and $U(\cdot,\cdot)$ are positive functions defined in (4.20). Then the right-hand side is less than or equal to zero as shown in (4.13). Since $\partial Q/\partial x$ and $\partial Q/\partial y$ are continuous across the line $x = c$, the inequality also holds on $x = c$.

Hence $(\partial Q/\partial x)(x, y) - (1+\alpha)(\partial Q/\partial y)(x, y) \le 0$ on E.

Step 3. To show $(1-\lambda)(\partial Q/\partial y)(x, y) - (\partial Q/\partial x)(x, y) \le 0$ on E:

(a) If $0 \le x \le c$ and $0 \le y \le y_0$, then $(1-\lambda)(\partial Q/\partial y)(x, y) - (\partial Q/\partial x)(x, y) = (A(y)/U(c, y_0))J(x)$ where $A(\cdot)$ and $U(\cdot,\cdot)$ are given in (4.20) and $J(x) = ((1-\lambda)/(1+\alpha)) - 2\rho(c)S(x) - S'(x)$ for $0 \le x \le c$. Now $J(0) = ((1-\lambda)/(1+\alpha)) - 1 < 0$, $J(c) = 0$ and $J'(x) = 2(\rho(x) - \rho(c))S'(x) \ge 0$. Hence $J(x) \le 0$ on $[0, c]$ and $(1-\lambda)(\partial Q/\partial y) - (\partial Q/\partial x) \le 0$ for $0 \le x \le c$ and $0 \le y \le y_0$.

(b) If $x > c$ then by (4.20), we see that $Q(x, y)$ is a function of $(1-\lambda)x + y$ and hence $(1-\lambda)(\partial Q/\partial y) - (\partial Q/\partial x) = 0$.

(c) If $y > y_0$, by direct computation one can observe that

$$(1-\lambda)\frac{\partial Q}{\partial y}(x, y) - \frac{\partial Q}{\partial x}(x, y) = \frac{S(c)A(y_0)g(y)}{U(c, y_0)S((a-y)/(1-\lambda))}\left[f(x) - f\left(\frac{a-y}{1-\lambda}\right)\right],$$

where $f(x) = [((1-\lambda)/(1+\alpha)) - S'(x)]/S(x)$ for $0 \le x \le (a-y)/(1-\lambda)$. Then as shown in the argument following (4.16), the function f is increasing. Hence $(1-\lambda)(\partial Q/\partial y)(x, y) - (\partial Q/\partial x)(x, y) \le 0$ if $y > y_0$ and (x, y) in E. Again, we use the continuity of $\partial Q/\partial x$ and $\partial Q/\partial y$ across the line $x = c$ to verify the inequality on the line $x = c$.

Now Steps 1, 2 and 3 together imply condition (iii) of the verification lemma. Hence we can conclude that $Q(x, y) \ge V(x, y)$ for all (x, y) in E. But (X^*, Y^*) is in $\Sigma(x, y)$, therefore $Q(x, y) \le V(x, y)$ on E. Consequently, (X^*, Y^*) is an optimal strategy and $Q(x, y) = V(x, y)$.

This completes the proof. □

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