The Annals of Applied Probability
1998, Vol. 8, No. 1, 246–280

DIRECTIONAL DECAY OF THE GREEN'S FUNCTION FOR A RANDOM NONNEGATIVE POTENTIAL ON $\mathbb{Z}^d$

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We derive a shape theorem type result for the almost sure exponential decay of the Green's function of $-\Delta + V$, where the potentials $V(x)$, $x \in \mathbb{Z}^d$, are i.i.d. nonnegative random variables. This result implies a large deviation principle governing the position of a d-dimensional random walk moving in the same potential.

0. Introduction and notation. We consider the following model of a random walk evolving in a random environment on $\mathbb{Z}^d$ $(d \geq 1)$. Let $(S_n)_{n\in\mathbb{N}_0}$ be a time discrete, symmetric, nearest-neighbor random walk on the hypercubic lattice $\mathbb{Z}^d$ with start in x and denote by P_x and E_x the probability measure and the expectation, respectively, of the underlying probability space.

The environment is assumed to be independent of the random walk and is given by the potentials $\omega(x)$, $x \in \mathbb{Z}^d$, which are supposed to be independent and identically distributed nonnegative random variables. Then for a fixed realization ω of the environment, the Green's function of $x, y \in \mathbb{Z}^d$ is defined as

$$g(x, y, \omega) = \sum_{n=0}^{\infty} E_x\left[\exp\left(-\sum_{m=0}^{n} \omega(S_m)\right)1_{\{S_n=y\}}\right]. \tag{1}$$

This is the Green's function of $-\Delta+V$ in the usual sense, if $\omega(z) = \ln(V(z)+1)$ for all $z \in \mathbb{Z}^d$ [see (6) below]. Here $\triangle f(x) = (\sum_{|e|=1} f(x+e))/(2d) - f(x)$ denotes the discrete Laplacian on $\mathbb{Z}^d$. We want to stress at this point that it is no loss of generality that we used a discrete time random walk instead of a continuous time one for the representation of the Green's function, as will be shown below.

We are mainly interested in almost sure decay rates of $g(x, y, \omega)$ when $|x - y|$ tends to infinity. Exponential decay properties of Green's functions have been studied before in the context of random Schrödinger operators and localization theory; see for instance, [2], Chapter IX.1, and [15], Chapter II. Our problem here has different motivations, since we investigate the directional exponential decay rates of the Green's function and work "below the spectrum" of $-\Delta + V$, as our potentials are nonnegative. We also stress the fact that, unlike localization type results, we need few assumptions on the distribution of our potentials.

Our first main result is the following theorem.

Received February 1997; revised July 1997.
AMS 1991 *subject classifications*. 60K35, 82D30.
Key words and phrases. Random walk, random potential, Green's function, asymptotic shape, first passage percolation, Lyapounov exponent, large deviations.

THEOREM A (Shape theorem). *Let $\omega = (\omega(x))_{x\in\mathbb{Z}^d}$ be a family of i.i.d. nonnegative random variables such that $\omega(x)$ is not concentrated in zero and has finite dth moment. Then there exists a nondegenerate finite norm $\alpha(\cdot)$ on $\mathbb{R}^d$ such that for almost all configurations ω,*

$$\lim_{n\to\infty} \frac{-\ln g(0, x_n, \omega)}{\alpha(x_n)} = 1$$

for all sequences $x_n \in \mathbb{Z}^d$ with $|x_n| \to \infty$.

Thus the constant $\alpha(x)$ governs the almost sure exponential decay of the Green's function in the direction x. Hence it can be viewed as a higher-dimensional analogue of the Lyapounov exponent in the one-dimensional situation (see [2]). This result is in some sense the discrete space counterpart of Sznitman's shape theorem for a Brownian motion evolving in a Poissonian potential [20]. However, as far as the proof is concerned, it is slightly closer to the shape theorem of first passage percolation (see, e.g., [1, 5, 9]). Here the random function $-\ln g(\cdot, \cdot, \omega)$ corresponds to the point-to-point travel times and the Lyapounov exponent $\alpha(x)$ to the time constant in direction x. The main difference between first passage percolation and our model is that in first passage percolation the random distance between two points is the infimum of the passage times of all paths connecting these points, whereas in our model a weighted average over all these paths is taken. We shall show that for large potentials, the Lyapounov exponent approximates the time constant of the corresponding first passage model (cf. [20], page 1657).

Besides Theorem A, we prove a new type of shape theorem which we call "uniform shape theorem." It generalizes Theorem A except in the case $d = 2$.

THEOREM B (Uniform shape theorem). *In addition to the hypotheses of Theorem* A, *assume that the support of $\omega(x)$ does not contain zero if $d = 2$. Then for almost all configurations ω,*

$$\lim_{n\to\infty} \frac{-\ln g(x_n, y_n, \omega)}{\alpha(x_n - y_n)} = 1$$

for all sequences $x_n, y_n \in \mathbb{Z}^d$ such that $c \max\{|x_n|, |y_n|\} \le |x_n - y_n| \to \infty$ as $n \to \infty$ for some $c > 0$.

The reason for the restriction that the distance between x_n and y_n must grow at least as fast as their distance from the origin is that in almost all configurations ω there is somewhere far from the origin an arbitrary large region in which the potentials are rather untypical, for example, very large. If x_n and y_n were located in such a region then the random walk on its way from x_n to y_n would mainly see this untypical environment and thus make $g(x_n, y_n, \omega)$ deviate from its usual behavior. The condition $c \max\{|x_n|, |y_n|\} \le |x_n - y_n|$ ensures that this cannot happen.

The idea of the proof of Theorem B is also applicable, for example, to first passage percolation. It is based on an upper bound for the variance of

$-\ln g(x, y, \omega)$. An equivalent estimate of this kind is known to hold also for first passage percolation [10].

THEOREM C (Upper bound on fluctuations). *Under the assumptions of Theorem B there is a constant C just depending on the distribution of $\omega(x)$ and the dimension d such that*

$$\mathrm{Var}(-\ln g(x, y, \omega)) \leq C|x - y| \quad \textit{for all } x, y \in \mathbb{Z}^d.$$

Theorem B enables us to study the large deviations of S_n/n, where the random walk is governed by the random path measures

$$Q_{n,x,\omega} = \frac{1}{Z_{n,x,\omega}} \exp\Big(-\sum_{m=0}^{n-1} \omega(S_m)\Big) P_{nx}$$

for $x \in \mathbb{R}^d$ and typical ω. Here $Z_{n,x,\omega}$ is the normalizing constant and the random walk is supposed to start at some neighboring lattice site $[nx]$ of nx if $nx \notin \mathbb{Z}^d$, that is, $P_{nx} := P_{[nx]}$.

THEOREM D (Large deviation principle). *Under the assumptions of Theorem A for almost all configurations ω,*

$$-\inf_{y \in A^o} I(y - x) \leq \overline{\underline{\lim_{n\to\infty}}} \frac{1}{n} \ln Q_{n,x,\omega}[S_n \in nA] \leq -\inf_{y \in \bar{A}} I(y - x)$$

for any $x \in \mathbb{R}^d$ and any Borelian subset A of $\mathbb{R}^d$. Here the rate function I is given by

$$I(x) := \sup_{\lambda \geq 0} (\alpha_{\lambda - \underline{\nu}}(x) - \lambda),$$

where α_λ denotes the norm according to Theorem A that belongs to the potentials $\omega + \lambda$ and $\underline{\nu}$ is the minimum of the support of $\omega(x)$.

In the case $d = 1$ this is reminiscent of ideas of Gärtner and Freidlin (e.g., [6]). Sznitman considered the corresponding problem for a Brownian motion starting at $x = 0$ in a Poissonian potential [20], Theorem 2.1. Due to Theorem B, we are able to simplify Sznitman's method and extend it to all $x \in \mathbb{R}^d$.

Theorem D can be used to compute the Lyapounov exponents if the potentials are equal to a nonrandom positive constant λ. In this case the Green's function is essentially the generating function of the site occupancy probabilities, a comprehensive account of which is given in [7], Chapters 3.2, 3.3, A.3. However, to the best of our knowledge, the exponential decay rate $\alpha(x, \delta_\lambda)$ of $g(0, kx, \lambda)$ for $k \to \infty$ has not yet been determined explicitly in the literature except for directions x along the coordinate axes; see [14], Theorem A.2. We give a fairly explicit expression for $\alpha(x)$ for arbitrary directions, which also shows that at least in this case $\alpha(x)$ is analytic in $x \neq 0$.

We have already mentioned the relations of this model to first passage percolation. In order to give some further applications and to clarify the meaning of the Green's function, we briefly describe two models that give different interpretations of this quantity.

Random walk with random killing. Fix $\omega \in \Omega$ and consider a Markov chain with state space $\mathbb{Z}^d \cup \{\dagger\}$ and transition probability

$$p_\omega(x, y) := \begin{cases} \exp(-\omega(x))/2d, & \text{if } |x - y| = 1, \\ 1 - \exp(-\omega(x)), & \text{if } x \neq \dagger = y, \\ 1, & \text{if } x = y = \dagger, \\ 0, & \text{otherwise.} \end{cases}$$

It can be shown that $g(x, y, \omega)$ is the expected total number of visits to y provided the Markov chain starts in x (cf. [7], Section 3.2.4, and [12], page 34, for constant nonrandom ω).

Random electric network. This interpretation is in the spirit of [4]. Turn the lattice $\mathbb{Z}^d$ into an electrical network by replacing each edge by a resistor of size $2d$. Furthermore, fix $\omega \in \Omega$ and ground each lattice site via a resistance of size $(\exp(\omega(x)) - 1)^{-1}$. Now inject a unit current at site y and denote by $U(x)$ the resulting voltage at site x. Then $U(x) = g(x, y, \omega)$ (cf. [4], pages 47, 52).

Let us now describe how the present article is organized. In Section 1 we introduce two additional two-point functions (Definitions 1, 2) that have the same asymptotic behavior as the Green's function. Furthermore we state some basic properties of these functions (Proposition 2) and investigate the influence of the underlying continuous or discrete time structure on these quantities (Proposition 1). In Section 2 we first use subadditivity to prove the existence of the Lyapounov exponents for each direction (Proposition 4) and then derive the maximal lemma (Lemma 7) that enables us to patch these limits and get the desired shape theorem (Theorems A and 8). Section 3 describes in detail the relations between first passage percolation and our model. In Section 4 we use a rank-one perturbation formula (Lemma 12) and a martingale method to prove the upper bound on the fluctuations of our two-point functions (Theorems C and 11). This is our main tool for the proof of the uniform shape theorem in Section 5 (Theorems B and 13). We also give a version of this result, which is phrased in terms of asymptotic shapes and thus really deserves the name "shape theorem" (Theorem 15). Furthermore, we show that this result also holds for point-to-set distances, which are more general than point-to-point functions (Corollary 16). Section 6 contains the promised large deviation estimates (Theorems D and 19). This result is exploited in Section 7, in which we compute the Lyapounov exponents for the unperturbed case of constant potentials (Theorem 21).

We finally introduce further notations and conventions. We shall assume throughout the whole paper that the potentials $\omega(x)$ are nonnegative i.i.d. random variables whose common distribution will be called ν; that is, ν is a probability measure supported on the nonnegative real half line. To avoid trivialities we furthermore suppose that ν is not concentrated in zero, that is, $\nu \neq \delta_0$. Hence the canonical probability space that governs the environment is

$\Omega = [0, \infty[^{\mathbb{Z}^d}$ endowed with the usual product sigma algebra and the product measure $\mathbb{P} = \bigotimes_{x\in\mathbb{Z}^d} \nu$ as probability measure. Consequently, the random variables $\omega(x)\colon \Omega \to [0, \infty[$ may be chosen as the projections. $\mathbb{E}$ is the expectation with respect to $\mathbb{P}$.

We close this section with some general notation. By $|x|$ for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we always mean the ℓ_1-norm of x, that is, $|x_1| + \ldots + |x_d|$. We shall not use the Euclidean norm on $\mathbb{R}^d$. The $|\cdot|$-unit sphere will be called S^{d-1}. The L^2-norm for random variables is designated by $\|\cdot\|_2$. By $[x]$ we denote the lattice site with minimal $|\cdot|$-distance from x with some deterministic rule for breaking ties. Note that always $|[x] - x| \le d/2$. If we apply a function $f(x, y, \ldots)$ which has originally been defined solely for $x, y, \ldots \in \mathbb{Z}^d$ to $x, y, \ldots \in \mathbb{R}^d$, we always mean $f([x], [y], \ldots)$. The canonical unit vectors of $\mathbb{R}^d$ are $e_1, \ldots, e_d$. The cardinality of a set M is designated by $\sharp M$. The largest integer less than or equal to u is $\lfloor u \rfloor$ and $u \vee w$ is the maximum of u and w. To simplify the notation of expectations, we use the abbreviation $E[Z, A] := E[Z \cdot 1_A]$. We use $C, c, c', c_1, c_2, \ldots$ to denote arbitrary positive constants, which may change from line to line and depend only on dimension d and the underlying measure ν. If a constant is to depend on some other quantity, this will be made explicit.

1. Further two-point functions and time structures. In this section we first introduce in addition to the Green's function three further random two-point functions that will play an important role throughout the whole paper.

DEFINITION 1. For any $\omega \in \Omega$ and $x, y \in \mathbb{Z}^d$, define $g(x, y, \omega)$ as done in (1). Let $0 \le H(y) := \inf\{n \ge 0\colon S_n = y\} \le \infty$ be the first passage time through y, that is, the first time at which the random walk visits y. Then define

$$e(x, y, \omega) := E_x\left[\exp\left(-\sum_{n=0}^{H(y)-1} \omega(S_n)\right),\ H(y) < \infty\right] \in]0, 1], \tag{2}$$

$$a(x, y, \omega) := -\ln e(x, y, \omega) \in [0, \infty[,$$

$$d(x, y, \omega) := \max\{a(x, y, \omega), a(y, x, \omega)\} \in [0, \infty[.$$

In the model of a random walk with random killing in the introduction, $e(x, y, \omega)$ equals the probability that the particle with start in x will return to y before it reaches †. In the random electric network model described thereafter, it equals $U(x)$ if we apply a unit voltage between y and the ground.

Before we list some basic properties of these functions, we give the corresponding definitions for a wide class of continuous time models. To this end let $(S_t)_{t\ge 0}$ be a symmetric, nearest-neighbor random walk on $\mathbb{Z}^d$ with right-continuous paths starting at $x \in \mathbb{Z}^d$ for which the times τ_i, $i \in \mathbb{N}$ between successive steps are independent, identically distributed random variables. The common distribution of the τ_i is denoted by ψ, for which we assume

$\psi(]0,\infty[) = 1$. Probabilities and expected values for this random walk are designated by $P_{x,\psi}$ and $E_{x,\psi}$.

DEFINITION 2. For any $V \in \Omega$ and $x, y \in \mathbb{Z}^d$ and any distribution ψ on $\mathbb{R}$ with $\psi(]0,\infty[) = 1$ define

$$g^*(x, y, V, \psi) := \int_0^\infty E_{x,\psi}\left[\exp\left(-\int_0^t V(S_u)\,du\right),\ S_t = y\right] dt,$$

$$e^*(x, y, V, \psi) := E_{x,\psi}\left[\exp\left(-\int_0^{H(y)} V(S_t)\,dt\right),\ H(y) < \infty\right].$$

Furthermore, define $a^*(x, y, V, \psi)$ and $d^*(x, y, V, \psi)$ analogously to Definition 1.

The relation between the discrete and the continuous time functions is as follows.

PROPOSITION 1. *Let ψ be a probability measure on $\mathbb{R}$ with $\psi(]0,\infty[) = 1$. Furthermore, let $\omega, V \in \Omega$ with $\omega(z) = -\ln \int \exp(-V(z)t)\,d\psi(t)$ for all $z \in \mathbb{Z}^d$. Then for all $x, y, \in \mathbb{Z}^d$,*

$$f^*(x, y, V, \psi) = f(x, y, \omega) \quad \text{for all } f \in \{a, d, e\}, \tag{3}$$

$$g^*(x, y, V, \psi) = \begin{cases} (\exp(\omega(y)) - 1)V(y)^{-1} g(x, y, \omega), & \text{if } V(y) \neq 0, \\ \int t\,d\psi(t) g(x, y, \omega), & \text{if } V(y) = 0. \end{cases} \tag{4}$$

For constant V, see [7], Chapters 5.2.1, 5.2.2.

PROOF. For the proof of (3) it suffices to consider the case $f = e$. For $f \in \{a, d\}$ the statement then follows from the definition of these quantities. Assume $x \neq y$ and let $\pi = (x = x_0, x_1, \dots, x_k = y)$ be a nearest-neighbor path from x to y with $x_n \neq y$ for $n < k$. Then

$$\begin{aligned} E_{x,\psi}&\left[\exp\left(-\int_0^{H(y)} V(S_t)\,dt\right),\ S_t \text{ follows } \pi\right] \\ &= E_{x,\psi}\left[\prod_{n=0}^{k-1} \exp(-V(x_n)\tau_n),\ S_t \text{ follows } \pi\right] \\ &= P_{x,\psi}[S_t \text{ follows } \pi] \prod_{n=0}^{k-1} E_{x,\psi}[\exp(-V(x_n)\tau_n)] \\ &= P_x[S_n \text{ follows } \pi] \prod_{n=0}^{k-1} \exp(-\omega(x_n)) \\ &= E_x\left[\prod_{n=0}^{k-1} \exp(-\omega(x_n)),\ S_n \text{ follows } \pi\right] \\ &= E_x\left[\exp\left(-\sum_{n=0}^{H(y)-1} \omega(S_n)\right),\ S_n \text{ follows } \pi\right]. \end{aligned}$$

Summing over all possible paths π yields the assertion. For the proof of (4) denote by H_n the time of the nth visit of y and by L_n the first time after H_n at which y is left. Then

$$\begin{aligned}
g^*(x, y, V, \psi) &= \int_0^\infty \sum_{n=1}^\infty E_{x,\psi}\left[\exp\left(-\int_0^t V(S_u)\,du\right), H_n(y) \le t < L_n(y)\right] dt \\
&= \sum_{n=1}^\infty E_{x,\psi}\Bigg[\exp\left(-\int_0^{H_n(y)} V(S_u)\,du\right) \\
&\qquad\qquad \times \int_0^\infty \exp\left(-\int_{H_n(y)}^t V(S_u)\,du\right) dt,\ H_n(y) \le t < L_n(y)\Bigg] \\
&= \sum_{n=1}^\infty E_{x,\psi}\Bigg[\exp\left(-\int_0^{H_n(y)} V(S_u)\,du\right) \\
&\qquad\qquad \times \int_{H_n(y)}^{L_n(y)} \exp(-(t - H_n(y))V(y))\,dt,\ H_n(y) < \infty\Bigg] \\
&= E_{x,\psi}\left[\int_0^{\tau_1} \exp(-tV(y))\,dt\right] \\
&\qquad \times \sum_{n=1}^\infty E_{x,\psi}\left[\exp\left(-\int_0^{H_n(y)} V(S_u)\,du\right),\ H_n(y) < \infty\right].
\end{aligned}$$

An argument similar to that in the proof of (3) and a simple calculation show that the second factor in the line above equals, when divided by $\exp(\omega(y))$,

$$\sum_{n=1}^\infty E_x\left[\exp\left(-\sum_{m=0}^{H_n(y)} \omega(S_m)\right),\ H_n(y) < \infty\right] = g(x, y, \omega), \tag{5}$$

whereas the first factor, when multiplied by $\exp(\omega(y))$, is the prefactor in (4). □

The canonical example for ψ is the exponential distribution with mean, say m. In this case (3) and (4) hold if ω and V are related by the one-to-one transformation $\omega = \ln(mV + 1)$. Then (4) can be reduced to $g^*(x, y, V, \psi) = m\ g(x, y, \omega)$.

Thus the question of which time structure should govern the random walk is irrelevant for the study of the functions $f(x, y, V)$ with $f \in \{a, d, e\}$. At least in the important case of exponential distributed waiting times, the same holds also for $f = g$. We have a slight preference for the discrete time model, since it seems to make some details less technical and its relation to first passage percolation is more obvious. Thus we will restrict ourselves to the discrete time model.

Some fundamental properties of the two-point functions introduced in Definition 1 are listed in Proposition 2.

PROPOSITION 2. *Fix $y \in \mathbb{Z}^d$ and $\omega, V \in \Omega$ with $\omega = \ln(V + 1)$. Then the functions $g_{y,\omega} := g(\cdot, y, \omega)$ and $e_{y,\omega} := e(\cdot, y, \omega)$ satisfy the relations*

$$(-\Delta + V)g_{y,\omega}(x) = \delta_{x,y} \tag{6}$$

and

$$(-\Delta + V)e_{y,\,\omega}(x) = 0 \quad \text{if } x \neq y, \tag{7}$$

$$e_{y,\,\omega}(y) = 1. \tag{8}$$

Furthermore, for any $x, y, z \in \mathbb{Z}^d$,

$$g(x, y, \omega) = g(y, x, \omega), \tag{9}$$

$$g(x, y, \omega) = e(x, y, \omega)g(y, y, \omega) = e(y, x, \omega)g(x, x, \omega), \tag{10}$$

$$e(x, z, \omega) \geq e(x, y, \omega)e(y, z, \omega), \tag{11}$$

$$a(x, z, \omega) \leq a(x, y, \omega) + a(y, z, \omega). \tag{12}$$

If $d \geq 3$ *or* $\omega(z) > 0$ *for some* $z \in \mathbb{Z}^d$ *then* $d(\cdot, \cdot, \omega)$ *is a metric on* $\mathbb{Z}^d$.

PROOF. It is well known that the function $g^*(\cdot, y, V, \psi)$ solves (6), if ψ is the exponential distribution with mean one. Hence (6) holds due to Proposition 1. However (6) can also be derived directly by partition over the first step; (7) follows from (6) and (10) while (8) is immediate from (2). Then (9) holds since each path from x to y contributing to $g(x, y, \omega)$ is also a possible path for the opposite direction from y to x. For the proof of (10) recall (5) and apply the strong Markov property to $H(y)$. The same property also implies (11). The triangle inequality (12) is a simple consequence of (11). By definition, $d(x, y, \omega) \geq 0$ is symmetric in x and y and fulfills the triangle inequality as well. Furthermore for $x \neq y$, $d(x, y, \omega) > 0$ if $d \geq 3$ or $\omega \not\equiv 0$ since transience or $\omega(z) > 0$ imply $e(x, y, \omega), e(y, x, \omega) < 1$. □

2. Lyapounov exponents and shape theorem. We start with some simple but useful estimates on $\mathbb{E}[a(0, x)]$ and on the size of certain lattice animals. So let us introduce for $\omega \in \Omega$ and $x, y \in \mathbb{Z}^d$ the path measure

$$\hat{P}^y_{x,\,\omega} = e(x, y, \omega)^{-1} 1_{\{H(y)<\infty\}} \exp\Bigg(-\sum_{m=0}^{H(y)-1} \omega(S_m)\Bigg) P_x$$

under which the process $(S_m)_{m\geq 0}$ is, roughly speaking, a random walk starting at x, conditioned to reach y and being partially killed by the potential ω. The expectation with respect to $\hat{P}^y_{x,\,\omega}$ is denoted by $\hat{E}^y_{x,\,\omega}$. We shall derive an estimate on the total expected number of sites this process visits before it reaches y. To this end we attach to each trajectory $(S_m)_{m\geq 0}$ with start in x the lattice animal (cf. [21])

$$\mathscr{A}(x, y, (S_m)_{m\geq 0}) = \{z \in \mathbb{Z}^d \colon H(z) < H(y)\}$$

consisting of all sites which are visited before y is reached.

LEMMA 3. *Suppose that* ν *has finite expectation. Then for any* $x \in \mathbb{Z}^d$,

$$c_1|x| \leq c_1\mathbb{E}\big[\hat{E}^x_{0,\,\omega}[\sharp\mathscr{A}(0, x)]\big] \leq \mathbb{E}[a(0, x)] \leq c_2|x|,$$

where

$$c_1 = -\ln \mathbb{E}[\exp(-\omega(0))] \quad \textit{and} \quad c_2 = \ln(2d) + \mathbb{E}[\omega(0)].$$

PROOF. The first inequality is obvious since any lattice animal $\mathscr{A}(0, x)$ connecting the origin to x contains at least $|x|$ vertices. The second inequality is obtained by Jensen's inequality and independence as follows:

$$\begin{aligned} c_1\mathbb{E}\big[\hat{E}^x_{0,\omega}[\sharp\mathscr{A}(0,x)]\big] &\\ &\le \mathbb{E}\big[\ln \hat{E}^x_{0,\omega}[\exp(c_1\sharp\mathscr{A}(0,x))]\big] \\ &\le \mathbb{E}\bigg[\alpha(0,x) + \ln E_0\bigg[\exp\bigg(c_1\sharp\mathscr{A}(0,x) - \sum_{s\in\mathscr{A}(0,x)}\omega(s)\bigg),\ H(x)<\infty\bigg]\bigg] \\ &\le \mathbb{E}[\alpha(0,x)] + \ln E_0\bigg[\prod_{s\in\mathscr{A}(0,x)}\mathbb{E}[\exp(c_1-\omega(s))]\bigg] = \mathbb{E}[\alpha(0,x)]. \end{aligned}$$

For the last inequality we choose some path of length $|x|$ from the origin to x and iterate the triangle inequality (12), thus getting, from the translation invariance and symmetry of $\mathbb{P}$,

$$\mathbb{E}[\alpha(0,x)] \le |x|\,\mathbb{E}[\alpha(0,e_1)] \le |x|\,\mathbb{E}\big[-\ln E_0[\exp(-\omega(0)),\ H(e_1)=1]\big] = c_2|x|. \qquad \square$$

We are now ready to prove the existence and some basic properties of the Lyapounov exponents α.

PROPOSITION 4. *Suppose that ν has finite expectation. Then there exist a norm $\alpha(\cdot) = \alpha(\cdot, \nu, d)$ on $\mathbb{R}^d$ such that on a set Ω_0 of full $\mathbb{P}$-measure and in $L^1(\mathbb{P})$ for all $x \in \mathbb{Z}^d$,*

$$(13) \qquad \lim_{n\to\infty}\frac{1}{n}\alpha(0,nx,\omega) = \lim_{n\to\infty}\frac{1}{n}\mathbb{E}[\alpha(0,nx)] = \inf_{n\in\mathbb{N}}\frac{1}{n}\mathbb{E}[\alpha(0,nx)] = \alpha(x);$$

$\alpha(x)$ is invariant under permutations of the coordinates and under reflections in the coordinate hyperplanes and satisfies

$$(14) \qquad -\ln\mathbb{E}[\exp(-\omega(0))] \le \frac{\alpha(x)}{|x|} \le \ln(2d) + \mathbb{E}[\omega(0)]$$

for all $x \in \mathbb{R}^d\backslash\{0\}$. Moreover, for fixed x and d, the functional $\alpha(x,\cdot,d)$ is concave in the sense that for arbitrary random variables $\tau_i \ge 0$ and real numbers $\lambda_i \ge 0$ $(i = 1,\dots,n)$ with $\lambda_1 + \cdots + \lambda_n = 1$,

$$(15) \qquad \alpha\big(\mathrm{law}\big(\lambda_1\tau_1 + \cdots + \lambda_n\tau_n\big)\big) \ge \sum_{i=1}^n \lambda_i\alpha(\mathrm{law}(\tau_i)).$$

Furthermore, $\alpha(\nu_1) \le \alpha(\nu_2)$ if ν_1 is more variable than ν_2, that is, if $\int h\,d\nu_1 \le \int h\,d\nu_2$ for all increasing, concave h: $\mathbb{R} \to \mathbb{R}$.

REMARKS. (i) The partial order "ν_1 is more variable than ν_2" has been discussed, for example, in [18], Section 8.5, and [19], Section 1.4. It has been used in the context of first passage percolation by van den Berg and Kesten [23].

(ii) It is easy to see from (14) that for $d \geq 2$, α is at least in general not rotationally invariant. Indeed, if c is chosen such that $c(\sqrt{2}-1) > \ln 2d$ then for $\nu = \delta_c$,

$$\alpha(e_1) \leq c + \ln 2d < \sqrt{2}c \leq \alpha\big((e_1+e_2)/\sqrt{2}\big),$$

although e_1 and $(e_1+e_2)/\sqrt{2}$ both have Euclidean norm one. Thus the unit sphere of α is in general not a ball. An explicit expression for α in the case $\nu = \delta_c$ is given in the last section.

PROOF OF PROPOSITION 4. We may assume $x \neq 0$ and consider the process $a(nx, mx)$, $0 \leq n \leq m$, $n, m \in \mathbb{N}_0$. These random variables are integrable due to Lemma 3. Therefore and by ergodicity properties and translation invariance of $\mathbb{P}$ the assumptions of the subadditive ergodic theorem (see [13], page 277) are fulfilled. Consequently there is some constant $\alpha(x)$ such that (13) is satisfied $\mathbb{P}$-a.s. and in $L^1(\mathbb{P})$. This together with Lemma 3 implies (14). Furthermore, it is easy to conclude from (13) that

$$\begin{aligned} &\alpha(qx) = q\alpha(x), \\ &\alpha(x+y) \leq \alpha(x) + \alpha(y), \qquad (16) \\ &\alpha(\pi(x)) = \alpha(x) \end{aligned}$$

are satisfied for any $q \in \mathbb{N}$, $x, y \in \mathbb{Z}^d$ and any composition π of permutations and reflections of the coordinates. Here we used (12) and the fact that $a(0, \pi(x), \omega)$ and $a(0, x, \omega)$ have the same law under $\mathbb{P}$.

By setting $\alpha(x/q) := \alpha(x)/q$, we extend α well-defined at first to a function on $\mathbb{Q}^d$ and then by continuity to a function on $\mathbb{R}^d$, which satisfies (14) and (16) for any $q > 0$ and $x, y \in \mathbb{R}^d$. In particular, α is a finite norm, which is nondegenerate since $\nu \neq \delta_0$.

For the proof of (15) we use Hölder's inequality to get

$$a(x, y, \lambda_1\omega_1 + \cdots + \lambda_n\omega_n) \geq \sum_{i=1}^{n} \lambda_i a(x, y, \omega_i)$$

for all $x, y \in \mathbb{Z}^d$ and $\omega_i \in \Omega$, which implies (15).

The proof of the monotonicity of $\alpha(\nu)$ goes along the same lines as the proof of [23], Theorem 2.9.a. Indeed, define for $N \in \mathbb{N}$

$$a_N(x, y, \omega) := -\ln E_x\left[\exp\left(-\sum_{m=0}^{H(y)-1} \omega(S_m)\right), \ |S_m| \leq N \text{ for all } m < H(y)\right].$$

Then a_N is a concave increasing function of $\omega(z)$, $|z| \leq N$. Hence by [18], Propositon 8.5.4 or [19], (1.10.5),

$$\mathbb{E}_{\nu_1}[a_N(x, y, \omega)] \leq \mathbb{E}_{\nu_2}[a_N(x, y, \omega)],$$

where $\mathbb{E}_{\nu_i}$ denotes the expectation with respect to the product measure $\mathbb{P}_{\nu_i}$ that belongs to ν_i. It follows by letting $N \to \infty$ that

$$\mathbb{E}_{\nu_1}[a(x, y, \omega)] \le \mathbb{E}_{\nu_2}[a(x, y, \omega)],$$

which together with (13) implies $\alpha(\nu_1) \le \alpha(\nu_2)$. □

For the main result, Theorem 8 of this section, we need essentially two lemmas. The first one gives some upper bound estimates on $g(x, x)$. This quantity differs in general from 1 unlike $a(x, x)$ and $d(x, x)$. In dimension $d \ge 3$ there are easier bounds due to transience of the walk.

LEMMA 5. *The random variable* $\omega(0) + \ln g(0, 0, \omega)$ *is* $\mathbb{P}$-*a.s. nonnegative and all its moments are finite. Furthermore, for some* $C > 0$ $\mathbb{P}$-*a.s.*,

$$\limsup_{|x|\to\infty} \frac{\omega(x) + \ln g(x, x, \omega)}{(\ln |x|)^{1/d}} < C. \tag{17}$$

PROOF. The nonnegativity is obvious from $g(0, 0, \omega) \ge \exp(-\omega(0))$. By the strong Markov property, $\exp(\omega(0))g(0, 0, \omega)$ can be represented as a geometric series such that

$$e^{\omega(0)} g(0, 0, \omega) = \left(1 - E_0\left[\exp\left(-\sum_{m=0}^{H_2(0)-1} \omega(S_m)\right),\ H_2(0) < \infty\right]\right)^{-1}, \tag{18}$$

where $H_2(y)$ denotes the time of the second visit of y. Now let $\varepsilon > 0$ be such that $\nu([\varepsilon, \infty)) > \varepsilon$ and let $z(\omega)$ be some site with minimal norm such that $\omega(z(\omega)) > \varepsilon$. Then the right side of (18) is smaller than $(2d)^{2|z(\omega)|}(1 - e^{-\varepsilon})^{-1}$. Hence for $t \ge c_1 := -\ln(1 - \exp(-\varepsilon)) > 0$,

$$\mathbb{P}\big[\omega(0) + \ln g(0, 0, \omega) \ge t\big] \le \mathbb{P}\left[|z(\omega)| > \frac{t - c_1}{2 \ln 2d}\right] \le (1 - \varepsilon)^{c_2(t-c_1)^d},$$

which decays geometrically fast as $t \to \infty$. Thus all moments of finite order exist. Furthermore this shows that c_3 may be chosen large enough, such that $\sum_{x \ne 0} \mathbb{P}[\omega(x) + \ln g(x, x) > c_3(\ln |x|)^{1/d}]$ is finite. Now (17) follows from the Borel–Cantelli lemma. □

This lemma enables us to show that our two-point functions a, d and $-\ln g$ are of the same order in the following sense.

COROLLARY 6. *Suppose that the mth moment of* ν *is finite for some* $m \ge 1$. *Then on a set* Ω_1 *of full* $\mathbb{P}$-*measure and in* $L^1(\mathbb{P})$,

$$\lim_{c(|x|\vee|y|)\le|x-y|\to\infty} \frac{\operatorname{diam}\{a(x, y), d(x, y), -\ln g(x, y)\}}{|x - y|^{d/m}} = 0 \quad \textit{for all } c > 0,$$

where diam M *denotes the diameter of the set* M.

PROOF. Using (10) we get

$$
\begin{aligned}
|a(x, y, \omega) + \ln g(x, y, \omega)| &= |\ln g(y, y, \omega)| \\
&\le \omega(y) + (\omega(y) + \ln g(y, y, \omega)).
\end{aligned} \tag{19}
$$

The right-hand side of (19) tends, when divided by $|x-y|^{d/m}$, to zero $\mathbb{P}$-a.s. This is true for $\omega(y)$ due to the existence of the mth moment and follows for $\omega(y)+\ln g(y, y, \omega)$ from Lemma 5. Moreover, the distribution of the right-hand side of (19) has finite mean due to $m \ge 1$ and Lemma 5 and does not depend on y. Hence convergence takes also place in $L^1(\mathbb{P})$. An equivalent statement holds for $|a(y, x) + \ln g(x, y)|$ and consequently for $|d(x, y) + \ln g(x, y)|$ as well. □

The second lemma plays the role of the maximal lemmas used in the context of first passage percolation (e.g. [9], Lemma (3.5), (3.6), [1]) or Brownian motion in a Poissonian potential ([20], Lemma 1.3).

LEMMA 7. *Suppose $d \ge 2$. If the second moment of ν is finite then there are some constants c_1, c_2 such that for any $x, y \in \mathbb{Z}^d$ and any $t > 0$,*

$$
\mathbb{P}[a(x, y, \omega) - \omega(x) > t] \le \frac{c_1|y-x|^{2d}}{(t - c_2|y-x|)_+^{4d}} \quad (\le \infty). \tag{20}
$$

Suppose $d \ge 1$. If the dth moment of ν exists then there are a constant c_3 and a set Ω_2 of full $\mathbb{P}$-measure such that for all $\omega \in \Omega_2$,

$$
\begin{aligned}
&\sup\{d(x, y, \omega)\colon y \in \mathbb{R}^d, |y-x| < \varepsilon|x|\} < c_3\varepsilon|x| \\
&\qquad \textit{for any } \varepsilon \in \mathbb{Q} \cap (0, \infty) \textit{ and for sufficiently large } x \in \mathbb{R}^d.
\end{aligned} \tag{21}
$$

PROOF. Let $x, y \in \mathbb{Z}^d$ and assume $x \ne y$. Observe that (due to arguments similar to those in [9], page 135), in dimension $d \ge 2$ there exist $2d$ self-avoiding nearest-neighbor paths $r_i = (r_{i,n})_{n=0}^{m_i}$ $(i = 1, \dots, 2d)$ from x to y, each containing $m_i \le |y-x| + 8$ edges and being pairwise site disjoint except for the starting and the end point. Now for any $i = 1, \dots, 2d$,

$$
\begin{aligned}
a(x, y, \omega) - \omega(x) &\le -\ln E_x\left[\exp\left(-\sum_{n=1}^{H(y)-1} \omega(S_n)\right),\ (S_n)_{n=0}^{m_i} = r_i\right] \\
&= m_i \ln 2d + \sum_{n=1}^{m_i-1} \omega(r_{i,n}).
\end{aligned}
$$

Since we get the corresponding assertion for $a(y, x, \omega) - \omega(y)$ by reversing the direction of the paths, we have for all $t > 0$ by pairwise disjointness of the paths and independence,

$$
\mathbb{P}[\bar{d}(x, y) > t] \le \mathbb{P}\left[\sum_{n=1}^{|y-x|+8} (\omega(ne_1) - \mathbb{E}[\omega(0)]) > t - c(|y-x|+8)\right]^{2d},
$$

where $\bar{d}(x, y, \omega) := \max\{a(x, y, \omega) - \omega(x), a(y, x, \omega) - \omega(y)\}$ and $c = \ln 2d + \mathbb{E}[\omega(0)]$. Using Chebyshev's inequality we get

$$\mathbb{P}[\bar{d}(x, y) > t] \leq \left(\frac{(|y-x|+8)\mathrm{Var}\,[\omega(0)]}{(t - c(|y-x|+8))_+^2}\right)^{2d} \leq \frac{c_1|y-x|^{2d}}{(t - c_2|y-x|)_+^{4d}} \tag{22}$$

and therefore (20).

Since (20) only holds for $d \geq 2$, we have to divide the proof of (21) into two parts. In the first part we treat the case $d \geq 2$. Fix $\varepsilon \in \mathbb{Q} \cap (0, \infty)$. Let Z be a finite subset of the $|\cdot|$-unit sphere S^{d-1} such that the closed balls $B(z, \varepsilon)$ with center $z \in Z$ and radius ε cover S^{d-1}. For $z \in \mathbb{R}^d$ set $Y_z := \{[y]\colon y \in \mathbb{R}^d,\ |y - z| \leq 3\varepsilon|z|\} \subseteq \mathbb{Z}^d$. Then for $n \geq d/\varepsilon$, due to (22),

$$\begin{aligned}
\mathbb{P}\Big[\sup_{z\in nZ}\sup_{y\in Y_z} \bar{d}(z, y, \omega) > 5c_2\varepsilon n\Big] &\leq \sum_{z\in nZ}\sum_{y\in Y_z} \frac{c_1|[z]-y|^{2d}}{(5c_2\varepsilon n - c_2|[z]-y|)_+^{4d}} \\
&\leq \sum_{z\in nZ}\sum_{y\in Y_z} \frac{c_1(3\varepsilon|z|+d)^{2d}}{(5c_2\varepsilon n - c_2(3\varepsilon|z|+d))_+^{4d}} \\
&\leq \sum_{z\in nZ} \sharp Y_z \frac{c_1(4\varepsilon n)^{2d}}{(c_2\varepsilon n)^{4d}} \leq c_5(\varepsilon)\sharp Z n^{-d}.
\end{aligned}$$

Therefore, due to the Borel–Cantelli lemma, there exists for $\mathbb{P}$-almost all $\omega \in \Omega$ some N such that

$$\sup_{z\in nZ}\sup_{y\in Y_z} \bar{d}(z, y, \omega) \leq 5c_2\varepsilon n \quad \text{for all } n \geq N. \tag{23}$$

Furthermore, by the existence of the dth moment of ν we have again by the Borel–Cantelli lemma,

$$\lim_{|x|\to\infty} \frac{\omega(x)}{|x|} = 0 \tag{24}$$

on a set Ω_3 of full $\mathbb{P}$-measure. Thus we can choose $N \geq 1 + 3\varepsilon$ big enough such that

$$\omega(x) \leq \frac{\varepsilon|x|}{1+\varepsilon} \vee N \quad \text{for all } x. \tag{25}$$

We shall show that there is a constant c_3 such that for any ω which fulfills (23) and (25), (21) is valid if $|x| \geq 2N/\varepsilon$. To this end, set $n = \lfloor |x| \rfloor$; let $z' \in Z$ such that $x/|x| \in B(z', \varepsilon)$ and set $z = nz'$. Then

$$|x - z| \leq \big|x - |x|z'\big| + \big||x|z' - z\big| \leq \varepsilon|x| + 1 \leq 2\varepsilon|z|$$

and for any $y \in \mathbb{R}^d$ with $|x - y| < \varepsilon|x|$,

$$|y - z| \leq |y - x| + |x - z| < 2\varepsilon|x| + 1 \leq 3\varepsilon|z|,$$

that is, $[x], [y] \in Y_z$. Consequently, from (23), if $|x-y| < \varepsilon|x|$,

$$\begin{aligned} d(x, y, \omega) &\le d(x, z, \omega) + d(z, y, \omega) \\ &\le \bar{d}(x, z, \omega) + \max\{\omega(x), \omega(z)\} + \bar{d}(z, y, \omega) + \max\{\omega(z), \omega(y)\} \\ &\le 10c_2\varepsilon n + 2\max\{\omega(x'): |x'-x| \le \varepsilon|x| + 1\}. \end{aligned}$$

Since $|x'| \le (1+\varepsilon)|x| + 1$, we get by (25) $\omega(x') \le (\varepsilon|x|+1) \vee N \le 3\varepsilon|x|/2$. Therefore $d(x, y, \omega) \le (10c_2 + 3)\varepsilon|x| =: c_3\varepsilon|x|$.

Now we come to the case $d = 1$. The main reason why the one-dimensional case is different from the higher-dimensional one is that in one dimension there is only one way to get from x to y: one has to pass all the sites between x and y. For this reason the proof of (20) fails for $d = 1$. On the other hand, the result is that the process $a(x, y)$, $x \le y$, is not only subadditive but additive, that is,

$$a(x, z, \omega) = a(x, y, \omega) + a(y, z, \omega) \quad \text{if } x \le y \le z \text{ and } d = 1. \tag{26}$$

Thus the supremum in (21) can be simplified considerably. Indeed, let $x \in \mathbb{R}$, $\omega \in \Omega$ and $\varepsilon \in \mathbb{Q} \cap (0, \infty)$. Then

$$\sup\{d(x, y, \omega): y \in \mathbb{R},\ |y-x| < \varepsilon|x|\} \le d(x, x-\varepsilon|x|, \omega) \vee d(x, x+\varepsilon|x|, \omega).$$

Thus we shall construct some constant c_3 such that

$$d(x, x+\varepsilon|x|, \omega) < c_3\varepsilon|x| \tag{27}$$

for $\mathbb{P}$-almost all $\omega \in \Omega$ and large $|x|$. An analogous statement with $-\varepsilon$ instead of $+\varepsilon$ can be derived similarly. Without loss of generality, we assume $x > 0$. We define recursively $x_0 := 0$ and $x_{n+1} := [(1+\varepsilon)x_n] + 1$. Then the sets $\{x_n, \dots, x_{n+1} - 1\}$, $n \ge 0$, partition the nonnegative integers. We want to control $d(x_n, x_{n+1} - 1, \omega)$. By choosing the shortest path from x_n to $x_{n+1} - 1$ for the random walk, we get

$$d(x_n, x_{n+1} - 1, \omega) \le (x_{n+1} - x_n)\ln 2 + \omega(x_n) + \cdots + \omega(x_{n+1} - 1). \tag{28}$$

Now since the sequence x_n grows geometrically fast, we have

$$\sum_{n \ge 0} \mathbb{P}[\omega(x_n) + \cdots + \omega(x_{n+1} - 1) \ge 2(x_{n+1} - x_n)\mathbb{E}[\omega(0)]] < \infty \tag{29}$$

(see, e.g., [17], Section 6.8.5). Therefore, by the Borel–Cantelli lemma, the right side of (28) is $\mathbb{P}$-a.s. for large n less than $c'(x_{n+1} - x_n)$. Now let n be such that $x_n < x \le x_{n+1}$. Then if x is large enough,

$$d(x, (1+\varepsilon)x, \omega) \le d(x_n, x_{n+2} - 1, \omega) \le c'(x_{n+2} - x_n) < c_3\varepsilon x,$$

which proves (27). $\square$

We are now ready to prove the analogue of the shape theorems of first passage percolation (e.g., [9], Theorem 3.1) and Brownian motion in a Poissonian potential [20]. A strengthened version will be given in Theorem 13.

THEOREM 8. *Suppose that the dth moment of ν is finite. Then on a set Ω_4 of full $\mathbb{P}$-measure and in $L^1(\mathbb{P})$,*

$$\lim_{|x|\to\infty} \frac{a(0,x,\omega)}{\alpha(x)} = 1. \tag{30}$$

The same identity holds for $a(x,0,\omega)$, $d(0,x,\omega)$ and $-\ln g(0,x,\omega)$ instead of $a(0,x,\omega)$ as well.

PROOF. For the proof of the $\mathbb{P}$-a.s. convergence in (30), it suffices to show

$$\lim_{k\to\infty} \frac{1}{|x_k|}\big|a(0,x_k,\omega) - \alpha(x_k)\big| = 0 \tag{31}$$

for all $\omega \in \Omega_4 := \Omega_0 \cap \Omega_1 \cap \Omega_2 \cap \Omega_3$ [see (13), Corollary 6, (21), (24)] and for all sequences x_k tending to infinity such that $x_k/|x_k| \to e \in S^{d-1}$. To this end, let $\varepsilon \in \mathbb{Q} \cap (0,1)$ and choose $v \in \mathbb{Q}^d$ and $M \in \mathbb{N}$ such that $Mv \in \mathbb{Z}^d$ and $|v-e| < \varepsilon$ as well as $|\alpha(v) - \alpha(e)| < \varepsilon$. We approximate x_k by the lattice site

$$x'_k := \left\lfloor \frac{|x_k|}{M} \right\rfloor Mv \in \mathbb{Z}^d$$

where $\lfloor z \rfloor$ denotes the largest integer less than or equal to z. Then for k large enough,

$$\begin{aligned}
|x_k - x'_k| &\le \left| x_k - \left\lfloor \frac{|x_k|}{M} \right\rfloor M \frac{x_k}{|x_k|} \right| + \left| \left\lfloor \frac{|x_k|}{M} \right\rfloor M \frac{x_k}{|x_k|} - x'_k \right| \\
&= \left| 1 - \left\lfloor \frac{|x_k|}{M} \right\rfloor \frac{M}{|x_k|} \right| |x_k| + \left\lfloor \frac{|x_k|}{M} \right\rfloor M \left| \frac{x_k}{|x_k|} - v \right| \\
&< \varepsilon |x_k|.
\end{aligned} \tag{32}$$

Using the triangle inequality (12), we get

$$\begin{aligned}
\left| \frac{a(0,x_k)}{|x_k|} - \alpha\left(\frac{x_k}{|x_k|}\right) \right| &\le \frac{d(x'_k, x_k)}{|x_k|} + \left| \frac{a(0,x'_k)}{|x_k|} - \frac{a(0,x'_k)}{|x'_k|} \right| \\
&\quad + \left| \frac{a(0,x'_k)}{|x'_k|} - \alpha(v) \right| + \left| \alpha(v) - \alpha\left(\frac{x_k}{|x_k|}\right) \right|.
\end{aligned} \tag{33}$$

The first summand on the right-hand side of (33) is bounded above by $c_3\varepsilon$ due to (32) and (21) for k large enough. The second summand is finally less than $(\alpha(v)+\varepsilon)\varepsilon < (\alpha(e)+2\varepsilon)\varepsilon$ due to (32) and the choice of e. The third summand tends to zero for k going to infinity by the definition of x'_k and (13). Finally the last term is smaller than ε. Hence letting $\varepsilon \searrow 0$ finishes the proof of (31) and of the almost sure convergence in (30). Since for $d \ge 2$, (20) implies that the family $a(0,x)/|x|$ is uniformly integrable, which is also clear for $d=1$ due to $a(0,x) \le c|x| + \omega(0) + \cdots + \omega(x)$, the $L^1(\mathbb{P})$ convergence follows as well.

It remains to show that (30) holds also for $a(x,0,\omega)$, $d(0,x,\omega)$ and $-\ln g(0,x,\omega)$ in place of $a(0,x,\omega)$. But this is immediate from Corollary 6. □

REMARK. If the dth moment of ν is infinite, then by the Borel–Cantelli lemma $\sup_x \omega(x)/|x| = \infty$. Thus in this case (30) does not hold, at least not for $a(x,0)$, $d(0,x)$ or $-\ln g(0,x)$ in place of $a(0,x)$ due to $d(0,x) \geq a(x,0) \geq \omega(x)$ and (17).

3. Relations to first passage percolation. In the following we shall describe how this model of a random walk in a random potential is related to first passage percolation. In standard bond first passage percolation one assigns to each edge e between adjacent vertices of $\mathbb{Z}^d$ a random nonnegative variable $\omega(e)$ which is interpreted as the passage time of e. However, in our model the potentials $\omega(x)$ are attached to the vertices $x \in \mathbb{Z}^d$ themselves. Thus it is convenient in this context to consider site instead of bond first passage percolation. That is, we interpret the potential $\omega(x)$ as the time it takes a particle to pass the vertex x, no matter where it comes from and where it goes to. The passage time of a directed nearest neighbor path $r = (x = x_0, x_1, \ldots, x_n = y)$ is then defined as

$$T(r, \omega) := \sum_{i=0}^{n-1} \omega(x_i).$$

Thus the travel time from x to y, namely,

$$T(x, y, \omega) := \inf\{T(r, \omega)\colon r \text{ is a path from } x \text{ to } y\}$$

is the minimal time in which a particle starting at x can reach y. Just as in bond first passage percolation it follows from the subadditive ergodic theorem that there exists some constant $\mu(\cdot)$ such that $\mathbb{P}$-a.s. and in $L^1(\mathbb{P})$,

$$\mu(x) = \lim_{n\to\infty} \frac{T(0, nx, \omega)}{n} = \inf_{n\in\mathbb{N}} \frac{\mathbb{E}[T(0, nx, \omega)]}{n} \quad \text{for all } x \in \mathbb{Z}^d, \tag{34}$$

where we assume $\mathbb{E}[\omega(0)]$ to be finite. The difference between $a(x, y, \omega)$ and $T(x, y, \omega)$, is that a is, roughly speaking, a weighted average over $T(r)$ for paths r from x to y, whereas T is just the infimum of all $T(r)$. However, this difference vanishes when ω is multiplied with a large positive number, since in this case the dominant part of a is provided by the paths with minimal passage time. This is the content of the following proposition.

PROPOSITION 9. *For all $x, y \in \mathbb{Z}^d$ and $\omega \in \Omega$,*

$$\frac{a(x, y, M\omega)}{M} \searrow T(x, y, \omega), \qquad M \to \infty. \tag{35}$$

Let $\alpha^{(M)}$ $(M > 0)$ denote the Lyapounov exponent belonging to the distribution of the potential $M\omega(0)$. Then for all $x \in \mathbb{R}^d$,

$$\frac{\alpha^{(M)}(x)}{M} \searrow \mu(x), \qquad M \to \infty. \tag{36}$$

PROOF. First observe, that for any $M > 0$,

$$M^{-1}a(x, y, M\omega) \geq -M^{-1} \ln E_x\big[\exp(-T(x, y, M\omega)), H(y) < \infty\big] \geq T(x, y, \omega).$$

If $0 < M_1 < M_2$, then by Jensen's inequality,

$$\begin{aligned} &M_2^{-1}a(x, y, M_2\omega) \\ &\quad = -M_2^{-1} \ln E_x\left[\left(\exp\left(-\sum_{m=0}^{H(y)-1} \omega(S_m)\right)\right)^{M_1 M_2/M_1},\ H(y) < \infty\right] \\ &\quad \leq M_1^{-1}a(x, y, M_1\omega). \end{aligned} \tag{37}$$

Consequently (35) follows from

$$\lim_{M\to\infty} M^{-1}a(x, y, M\omega) \leq T(x, y, \omega) + \varepsilon \quad \text{for all } \varepsilon > 0. \tag{38}$$

For the proof of (38), let $r = (x = x_0, x_1, \dots, x_n = y)$ be a path from x to y such that $T(r, \omega) \leq T(x, y, \omega) + \varepsilon$. Then

$$\begin{aligned} &M^{-1}a(x, y, M\omega) \\ &\quad \leq -M^{-1} \ln E_x\left[\exp\left(-\sum_{m=0}^{H(y)-1} M\omega(S_m)\right),\ S_m = x_m\ (m = 0, \dots, n)\right] \\ &\quad = M^{-1}n \ln(2d) + T(r, \omega) \leq M^{-1}n\ln(2d) + T(x, y, \omega) + \varepsilon, \end{aligned}$$

which implies (38). For the proof of (36) we may assume $x \in \mathbb{Z}^d$. It follows then from (37) that $\alpha^{(M)}(x)/M$ decreases as $M \to \infty$ to

$$\begin{aligned} \inf_{M\in\mathbb{N}} \frac{\alpha^{(M)}(x)}{M} &= \inf_{M\in\mathbb{N}} \inf_{n\in\mathbb{N}} \frac{1}{nM} \mathbb{E}[a(0, nx, M\omega)] \qquad \text{[by (13)]} \\ &= \inf_{n\in\mathbb{N}} \frac{1}{n} \inf_{M\in\mathbb{N}} \mathbb{E}\left[\frac{a(0, nx, M\omega)}{M}\right] \\ &= \inf_{n\in\mathbb{N}} \frac{1}{n} \mathbb{E}[T(0, nx, \omega)] \qquad \text{[by (35)]} \\ &= \mu(x) \qquad \text{[by (34)].} \end{aligned}$$

□

Let us finally mention another relation, this time to bond first passage percolation. Here we assign to each nearest-neighbor edge with endpoints x and y the passage time $d(x, y, \omega)$. Then

$$L(r, \omega) = \sum_{i=1}^{n} d(x_{i-1}, x_i, \omega)$$

stands for the length of the path $r = (x = x_0, x_1, \dots, x_n = y)$ with respect to the random metric d. This induces a new random metric, the so-called interior metric of d, namely,

$$L(x, y, \omega) := \inf\{L(r, \omega)\colon r \text{ is a path from } x \text{ to } y\}.$$

Of course the passage times for different edges are no longer independent, but they are still stationary and ergodic. It follows from [1], that this together with some mild moment conditions ensures that the set of vertices x with $L(0, x) \leq t$ has still an asymptotic shape for $t \to \infty$.

4. Fluctuations around the mean value. This section studies the fluctuations of $a(0, y, \omega)$ around $\mathbb{E}[a(0, y)]$. Again we treat the cases $d = 1$ and $d \geq 2$ separately, since in one dimension the process $a(n, m)$, $0 \leq n \leq m$, is not only subadditive but additive, that is, (26) holds. This enables us to derive the following result.

PROPOSITION 10. *Suppose $d = 1$. Then*

$$\alpha(n) = n\ \mathbb{E}[a(0, 1)] = \mathbb{E}[a(0, n)]. \tag{39}$$

If in addition the variance of ν is positive and finite then

$$\frac{a(0, n) - \alpha(n)}{\sigma\sqrt{n}} \overset{\text{law}}{\longrightarrow} \mathscr{N}(0, 1), \qquad n \to \infty, \tag{40}$$

where

$$\sigma^2 = \operatorname{Var}[a(0, 1)] + 2 \sum_{j=1}^{\infty} \operatorname{Cov}[a(0, 1), a(j, j+1)] < \infty;$$

(40) *also holds with d or $-\ln g$ instead of a with the same σ.*

Equation (39) is the counterpart to [20], (1.30) and appears in a similar context in [6], (5.3) and (5.4), page 502.

PROOF. From (26), (39) follows from the ergodic theorem. Then (40) is a consequence of the central limit theorem for functionals of mixing sequences [8], Theorem 18.6.1. Indeed, we only have to check that

$$\sum_{k=1}^{\infty} \big\| a(0, 1) - \mathbb{E}\big[a(0, 1) \mid \omega(-k), \dots, \omega(-1), \omega(0)\big]\big\|_2 \tag{41}$$

converges. To show this we introduce

$$\begin{aligned}
e_{-k}(0, 1) := E_0\Bigg[\exp\Bigg(-\sum_{m=0}^{H(1)-1} \omega(S_m)\Bigg),\ H(1) < H(-k)\Bigg] &\leq e(0, 1) \\
= e_{-k}(0, 1) + E_0\Bigg[\exp\Bigg(-\sum_{m=0}^{H(1)-1} \omega(S_m)\Bigg),\ H(1) > H(-k)\Bigg]& \\
\leq e_{-k}(0, 1) + \exp(-\omega(0) - \omega(-1) \cdots - \omega(-k))& \\
\leq e_{-k}(0, 1)\big(1 + 2\exp(-\omega(-1) - \cdots - \omega(-k))\big)&
\end{aligned}$$

since $e_{-k}(0,1) \geq \exp(-\omega(0))/2$ and get

$$0 \geq a(0,1) + \ln e_{-k}(0,1) \geq -2\exp(-\omega(-1) - \cdots - \omega(-k)),$$

which implies

$$\begin{aligned}
&\big\| a(0,1) - \mathbb{E}[a(0,1) \mid \omega(-k), \ldots, \omega(0)] \big\|_2 \\
&\quad \leq \big\| a(0,1) + \ln e_{-k}(0,1) \big\|_2 \\
&\qquad + \big\| \mathbb{E}\big[-\ln e_{-k}(0,1) - a(0,1) \mid \omega(-k), \ldots, \omega(0)\big] \big\|_2 \\
&\quad \leq 2\big\| 2\exp(-\omega(-1) - \cdots - \omega(-k)) \big\|_2 = 4\mathbb{E}[\exp(-2\omega(0))]^{k/2}
\end{aligned}$$

thus proving convergence in (41). The last statement of the proposition follows from Corollary 6. □

As in related models like first passage percolation or Brownian motion in a Poissonian potential, an analogous result for $d \geq 2$ seems to be out of reach. For the purposes of the next section we shall derive a rough upper bound on the variance of $a(x, y, \omega)$.

THEOREM 11. *Suppose that the second moment of ν is finite. Furthermore, assume that the minimum $\underline{\nu}$ of the support of ν is strictly positive if $d = 2$. Then for some finite constant C*

$$\operatorname{Var}(a(x, y, \omega)) \leq C|x - y| \quad \text{for all } x, y \in \mathbb{Z}^d. \tag{42}$$

The same holds for d and $-\ln g$ instead of a.

This is the analogue to [10], (1.13), in first passage percolation. We do not think that the hypothesis $\underline{\nu} > 0$ if $d = 2$ is necessary. For the proof of the theorem we use the following rank one perturbation formula that gives an upper bound on how much $a(0, y, \omega)$ may change when ω is changed at a single site.

LEMMA 12. *Let $z \in \mathbb{Z}^d$ and $\omega_1, \omega_2 \in \Omega$ such that $\omega_1(x) = \omega_2(x)$ for $x \neq z$ and $\omega_1(z) \leq \omega_2(z)$. Furthermore, suppose $\omega_1(z) \geq c_1 > 0$ if $d \leq 2$. Then there is some constant $c_2 > 0$ which may depend on c_1 such that for any $y \in \mathbb{Z}^d$,*

$$0 \leq a(0, y, \omega_2) - a(0, y, \omega_1) \leq \min\big\{-\ln \hat{P}^y_{0,\omega_1}[H(y) \leq H(z)],\ \omega_2(z) + c_2\big\}. \tag{43}$$

PROOF. The statement is obvious for $y = z$. Hence we assume $y \neq z$. The left inequality in (43) is immediate from $\omega_2 \geq \omega_1$. For the right inequality, observe that the strong Markov property implies

$$\frac{e(0, y, \omega_2)}{e(0, y, \omega_1)} = \hat{P}^y_{0,\omega_1}[H(y) < H(z)] + \frac{e(z, y, \omega_2)}{e(z, y, \omega_1)} \hat{P}^y_{0,\omega_1}[H(y) > H(z)] \tag{44}$$

(cf. [22], (2.10)) which yields the first part of the right inequality in (43). For the second part note that (44) and $\omega_2 \ge \omega_1$ imply

$$\frac{e(0, y, \omega_2)}{e(0, y, \omega_1)} \ge \frac{e(z, y, \omega_2)}{e(z, y, \omega_1)}.$$

Now again from the strong Markov property for $i = 1, 2$,

$$e(z, y, \omega_i) = \frac{\exp(-\omega_i(z)) E_z[\exp(-\sum_{m=1}^{H(y)-1} \omega_i(S_m)),\ H(y) < H_2(z)]}{1 - E_z[\exp(-\sum_{m=0}^{H_2(z)-1} \omega_i(S_m)),\ H_2(z) < H(y)]}. \tag{45}$$

By $\omega_1(z) \ge c_1 > 0$ for $d \le 2$ and transience for $d \ge 3$, respectively, the denominator in (45) is greater than a positive constant such that

$$e(z, y, \omega_1) \le c_3 E_z\left[\exp\left(-\sum_{m=1}^{H(y)-1} \omega_1(S_m)\right),\ H(y) < H_2(z)\right]$$

for some finite c_3. On the other hand, by (45),

$$e(z, y, \omega_2) \ge \exp(-\omega_2(z)) E_z\left[\exp\left(-\sum_{m=1}^{H(y)-1} \omega_1(S_m)\right),\ H(y) < H_2(z)\right],$$

where we used that ω_1 and ω_2 coincide outside z. The assertion now easily follows. □

PROOF OF THEOREM 11. Due to Proposition 10 we only have to consider the case $d \ge 2$. Furthermore, we may assume that the variance of ν is positive.

The proof is based on the martingale method as in [10]. Let x_k, $k \ge 1$, be an enumeration of $\mathbb{Z}^d$ and $\mathscr{F}_k$ be the σ-field generated by $\omega(x_1), \dots, \omega(x_k)$. Here $\mathscr{F}_0$ denotes the trivial σ-field $\{\varnothing, \Omega\}$. Then for fixed $x, y \in \mathbb{Z}^d$,

$$M_k = \mathbb{E}[a(x, y) | \mathscr{F}_k], \qquad k \ge 0$$

defines a martingale with respect to the filtration $\mathscr{F}_k$, $k \ge 0$, that converges $\mathbb{P}$-a.s. and in $L^1(\mathbb{P})$, $L^2(\mathbb{P})$ to $a(x, y)$. We denote the increments $M_k - M_{k-1}$ of the martingale by Δ_k. Since they are pairwise uncorrelated, we get

$$\operatorname{Var}(a(x, y)) = \mathbb{E}\left[\left(\sum_{k\ge1} \Delta_k\right)^2\right] = \sum_{k\ge1} \mathbb{E}[\Delta_k^2]. \tag{46}$$

To simplify the notation we denote by $\mathbb{E}_\sigma$ the expectation over the variable σ with respect to the measure $\mathbb{P}$. Furthermore, for $\omega, \sigma \in \Omega$ and $k \ge 0$ let $[\omega, \sigma]_k \in \Omega$ be the configuration that agrees with ω on the first k coordinates and with σ on the coordinates after k. Then we can represent the increments as

$$\Delta_k(\omega) = \mathbb{E}_\sigma[a(x, y, [\omega, \sigma]_k) - a(x, y, [\omega, \sigma]_{k-1})]$$

and get that the rightmost side of (46) is less than

$$\sum_{k\geq 1}\mathbb{E}_\omega\big[\mathbb{E}_\sigma\big[(a(x,y,[\omega,\sigma]_k)-a(x,y,[\omega,\sigma]_{k-1}))^2\big]\big]. \tag{47}$$

By symmetry we can restrict the domain of integration in (47) to those configurations with $\sigma(x_k)\leq\omega(x_k)$ such that $a(x,y,[\omega,\sigma]_k)\geq a(x,y,[\omega,\sigma]_{k-1})$. Consequently by Lemma 12, (47) is smaller than

$$\begin{aligned}2\sum_{k\geq 1}\Big(&\mathbb{E}_\omega\Big[\mathbb{E}_\sigma\Big[(\omega(x_k)+c_2)^2,\ \hat{P}^y_{x,[\omega,\sigma]_{k-1}}[H(y)\leq H(x_k)]<\tfrac12\Big]\Big]\\&+\mathbb{E}_\omega\Big[\mathbb{E}_\sigma\Big[(\ln\hat{P}^y_{x,[\omega,\sigma]_{k-1}}[H(y)\leq H(x_k)])^2,\\&\qquad\hat{P}^y_{x,[\omega,\sigma]_{k-1}}[H(y)\leq H(x_k)]\geq\tfrac12\Big]\Big]\Big).\end{aligned}$$

Using $(\ln t)^2\leq 1-t$ for $1/2\leq t\leq 1$, one sees that this is less than

$$\begin{aligned}2\sum_{k\geq 1}\Big(&\mathbb{E}_\omega[(\omega(0)+c_2)^2]\mathbb{P}_\omega\big[\hat{P}^y_{x,\omega}[H(y)>H(x_k)]\geq\tfrac12\big]\\&+\mathbb{E}_\omega\big[\hat{P}^y_{x,\omega}[H(y)>H(x_k)]\big]\Big)\\&\leq 2\big(2\mathbb{E}_\omega[(\omega(0)+c_2)^2]+1\big)\sum_{k\geq 1}\mathbb{E}_\omega\big[\hat{P}^y_{x,\omega}[H(y)>H(x_k)]\big]\\&=c_3\mathbb{E}_\omega\big[\hat{E}^y_{x,\omega}[\sharp\mathscr{A}]\big]\\&\leq c_4|x-y|\end{aligned}$$

by Lemma 3. This proves (42). For the last statement of the theorem, observe that due to Lemma 5, $\ln g(0,0)$ has finite variance. Let $(x=x_0,x_1,\dots,x_n=y)$ be a nearest-neighbor path from x to y of length $n=|x-y|$. Then by (10), (12), Lemma 3 and the Cauchy–Schwarz inequality,

$$\begin{aligned}\operatorname{Var}(-\ln g(x,y))&=\operatorname{Var}(a(x,y))+\operatorname{Var}(\ln g(y,y))\\&\quad+2\big(\mathbb{E}[-a(x,y)\ln g(y,y)]+\mathbb{E}[a(x,y)]\mathbb{E}[\ln g(y,y)]\big)\\&\leq Cn+C_1+2\sum_{i=1}^n\mathbb{E}[a(x_{i-1},x_i,\omega)\omega(y)]+C_2n\\&\leq C_3n+2n\ \|a(0,e_1)\|_2\ \|\omega(0)\|_2=C_4n.\end{aligned}$$

The variance of $d(x,y,\omega)$ can be estimated similarly. □

5. A uniform shape theorem. The results of the previous section enable us to improve Theorem 8 as follows.

THEOREM 13. *Suppose that the dth moment of ν is finite. If $d=2$, then assume additionally that $\underline{\nu}>0$. Then on a set Ω_5 of full $\mathbb{P}$-measure and in $L^1(\mathbb{P})$,*

$$\lim_{c(|x|\vee|y|)\leq|x-y|\to\infty}\frac{a(x,y,\omega)}{\alpha(x-y)}=1\quad\textit{for all } c>0. \tag{48}$$

The same identity holds with d as well or $-\ln g$ instead of a.

For the proof we have to replace (13) by the stronger statement of the following lemma.

LEMMA 14. *Assume that the dth moment of* ν *is finite. If* $d = 2$, *suppose furthermore* $\underline{\nu} > 0$. *Then on a set* Ω_6 *of full* $\mathbb{P}$-*measure for all* $x, y \in \mathbb{Z}^d$,

$$\lim_{n\to\infty} \frac{a(nx, ny, \omega)}{n} = \alpha(x - y). \tag{49}$$

The same holds for d *and* $-\ln g$ *instead of* a.

The proof is based on an idea of Newman [16] in first passage percolation.

PROOF. From Corollary 6 it suffices to prove the statement for the function a. For $d = 1$, the claim follows directly from (13) and (26). In the case $d \geq 2$ note that (49) would follow from

$$\frac{|a(nx, ny, \omega) - \mathbb{E}[a(nx, ny)]|}{n} \to 0, \qquad n \to \infty, \ \mathbb{P}\text{-a.s.} \tag{50}$$

due to (13). Intending to use the Borel–Catelli lemma, we let $\delta \in \mathbb{Q} \cap (0, 1)$. Then for any $n \in \mathbb{N}$ by Chebyshev's inequality and Theorem 11,

$$\mathbb{P}\left[\frac{|a(nx, ny) - \mathbb{E}[a(nx, ny)]|}{n} > \delta\right] \leq \frac{\operatorname{Var}(a(nx, ny))}{n^2\delta^2} \leq \frac{C|x - y|}{n\delta^2}. \tag{51}$$

Since the rightmost side of (51) is not summable over $n \in \mathbb{N}$, we cannot apply the Borel–Cantelli lemma directly to obtain (50). However, we see that the convergence in (50) takes place for any sequence of integers with summable reciprocals. For example, we get that (50) holds with n replaced by n^2. Now for any $n \in \mathbb{N}$ the left-hand side of (50) is less than

$$\begin{aligned} &\frac{1}{n}\left|a\left(\lfloor\sqrt{n}\rfloor^2 x, \lfloor\sqrt{n}\rfloor^2 y\right) - \mathbb{E}\left[a\left(0, \lfloor\sqrt{n}\rfloor^2 (x - y)\right)\right]\right| + \frac{1}{n} d\left(nx, \lfloor\sqrt{n}\rfloor^2 x\right) \\ &\quad + \frac{1}{n} d\left(\lfloor\sqrt{n}\rfloor^2 y, ny\right) + \frac{1}{n}\left|\mathbb{E}\left[a\left(0, \lfloor\sqrt{n}\rfloor^2 (x - y)\right)\right] - \mathbb{E}[a(0, n(x - y))]\right|. \end{aligned}$$

The first summand tends to zero $\mathbb{P}$-a.s. according to the preceding explanation. The last term also vanishes due to (13). For the second term, observe that $|nx - \lfloor\sqrt{n}\rfloor^2 x| \leq 2\sqrt{n}|x|$. Consequently, from to (21) the second summand goes to zero $\mathbb{P}$-a.s. The same holds for the third term. □

PROOF OF THEOREM 13. Following Corollary 6, it suffices to prove (48) with a replaced by d. The convergence in $L^1(\mathbb{P})$ follows from the $\mathbb{P}$-a.s. convergence again by uniform integrability. For the proof of the $\mathbb{P}$-a.s. convergence, it suffices to show

$$\frac{|d(x_k, y_k, \omega) - \alpha(x_k - y_k)|}{|x_k - y_k|} \longrightarrow 0, \qquad k \to \infty \tag{52}$$

for $\omega \in \Omega_5 := \Omega_2 \cap \Omega_6$ [see (21) and (49)], $c > 0$, and sequences x_k, y_k with $|x_k - y_k| \geq c(|x_k| \vee |y_k|) \to \infty$ which have the following properties: by symmetry of d we may assume $|y_k| \leq |x_k|$. Due to the compactness of S^{d-1} and of the unit interval we can furthermore assume $x_k/|x_k| \to e_x \in S^{d-1}$ and $|y_k|/|x_k| \to r \in [0, 1]$. If there are infinitely many $y_k \neq 0$, let us assume $y_k/|y_k| \to e_y \in S^{d-1}$ for those k with $y_k \neq 0$; otherwise let $e_y \in S^{d-1}$ be arbitrary.

Now let $\varepsilon \in \mathbb{Q} \cap (0, 1)$ and choose $v_x, v_y \in S^{d-1} \cap \mathbb{Q}^d$, $M \in \mathbb{N}$ and $q = q_1/q_2 \in (0, 1]$ with $q_1, q_2 \in \mathbb{N}$ such that $Mv_x, Mv_y \in \mathbb{Z}^d$, $|v_x - e_x|, |v_y - e_y| < \varepsilon$ and $|q - r| < \varepsilon$. We approximate x_k and y_k by the lattice vertices

$$x'_k := \left\lfloor \frac{|x_k|}{q_2 M} \right\rfloor q_2 M v_x \quad \text{and} \quad y'_k := \left\lfloor \frac{|x_k|}{q_2 M} \right\rfloor q_1 M v_y. \tag{53}$$

For k large enough we have

$$|x_k - x'_k| \leq \big|x_k - |x_k| v_x\big| + \big||x_k| v_x - x'_k\big| < \varepsilon |x_k| + q_2 M < 2\varepsilon |x_k| \tag{54}$$

and

$$\begin{aligned} |y_k - y'_k| &\leq \big|y_k - |y_k| v_y\big| + \big||y_k| v_y - |x_k| q v_y\big| + \big||x_k| q v_y - y'_k\big| \\ &< \varepsilon |y_k| + \varepsilon |x_k| + q_1 M < 3\varepsilon |x_k| \leq \frac{4\varepsilon}{q} |y'_k| \leq 4\varepsilon |x_k|. \end{aligned} \tag{55}$$

The left-hand side of (52) is at most

$$\begin{aligned} &\frac{d(x_k, x'_k)}{|x_k - y_k|} + \frac{d(y_k, y'_k)}{|x_k - y_k|} + \frac{|d(x'_k, y'_k) - \alpha(x'_k - y'_k)|}{|x_k - y_k|} \\ &\quad + \frac{|\alpha(x'_k - y'_k) - \alpha(x_k - y_k)|}{|x_k - y_k|}. \end{aligned} \tag{56}$$

The third term in (56) vanishes for k tending to infinity, due to Lemma 14. Thus by (21), (54), (55), (14) and $|x_k - y_k| \geq c|x_k|$, (56) is less than

$$\frac{2c_3 \varepsilon |x_k|}{c|x_k|} + \frac{4c_3 \varepsilon |y'_k|}{qc|x_k|} + \frac{\alpha(x'_k - x_k) + \alpha(y'_k - y_k)}{c|x_k|} \leq c_4 \varepsilon$$

for large k. Letting $\varepsilon \searrow 0$ proves (52). □

To phrase this result in terms of asymptotic shapes, we introduce

$$A(x, r) := \{ y \in \mathbb{R}^d : \alpha(x - y) \leq r \} \quad \text{and}$$
$$D(x, r, \omega) := \{ y \in \mathbb{R}^d : d(x, y, \omega) \leq r \}$$

for $x \in \mathbb{R}^d$, $r \geq 0$, and $\omega \in \Omega$.

THEOREM 15. *Suppose that the dth moment of ν is finite. If $d = 2$, then assume furthermore $\underline{\nu} > 0$. Then there is $\mathbb{P}$-a.s. for all $\varepsilon > 0$ and all $K \in \mathbb{N}$ some R, such that for all $r \geq R$ and all $x \in \mathbb{R}^d$ with $|x| \leq rK$ the following holds*:

$$A(x, (1 - \varepsilon) r) \subseteq D(x, r, \omega) \subseteq A(x, (1 + \varepsilon) r). \tag{57}$$

PROOF. Let $\omega \in \Omega_2 \cap \Omega_5$ [see (21), (48)] and assume that the statement is false for this ω. Then there are some $\varepsilon > 0$, $K \in \mathbb{N}$, a sequence r_n tending to infinity and a sequence x_n with $|x_n| \le r_n K$ such that the left or the right relation in (57) fails to hold for all $x = x_n$ and $r = r_n$, $n \in \mathbb{N}$.

We first consider the case that the left relation is false. In this case there are $y_n \in A(x_n, (1-\varepsilon)r_n)$ with $y_n \notin D(x_n, r_n, \omega)$. Therefore $|x_n| \vee |y_n|$ tends to infinity since $d(x_n, y_n) \ge r_n$ does so. In addition,

$$\frac{d(x_n, y_n)}{\alpha(x_n - y_n)} \ge \frac{r_n}{(1-\varepsilon)r_n} = \frac{1}{(1-\varepsilon)} > 1.$$

This yields the desired contradiction to Theorem 13. Indeed, the only thing that remains to be shown is that

$$|x_n - y_n| \ge c(|x_n| \vee |y_n|) \tag{58}$$

holds for some $c > 0$ and infinitely many n. If the set $\{x_n: n \in \mathbb{N}\}$ is bounded, then (58) is immediate from $|x_n| \vee |y_n| \to \infty$. If the set is unbounded, then due to $d(x_n, y_n) \ge r_n \ge |x_n|/K$, (21) implies $|x_n - y_n| \ge |x_n|/(Kc_3)$ for infinitely many n. Hence for these n we have $|y_n| \le |x_n - y_n| + |x_n| \le c'|x_n - y_n|$ and consequently (58).

Now we treat the case when the relation on the right side of (57) fails to hold. Then there exist $y_n \in D(x_n, r_n, \omega)$ with $y_n \notin A(x_n, (1+\varepsilon)r_n)$. Again $|x_n| \vee |y_n|$ tends to infinity, since $\alpha(x_n - y_n) \ge (1+\varepsilon)r_n$ does so. Furthermore

$$\frac{d(x_n, y_n)}{\alpha(x_n - y_n)} \le \frac{r_n}{(1+\varepsilon)r_n} = \frac{1}{1+\varepsilon} < 1.$$

Thus again we just have to show that (58) holds for some $c > 0$ and infinitely many n to produce a contradiction to Theorem 13. This time (58) follows from

$$|x_n - y_n| \ge c_1 \alpha(x_n - y_n) \ge c_1 r_n \ge \frac{c_1}{K}|x_n|$$

and

$$|y_n| \le |x_n - y_n| + |x_n| \le \left(1 + \frac{c_1}{K}\right)|x_n - y_n|. \qquad \square$$

REMARK. The assumption $|x - y| \ge c(|x| \vee |y|)$ in Theorem 13 is optimal in the sense that it cannot be replaced by the weaker condition $|x - y| \ge c(|x| \vee |y|)^\gamma$ for any $\gamma < 1$. Indeed, let $\gamma < \gamma' < 1$ and ν such that the dth moment but not the d/γ'th moment of ν is finite. Then there exists due to the Borel–Cantelli lemma for $\mathbb{P}$-almost all ω some sequence x_n tending to infinity such that $\omega(x_n) \ge 2C|x_n|^{\gamma'}$, where $C := \sup\{\alpha(x): x \in S^{d-1}\}$. Furthermore one can choose y_n with the property $|x_n|^\gamma \le |x_n - y_n| \le |x_n|^{\gamma'}$ and $|y_n| \le |x_n|$. Consequently

$$\liminf_{n\to\infty} \frac{a(x_n, y_n, \omega)}{\alpha(x_n - y_n)} \ge \liminf_{n\to\infty} \frac{\omega(x_n)}{C|x_n - y_n|} \ge 2.$$

Similarly one can show that the condition $|x| \le rK$ in Theorem 15 cannot be replaced by $|x|^\gamma \le rK$ either.

Finally we consider a generalization of Theorem 13 for point-to-set distances instead of point-to-point distances. This includes the equivalent of the point-to-hyperplane constant in first passage percolation as a special case. To this end we allow the second argument K of $e(x, K, \omega)$ to be a nonempty subset of $\mathbb{R}^d$ and define $e(x, K, \omega)$ as done in (2), but replace $H(y)$ by $H(K) := \inf\{H(y)\colon y \in K\}$. Furthermore, we write $\alpha(x, K, \omega)$ for $-\ln e(x, K, \omega)$ and denote the distance between x and K by $\delta(x, K) := \inf\{|x-y|\colon y \in K\}$.

COROLLARY 16. *Under the assumptions of Theorem* 13 *on a set* Ω_5 *of full* $\mathbb{P}$-*measure*,

$$\lim_{n\to\infty} \frac{\alpha(x_n, K_n, \omega)}{\inf_{y\in K_n} \alpha(x_n - y)} = 1 \quad \text{for all } c > 0 \text{ and all sequences } x_n \in \mathbb{R}^d \text{ and } \varnothing \neq K_n \subseteq \mathbb{R}^d \text{ with } c|x_n| \le \delta(x_n, K_n) \to \infty \text{ as } n \to \infty. \tag{59}$$

PROOF. Let $\omega \in \Omega_5$ [see (48)]. Pick $y_n \in K_n$ such that $\alpha(x_n - y_n) \le \inf \alpha(x_n - K_n) + 1$. Then

$$\limsup_{n\to\infty} \frac{\alpha(x_n, K_n, \omega)}{\inf \alpha(x_n - K_n)} \le \limsup_{n\to\infty} \frac{\alpha(x_n, y_n, \omega)}{\alpha(x_n - y_n)} \frac{\alpha(x_n - y_n)}{\alpha(x_n - y_n) - 1} = 1$$

from (48). For the opposite inequality let c_1, c_2 be as in Lemma 3 and set

$$D_n := [K_n] \cup \big[B\big(x_n, c_2/c_1\ \delta(x_n, [K_n]) + d\big)^c\big]$$

whose interior boundary is

$$L_n := \big\{z \in D_n\colon |z - z'| = 1 \text{ for some } z' \in \mathbb{Z}^d \backslash D_n\big\}.$$

Hence

$$\begin{aligned} e(x_n, K_n, \omega) \le e(x_n, L_n, \omega) &\le \sum_{y\in L_n} e(x_n, y, \omega) \\ &\le \sharp L_n \max_{y\in L_n} e(x_n, y, \omega) = \sharp L_n e(x_n, y_n, \omega) \end{aligned}$$

for some $y_n \in L_n$. Since $\sharp L_n$ grows at most polynomially with $\delta(x_n, K_n)$, this implies

$$\begin{aligned} \liminf_{n\to\infty} \frac{\alpha(x_n, K_n, \omega)}{\inf \alpha(x_n - K_n)} &\ge \liminf_{n\to\infty} \frac{-\ln \sharp L_n + \alpha(x_n, y_n, \omega)}{\inf \alpha(x_n - K_n)} \\ &= \liminf_{n\to\infty} \frac{\alpha(x_n, y_n, \omega)}{\alpha(x_n - y_n)} \frac{\alpha(x_n - y_n)}{\inf \alpha(x_n - [K_n])} \frac{\inf \alpha(x_n - [K_n])}{\inf \alpha(x_n - K_n)} \\ &= \liminf_{n\to\infty} \frac{\alpha(x_n - y_n)}{\inf \alpha(x_n - [K_n])} \qquad \text{[by (48)]} \\ &\ge 1. \end{aligned}$$

The last inequality holds because of $\alpha(x_n - y_n) \geq \inf \alpha(x_n - [K_n])$. This is obvious in the case $y_n \in [K_n]$. Otherwise we have $y_n \in [B(x_n, c_2/c_1\delta(x_n, K_n)+d)^c]$ and therefore, due to (14),

$$\alpha(x_n - y_n) \geq c_1|x_n - y_n| \geq c_2\delta(x_n, [K_n]) \geq \inf \alpha(x_n - [K_n]). \qquad \square$$

6. Large deviation estimates. In this section we investigate, for fixed $x \in \mathbb{R}^d$ and typical $\omega \in \Omega$, large deviations of S_n/n under the path measures

$$Q_{n,x,\omega} = \frac{1}{Z_{n,x,\omega}} \exp\biggl(-\sum_{m=0}^{n-1} \omega(S_m)\biggr) P_{nx} \tag{60}$$

when n tends to infinity. Here

$$Z_{n,x,\omega} = E_{nx}\biggl[\exp\biggl(-\sum_{m=0}^{n-1} \omega(S_m)\biggr)\biggr] \tag{61}$$

is the normalizing constant.

We want to stress that the results of this section, which are developed in the setting of discrete time, do not automatically hold in continuous time by means of a lemma like Proposition 1. Instead, in order to derive the corresponding assertions for continuous time, one has to go through the proofs and modify some details.

The asymptotic exponential behavior of the normalizing constant depends only on the minimum $\underline{\nu}$ of the support of ν as stated in Proposition 17.

PROPOSITION 17. *If the dth moment of ν is finite, then on a set Ω_7 of full $\mathbb{P}$-measure for any $x \in \mathbb{R}^d$,*

$$\lim_{n\to\infty} \frac{-\ln Z_{n,x,\omega}}{n} = \underline{\nu}. \tag{62}$$

For Bernoulli measures ν, this has been observed before in [11], Theorem 3.

PROOF. It follows directly from the definitions of $Z_{n,x,\omega}$ and $\underline{\nu}$ that the left side of (62), if it exists, is greater than the right side. For the converse inequality, define $y = y(z, \varepsilon, R, \omega)$ for $z \in \mathbb{R}^d$, $\varepsilon > 0$, $R \in \mathbb{N}$, R even and $\omega \in \Omega$ to be some vertex with minimal distance from z such that the potentials $\omega(u)$ for all sites u inside the ball with radius R and center y are less than $\underline{\nu} + \varepsilon$. Note that y exists $\mathbb{P}$-a.s. for all z, ε, R. Moreover, it is not hard to show by the Borel–Cantelli lemma that on a set Ω_8 of full $\mathbb{P}$-measure there is a constant $c(\varepsilon, R, \omega)$ such that for $|z|$ large the distance between z and y is less than $c(\ln|z|)^{1/d}$. Now by the strong Markov property for any $\omega \in \Omega_7 := \Omega_2 \cap \Omega_8$ [see (21)],

$$\begin{aligned} Z_{n,x,\omega} &\geq E_{nx}\biggl[\exp\biggl(-\sum_{m=0}^{H(y(nx))+n-1} \omega(S_m)\biggr),\ H(y(nx)) < \infty\biggr] \\ &\geq e(nx, y(nx)) \exp(-(\underline{\nu} + \varepsilon)n) P_{y(nx)}\bigl[|S_m - y(nx)| \leq R,\ (m < n)\bigr]. \end{aligned}$$

Since for a random walk S_m starting at the origin, $|S_m| \le R$ is guaranteed for $m < n$ if $S_{kR} = 0$ for all $k < n/R$, this implies

$$\limsup_{n\to\infty} \frac{-\ln Z_{n,x,\omega}}{n} \le \limsup_{n\to\infty} \frac{a(nx, y(nx))}{n} + \underline{\nu} + \varepsilon - \frac{1}{R}\ln P_0[S_R = 0]. \tag{63}$$

The first term on the right side of (63) is zero $\mathbb{P}$-a.s. due to (21). Since $P_0[S_R = 0]$ decays just polynomially in R when R is even, we can choose R such that the last summand in (63) is less than ε. Letting $\varepsilon \searrow 0$ completes the proof. □

The rate function of the large deviation principle will be given by

$$I(x) := \sup_{\lambda\ge 0}\big(\alpha_{\lambda-\underline{\nu}}(x) - \lambda\big). \tag{64}$$

Here $\alpha_\lambda(x) := \alpha(x, \nu * \delta_\lambda)$ is the Lyapounov exponent that belongs to the distribution $\nu * \delta_\lambda$ of $\omega(0) + \lambda$. For fixed $x \in \mathbb{R}^d$, the map $\lambda \mapsto \alpha_\lambda(x)$ for $\lambda \ge -\underline{\nu}$ is continuous. Furthermore, it is also concave increasing. Indeed, concavity and monotonicity follow from (15) and the last assertion of Proposition 4, respectively. As a consequence, the map is lower semicontinuous. On the other hand, by dominated convergence $\mathbb{E}[a(0, x, \lambda + \omega)]$ depends continuously on λ. Therefore for $x \in \mathbb{Z}^d$ due to (13), $\alpha_\lambda(x) = \inf_n \mathbb{E}[a(0, nx, \lambda + \omega)]/n$ is upper semicontinuous. This implies continuity in $\lambda \ge -\underline{\nu}$ for arbitrary $x \in \mathbb{R}^d$. Moreover, it now follows from a Dini-type argument that $\alpha_\lambda(x)$ is jointly continuous in λ and x.

Observe that (14) provides the bounds

$$(\lambda + c_1)|x| \le \alpha_\lambda(x) \le (\lambda + c_2)|x|, \tag{65}$$

where $c_1 = -\ln \mathbb{E}[\exp(-\omega(0))]$ and $c_2 = \ln(2d) + \mathbb{E}[\omega(0)]$. Note that for $\lambda > -\nu$ the right-hand and left-hand derivatives of $\alpha_\lambda(x)$ with respect to λ exist with $\alpha'_{\lambda+}(x) \le \alpha'_{\lambda-}(x)$ and coincide except maybe at a countable number of locations. Figure 1 sketches $\alpha_\lambda(x)$ and the bounds of (65) as functions of λ for some x with $0 < |x| < 1$ and $\underline{\nu} = 0$.

From (64) and (65) we get the bounds

$$(c_1 - \underline{\nu})|x| \le I(x) \le (c_2 - \underline{\nu})|x| \quad \text{if } |x| \le 1 \tag{66}$$

and $I(x) = \infty$ otherwise. Since $I(x)$ is convex owing to the analogous property of $\alpha_\lambda(x)$ and lower semicontinuous as a supremum of continuous functions, we get from (66) that $I(x)$ is continuous on the closed $|\cdot|$-unit ball.

Let us mention at this point that (65), (66) and Figure 1 are typical for discrete time. In continuous time we would have got logarithmic bounds for the Lyapounov exponents due to Lemma 3 and (13). In particular $I(x)$ would be finite on the whole space.

Before coming to the main result, Theorem 19, we first extend the uniform shape theorem to all distributions $\nu * \delta_\lambda$ with $\lambda > 0$.

COROLLARY 18. *Suppose that the dth moment of ν is finite and replace in* (59) *ω by $\omega + \lambda$ and α by α_λ. Then on a set Ω_9 of full $\mathbb{P}$-measure*, (59) *holds for any $\lambda > 0$.*

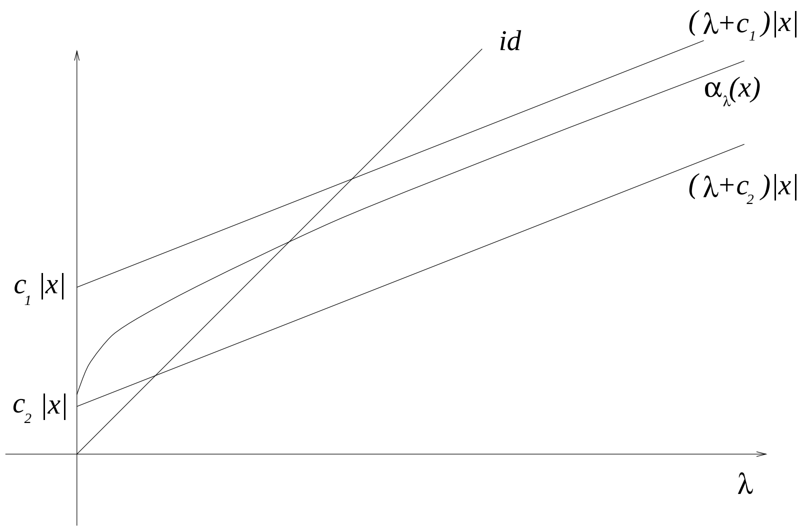

FIG. 1. $\alpha_\lambda(x)$ *in dependence of* λ.

PROOF. According to Corollary 16 there is a set Ω_9 of full $\mathbb{P}$-measure such that (59) is valid for all $0 < \lambda \in \mathbb{Q}$ and $\omega \in \Omega_9$. In order to prove (59) for arbitrary $\lambda > 0$, fix $\varepsilon > 0$ and pick $\lambda_1, \lambda_2 \in \mathbb{Q}$ such that

$$0 < \lambda_1 < \lambda < \lambda_2 \quad \text{and} \quad \sup_{|x| \le c_2/c_1} \alpha_{\lambda_2}(x) - \alpha_{\lambda_1}(x) < \varepsilon/2, \tag{67}$$

where c_1, c_2 are as in Lemma 3. This is possible due to the joint continuity of $\alpha_\lambda(x)$. Note that for any $\mu \ge 0$ and any x, y with $|x| > c_2/c_1$, $|y| = 1$ due to (14),

$$\alpha_\mu(x) \ge |x|(c_1 + \mu) \ge c_2 + \mu \ge \alpha_\mu(y)$$

and consequently for any $K \subseteq \mathbb{R}^d$ with $K \cap S^{d-1} \ne \varnothing$,

$$\inf \alpha_{\lambda_2}(K) - \inf \alpha_{\lambda_1}(K) < \varepsilon/2. \tag{68}$$

Now let $x_n \in \mathbb{R}^d$ and $\varnothing \ne K_n \subseteq \mathbb{R}^d$ with $c|x_n| \le \delta(x_n, K_n) \to \infty$ for some $c > 0$ and abbreviate $a_\mu := a(x_n, K_n, \omega + \mu)$ and $\alpha_\mu := \inf \alpha_\mu(x_n - K_n)$ for $\omega \in \Omega_9$ and n fixed. Then due to monotonicity of a_μ and α_μ in μ,

$$\begin{aligned} |a_\lambda - \alpha_\lambda| &\le (a_\lambda - a_{\lambda_1}) + |a_{\lambda_1} - \alpha_{\lambda_1}| + (\alpha_\lambda - \alpha_{\lambda_1}) \\ &\le (a_{\lambda_2} - \alpha_{\lambda_2}) + (\alpha_{\lambda_1} - a_{\lambda_1}) + |a_{\lambda_1} - \alpha_{\lambda_1}| + 2(\alpha_{\lambda_2} - \alpha_{\lambda_1}), \end{aligned}$$

which is after division by $\delta(x_n, K_n)$ less than ε for large n due to (59) and (68). The claim now follows. □

THEOREM 19. *Suppose that the dth moment of* ν *is finite. Then on a set* Ω_{10} *of full* $\mathbb{P}$*-measure for any* $x \in \mathbb{R}^d$, S_n/n *obeys a large deviation principle at rate* *n with rate function $I(\cdot - x)$ under $Q_{n,x,\omega}$ as n tends to infinity. Namely, for any $A, B \subseteq \mathbb{R}^d$, A closed, B open and $\omega \in \Omega_{10}$,*

$$\limsup_{n\to\infty} \frac{1}{n} \ln Q_{n,x,\omega}[S_n \in nA] \le -\inf_{y\in A} I(y-x), \tag{69}$$

$$\liminf_{n\to\infty} \frac{1}{n} \ln Q_{n,x,\omega}[S_n \in nB] \ge -\inf_{y\in B} I(y-x). \tag{70}$$

We follow with some modifications the line of proof of Sznitman [20], Theorem 2.1.

PROOF. First observe that $Q_{n,x,\omega}$ and I do not change when ω and ν are shifted by a constant $\lambda \ge -\underline{\nu}$, that is, replaced by $\lambda+\omega$ and $\nu * \delta_\lambda$, respectively. Hence we may assume $\underline{\nu} = 0$. Moreover, since $I(y-x) = \infty$ if $|y-x| > 1$ and $Q_{n,x,\omega}$-a.s., $|S_n - [nx]| \le n$ we can restrict to the case where A and B are closed and open, respectively, subsets of the closed $|\cdot|$-unit ball $B(x,1)$ with center x. Now let $\omega \in \Omega_{10} := \Omega_7 \cap \Omega_9$ [see (62), Corollary 18].

We start with the proof of (69). We may assume $x \notin A$; otherwise (69) is immediate from $I(0) = 0$. Then for $n \in \mathbb{N}$ and $\lambda > 0$,

$$\exp(-\lambda n) E_{nx}\left[\exp\left(-\sum_{m=0}^{n-1} \omega(S_m)\right),\ S_n \in nA\right] \le e(nx, nA, \lambda+\omega).$$

Since Corollary 18 applies because λ and $\delta(x, A)$ are positive, we get, by letting $\lambda > 0$ vary and using Proposition 17,

$$\limsup_{n\to\infty} \frac{1}{n} \ln Q_{n,x,\omega}[S_n \in nA] \le -\sup_{\lambda>0} \inf_{z\in A} (\alpha_\lambda(x-z) - \lambda). \tag{71}$$

For the proof of (69) we therefore need to exchange infimum and supremum in (71). To this end, let $\varepsilon > 0$. Thanks to compactness of A there are $\lambda_1, \dots, \lambda_m > 0$ such that the compact sets

$$A_i := \left\{z \in A\colon \alpha_{\lambda_i}(x-z) - \lambda_i \ge \inf_{y\in A} I(x-y) - \varepsilon\right\}, \qquad i = 1, \dots, m$$

cover A. Hence the left side of (69) is less than

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \ln Q_{n,x,\omega}[S_n \in n(A_1 \cup \dots \cup A_m)] \\
&\quad = \sup_{i=1,\dots,m} \limsup_{n\to\infty} \frac{1}{n} \ln Q_{n,x,\omega}[S_n \in nA_i] \\
&\quad \le -\inf_{i=1,\dots,m} \sup_{\lambda>0} \inf_{z\in A_i} (\alpha_\lambda(x-z) - \lambda) \qquad \text{[by (71)]} \\
&\quad \le -\inf_{i=1,\dots,m} \inf_{z\in A_i} (\alpha_{\lambda_i}(x-z) - \lambda_i) \le \varepsilon - \inf_{y\in A} I(x-y).
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this proves (69).

We now establish estimate (70). Since the restriction of I to $B(0,1)$ is continuous, it suffices to show

$$\liminf_{n\to\infty} \frac{1}{n} \ln Q_{n,x,\omega}[S_n \in nB(z,r)] \geq -I(z-x) \tag{72}$$

for all $r > 0$ and z with $|u| < 1$ where $u := z - x \neq 0$. To prove this, set $y_n = y(nz, \varepsilon, R, \omega)$ for fixed $\varepsilon > 0$ and $R \in \mathbb{N}$, R even, as defined in the proof of Proposition 17. Since the distance between nz and y_n grows not more than logarithmicly in n, the ball $B(y_n, R)$ is finally contained in $B(nz, nr)$. Thus, as in the proof of Proposition 17, by the strong Markov property the left member of (72) is bigger than

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n} \ln E_{nx}\left[\exp\left(-\sum_{m=0}^{H(y_n)-1} \omega(S_m)\right),\ H(y_n) \leq n\right] \\ -\varepsilon + \frac{1}{R} \ln P_0[S_R = 0],\end{aligned} \tag{73}$$

where the normalizing constant vanished due to Proposition 17 and $\underline{\nu} = 0$. Consequently it suffices to show that the first term in (73) is greater than $-I(u)$ because letting $\varepsilon \searrow 0$ and $R \to \infty$ then proves (72). To do this, we first determine some point λ_0 at which the continuous function $\lambda \mapsto \alpha_\lambda(u) - \lambda$ attains its maximum. Indeed, the function has a maximum due to $|u| < 1$ and (65). If $\alpha'_{\lambda-}(u) < 1$ for all $\lambda > 0$, the maximum is located at $\lambda_0 = 0$. Otherwise we can choose $\lambda_0 = \sup\{\lambda > 0\colon \alpha'_{\lambda-}(u) \geq 1\}$. In any case $I(u) = \alpha_{\lambda_0}(u) - \lambda_0$. Now the first term in (73) is bigger than

$$\sup_{0\leq\gamma<1} \liminf_{n\to\infty} \frac{1}{n} \ln E_{nx}\left[\exp\left(-\sum_{m=0}^{T_n(\gamma)-1} (\omega+\lambda_0)(S_m)\right) \exp(\lambda_0 T_n(\gamma)),\ T_n(\gamma) < n\right],$$

where $T_n(\gamma) = \min\{m > \gamma n\colon S_m = y_n\}$. This is greater than

$$\begin{aligned}\sup_{0\leq\gamma<1}\Biggl(\lambda_0\gamma + \sup_{\lambda\geq\lambda_0} \liminf_{n\to\infty} \frac{1}{n} \ln E_{nx} \\ \times\left[\exp\left(-\sum_{m=0}^{T_n(\gamma)-1} (\omega+\lambda)(S_m)\right),\ T_n(\gamma) < n\right]\Biggr).\end{aligned} \tag{74}$$

It is now time to explain the relation between $\alpha'_{\lambda\pm}(u)$ and the velocity at which the walk moves in direction u.

LEMMA 20. *Suppose that the dth moment of* ν *is finite. Then there is a set* Ω_9 *of full* $\mathbb{P}$*-measure such that for all* $\omega \in \Omega_9$ *the following holds. Let* $x, y, x_n, y_n \in \mathbb{R}^d$ $(n \in \mathbb{N})$ *with* $x_n/n \to x$, $y_n/n \to y$ *and* $u := x - y \neq 0$. *Furthermore, let* $\lambda > 0$ *and* $0 \leq \gamma < \delta < \infty$ *such that*

$$(\gamma, \delta) \cap [\alpha'_{\lambda+}(u), \alpha'_{\lambda-}(u)] \neq \varnothing. \tag{75}$$

Then

$$\liminf_{n\to\infty} \frac{1}{n} \ln E_{x_n}\left[\exp\left(-\sum_{m=0}^{T_n(\gamma)-1} (\omega+\lambda)(S_m)\right),\ T_n(\gamma) < \delta n\right] \geq -\alpha_\lambda(u). \tag{76}$$

Let us postpone the proof of this lemma until the end of the section. In the case $\lambda_0 = 0$, we have $0 < \alpha'_{\lambda-}(u) < 1$ for all $\lambda > 0$ and consequently (75) with $\gamma = 0$ and $\delta = 1$. Thus by Lemma 20, (74) is greater than $\sup_{\lambda>0} -\alpha_\lambda(u) = -\alpha_0(u) = -I(u)$ which had to be shown. In the case $\lambda_0 > 0$ there exists for any $\gamma < 1$ and any neighborhood U of λ_0 some $\lambda_0 \le \lambda \in U$ fulfilling (75), again with $\delta = 1$. This time Lemma 20 yields that (74) is bigger than $\lambda_0 - \alpha_{\lambda_0}(u)$, thus completing the proof. □

PROOF OF LEMMA 20. Let $\omega \in \Omega_9$ (see Corollary 18). First we show that

$$\lim_{n\to\infty} \hat{P}^{y_n}_{x_n,\,\omega+\lambda}\big[H(y_n)/n \in (\gamma_1, \gamma_2)\big] = 1 \tag{77}$$

if $\gamma_1 < \alpha'_{\lambda+}(u) \le \alpha'_{\lambda-}(u) < \gamma_2$. To this end, let $0 < \mu < \lambda$. Then due to Corollary 18,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \ln \hat{P}^{y_n}_{x_n,\,\omega+\lambda}[H(y_n) \ge \gamma_2 n] \\
&\quad = \alpha_\lambda(u) + \limsup_{n\to\infty} \frac{1}{n} \ln E_{x_n}\Bigg[\exp\Bigg(-\sum_{m=0}^{H(y_n)-1} (\omega + \lambda - \mu)(S_m)\Bigg) \\
&\qquad\qquad \times \exp(-\mu H(y_n)),\ \gamma_2 n \le H(y_n) < \infty\Bigg] \\
&\quad \le \alpha_\lambda(u) - \alpha_{\lambda-\mu}(u) - \mu\gamma_2,
\end{aligned}$$

which is negative for μ small enough. A similar statement holds for the event $\{H(y_n) \le \gamma_1 n\}$, which implies (77).

Now we are ready for the proof of (76). Let $\rho \in (0, 1)$ and $\eta > 0$ such that

$$\rho\alpha'_{\lambda+}(u) + (1-\rho)\alpha'_{\lambda-}(u) + [-\eta, \eta] \subseteq (\gamma, \delta) \quad \text{and set } \xi_n = (1-\rho)x_n + \rho y_n.$$

Then by the strong Markov property, the left-hand side of (76) is bigger than

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n} \ln E_{x_n}\Bigg[\exp\Bigg(-\sum_{m=0}^{H(\xi_n)-1} (\omega + \lambda)(S_m)\Bigg),\ \frac{H(\xi_n)}{\rho n} \in \alpha'_{\lambda+}(u) + [-\eta, \eta]\Bigg] \\
&\quad + \liminf_{n\to\infty} \frac{1}{n} \ln E_{\xi_n}\Bigg[\exp\Bigg(-\sum_{m=0}^{H(y_n)-1} (\omega + \lambda)(S_m)\Bigg),\ \frac{H(y_n)}{(1-\rho)n} \in \alpha'_{\lambda-}(u) + [-\eta, \eta]\Bigg],
\end{aligned}$$

which is, due Corollary 18, greater than

$$\begin{aligned}
&\sup_{\lambda_1>\lambda}\Bigg(\liminf_{n\to\infty} \frac{1}{n} \ln \hat{P}^{\xi_n}_{x_n,\,\omega+\lambda_1}\Bigg[\frac{H(\xi_n)}{\rho n} \in \alpha'_{\lambda+}(u) + [-\eta, \eta]\Bigg] - \alpha_{\lambda_1}(\rho u)\Bigg) \\
&\quad + \sup_{0<\lambda_2<\lambda}\Bigg(\liminf_{n\to\infty} \frac{1}{n} \ln \hat{P}^{y_n}_{\xi_n,\,\omega+\lambda_2}\Bigg[\frac{H(y_n)}{(1-\rho)n} \in \alpha'_{\lambda-}(u) + [-\eta, \eta]\Bigg] \\
&\qquad\qquad - \alpha_{\lambda_2}((1-\rho)u) + (\lambda_2 - \lambda)(\alpha'_{\lambda-}(u) + \eta)\Bigg).
\end{aligned} \tag{78}$$

Now in any neighborhood of λ there is some $\lambda_1 > \lambda$ such that $\alpha'_{\lambda_1}(u)$ exists and belongs to $\alpha'_{\lambda+}(u) + [-\eta, 0]$. A similar assertion holds for $\lambda_2 < \lambda$ and $\alpha'_{\lambda-}(u) + [0, \eta]$. Thus applying (77) twice, we see that (78) is greater than $-\alpha_\lambda(\rho u) - \alpha_\lambda((1-\rho)u) = -\alpha_\lambda(u)$. □

7. The constant nonrandom case. If the potentials $\omega(x)$ are equal to a nonrandom positive constant λ, our basic two-point functions e and g can be written as

$$e(0, x, \lambda) = E_0[\exp(-\lambda H(x))] \quad \text{and} \quad g(0, x, \lambda) = \sum_{n=0}^{\infty} e^{-\lambda(n+1)} P_0[S_n = x].$$

For a detailed study of Green's function, which is, up to the factor $e^{-\lambda}$, the generating function of the site occupancy probabilities, see [7], Chapters 3.2, 3.3, A.3; see also [12], Chapter 1.5. The exponential decay rate $\alpha(e_1, \delta_\lambda, d)$ of the Green's function $g(0, ke_1, \lambda)$ in the direction of the coordinate axes has previously been computed by Madras and Slade in [14], Theorem A.2, giving

$$\alpha(e_1, \delta_\lambda) = \operatorname{arccosh}\left(de^\lambda - d + 1\right).$$

Unfortunately this computation cannot be directly extended to other directions. However, the large deviation estimates of the preceding section enable us to give a quick derivation of the decay rates for arbitrary directions. They are as follows.

THEOREM 21. *Suppose $\nu = \delta_\lambda$ for some $\lambda > 0$. Then for any $x = (x_1, \dots, x_d) \in \mathbb{R}^d \setminus \{0\}$,*

$$\alpha(x) = \sum_{j=1}^{d} x_j \operatorname{arcsinh}(x_j s), \tag{79}$$

where $s > 0$ solves

$$e^\lambda d = \sum_{j=1}^{d} \sqrt{1 + (x_j s)^2}. \tag{80}$$

In particular, for any $n \in \{1, \dots, d\}$,

$$\alpha(e_1 + \cdots + e_n) = n \operatorname{arccosh} \frac{d(e^\lambda - 1) + n}{n}.$$

Figure 2 shows some unit spheres of α for various λ. Observe that for large λ the shape approximates the diamond which is the asymptotic shape arising in first passage percolation for constant passage times. This is the only explicit example for (36) we know.

PROOF. Note that $Q_{n,0,\lambda}$ is just P_0 and thus does not depend on λ. Consequently, Theorem 19 tells us that the distribution of S_n/n under P_0 satisfies a large deviation principle with convex rate function

$$I(x) = \sup_{\lambda \geq 0} \alpha(x, \delta_\lambda) - \lambda. \tag{81}$$

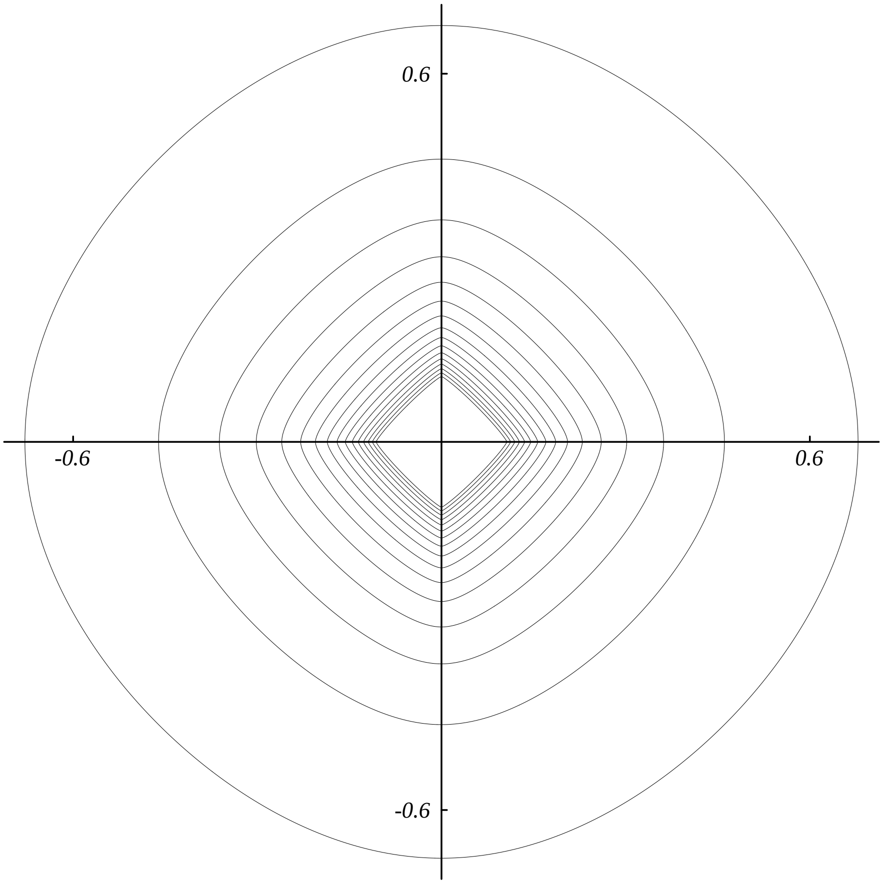

FIG. 2. *The unit sphere of α in two dimensions for the constant potentials* $\lambda = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots, 8$ (*from outside inward*).

Observe that this equation characterizes $\alpha(x, \delta_\lambda)$ completely, provided we know that $\alpha(x, \delta_\lambda)$ is homogeneous in x and concave increasing in λ. On the other hand, because of the well-known theorem of Cramér (see, e.g., [3], Theorem 2.2.30), the same process satisfies a large deviation principle with convex rate function

$$\tilde{I}(x) = \sup_{y \in \mathbb{R}^d} x \cdot y - \ln E_0[\exp(S_1 \cdot y)]. \tag{82}$$

Since I and $\tilde{I}$ are continuous on the set where they are finite, they coincide. Thus we get from (81) and (82),

$$\sup_{\lambda \geq 0} \alpha(x, \delta_\lambda) - \lambda = \sup_{y \in \mathbb{R}^d} x \cdot y - \ln E_0[\exp(S_1 \cdot y)]. \tag{83}$$

This implies

$$\alpha(x, \delta_\lambda) = \sup\{x \cdot y\colon \ln E_0[\exp(S_1 \cdot y)] = \lambda\}, \tag{84}$$

since the right side of (84) satisfies (83), is homogeneous in x and concave increasing in λ. Now fix $x = (x_1, \dots, x_d) \neq 0$ and let $y = (y_1, \dots, y_d)$ maximize $x \cdot y$ under the constraint $E_0[\exp(S_1 \cdot y)] = e^\lambda$ such that $\alpha(x) = x \cdot y$. Then an elementary calculation yields that there is some $t \in \mathbb{R}$ with

$$0 = x_i + tE_0[S_1 \cdot e_i \exp(S_1 \cdot y)] = x_i + \frac{t}{d} \sinh y_i, \qquad i = 1, \dots, d, \tag{85}$$

$$e^\lambda = E_0[\exp(S_1 \cdot y)] = \frac{1}{d} \sum_{i=1}^{d} \cosh y_i = \frac{1}{d} \sum_{i=1}^{d} \sqrt{1 + \sinh^2 y_i}. \tag{86}$$

Since $\alpha(x) = x \cdot y$, (85) and (86) prove (79) and (80), respectively, with $s = -d/t$. Observe that s must be positive because otherwise the right-hand side of (79) would be negative. □

This result shows that $\alpha(x)$ is analytic in $x \neq 0$ if $\nu = \delta_\lambda$ and of course raises the question whether this is also true in the random case.

Acknowledgments. I am indebted and grateful to Prof. A.-S. Sznitman, who proposed this subject, for many helpful suggestions and his constant interest in my work. Special thanks are due to Franz Merkl, in particular for his help with Theorem 21. For [16] I am glad to thank Prof. C. M. Newman.

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