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# ERROR ESTIMATES FOR THE BINOMIAL APPROXIMATION OF AMERICAN PUT OPTIONS

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We establish some error estimates for the binomial approximation of American put prices in the Black–Scholes model. Namely, we prove that if $P$ is the American put price and $P_n$ its $n$-step binomial approximation, there exist positive constants $c$ and $C$ such that $-c/n^{2/3} \le P_n - P \le C/n^{3/4}$. With an additional assumption on the interest rate and the volatility, a better upper bound is derived.

**1. Introduction.** The purpose of this paper is to derive some error estimates for the binomial approximation of American put prices. Recall that an American put on a stock is the right to *sell* one share at a specified price $K$ (called the exercise price) at *any* instant until a given future date $T$ (called the expiration date or date of maturity). In the Black–Scholes model, the value at time $t$ ($0 \le t \le T$) of such an option can be written as a function $F(t, S_t)$ of time and the current stock price, with

$$F(t,x) = \sup_{\tau \in \mathscr{T}_{0,\,T-t}} \mathbf{E} \exp(-r\tau)\left(K - x\exp\left(\left(r - \frac{\sigma^2}{2}\right)\tau + \sigma B_\tau\right)\right)^+,$$

where $B = (B_t)_{0 \le t \le T}$ is a standard Brownian motion, $r$ is the interest rate (assumed to be a positive constant), $\sigma$ is the so-called volatility (also assumed to be a positive constant) and $\mathscr{T}_{0,\,T-t}$ denotes the set of all stopping times of the natural filtration of $B$, with values in the interval $[0, T-t]$. We refer the reader to [14, 15, 25, 21] for basic results on American options.

The above formula relates the price function $F$ to an optimal stopping problem along the paths of Brownian motion. Indeed, we have $F(t,x) = P(t, \log(x))$, with

$$P(t,x) = \sup_{\tau \in \mathscr{T}_{0,\,T-t}} \mathbf{E} e^{-r\tau} \psi(x + \mu\tau + \sigma B_\tau), \tag{1}$$

where $\psi(x) = (K - e^x)^+$ and $\mu = r - \sigma^2/2$.

A natural numerical method to compute the function $P$ defined by (1) is to approximate the underlying Brownian motion $B$ by a random walk and apply dynamic programming. Various approximations of $B$ have been considered in the financial literature (see [10, 26, 13, 9, 30, 23]).

In this paper, we will concentrate on the following approximation (which, in this financial context, appears for the first time in [13]). Let $(\varepsilon_k)_{k \ge 1}$ be a

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sequence of i.i.d. random variables, satisfying $\mathbf{P}(\varepsilon_k = 1) = \mathbf{P}(\varepsilon_k = -1) = 1/2$ and let

$$B_t^{(n)} = \frac{\sqrt{T}\sum_{k=1}^{[nt/T]} \varepsilon_k}{\sqrt{n}}.$$

For $t = kT/n$, $k = 0, 1, \dots, n$, denote

$$P^{(n)}(t, x) = \sup_{\tau \in \mathscr{T}_{0,T-t}^{(n)}} \mathbf{E} e^{-r\tau} \psi(x + \mu\tau + \sigma B_\tau^{(n)}),$$

where $\mathscr{T}_{0,T-t}^{(n)}$ is the set of stopping times (with respect to the natural filtration of $B^{(n)}$), with values in $[0, T-t] \cap \{0, T/n, 2T/n, \dots, (n-1)T/n, T\}$. From the classical results of Kushner [19], it follows that $\lim_{n\to\infty} P^{(n)}(0, x) = P(0, x)$ (see [1, 22] for related results). To our knowledge, the rate of convergence is not known. Our main result is the following.

THEOREM 1. *For any real number $x$, there exist positive constants $c$ and $C$ such that*

$$\forall\, n \in \mathbf{N}, \qquad -\frac{c}{n^{2/3}} \le P^{(n)}(0, x) - P(0, x) \le \frac{C}{n^{3/4}}.$$

*Moreover, if $\mu \le 0$ (i.e., $r \le \sigma^2/2$),*

$$\forall\, x \in \mathbf{R},\ \exists\, \tilde{C} > 0,\ \forall\, n \in \mathbf{N}, \qquad P^{(n)}(0, x) - P(0, x) \le \tilde{C}\left(\frac{\sqrt{\log n}}{n}\right)^{4/5}.$$

These estimates are probably not optimal. Indeed, numerical experiments seem to suggest that the error is $O(1/n)$ (cf. [8]), as in the approximation of European options (cf. [23]). We will comment on the limits of our methods in Remark 3. The constants in Theorem 1 may depend on $x$, but a careful examination of the proof will show that they can be chosen independently of $x$, as long as $x$ remains in a fixed bounded set.

REMARK 1. In the past few years, many numerical methods have been developed to price American options (see [8] for a recent comparison of various methods). The only one for which error bounds are known is the Gaussian approximation, which uses standard normal variables as $\varepsilon_k$'s instead of Bernoulli trials. The error bound in that case is $O(1/n)$ (as shown in [9]) but the method can be implemented for small values of $n$ only. Our methods could probably be applied to other binomial approximations, but, since we are not able to derive sharp results, we prefer to focus on the simplest binomial approximation, for which the mathematical analysis is easier. We also note that an adaptation of the methods of Baiocchi and Pozzi [2] might lead to error estimates for the finite difference method of Brennan and Schwartz [7].

REMARK 2. In the context of discretization of stochastic differential equations, a lot of work has been devoted to the derivation of error estimates for quantities such as $|\mathbf{E}f(X_T) - \mathbf{E}f(X^n_T)|$, where $X$ is a diffusion process, $X^n$ some approximation, and $T$ a *deterministic* time (see [18, 28]). The method for deriving these estimates consists of relating the error to the solution of a parabolic partial differential equation (cf. [24, 28, 29, 3]). Our method is similar in the sense that we will use the parabolic variational inequality satisfied by the function price $P$.

REMARK 3. The main difficulty that we face in trying to adapt the techniques used in the discretization of SDE's is the lack of regularity of the payoff function $\psi$ and of the price function $P$. Indeed, the solution of a variational inequality is typically less regular than the solution of the corresponding parabolic PDE. The strongest regularity results for variational inequalities that we could find in the literature are quadratic estimates for the second order derivatives, due to Friedman and Kinderlehrer (see Theorem 3 below). The original estimates of Friedman and Kinderlehrer (see [11] and [17]) require the boundedness of the second derivative of the payoff function, an assumption which is not satisfied in our case, and, when deriving quadratic estimates applicable to the American put, we end up with constants blowing up near $T$. This is the main reason why we are not able to derive sharper estimates. A better understanding of the behavior of the partial derivatives of the American put price would be needed to improve our results. Another way of evaluating our approach is to apply it to $C^2$ payoff functions. We are able to prove that if $f$ is a bounded function, with bounded first and second derivatives, then, with the same notations as above,

$$\sup_{\tau\in\mathcal{T}^{(n)}_{0,T}} \mathbf{E}e^{-r\tau}f(B^{(n)}_\tau) - \sup_{\tau\in\mathcal{T}_{0,T}} \mathbf{E}e^{-r\tau}f(B_\tau) = O\left(\frac{\sqrt{\log n}}{n}\right).$$

This result (which will be published in a separate paper) is probably not yet optimal, but seems to be close to the right order of convergence.

The paper is organized as follows. In Section 2, we state some preliminary results concerning partial derivatives of the function $P(t, x)$ and the exercise boundary. In Section 3 we introduce the approximating process $X^{(n)} = x + \sigma B^{(n)}$, state the discrete version of Itô's formula and relate the finite-difference operator appearing in this formula to the infinitesimal generator of the limit process. The results of this section include estimates to be used in the proof of Theorem 1. The proof, which is given in Section 4, consists of looking at the continuous price function along the paths of the approximating process and breaking the set $[0, T] \times \mathbf{R}$ into three separate regions $\bar{S}$, $\bar{C}$ and $\bar{B}$, which are approximations of the stopping region, the continuation region and the free boundary. The Appendix is devoted to the proof of an analytic result stated in Section 2.

Throughout the paper the letter $C$ will be used to denote a positive constant depending on the parameters of the problem (but not on $n$). The value of $C$ may vary from line to line.

## 2. Preliminary results.

2.1. *Partial derivatives of P.* Consider the function $P(t, x)$ defined, for $(t, x) \in [0, T] \times \mathbf{R}$, by (1). It is easy to check that $P$ is continuous on $[0, T] \times \mathbf{R}$ and that $x \mapsto P(t, x)$ is a Lipschitz function, with Lipschitz constant independent of $t$, so that $\partial P/\partial x$ is bounded on $[0, T] \times \mathbf{R}$. Further regularity properties of $P$ can be proved using variational inequalities.

It is well known that $P$ satisfies the following variational inequality:

$$\max\left(\frac{\partial P}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial x^2} + \mu\frac{\partial P}{\partial x} - rP, \psi - P\right) = 0,$$

with $P(T, \cdot) = \psi$. Theorem 2 can be derived from [12] (Theorem 3.6 and its proof; see also [21], Section 3).

THEOREM 2. *The partial derivatives $\partial P/\partial t$ and $\partial^2 P/\partial x^2$ are locally bounded on $[0, T) \times \mathbf{R}$ and there exists a positive constant $C$ such that*

$$\forall (t, x) \in [0, T) \times \mathbf{R}, \qquad \left|\frac{\partial P}{\partial t}(t, x)\right| + \left|\frac{\partial^2 P}{\partial x^2}(t, x)\right| \le \frac{C}{\sqrt{T - t}}.$$

In the sequel we will need estimates for $(\partial^2 P/\partial t^2)$ and $(\partial^2 P/\partial t \partial x)$ given in the following theorem.

THEOREM 3. *There exists a positive constant $C$ such that, for $0 \le t_1 < T' < T$,*

$$\int_{t_1}^{T'} dt (T' - t) \int_{-\infty}^{+\infty} \frac{dx}{(1 + x^2)^{3/2}} \left(\frac{\partial^2 P}{\partial t^2}(t, x)\right)^2 + (T' - t_1) \int_{-\infty}^{+\infty} \frac{dx}{(1 + x^2)^{3/2}} \left(\frac{\partial^2 P}{\partial t \partial x}(t_1, x)\right)^2 \le \frac{C}{\sqrt{T - T'}}.$$

This theorem is essentially a variant of results of Friedman and Kinderlehrer (see [11] and [17], Chapter VIII). Its proof is given in the Appendix.

2.2. *The free boundary.* For each $t \in [0, T)$, there exists a real number $s(t)$ such that

$$\forall\ x \le s(t),\ P(t, x) = \psi(x) \quad \text{and} \quad \forall\ x > s(t),\ P(t, x) > \psi(x).$$

In the finance literature, $e^{s(t)}$ is called the *critical* price at time $t$; in the context of variational inequalities, the curve $(s(t))_{0 \le t < T}$ is called the free boundary. Note that, because of the variational inequality satisfied by $P$, we have

$$\forall\ t \in [0, T),\ \forall\ x > s(t),$$

$$\frac{\partial P}{\partial t}(t, x) + \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial x^2}(t, x) + \mu\frac{\partial P}{\partial x}(t, x) - rP(t, x) = 0. \tag{2}$$

The set

$$\mathscr{C} = \{(t, x) \in [0, T) \times \mathbf{R} \mid x > s(t)\}$$

is called the *continuation* region. Its complement is the *stopping* region. Since $P(t, x)$ is a nonincreasing function of $t$, $t \mapsto s(t)$ is nondecreasing. It can be proved (see [11]) that the function $s$ is $C^\infty$ on the interval $[0, T)$. The behavior of $s(t)$ for $t$ close to $T$ has been studied in [16] and [4, 5] (see also [20]); in particular, we have $\lim_{t\to T} s(t) = \log(K)$. We will need the following estimate for the modulus of continuity of $s$.

PROPOSITION 1. *There exists a constant $C$ such that*

$$\forall\, t_1, t_2 \in [0, T), \qquad (s(t_2) - s(t_1))^2 \le C \sup_{x\in\mathbf{R}} \big|P(t_2, x) - P(t_1, x)\big|.$$

PROOF. The proof is inspired by [11] (especially page 164). We may (and shall) assume that $0 \le t_1 < t_2 < T$. Introduce the differential operator

$$A = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \mu\frac{\partial}{\partial x} - r.$$

If $s(t_1) < x < s(t_2)$, we have $(\partial P/\partial t)(t_1, x) + AP(t_1, x) = 0$ and $AP(t_2, x) = A\psi(x) = -rK$. Hence

$$\begin{aligned}\forall\, x \in (s(t_1), s(t_2)), \qquad rK &= \frac{\partial P}{\partial t}(t_1, x) + AP(t_1, x) - AP(t_2, x)\\ &\le AP(t_1, x) - AP(t_2, x),\end{aligned}$$

since $t \mapsto P(t, x)$ is nonincreasing. Now let $\phi$ be a nonnegative $C^\infty$ function compactly supported in the open interval $(s(t_1), s(t_2))$. Integrating the last inequality yields

$$\begin{aligned} rK\int_{s(t_1)}^{s(t_2)} \phi(x)\,dx &\le \int_{s(t_1)}^{s(t_2)} (AP(t_1, x) - AP(t_2, x))\phi(x)\,dx\\ &= \int_{s(t_1)}^{s(t_2)} (P(t_1, x) - P(t_2, x))A^*\phi(x)\,dx,\end{aligned}$$

with $A^*\phi = (\sigma^2/2)\phi'' - \mu\phi' - r\phi$. Choosing

$$\phi(x) = \rho\left(\frac{x - s(t_1)}{s(t_2) - s(t_1)}\right),$$

with $\rho$ smooth, nonnegative, compactly supported in $(0, 1)$, we obtain

$$\begin{aligned} &rK(s(t_2) - s(t_1))\int_0^1 \rho(y)\,dy\\ &\quad\le \sup_{x\in\mathbf{R}}\big|P(t_2, x) - P(t_1, x)\big|\left(\frac{C_1}{s(t_2) - s(t_1)} + C_2 + C_3(s(t_2) - s(t_1))\right),\end{aligned}$$

where $C_1$, $C_2$, $C_3$ are positive constants. From this inequality [and the fact that $s(t)$ is bounded] the proposition follows easily. □

We complete this section with a technical result on the difference between the price function and the payoff near the free boundary.

PROPOSITION 2. *There exists a positive constant $C$ such that*

$$\forall\, t \in [0, T),\ \forall\, y \geq s(t), \qquad P(t, y) - \psi(y) \leq \frac{C}{\sqrt{T-t}}(y - s(t))^2.$$

PROOF. Let $\hat{\psi}: \mathbf{R} \to \mathbf{R}$ be a bounded $C^2$ function, with bounded derivatives, satisfying $\hat{\psi}(y) = \psi(y)$ for $y \leq \log(K)$ and $\hat{\psi}(y) \leq 0 = \psi(y)$ for $y > \log K$. It is clear that such a function exists. Now, fix $t \in [0, T)$. We know that the function $y \mapsto P(t, y) - \hat{\psi}(y)$ is of class $C^1$ and that $P(t, s(t)) - \hat{\psi}(s(t)) = (\partial P/\partial x)(t, s(t)) - \hat{\psi}'(s(t)) = 0$. This is the well-known *smooth fit* property (see, for instance, [12], Corollary 3.7). Therefore, using Taylor's formula, we have

$$\forall\, y \geq s(t), \qquad P(t, y) - \hat{\psi}(y) \leq \frac{(y - s(t))^2}{2} \sup_{x > s(t)} \left( \frac{\partial^2 P}{\partial x^2}(t, x) - \hat{\psi}''(x) \right).$$

The proposition is now a consequence of the inequality $P(t, y) - \psi(y) \leq P(t, y) - \hat{\psi}(y)$ and the estimate for $\partial^2 P/\partial x^2$ given in Theorem 2. □

**3. The discrete approximation.** Recall that $(\varepsilon_k)_{k \geq 1}$ is a sequence of i.i.d. random variables, satisfying $\mathbf{P}(\varepsilon_k = 1) = \mathbf{P}(\varepsilon_k = -1) = 1/2$. Let $n$ be a fixed positive integer. We will denote by $X^{(n)} = (X_t^{(n)})_{0 \leq t \leq T}$ the stochastic process defined by

$$X_t^{(n)} = x + \frac{\sigma\sqrt{T}}{\sqrt{n}} \sum_{k=1}^{[nt/T]} \varepsilon_k = x + \sigma B_t^{(n)}.$$

For notational convenience, we set

$$h = \frac{T}{n}.$$

Let $u(t, x)$ be a continuous function on $[0, T] \times \mathbf{R}$. For $0 \leq t \leq T - h$ and $x \in \mathbf{R}$, let

$$\mathscr{D}u(t, x) = \tfrac{1}{2}\big[u\big(t + h, x + \sigma\sqrt{h}\big) + u\big(t + h, x - \sigma\sqrt{h}\big)\big] - u(t, x).$$

The operator $(n/T) \times \mathscr{D} = (1/h)\mathscr{D}$ can be viewed as a finite difference approximation of the differential operator $(\partial/\partial t) + (\sigma^2/2)(\partial^2/\partial x^2)$. Indeed, if $u$ is smooth, $(1/h) \times \mathscr{D}u(t, y) = (\partial u/\partial t)(t, y) + (\sigma^2/2)(\partial^2 u/\partial x^2)(t, y) + O(h)$. The following elementary proposition can be viewed as the discrete time version of Itô's formula.

PROPOSITION 3. *There exists a martingale $(M_t)_{0 \leq t \leq T}$ (with respect to the natural filtration of $X^{(n)}$), such that $M_0 = 0$ and, for all $t \in \{0, h, 2h, \ldots,$*

$(n-1)h, nh = T\}$,

$$u(t, X_t^{(n)}) = u(0, x) + M_t + \sum_{j=1}^{nt/T} \mathscr{D}u((j-1)h, X_{(j-1)h}^{(n)}).$$

PROOF. Let $Y_t = u(t, X_t^{(n)})$ and $t = kT/n$. We have

$$Y_t = Y_0 + \sum_{j=1}^{k} (Y_{jh} - Y_{(j-1)h}),$$

and, if $(\mathscr{F}_t^n)_{0\le t\le T}$ denotes the natural filtration of $X^{(n)}$,

$$\begin{aligned} Y_{jh} - Y_{(j-1)h} &= Y_{jh} - \mathbf{E}(Y_{jh} \mid \mathscr{F}_{(j-1)h}^n) + \mathbf{E}(Y_{jh} \mid \mathscr{F}_{(j-1)h}^n) - Y_{(j-1)h} \\ &= M_{jh} - M_{(j-1)h} + \mathscr{D}u((j-1)h, X_{(j-1)h}^{(n)}), \end{aligned}$$

where $M$ is the martingale defined by

$$M_t = \sum_{j=1}^{[nt/T]} \big(Y_{jh} - \mathbf{E}(Y_{jh} \mid \mathscr{F}_{(j-1)h}^n)\big). \qquad \square$$

In Section 4 we will apply Proposition 3 to the function $u(t, y) = e^{-rt} P(t, y + \mu t)$. Proposition 4 will be used to control $\mathscr{D}u$.

PROPOSITION 4. *Let $0 \le t \le T - h$ and $x \in \mathbf{R}$. Assume $v$ is a $C^2$ function on $[t, t+h] \times [x - \sigma\sqrt{h}, x + \sigma\sqrt{h}]$. Then we have*

$$\mathscr{D}v(t, x) = \frac{1}{\sigma} \int_0^{\sqrt{h}} d\xi \int_{-\sigma\xi}^{\sigma\xi} dz \left( z \frac{\partial^2 v}{\partial t \partial x}(t + \xi^2, x + z) + \delta(t + \xi^2, x + z) \right),$$

*where*

$$\delta(\tau, \zeta) = \frac{\partial v}{\partial t}(\tau, \zeta) + \frac{\sigma^2}{2} \frac{\partial^2 v}{\partial x^2}(\tau, \zeta).$$

PROOF. We may assume that $t = x = 0$ without loss of generality. We have

$$\begin{aligned} \mathscr{D}v(0, 0) &= \tfrac{1}{2}\big(v(h, \sigma\sqrt{h}) + v(h, -\sigma\sqrt{h})\big) - v(0, 0) \\ &= \phi(\sqrt{h}) - \phi(0) = \int_0^{\sqrt{h}} \phi'(\xi)\, d\xi, \end{aligned}$$

where $\phi(\xi) = (1/2)(v(\xi^2, \sigma\xi) + v(\xi^2, -\sigma\xi))$. We compute $\phi'$:

$$\begin{aligned} \phi'(\xi) &= \xi \left( \frac{\partial v}{\partial t}(\xi^2, \sigma\xi) + \frac{\partial v}{\partial t}(\xi^2, -\sigma\xi) \right) + \frac{\sigma}{2} \left( \frac{\partial v}{\partial x}(\xi^2, \sigma\xi) - \frac{\partial v}{\partial x}(\xi^2, -\sigma\xi) \right) \\ &= \xi \left( \frac{\partial v}{\partial t}(\xi^2, \sigma\xi) + \frac{\partial v}{\partial t}(\xi^2, -\sigma\xi) \right) + \frac{\sigma^2}{2} \int_{-\xi}^{\xi} \frac{\partial^2 v}{\partial x^2}(\xi^2, \sigma\eta)\, d\eta \\ &= \xi \left( \frac{\partial v}{\partial t}(\xi^2, \sigma\xi) + \frac{\partial v}{\partial t}(\xi^2, -\sigma\xi) \right) + \int_{-\xi}^{\xi} \left( -\frac{\partial v}{\partial t}(\xi^2, \sigma\eta) + \delta(\xi^2, \sigma\eta) \right) d\eta. \end{aligned}$$

Now, writing $\xi = \int_0^\xi d\eta$ and

$$\int_{-\xi}^{\xi} \frac{\partial v}{\partial t}(\xi^2, \sigma\eta)\, d\eta = \int_0^{\xi} \left( \frac{\partial v}{\partial t}(\xi^2, \sigma\eta) + \frac{\partial v}{\partial t}(\xi^2, -\sigma\eta) \right) d\eta,$$

we obtain

$$\begin{aligned}
\phi'(\xi) &= \int_0^{\xi} \left( \frac{\partial v}{\partial t}(\xi^2, \sigma\xi) + \frac{\partial v}{\partial t}(\xi^2, -\sigma\xi) - \left( \frac{\partial v}{\partial t}(\xi^2, \sigma\eta) + \frac{\partial v}{\partial t}(\xi^2, -\sigma\eta) \right) \right) d\eta \\
&\quad + \int_{-\xi}^{\xi} \delta(\xi^2, \sigma\eta)\, d\eta \\
&= \int_0^{\xi} d\eta \int_{\sigma\eta}^{\sigma\xi} dz \left( \frac{\partial^2 v}{\partial t \partial x}(\xi^2, z) - \frac{\partial^2 v}{\partial t \partial x}(\xi^2, -z) \right) + \int_{-\xi}^{\xi} \delta(\xi^2, \sigma\eta)\, d\eta \\
&= \int_0^{\sigma\xi} \frac{z}{\sigma} \left( \frac{\partial^2 v}{\partial t \partial x}(\xi^2, z) - \frac{\partial^2 v}{\partial t \partial x}(\xi^2, -z) \right) dz + \frac{1}{\sigma} \int_{-\sigma\xi}^{\sigma\xi} \delta(\xi^2, z)\, dz \\
&= \frac{1}{\sigma} \int_{-\sigma\xi}^{\sigma\xi} z \frac{\partial^2 v}{\partial t \partial x}(\xi^2, z)\, dz + \frac{1}{\sigma} \int_{-\sigma\xi}^{\sigma\xi} \delta(\xi^2, z)\, dz. \qquad \square
\end{aligned}$$

We will also use the following result, in particular for estimates near maturity.

PROPOSITION 5. *Assume $u(t, x)$ is continuous on $[0, T] \times \mathbf{R}$ and admits locally bounded derivatives $(\partial u/\partial t)$, $(\partial u/\partial x)$, $(\partial^2 u/\partial x^2)$ on $[0, T) \times \mathbf{R}$, such that we have the following*:

(i) *$\partial u/\partial x$ and $(\partial u/\partial t) + (\sigma^2/2)(\partial^2 u/\partial x^2)$ are bounded on $[0, T) \times \mathbf{R}$;*
(ii) $\sup_{0 \le t < T} \int_{-\infty}^{+\infty} |(\partial^2 u/\partial x^2)(t, x)|\, dx < \infty$.

*Then there exists a positive constant $C$ such that, for any integer $j$ satisfying $1 \le j \le n$,*

$$\mathbf{E}\big|\mathscr{D}u\big((j-1)h, X^{(n)}_{(j-1)h}\big)\big| \le \frac{C}{\sqrt{jn}}.$$

For the proof of this proposition, we need the following elementary lemma.

LEMMA 1. *Let $\phi: \mathbf{R} \to \mathbf{R}$ be a function with locally bounded second derivative $\phi''$. For any real number $y$ and for any $\delta > 0$, we have*

$$\left| \frac{1}{2}\big(\phi(y+\delta) + \phi(y-\delta)\big) - \phi(y) \right| \le \frac{\delta}{2} \int_{y-\delta}^{y+\delta} |\phi''(z)|\, dz.$$

PROOF. We have

$$\begin{aligned}
\tfrac{1}{2}\big(\phi(y+\delta) + \phi(y-\delta)\big) - \phi(y) &= \tfrac{1}{2} \int_0^{\delta} (\phi'(y+a) - \phi'(y-a))\, da \\
&= \tfrac{1}{2} \int_0^{\delta} da \int_{y-a}^{y+a} \phi''(z)\, dz.
\end{aligned}$$

The result follows easily. $\square$

PROOF OF PROPOSITION 5. Let

$$\mathscr{D}_1 u(t, y) = \tfrac{1}{2}\big(u(t+h, y+\sigma\sqrt{h}) + u(t+h, y-\sigma\sqrt{h})\big) - u(t+h, y)$$

and

$$\mathscr{D}_2 u(t, y) = u(t+h, y) - u(t, y).$$

Obviously, $\mathscr{D}u(t, y) = \mathscr{D}_1 u(t, y) + \mathscr{D}_2 u(t, y)$. Note that $\mathscr{D}_1$ corresponds to $(\sigma^2/2)\partial^2/\partial x^2$ and $\mathscr{D}_2$ to $\partial/\partial t$. It follows from Lemma 1 that

$$\begin{aligned}\mathbf{E}\big|\mathscr{D}_1 u(t, X_t^{(n)})\big| &\le \frac{\sigma\sqrt{h}}{2}\mathbf{E}\bigg(\int_{X_t^{(n)}-\sigma\sqrt{h}}^{X_t^{(n)}+\sigma\sqrt{h}} \bigg|\frac{\partial^2 u}{\partial x^2}(t+h, z)\bigg|\, dz\bigg)\\ &= \frac{\sigma\sqrt{h}}{2}\int_{-\infty}^{+\infty}\bigg|\frac{\partial^2 u}{\partial x^2}(t+h, z)\bigg|\mathbf{P}\big(|X_t^{(n)} - z| < \sigma\sqrt{h}\big)\, dz.\end{aligned} \tag{3}$$

Assume $j > 1$. Then

$$\begin{aligned}\mathbf{P}\big(|X_{(j-1)h}^{(n)} - z| < \sigma\sqrt{h}\big) &= \mathbf{P}\bigg(\bigg|\sigma\sqrt{h}\sum_{k=1}^{j-1}\varepsilon_k - (z-x)\bigg| < \sigma\sqrt{h}\bigg)\\ &= \mathbf{P}\bigg(\bigg|\frac{1}{\sqrt{j-1}}\sum_{k=1}^{j-1}\varepsilon_k - \frac{z-x}{\sigma\sqrt{h(j-1)}}\bigg| < \frac{1}{\sqrt{j-1}}\bigg).\end{aligned}$$

Now recall the classical Berry–Esseen estimate (cf. [27], Chapter III)

$$\exists\, C > 0,\ \forall\, j > 1,\ \forall\, y \in \mathbf{R}, \qquad \bigg|\mathbf{P}\bigg(\frac{1}{\sqrt{j-1}}\sum_{k=1}^{j-1}\varepsilon_k \le y\bigg) - \mathbf{P}(g \le y)\bigg| \le \frac{C}{\sqrt{j}},$$

where $g$ is a standard normal random variable. It then follows that

$$\begin{aligned}\mathbf{P}\big(\big|X_{(j-1)h}^{(n)} - z\big| < \sigma\sqrt{h}\big) &\le \frac{C}{\sqrt{j}} + \mathbf{P}\bigg(\bigg|g - \frac{z-x}{\sigma\sqrt{h(j-1)}}\bigg| < \frac{1}{\sqrt{j-1}}\bigg)\\ &\le \frac{C}{\sqrt{j}} + \frac{2}{\sqrt{2\pi}\sqrt{j-1}}.\end{aligned}$$

Going back to (3) and using the second assumption of the proposition, we obtain

$$\mathbf{E}\big|\mathscr{D}_1 u((j-1)h, X_{(j-1)h}^{(n)})\big| \le \frac{C}{\sqrt{nj}},$$

for some constant $C$ and for $j > 1$. The estimate is clearly also valid for $j = 1$.

It remains to estimate $\mathscr{D}_2 u$. Using Itô's formula and the boundedness of $\partial u/\partial t + (\sigma^2/2)(\partial^2 u/\partial x^2)$, we have

$$\big|u(t, y) - \mathbf{E}\big(u(t+h, y+\sigma B_h)\big)\big| \le Ch,$$

where $(B_t)_{t\geq 0}$ is standard Brownian motion. It follows that, if we set

$$\bar{\mathscr{D}}_2 u(t,y) = u(t+h,y) - \mathbf{E}\left(u(t+h, y+\sigma B_h)\right),$$

then

$$\left|\mathscr{D}_2 u(t,y) - \bar{\mathscr{D}}_2 u(t,y)\right| \leq Ch.$$

Therefore, it suffices to estimate $\mathbf{E}|\bar{\mathscr{D}}_2 u((j-1)h, X^{(n)}_{(j-1)h})|$. Now

$$\bar{\mathscr{D}}_2 u(t,y) = \mathbf{E}\big(u(t+h,y) - \tfrac{1}{2}\big(u(t+h, y+\sigma\sqrt{h}|g|) + u(t+h, y-\sigma\sqrt{h}|g|)\big)\big),$$

where $g$ denotes a standard normal random variable, which we can assume to be independent of the process $X^{(n)}$. Hence, using Lemma 1,

$$|\bar{\mathscr{D}}_2 u(t,y)| \leq \frac{1}{2}\int \left|\frac{\partial^2 u}{\partial x^2}(t+h,z)\right| \mathbf{E}\left(\sigma\sqrt{h}|g|\mathbf{1}_{\{|z-y|<\sigma\sqrt{h}|g|\}}\right) dz.$$

Therefore, using the independence of $g$ and $X^{(n)}$,

$$\mathbf{E}\left|\bar{\mathscr{D}}_2 u((j-1)h, X^{(n)}_{(j-1)h})\right| \leq \frac{1}{2}\int \left|\frac{\partial^2 u}{\partial x^2}(jh,z)\right| \mathbf{E}\big(\sigma\sqrt{h}|g|\mathbf{1}_{\{|z-X^{(n)}_{(j-1)h}|<\sigma\sqrt{h}|g|\}}\big) dz.$$

Conditioning with respect to $g$ and using the Berry–Esseen estimate again, we obtain

$$\mathbf{E}\left|\bar{\mathscr{D}}_2 u((j-1)h, x+X^{(n)}_{(j-1)h})\right| \leq \frac{C}{\sqrt{nj}}. \qquad \square$$

## 4. Proof of the main result.

4.1. *Orientation.* We will apply the results of Section 3 to the function $u$ defined by

$$u(t,y) = e^{-rt}P(t, y+\mu t).$$

With this definition we have, using (2) and the equality $P(t,x) = K - e^x$, for $x \leq s(t)$,

$$\begin{aligned}\left(\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}\right)(t,y) &= e^{-rt}\left(\frac{\partial P}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial x^2} + \mu\frac{\partial P}{\partial x} - rP\right)(t, y+\mu t) \\ &= -rKe^{-rt}\mathbf{1}_{\{y+\mu t\leq s(t)\}}.\end{aligned} \tag{4}$$

In particular, the first assumption in Proposition 5 is satisfied. We claim that the second assumption is also satisfied. Indeed, it follows from the convexity of $x \mapsto P(t, \log x)$ that

$$\frac{\partial^2 P}{\partial x^2} - \frac{\partial P}{\partial x} \geq 0.$$

Hence, using the fact that $x \mapsto P(t, x)$ is nonincreasing,

$$\int_{-\infty}^{+\infty} \left| \frac{\partial^2 P}{\partial x^2}(t, x) \right| dx \leq \int_{-\infty}^{+\infty} \left( \frac{\partial^2 P}{\partial x^2}(t, x) - \frac{\partial P}{\partial x}(t, x) \right) dx + \int_{-\infty}^{+\infty} \left| \frac{\partial P}{\partial x}(t, x) \right| dx$$
$$= \int_{-\infty}^{+\infty} \frac{\partial^2 P}{\partial x^2}(t, x)\, dx - 2 \int_{-\infty}^{+\infty} \frac{\partial P}{\partial x}(t, x)\, dx,$$

and these integrals are controlled by the sup norms of $\partial P/\partial x$ and $P$.

We can now explain our method. Observe that, for any stopping time $\tau \in \mathscr{T}_{0,T}^{(n)}$, we have

$$\text{(5)} \qquad \begin{aligned} \mathbf{E} e^{-r\tau} \psi(x + \mu\tau + \sigma B_\tau^{(n)}) &\leq \mathbf{E} e^{-r\tau} P(\tau, x + \mu\tau + \sigma B_\tau^{(n)}) \\ &= \mathbf{E}(u(\tau, X_\tau^{(n)})), \end{aligned}$$

where, according to the notations of Section 3, $X^{(n)} = x + \sigma B^{(n)}$. It follows from Proposition 3 that

$$\text{(6)} \qquad \mathbf{E}(u(\tau, X_\tau^{(n)})) = u(0, x) + \mathbf{E}\left( \sum_{j=1}^{n\tau/T} \mathscr{D}u((j-1)h, X_{(j-1)h}^{(n)}) \right).$$

Note that $u(0, x) = P(0, x)$, so that, putting (5) and (6) together,

$$\text{(7)} \qquad P^{(n)}(0, x) - P(0, x) \leq \sup_{\tau \in \mathscr{T}_{0,T}^{(n)}} \mathbf{E}\left( \sum_{j=1}^{n\tau/T} \mathscr{D}u((j-1)h, X_{(j-1)h}^{(n)}) \right).$$

The upper bounds in Theorem 1 will follow from estimating the right-hand side of (7).

To prove the lower bound, we will select a stopping time $\tau$ in $\mathscr{T}_{0,T}^{(n)}$ for which $\mathbf{E}(u(\tau, X_\tau^{(n)}))$ is close to $\mathbf{E} e^{-r\tau} \psi(x + \mu\tau + \sigma B_\tau^{(n)})$.

We now introduce two subsets of $[0, T] \times \mathbf{R}$:

$$\bar{C} = \left\{ (t, y) \in [0, T-h] \times \mathbf{R} \mid \mu t + y > s(t+h) + |\mu| h + \sigma\sqrt{h} \right\}$$

and

$$\bar{S} = \left\{ (t, y) \in [0, T-h] \times \mathbf{R} \mid \mu t + y < s(t) - |\mu| h - \sigma\sqrt{h} \right\}.$$

Here, the letter $C$ refers to "continuation region" and the letter $S$ to "stopping region."

4.2. *An estimate for the continuation region.* We have the following estimate, when $(t, X_t^{(n)})$ is restricted to the continuation region.

PROPOSITION 6. *There exists a positive constant C such that, for all integers n,*

$$\mathbf{E}\left( \sum_{j=0}^{n-1} \left| \mathscr{D}u(jh, X_{jh}^{(n)}) \right| \mathbf{1}_{\{(jh, X_{jh}^{(n)}) \in \bar{C}\}} \right) \leq C \left( \frac{\sqrt{\log n}}{n} \right)^{4/5}.$$

PROOF. If $(t, y) \in \bar{C}$, then, whenever $t < \tau < t+h$ and $y - \sigma\sqrt{h} < z < y + \sigma\sqrt{h}$, we have

$$\begin{aligned}\mu\tau + z &> y + \mu t - |\mu| h - \sigma\sqrt{h} \\ &> s(t+h) \geq s(\tau).\end{aligned}$$

It follows that, on the open set $(t, t+h) \times (y - \sigma\sqrt{h}, y + \sigma\sqrt{h})$,

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} = 0.$$

This implies, in particular, that $u$ is smooth on $(t, t+h) \times (y - \sigma\sqrt{h}, y + \sigma\sqrt{h})$. Applying Proposition 4, we have, for $(t, y) \in \bar{C}$,

$$\begin{aligned}\mathscr{D}u(t, y) &= \frac{1}{\sigma}\int_0^{\sqrt{h}} d\xi \int_{-\sigma\xi}^{\sigma\xi} dz \left( z \frac{\partial^2 u}{\partial t \partial x}(t+\xi^2, y+z)\right) \\ &= \frac{\sigma}{2}\int_0^{\sqrt{h}} d\xi \int_{-\sigma\xi}^{\sigma\xi} dz \left(\xi^2 - \frac{z^2}{\sigma^2}\right)\frac{\partial^3 u}{\partial t \partial x^2}(t+\xi^2, y+z) \\ &= -\frac{1}{\sigma}\int_0^{\sqrt{h}} d\xi \int_{-\sigma\xi}^{\sigma\xi} dz \left(\xi^2 - \frac{z^2}{\sigma^2}\right)\frac{\partial^2 u}{\partial t^2}(t+\xi^2, y+z),\end{aligned}$$

where we have integrated by parts and used the fact that

$$\frac{\partial^3 u}{\partial t \partial x^2} = -\frac{2}{\sigma^2}\frac{\partial^2 u}{\partial t^2}$$

on the open set $(t, t+h) \times (y - \sigma\sqrt{h}, y + \sigma\sqrt{h})$. Hence, using the inequalities $\xi^2 - (z^2/\sigma^2) \leq \xi^2 \leq \xi\sqrt{h}$,

$$\begin{aligned}\left|\mathscr{D}u(t, y)\right| &\leq \frac{1}{\sigma}\int_0^{\sqrt{h}} d\xi \int_{-\sigma\xi}^{\sigma\xi} dz\, \xi^2 \left|\frac{\partial^2 u}{\partial t^2}(t+\xi^2, y+z)\right| \\ &\leq \frac{\sqrt{h}}{\sigma}\int_0^{\sqrt{h}} \xi\, d\xi \int_{-\sigma\sqrt{h}}^{\sigma\sqrt{h}} dz \left|\frac{\partial^2 u}{\partial t^2}(t+\xi^2, y+z)\right| \\ &= \frac{\sqrt{h}}{2\sigma}\int_t^{t+h} ds \int_{y-\sigma\sqrt{h}}^{y+\sigma\sqrt{h}} dz \left|\frac{\partial^2 u}{\partial t^2}(s, z)\right|.\end{aligned}$$

Therefore, for $0 \leq j \leq n-1$,

$$\begin{aligned}&\mathbf{E}\left(\left|\mathscr{D}u(jh, X_{jh}^{(n)})\right| \mathbf{1}_{\{(jh, X_{jh}^{(n)}) \in \bar{C}\}}\right) \\ &\quad \leq \frac{\sqrt{h}}{2\sigma}\int_{jh}^{(j+1)h} d\tau \int_{-\infty}^{+\infty} dz \left|\frac{\partial^2 u}{\partial t^2}(\tau, z)\right| \mathbf{P}\left(|X_{jh}^{(n)} - z| < \sigma\sqrt{h}\right).\end{aligned}$$

Now, if $j \geq 1$,

$$\mathbf{P}\big(|X_{jh}^{(n)} - z| < \sigma\sqrt{h}\big) = \mathbf{P}\left(\left|\sigma\sqrt{h}\sum_{k=1}^{j}\varepsilon_k - (z-x)\right| < \sigma\sqrt{h}\right)$$
$$= \mathbf{P}\left(\left|\frac{1}{\sqrt{j}}\sum_{k=1}^{j}\varepsilon_k - \frac{z-x}{\sigma\sqrt{hj}}\right| < \frac{1}{\sqrt{j}}\right).$$

Now recall the nonuniform Berry–Esseen estimate (see [27], Chapter III, Exercise 2)

$$\left|\mathbf{P}\left(\frac{1}{\sqrt{j}}\sum_{k=1}^{j}\varepsilon_k < \xi\right) - \mathbf{P}(g < \xi)\right| \leq \frac{C}{\sqrt{j}(1+|\xi|^3)}. \tag{8}$$

We have

$$\begin{aligned}\mathbf{P}\big(|X_{jh}^{(n)} - z| < \sigma\sqrt{h}\big) &= \mathbf{P}\left(\left|\frac{1}{\sqrt{j}}\sum_{k=1}^{j}\varepsilon_k - \frac{z-x}{\sigma\sqrt{hj}}\right| < \frac{1}{\sqrt{j}}\right) \\ &\leq \mathbf{P}\left(\left|g - \frac{z-x}{\sigma\sqrt{hj}}\right| < \frac{1}{\sqrt{j}}\right) + a^+ + a^-,\end{aligned} \tag{9}$$

with

$$a^+ = \left|\mathbf{P}\left(\frac{1}{\sqrt{j}}\sum_{k=1}^{j}\varepsilon_k < \frac{z-x}{\sigma\sqrt{hj}} + \frac{1}{\sqrt{j}}\right) - \mathbf{P}\left(g < \frac{z-x}{\sigma\sqrt{hj}} + \frac{1}{\sqrt{j}}\right)\right|$$

and

$$a^- = \left|\mathbf{P}\left(\frac{1}{\sqrt{j}}\sum_{k=1}^{j}\varepsilon_k < \frac{z-x}{\sigma\sqrt{hj}} - \frac{1}{\sqrt{j}}\right) - \mathbf{P}\left(g < \frac{z-x}{\sigma\sqrt{hj}} - \frac{1}{\sqrt{j}}\right)\right|.$$

Using (8), we derive

$$a^{\pm} \leq \frac{C}{\sqrt{j}(1+|\xi_j \pm \frac{1}{\sqrt{j}}|^3)} \leq \frac{2C}{\sqrt{j}(2+|\xi_j \pm \frac{1}{\sqrt{j}}|^3)},$$

with $\xi_j = (z-x)/\sigma\sqrt{hj}$. Using the elementary inequality $|a+b|^3 \leq 4(|a|^3 + |b|^3)$, we obtain

$$\begin{aligned}\left|\xi_j \pm \frac{1}{\sqrt{j}}\right|^3 &\geq \frac{1}{4}|\xi_j|^3 - \frac{1}{j^{3/2}} \\ &= \frac{1}{4}\left|\frac{z-x}{\sigma\sqrt{hj}}\right|^3 - \frac{1}{j^{3/2}} \\ &\geq \frac{|z-x|^3}{4(\sigma\sqrt{T})^3} - 1,\end{aligned}$$

where, for the last inequality, we have used $\sqrt{hj} \le \sqrt{T}$. It follows that

$$a^{\pm} \le \frac{C}{\sqrt{j}(1+|x-z|^3)}. \tag{10}$$

On the other hand,

$$\mathbf{P}\left(\left|g - \frac{z-x}{\sigma\sqrt{hj}}\right| < \frac{1}{\sqrt{j}}\right) = \int_{\xi_j-(1/\sqrt{j})}^{\xi_j+(1/\sqrt{j})} \exp(-y^2/2)\frac{dy}{\sqrt{2\pi}}.$$

Note that, if $\xi_j - (1/\sqrt{j}) < y < \xi_j + (1/\sqrt{j})$, we have $\exp(-y^2/2) \le \exp(-\xi_j^2/2)\exp(|\xi_j|/\sqrt{j}) \le \exp(-\xi_j^2/2)\exp(|\xi_j|)$. Therefore,

$$\mathbf{P}\left(\left|g - \frac{z-x}{\sigma\sqrt{hj}}\right| < \frac{1}{\sqrt{j}}\right) \le \frac{2}{\sqrt{j}}\frac{M}{1+|\xi_j|^3},$$

where $M$ is a constant such that

$$\forall\ \xi \in \mathbf{R}, \qquad \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\xi^2}{2} + |\xi|\right) \le \frac{M}{1+|\xi|^3}.$$

This yields

$$\mathbf{P}\left(\left|g - \frac{z-x}{\sigma\sqrt{hj}}\right| < \frac{1}{\sqrt{j}}\right) \le \frac{C}{\sqrt{j}(1+|x-z|^3)},$$

and, with (9) and (10),

$$\mathbf{P}\big(|X_{jh}^{(n)} - z| < \sigma\sqrt{h}\big) \le \frac{C}{\sqrt{j}(1+|x-z|^3)},$$

for $j \ge 1$. Consequently,

$$\begin{aligned}
&\mathbf{E}\big(|\mathscr{D}u(jh, X_{jh}^{(n)})|\mathbf{1}_{\{(jh,X_{jh}^{(n)})\in\bar{C}\}}\big)\\
&\quad\le \frac{C}{\sqrt{n}}\int_{jh}^{(j+1)h} d\tau \int_{-\infty}^{+\infty} dz\left|\frac{\partial^2 u}{\partial t^2}(\tau, z)\right|\frac{1}{\sqrt{j}(1+|x-z|^3)}\\
&\quad\le \frac{C\sqrt{h}}{\sqrt{n}}\int_{jh}^{(j+1)h} \frac{d\tau}{\sqrt{jh}} \int_{-\infty}^{+\infty} dz\left|\frac{\partial^2 u}{\partial t^2}(\tau, z)\right|\frac{1}{(1+|x-z|^3)}\\
&\quad\le \frac{C}{n}\int_{jh}^{(j+1)h} \frac{d\tau}{\sqrt{\tau}} \int_{-\infty}^{+\infty} \frac{dz}{1+|z|^3}\left|\frac{\partial^2 u}{\partial t^2}(\tau, z)\right|,
\end{aligned} \tag{11}$$

with a constant $C$ which may depend on $x$.

We now write $\mathbf{E}(\sum_{j=1}^{n-1}|\mathscr{D}u(jh, X_{jh}^{(n)})|\mathbf{1}_{\{(jh,X_{jh}^{(n)})\in\bar{C}\}})$ as the sum $I(\theta) + J(\theta)$, where $\theta \in [1/2, 1)$,

$$I(\theta) = \mathbf{E}\left(\sum_{j=1}^{[n\theta]-2}|\mathscr{D}u(jh, X_{jh}^{(n)})|\mathbf{1}_{\{(jh,X_{jh}^{(n)})\in\bar{C}\}}\right)$$

and

$$J(\theta)=\mathbf{E}\bigg(\sum_{j=[n\theta]-1}^{n-1}\big|\mathscr{D}u(jh,X_{jh}^{(n)})\big|\mathbf{1}_{\{(jh,X_{jh}^{(n)})\in\bar{C}\}}\bigg).$$

Using (11) and the Cauchy–Schwarz inequality, we get

$$\begin{aligned}I(\theta)&\le\frac{C}{n}\sum_{j=1}^{[n\theta]-2}\int_{jh}^{(j+1)h}\frac{d\tau}{\sqrt{\tau}}\int_{-\infty}^{+\infty}\frac{dz}{1+|z|^3}\left|\frac{\partial^2u}{\partial t^2}(\tau,z)\right|\\&\le\frac{C}{n}\sqrt{A}\sqrt{B},\end{aligned}$$

with

$$A=\sum_{j=1}^{[n\theta]-2}\int_{jh}^{(j+1)h}\frac{d\tau}{\tau(\theta T-\tau)}\int_{-\infty}^{+\infty}\frac{dz}{1+|z|^3}$$

and

$$B=\sum_{j=1}^{[n\theta]-2}\int_{jh}^{(j+1)h}d\tau\,(\theta T-\tau)\int_{-\infty}^{+\infty}\frac{dz}{1+|z|^3}\left|\frac{\partial^2u}{\partial t^2}(\tau,z)\right|^2.$$

Clearly,

$$\begin{aligned}A&\le C\int_{h}^{\theta T-h}\frac{d\tau}{\tau(\theta T-\tau)}=\frac{2C}{\theta T}\log\left(\frac{\theta T-h}{h}\right)\\&\le C\log(1/h),\end{aligned}$$

since we have assumed $\theta\in[1/2,1)$. Therefore, we have $A\le C/\log n$. We now estimate $B$. We have

$$B\le\int_0^{\theta T}d\tau(\theta T-\tau)\int_{-\infty}^{\infty}\frac{dz}{1+|z|^3}\left|\frac{\partial^2u}{\partial t^2}(\tau,z)\right|^2.$$

Now, from the definition of $u$ in terms of $P$, we can compute $\partial^2u/\partial t^2$. For $(t,z)\in[0,T)\times\mathbf{R}$, we have

$$\frac{\partial^2u}{\partial t^2}(t,z)=e^{-rt}\left(\frac{\partial^2P}{\partial t^2}+\mu^2\frac{\partial^2P}{\partial x^2}+r^2P+2\mu\frac{\partial^2P}{\partial t\partial x}-2r\frac{\partial P}{\partial t}-2r\mu\frac{\partial P}{\partial x}\right)(t,z+\mu t).$$

From this expression we can derive, using Theorems 2 and 3, that

$$B\le\frac{C}{\sqrt{T-\theta T}}.$$

Hence,

$$I(\theta)\le\frac{C\sqrt{\log n}}{n}\frac{1}{(1-\theta)^{1/4}}.$$

We now derive an upper bound for $J(\theta)$ by using Proposition 5:

$$J(\theta) \le \sum_{j=[n\theta]-1}^{n-1} \frac{C}{\sqrt{nj}} \le C\left(1-\theta+\frac{1}{n}\right).$$

Putting things together, we have

$$I(\theta)+J(\theta) \le C\left[\frac{\sqrt{\log n}}{n}\frac{1}{(1-\theta)^{1/4}}+(1-\theta)\right].$$

Taking $1-\theta = (\sqrt{\log n}/n)^{4/5}$, we obtain

$$\mathbf{E}\left(\sum_{j=1}^{n-1} \left|\mathscr{D}u(jh, X_{jh}^{(n)})\right| \mathbf{1}_{\{((j-1)h,\, X_{(j-1)h}^{(n)}) \in \bar{C}\}}\right) \le C\left(\frac{\sqrt{\log n}}{n}\right)^{4/5},$$

which yields the desired result [since the term corresponding to $j = 0$ is $O(1/n)$, as follows easily from the boundedness of $(\partial u/\partial t)(t,x)$ and $(\partial^2 u/\partial x^2)(t,x)$ for small $t$]. □

4.3. *Proof of the upper bound.* We first state the following estimate in the stopping region.

LEMMA 2. *For $n$ large enough, we have*

$$\forall\, (t,y) \in \bar{S}, \qquad \mathscr{D}u(t,y) \le 0.$$

PROOF. The condition $(t,y) \in \bar{S}$ implies $\mu(t+h)+y\pm\sigma\sqrt{h} \le s(t) \le s(t+h)$ and $\mu t + y \le s(t)$, hence

$$\mathscr{D}u(t,y) = \tfrac{1}{2}\big(e^{-r(t+h)}\psi(y+\mu(t+h)+\sigma\sqrt{h}) + e^{-r(t+h)}\psi(y+\mu(t+h)-\sigma\sqrt{h})\big) - e^{-rt}\psi(y+\mu t).$$

Now, if $z \le s(t)$, $\psi(z) = K - e^z$. Therefore,

$$\begin{aligned}\mathscr{D}u(t,y) &= e^{-rt}\left[\frac{e^{-rh}}{2}\big(2K - e^{y+\mu(t+h)+\sigma\sqrt{h}} - e^{y+\mu(t+h)-\sigma\sqrt{h}}\big) - K + e^{y+\mu t}\right] \\ &= e^{-rt}\big[K(e^{-rh}-1) + e^{y+\mu t}\big(1-e^{-\sigma^2 h/2}\cosh(\sigma\sqrt{h})\big)\big],\end{aligned}$$

where we haved used $\mu = r-(\sigma^2/2)$. From this expression the lemma follows easily. □

We are now in a position to derive the upper bound in Theorem 1. We introduce the set

$$\bar{B} = \big\{(t,y) \in [0, T-h]\times\mathbf{R} \mid s(t)-|\mu|h-\sigma\sqrt{h} \le \mu t + y \le s(t+h)+|\mu|h+\sigma\sqrt{h}\big\}.$$

The letter $B$ obviously refers to "boundary." Note that

$$\bar{B} = [0, T-h] \times \mathbf{R} \backslash (\bar{C} \cup \bar{S}).$$

Now, using (7), we want to estimate

$$\sup_{\tau \in \mathscr{T}_{0,T}^{(n)}} \mathbf{E}\left( \sum_{j=1}^{n\tau/T} \mathscr{D}u((j-1)h, X_{(j-1)h}^{(n)}) \right).$$

For $0 \leq j \leq n-1$, we can write

$$\mathscr{D}u(jh, X_{jh}^{(n)}) = \mathscr{D}u(jh, X_{jh}^{(n)}) \mathbf{1}_{\{(jh, X_{jh}^{(n)}) \in \bar{C}\}} + \mathscr{D}u(jh, X_{jh}^{(n)}) \mathbf{1}_{\{(jh, X_{jh}^{(n)}) \in \bar{S}\}}$$
$$+ \mathscr{D}u(jh, X_{jh}^{(n)}) \mathbf{1}_{\{(jh, X_{jh}^{(n)}) \in \bar{B}\}}.$$

Using Proposition 6 and Lemma 2, we have, for $n$ large enough and for any $\tau \in \mathscr{T}_{0,T}^{(n)}$,

$$\mathbf{E}\left( \sum_{j=0}^{(\tau/h)-1} \mathscr{D}u(jh, X_{jh}^{(n)}) \right) \leq C \left( \frac{\sqrt{\log n}}{n} \right)^{4/5}$$
$$+ \mathbf{E}\left( \sum_{j=0}^{(\tau/h)-1} \mathscr{D}u(jh, X_{jh}^{(n)}) \mathbf{1}_{\{(jh, X_{jh}^{(n)}) \in \bar{B}\}} \right)$$

It follows from the boundedness of $(\partial u/\partial t)(t, \cdot)$ and $(\partial^2 u/\partial x^2)(t, \cdot)$ for small $t$ that

$$\mathscr{D}u(0, X_0^{(n)}) \leq \frac{C}{n}.$$

On the other hand, we know from Proposition 5 that

$$\mathbf{E}\left| \mathscr{D}u\big((n-1)h, X_{(n-1)h}^{(n)}\big) \right| \leq \frac{C}{n}.$$

We will now estimate $\mathscr{D}u(jh, X_{jh}^{(n)}) \mathbf{1}_{\{(jh, X_{jh}^{(n)}) \in \bar{B}\}}$ for $1 \leq j \leq n-2$. Applying Proposition 4 to suitable $C^2$ approximations of $u$, we have, since $(\partial u/\partial t) + (\sigma^2/2)(\partial^2 u/\partial x^2) \leq 0$,

$$\begin{aligned}
\mathscr{D}u(t, y) &\leq \frac{1}{\sigma} \int_0^{\sqrt{h}} d\xi \int_{-\sigma\xi}^{\sigma\xi} dz \left( z \frac{\partial^2 u}{\partial t \partial x}(t + \xi^2, y + z) \right) \\
&\leq \int_0^{\sqrt{h}} \xi \, d\xi \int_{-\sigma\sqrt{h}}^{\sigma\sqrt{h}} dz \left| \frac{\partial^2 u}{\partial t \partial x}(t + \xi^2, y + z) \right| \\
&= \frac{1}{2} \int_t^{t+h} ds \int_{y-\sigma\sqrt{h}}^{y+\sigma\sqrt{h}} dz \left| \frac{\partial^2 u}{\partial t \partial x}(s, z) \right|.
\end{aligned}$$

Note that when $(t, y) \in \bar{B}$ and $|z - y| \leq \sigma\sqrt{h}$, we have

$$s(t) - |\mu|h - 2\sigma\sqrt{h} - \mu t \leq z \leq s(t+h) + |\mu|h + 2\sigma\sqrt{h} - \mu t.$$

Therefore, if we assume $h \le 1$ and set $\lambda = |\mu| + 2\sigma$, we can write

$$\mathscr{D}u(t, y)\mathbf{1}_{\{(t,y)\in\bar{B}\}} \le \frac{1}{2}\int_t^{t+h} ds \int_{s(t)-\lambda\sqrt{h}-\mu t}^{s(t+h)+\lambda\sqrt{h}-\mu t} dz\, \mathbf{1}_{\{|z-y|\le\sigma\sqrt{h}\}}\left|\frac{\partial^2 u}{\partial t\partial x}(s, z)\right|.$$

Hence

$$\mathbf{E}\left(\sum_{j=1}^{(\tau/h)-1}\mathscr{D}u(jh, X_{jh}^{(n)})\mathbf{1}_{\{(jh,X_{jh}^{(n)})\in\bar{B}\}}\right)$$
$$\le \frac{1}{2}\left(\sum_{j=1}^{n-2}\int_{jh}^{(j+1)h} d\tau \int_{s(jh)-\lambda\sqrt{h}-\mu jh}^{s(jh+h)+\lambda\sqrt{h}-\mu jh} dz\mathbf{P}\big(|X_{jh}^{(n)} - z| \le \sigma\sqrt{h}\big)\left|\frac{\partial^2 u}{\partial t\partial x}(\tau, z)\right|\right) + Ch.$$

As the proof of Proposition 6 shows, we have

$$\mathbf{P}\big(|X_{jh}^{(n)} - z| \le \sigma\sqrt{h}\big) \le \frac{C}{\sqrt{j}(1+|z|^3)}.$$

Hence, using $\tau \le 2jh$, for $jh \le \tau \le (j+1)h$ and $j \ge 1$, and applying the Cauchy–Schwarz inequality,

$$\mathbf{E}\left(\sum_{j=1}^{(\tau/h)-1}\mathscr{D}u(jh, X_{jh}^{(n)})\mathbf{1}_{\{(jh,X_{jh}^{(n)})\in\bar{B}\}}\right)$$
$$\le C\left(\sum_{j=1}^{n-2}\sqrt{h}\int_{jh}^{(j+1)h}\frac{d\tau}{\sqrt{\tau}}\int_{s(jh)-\lambda\sqrt{h}-\mu jh}^{s(jh+h)+\lambda\sqrt{h}-\mu jh}\frac{dz}{1+|z|^3}\left|\frac{\partial^2 u}{\partial t\partial x}(\tau, z)\right|\right) + Ch$$
$$\le C\sqrt{h}\Bigg[\sum_{j=1}^{n-1}\int_{jh}^{(j+1)h}\frac{d\tau}{\sqrt{\tau}}\big(s(jh+h) - s(jh) + 2\lambda\sqrt{h}\big)^{1/2}$$
$$\times\left(\int\frac{dz}{1+|z|^3}\left|\frac{\partial^2 u}{\partial t\partial x}(\tau, z)\right|^2\right)^{1/2}\Bigg] + Ch.$$

It follows from Proposition 1 and Theorem 2 that, for $j \le n-2$,

$$s(jh+h) - s(jh) \le C\sqrt{h}\frac{1}{(T-jh-h)^{1/4}}$$
$$\le C\sqrt{h}\frac{1}{[(T-jh)/2]^{1/4}},$$

where we have used $j \le n-2$ in the last inequality.

On the other hand, we know from Theorem 2 and Theorem 3 (applied with $t_1 = \tau$ and $T' = (T+\tau)/2$) that

$$\left(\int\frac{dz}{1+|z|^3}\left|\frac{\partial^2 u}{\partial t\partial x}(\tau, z)\right|^2\right)^{1/2} \le \frac{C}{(T-\tau)^{3/4}}.$$

Therefore,

$$\mathbf{E}\left(\sum_{j=1}^{(\tau/h)-1} \mathscr{D}u(jh, X_{jh}^{(n)})\mathbf{1}_{\{(jh, X_{jh}^{(n)})\in\bar{B}\}}\right)$$
$$\leq C\sqrt{h}\int_0^T \frac{d\tau}{\sqrt{\tau}} \frac{h^{1/4}}{(T-\tau)^{1/8}} \frac{1}{(T-\tau)^{3/4}} + Ch$$
$$\leq Ch^{3/4},$$

which completes the proof of the upper bound in the general case. □

4.4. *The case* $\mu \leq 0$. In this section, we assume that $\mu \leq 0$. We will show that, in that case,

$$\mathbf{E}\left(\sum_{j=0}^{(\tau/h)-1} \mathscr{D}u(jh, X_{jh}^{(n)})\right) \leq C\left(\frac{\sqrt{\log n}}{n}\right)^{4/5}.$$

We first observe, as in Section 4.2, that

$$\mathscr{D}u(t, y) \leq \frac{1}{\sigma}\int_0^{\sqrt{h}} d\xi \int_{-\sigma\xi}^{\sigma\xi} dz\left(z\frac{\partial^2 u}{\partial t\partial x}(t+\xi^2, y+z)\right).$$

We would like to integrate by parts, as in the proof of Proposition 6. However, we no longer have $\partial^3 u/\partial t\partial x^2 = -(2/\sigma^2)\partial^2 u/\partial t^2$, since $(t, y)$ may not be in $\bar{C}$. However, we know from (4) that

$$\left(\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}\right)(t, y) = -rKe^{-rt}\mathbf{1}_{\{y\leq s(t)-\mu t\}}$$

since $\mu \leq 0$, $t \mapsto s(t) - \mu t$ is a nondecreasing function. Therefore

$$t \mapsto e^{rt}\left(\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}\right)(t, y)$$

is nonincreasing. It follows that

$$\frac{\partial}{\partial t}\left(e^{rt}\left(\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}\right)\right) \leq 0$$

in the sense of distributions. Hence

$$re^{rt}\left(\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}\right) + e^{rt}\left(\frac{\partial^2 u}{\partial t^2} + \frac{\sigma^2}{2}\frac{\partial^3 u}{\partial t\partial x^2}\right) \leq 0.$$

Therefore,

$$\frac{\partial^3 u}{\partial t\partial x^2} \leq \frac{2}{\sigma^2}\left(-\frac{\partial^2 u}{\partial t^2} + r^2 K\right).$$

Using suitable $C^2$ approximations of $u$, for which integration by parts is allowed, we obtain

$$\begin{aligned}\mathscr{D}u(t,y) &\le \frac{1}{\sigma}\int_0^{\sqrt{h}} d\xi \int_{-\sigma\xi}^{\sigma\xi} dz\Big(\xi^2 - \frac{z^2}{\sigma^2}\Big)\Big(\Big|\frac{\partial^2 u}{\partial t^2}(t+\xi^2, y+z)\Big| + r^2 K\Big)\\ &\le \frac{\sqrt{h}}{\sigma}\int_0^{\sqrt{h}} \xi\, d\xi \int_{-\sigma\xi}^{\sigma\xi} dz \Big|\frac{\partial^2 u}{\partial t^2}(t+\xi^2, y+z)\Big| + Ch^2\\ &\le \frac{\sqrt{h}}{2\sigma}\int_t^{t+h} d\tau \int_{y-\sigma\sqrt{h}}^{y+\sigma\sqrt{h}} \Big|\frac{\partial^2 u}{\partial t^2}(\tau, z)\Big|\, dz + Ch^2.\end{aligned}$$

We can now proceed exactly as in the proof of Proposition 6 to derive

$$\sup_{\tau\in\mathscr{T}_{0,T}^{(n)}} \mathbf{E}\sum_{j=0}^{\tau/h-1} \mathscr{D}u(jh, X_{jh}^{(n)}) \le C\Big(\frac{\sqrt{\log n}}{n}\Big)^{4/5},$$

which proves that, for $\mu \le 0$,

$$P^{(n)}(0,x) - P(0,x) \le C\Big(\frac{\sqrt{\log n}}{n}\Big)^{4/5}.$$

4.5. *Proof of the lower bound.* In order to derive the lower bound in Theorem 1, we consider the following stopping time:

$$\tau = \inf\{t\in[0, T-h] \mid \quad t/h \in \mathbf{N} \text{ and } (t, X_t^{(n)}) \notin \bar{C}\} \wedge T.$$

Using (6) and the definition of $\tau$, we have

$$\begin{aligned}\mathbf{E}\big(u(\tau, X_\tau^{(n)})\big) &= u(0,x) + \mathbf{E}\Big(\sum_{j=1}^{n\tau/T} \mathscr{D}u((j-1)h, X_{(j-1)h}^{(n)})\Big)\\ &= P(0,x) + \mathbf{E}\Big(\sum_{j=1}^{n\tau/T} \mathscr{D}u((j-1)h, X_{(j-1)h}^{(n)})\mathbf{1}_{\{((j-1)h, X_{(j-1)h}^{(n)})\in\bar{C}\}}\Big) \qquad (12)\\ &\ge P(0,x) - C\Big(\frac{\sqrt{\log n}}{n}\Big)^{4/5},\end{aligned}$$

where the last inequality follows from Proposition 6.

If $\tau < T$, $(\tau, X_\tau^{(n)})$ must be in $\bar{S}\cup\bar{B}$. Therefore, we should be able to estimate the difference $u(\tau, X_\tau^{(n)}) - e^{-r\tau}\psi(\mu\tau + X_\tau^{(n)})$ thanks to Proposition 2. However, since the estimate in Proposition 2 blows up as $t$ approaches $T$, we have to be careful when $\tau$ is close to $T$. Therefore, we introduce the following modified stopping time. For $0 < \alpha < T$ (to be chosen later), let

$$\tau_\alpha = \tau\mathbf{1}_{\{\tau+h<\alpha\}} + T\mathbf{1}_{\{\tau+h\ge\alpha\}}.$$

It is easy to check that $\tau_\alpha$ is a stopping time. We will prove below that, for $\alpha$ close to $T$ (say $\alpha > T/2$),

$$\left|\mathbf{E}\big(u(\tau, X_\tau^{(n)})\big) - \mathbf{E}\big(e^{-r\tau_\alpha}\psi(\mu\tau_\alpha + X_{\tau_\alpha}^{(n)})\big)\right| \le C\left(\frac{1}{n\sqrt{T-\alpha}} + (T-\alpha)\right). \tag{13}$$

Applying this inequality with $T-\alpha = 1/n^{2/3}$, together with (12), leads to

$$\mathbf{E}\big(\exp(-r\tau_\alpha)\psi(\mu\tau_\alpha + X_{\tau_\alpha}^{(n)})\big) - P(0,x) \ge -c/n^{2/3},$$

which implies the lower bound in Theorem 1.

It remains to prove (13). We have

$$\mathbf{E}\big(u(\tau, X_\tau^{(n)})\big) - \mathbf{E}\big(\exp(-r\tau_\alpha)\psi(\mu\tau_\alpha + X_{\tau_\alpha}^{(n)})\big) = E_1 + E_2,$$

where

$$E_1 = \mathbf{E}\big(u(\tau, X_\tau^{(n)}) - e^{-r\tau}\psi(\mu\tau + X_\tau^{(n)})\big)\mathbf{1}_{\{\tau+h<\alpha\}}$$

and

$$E_2 = \mathbf{E}\big(u(\tau, X_\tau^{(n)}) - u(T, X_T^{(n)})\big)\mathbf{1}_{\{\tau+h\ge\alpha\}}.$$

We first study $E_1$. We have

$$\begin{aligned} u(\tau, X_\tau^{(n)}) = {}& e^{-r\tau}\big(P(\tau, \mu\tau + X_\tau^{(n)}) - P(\tau+h, \mu\tau + X_\tau^{(n)})\big) \\ & + e^{-r\tau}P(\tau+h, \mu\tau + X_\tau^{(n)}). \end{aligned}$$

Hence

$$\begin{aligned} \left|u(\tau, X_\tau^{(n)}) - e^{-r\tau}\psi(\mu\tau + X_\tau^{(n)})\right| \le {}& h \sup_{\tau\le t\le\tau+h}\left|\left|\frac{\partial P}{\partial t}(t,\cdot)\right|\right|_{L^\infty(\mathbf{R})} \\ & + e^{-r\tau}\left|P(\tau+h, \mu\tau + X_\tau^{(n)}) - \psi(\mu\tau + X_\tau^{(n)})\right|. \end{aligned}$$

From Proposition 2, we know that, for all $t \in [0, T)$ and $y \ge s(t)$,

$$P(t,y) - \psi(y) \le \frac{C}{\sqrt{T-t}}(y - s(t))^2.$$

Note that, on the event $\{\tau < T\}$, $\mu\tau + X_\tau^{(n)} \le s(\tau+h) + |\mu|h + \sigma\sqrt{h}$. Hence

$$\begin{aligned} \left|P(\tau+h, \mu\tau + X_\tau^{(n)}) - \psi(\mu\tau + X_\tau^{(n)})\right| &\le \frac{C}{\sqrt{T-\tau-h}}(|\mu|h + \sigma\sqrt{h})^2 \\ &\le \frac{C}{n\sqrt{T-\alpha}} \quad \text{on } \{\tau+h<\alpha\}. \end{aligned}$$

On the set $\{\tau + h < \alpha\}$, we also have, due to Theorem 2,

$$\sup_{\tau \le t \le \tau+h} \left\| \frac{\partial P}{\partial t}(t, \cdot) \right\|_{L^\infty(\mathbf{R})} \le \frac{C}{\sqrt{T-\alpha}}.$$

Consequently,

$$|E_1| \le \frac{C}{n\sqrt{T-\alpha}}.$$

We now examine $E_2$. It follows from Proposition 3 and Proposition 5 that

$$\begin{aligned}
\left|\mathbf{E}\big(u(\tau, X_\tau^{(n)}) - u(T, X_T^{(n)})\big)\mathbf{1}_{\{\tau+h\ge\alpha\}}\right| &\le \mathbf{E}\left( \sum_{j=[\tau/h]}^{n-1} \left|\mathscr{D}u(jh, X_{jh}^{(n)})\right| \mathbf{1}_{\{\tau+h\ge\alpha\}} \right) \\
&\le C \sum_{j=[\alpha/h]}^{n} \frac{1}{\sqrt{nj}} \\
&\le \frac{C}{\sqrt{\alpha}}(T-\alpha).
\end{aligned}$$

Hence $|E_2| \le C(T-\alpha)$, for $\alpha$ close to $T$, which completes the proof of (13). □

## APPENDIX

PROOF OF THEOREM 3. Similar estimates have been proved by Friedman and Kinderlehrer (see [17], Chapter VIII, Theorem 3.4) for variational inequalities on a finite interval, when the obstacle $\psi$ has a bounded second derivative. Since the proof of Theorem 3 uses the same techniques, we will not give all the details.

We first introduce relevant function spaces. For $k > 1$, let $H_k = L^2(\mathbf{R}, dx/(1+x^2)^{k/2})$ and $V_k = \{f \in H_k \mid f' \in H_k\}$. The inner product on $H_k$ will be denoted by $(\cdot,\cdot)_k$ and the associated norm by $|\cdot|_k$. The natural norm on $V_k$ will be denoted by $\|\cdot\|_k$. Thus, we have

$$|f|_k^2 = \int_{-\infty}^{+\infty} f^2(x) \frac{dx}{(1+x^2)^{k/2}},$$

and $\|f\|_k^2 = |f|_k^2 + |f'|_k^2$.

Let $A$ be the partial differential operator defined by

$$A = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x} - r.$$

We associate with operator $A$ a bilinear functional on $V_k$, defined by

$$\begin{aligned}
a_k(f,g) = {}& \frac{\sigma^2}{2}\int_{-\infty}^{\infty} f'(x)g'(x)\frac{dx}{(1+x^2)^{k/2}} - \frac{k\sigma^2}{2}\int_{-\infty}^{\infty} f'(x)g(x)\frac{x}{(1+x^2)^{(k/2)+1}}\,dx \\
& - \mu\int_{-\infty}^{\infty} f'(x)g(x)\frac{dx}{(1+x^2)^{k/2}} + r\int_{-\infty}^{\infty} f(x)g(x)\frac{dx}{(1+x^2)^{k/2}},
\end{aligned}$$

so that, if $f' \in V_k$,

$$a_k(f, g) = -(Af, g)_k.$$

It will be convenient to write $a_k(f, g)$ as $a_k(f, g) = \tilde{a}_k(f, g) + \bar{a}_k(f, g)$, with

$$\tilde{a}_k(f, g) = \frac{\sigma^2}{2}\left[(f', g')_k + (f, g)_k\right] \quad \text{and} \quad \bar{a}_k(f, g) = a_k(f, g) - \tilde{a}_k(f, g). \tag{14}$$

With these notations, it is easy to check that $|\bar{a}_k(f, g)| \leq C||f||_k |g|_k$ and $|\bar{a}_k(f, g)| \leq C||g||_k |f|_k$ for some constant $C$ which does not depend on $f$ or $g$.

We will prove that if $\phi$ is a bounded continuous function with bounded derivatives $\phi'$ and $\phi''$, satisfying $A\phi \geq b$, where $b$ is a nonpositive constant and $||\phi||_{L^\infty} + ||\phi'||_{L^\infty} \leq a$ for some positive constant $a$, then the (bounded) solution of the variational inequality

$$(I) \qquad \max\left(\frac{\partial u}{\partial t} + Au, \phi - u\right) = 0, \qquad u(T, \cdot) = \phi,$$

satisfies the following: there exist positive constants $C_1$ and $C_2$ depending only on $a$ and $b$ such that

$$\forall\, t_1 \in [0, T), \; \int_{t_1}^{T} (T - t)\left|\frac{\partial^2 u}{\partial t^2}(t, \cdot)\right|_k^2 dt + (T - t_1)\left|\left|\frac{\partial u}{\partial t}(t_1, \cdot)\right|\right|_k^2 \leq C_1||\phi''||_{L^\infty} + C_2. \tag{15}$$

Applying this estimate with $k = 3$, $T$ replaced by $T'$, $\phi = P(T', \cdot)$ and $b = -rK$ yields Theorem 3 (recall that $[(\partial P/\partial t) + AP](T', x) = -rK\mathbf{1}_{\{x \leq s(T')\}}$, so that $AP(T', \cdot) \geq -rK$).

In order to prove (15), we introduce a family of penalty functions $\beta_\varepsilon\colon \mathbf{R} \to \mathbf{R}$ such that, for each $\varepsilon > 0$, $\beta_\varepsilon$ is a concave, nondecreasing, nonpositive $C^2$ function with bounded derivatives, satisfying $\beta_\varepsilon(u) = 0$, for $u \geq \varepsilon$ and $\beta_\varepsilon(0) = b$.

Let $u_\varepsilon$ be the solution of the parabolic semilinear equation

$$(I_\varepsilon) \qquad \frac{\partial u_\varepsilon}{\partial t} + Au_\varepsilon - \beta_\varepsilon(u_\varepsilon - \phi) = 0, \qquad u_\varepsilon(T, \cdot) = \phi,$$

Standard arguments show that problem $(I_\varepsilon)$ has a unique solution satisfying $u_\varepsilon \in L^2([0, T]; V_k)$ and $\partial u_\varepsilon/\partial t \in L^2([0, T]; H_k)$, with

$$\int_0^T \left|\frac{\partial u_\varepsilon}{\partial t}(t, \cdot)\right|_k^2 dt \leq K_1, \tag{16}$$

where $K_1$ depends only on $||\phi||_k$ (and not on $\varepsilon$). Moreover, as $\varepsilon$ tends to 0, $u_\varepsilon$ converges weakly to the solution of $(I)$. For details on the previous facts, we refer the reader to [6], Chapter 3, where similar results are proved. It is then sufficient to derive (15) for $u_\varepsilon$, making sure along the way that constants do not depend on $\varepsilon$.

For that purpose, we proceed essentially as in [17], Chapter VIII, Section 3. Let

$$v_\varepsilon = \frac{\partial u_\varepsilon}{\partial t}.$$

For notational convenience, we set $\beta = \beta_\varepsilon$. By differentiating $(I_\varepsilon)$ with respect to $t$, we have that $v_\varepsilon$ satisfies

$$\frac{\partial v_\varepsilon}{\partial t} + Av_\varepsilon - \beta'_\varepsilon(u_\varepsilon - \phi)v_\varepsilon = 0, \tag{17}$$

with terminal condition

$$v_\varepsilon(T, \cdot) = -A\phi + \beta(0) = -A\phi + b.$$

Observe that the condition $A\phi \geq b$, together with $\beta' \geq 0$, implies that $v_\varepsilon \leq 0$. Also note that, for $t \in [0, T]$, $||v_\varepsilon(t, \cdot)||_{L^\infty} \leq ||v_\varepsilon(T, \cdot)||_{L^\infty} \leq ||A\phi||_{L^\infty} + |b|$.

Now, we multiply (17) by $v_\varepsilon$ and integrate with respect to $dx/(1+x^2)^{k/2}$ to get

$$-\left(\frac{\partial v_\varepsilon}{\partial t}, v_\varepsilon\right)_k + a_k(v_\varepsilon, v_\varepsilon) + \big(\beta'(u_\varepsilon - \phi)v_\varepsilon, v_\varepsilon\big)_k = 0.$$

Since $\beta$ is nondecreasing, this implies

$$\left(\frac{\partial v_\varepsilon}{\partial t}, v_\varepsilon\right)_k - a_k(v_\varepsilon, v_\varepsilon) \geq 0.$$

We now integrate over the time interval $[t, T]$ and use (14) to obtain [with the notation $v_\varepsilon(t) = v_\varepsilon(t, \cdot)$]

$$\begin{aligned}\frac{1}{2}\big(|v_\varepsilon(T)|_k^2 - |v_\varepsilon(t)|_k^2\big) &\geq \int_t^T a_k(v_\varepsilon(t), v_\varepsilon(t))\,dt \\ &= \frac{\sigma^2}{2}\int_t^T ||v_\varepsilon(t)||_k^2\,dt + \int_t^T \bar{a}_k(v_\varepsilon(t), v_\varepsilon(t))\,dt \\ &\geq \frac{\sigma^2}{2}\int_t^T ||v_\varepsilon(t)||_k^2\,dt - C\int_t^T ||v_\varepsilon(t)||_k |v_\varepsilon(t)|_k\,dt.\end{aligned}$$

Hence, using (16),

$$\int_t^T ||v_\varepsilon(t)||_k^2\,dt \leq C\big(|A\phi|_k^2 + b^2 + K_1\big).$$

Note that

$$|A\phi|_k \leq |A\phi - b|_k + C|b|,$$

and, since $A\phi \geq b$, we can write

$$\begin{aligned}|A\phi - b|_k^2 &= \int_{-\infty}^{\infty} (A\phi(x) - b)^2 \frac{dx}{(1+x^2)^{k/2}} \\ &\leq \big(||A\phi||_{L^\infty} + |b|\big) \int_{-\infty}^{\infty} (A\phi(x) - b) \frac{dx}{(1+x^2)^{k/2}} \\ &\leq (C + |b|)\big(||A\phi||_{L^\infty} + |b|\big),\end{aligned}$$

since integrating by parts allows a control of $\int_{-\infty}^{\infty} A\phi(x)(dx/(1+x^2)^{k/2})$ by $||\phi||_{L^\infty}$ and $||\phi'||_{L^\infty}$. Therefore, we have the following inequality:

$$\int_t^T ||v_\varepsilon(t)||_k^2 dt \leq K_2\big(||A\phi||_{L^\infty} + 1\big), \tag{18}$$

where $K_2$ depends only on $b$ and $a$. Estimates involving $(\partial^2 u_\varepsilon/\partial t^2)$ can be obtained in the following way. Multiply (17) by $(\partial v_\varepsilon/\partial t)$ and integrate with respect to $dx/(1+x^2)^{k/2}$. Then

$$-\left(\frac{\partial v_\varepsilon}{\partial t}, \frac{\partial v_\varepsilon}{\partial t}\right)_k + a_k\left(v_\varepsilon, \frac{\partial v_\varepsilon}{\partial t}\right) + \left(\beta'(u_\varepsilon - \phi)v_\varepsilon, \frac{\partial v_\varepsilon}{\partial t}\right)_k = 0.$$

Note that

$$\begin{aligned}\left(\beta'(u_\varepsilon - \phi)v_\varepsilon, \frac{\partial v_\varepsilon}{\partial t}\right)_k &= \int_{-\infty}^{\infty} \beta'(u_\varepsilon - \phi)v_\varepsilon \frac{\partial v_\varepsilon}{\partial t} \frac{dx}{(1+x^2)^{k/2}} \\ &= \frac{1}{2}\frac{d}{dt}\big(\beta'(u_\varepsilon - \phi)v_\varepsilon, v_\varepsilon\big)_k - \frac{1}{2}\big(\beta''(u_\varepsilon - \phi)v_\varepsilon^2, v_\varepsilon\big)_k \\ &\leq \frac{1}{2}\frac{d}{dt}\Big(\beta'(u_\varepsilon - \phi)v_\varepsilon, v_\varepsilon\Big)_k,\end{aligned}$$

since $\beta$ is concave and $v_\varepsilon \leq 0$. Hence

$$\left(\frac{\partial v_\varepsilon}{\partial t}, \frac{\partial v_\varepsilon}{\partial t}\right)_k - a_k(v_\varepsilon, \frac{\partial v_\varepsilon}{\partial t}) \leq \frac{1}{2}\frac{d}{dt}\big(\beta'(u_\varepsilon - \phi)v_\varepsilon, v_\varepsilon\big)_k$$

$$\left(\frac{\partial v_\varepsilon}{\partial t}, \frac{\partial v_\varepsilon}{\partial t}\right)_k - \frac{\sigma^2}{4}\frac{d}{dt}||v_\varepsilon||_k^2 \leq \frac{1}{2}\frac{d}{dt}\big(\beta'(u_\varepsilon - \phi)v_\varepsilon, v_\varepsilon\big)_k + C\left|\frac{\partial v_\varepsilon}{\partial t}\right|_k ||v_\varepsilon||_k.$$

Now, let $0 < t_1 < t_2 < T$. Integrate from $t_1$ to $t_2$,

$$\begin{aligned}&\int_{t_1}^{t_2} \left|\frac{\partial v_\varepsilon}{\partial t}(t,\cdot)\right|_k^2 dt + \frac{\sigma^2}{4}||v_\varepsilon(t_1)||_k^2 \\ &\quad\leq \frac{\sigma^2}{4}||v_\varepsilon(t_2)||_k^2 + \frac{1}{2}\big(\beta'(u_\varepsilon(t_2) - \phi)v_\varepsilon(t_2), v_\varepsilon(t_2)\big)_k \\ &\qquad - \frac{1}{2}\big(\beta'(u_\varepsilon(t_1) - \phi)v_\varepsilon(t_1), v_\varepsilon(t_1)\big)_k + C\int_{t_1}^{t_2} \left|\frac{\partial v_\varepsilon}{\partial t}(t,\cdot)\right|_k ||v_\varepsilon(t)||_k \, dt\end{aligned}$$

$$\leq \frac{\sigma^2}{4}||v_\varepsilon(t_2)||_k^2 + \frac{1}{2}\big(\beta'(u_\varepsilon(t_2)-\phi)v_\varepsilon(t_2), v_\varepsilon(t_2)\big)_k + C\int_{t_1}^{t_2}\left|\frac{\partial v_\varepsilon}{\partial t}(t,\cdot)\right|_k ||v_\varepsilon(t)||_k\,dt.$$

We now integrate with respect to $t_2$ from $t_1$ to $T$ to get

$$\int_{t_1}^{T} dt_2\int_{t_1}^{t_2} dt\left|\frac{\partial v_\varepsilon}{\partial t}(t,\cdot)\right|_k^2 + \frac{\sigma^2}{4}(T-t_1)||v_\varepsilon(t_1)||_k^2$$
$$\leq \frac{\sigma^2}{4}\int_{t_1}^{T} dt_2||v_\varepsilon(t_2)||_k^2 + C\int_{t_1}^{T} dt_2\int_{t_1}^{t_2} dt\left|\frac{\partial v_\varepsilon}{\partial t}(t,\cdot)\right|_k ||v_\varepsilon(t)||_k + \frac{1}{2}\int_{t_1}^{T}\big(\beta'(u_\varepsilon(t_2)-\phi)v_\varepsilon(t_2), v_\varepsilon(t_2)\big)_k\,dt_2,$$

and, using Fubini's theorem,

$$\int_{t_1}^{T} dt(T-t)\left|\frac{\partial v_\varepsilon}{\partial t}(t,\cdot)\right|_k^2 + \frac{\sigma^2}{4}(T-t_1)||v_\varepsilon(t_1)||_k^2$$
$$\leq \frac{\sigma^2}{4}\int_{t_1}^{T} dt||v_\varepsilon(t)||_k^2 + C\int_{t_1}^{T} dt(T-t)\left|\frac{\partial v_\varepsilon}{\partial t}(t,\cdot)\right|_k ||v_\varepsilon(t)||_k \tag{19}$$
$$+ \frac{1}{2}\int_{t_1}^{T}\big(\beta'(u_\varepsilon(t)-\phi)v_\varepsilon(t), v_\varepsilon(t)\big)_k\,dt.$$

Now, observe that

$$\big(\beta'(u_\varepsilon(t)-\phi)v_\varepsilon(t), v_\varepsilon(t)\big)_k$$
$$\leq ||v_\varepsilon(t)||_{L^\infty}\int_{-\infty}^{\infty}\beta'(u_\varepsilon(t)-\phi)|v_\varepsilon(t)|\frac{dx}{(1+x^2)^{k/2}}$$
$$\leq \big(||A\phi||_{L^\infty}+|b|\big)\int_{-\infty}^{\infty}\beta'(u_\varepsilon(t)-\phi)(-v_\varepsilon(t))\frac{dx}{(1+x^2)^{k/2}}$$
$$= \big(||A\phi||_{L^\infty}+|b|\big)\frac{-d}{dt}\int_{-\infty}^{\infty}\beta(u_\varepsilon(t)-\phi)\frac{dx}{(1+x^2)^{k/2}}.$$

Hence

$$\int_{t_1}^{T}(\beta'(u_\varepsilon(t)-\phi)v_\varepsilon(t), v_\varepsilon(t))_k$$
$$\leq C\big(||A\phi||_{L^\infty}+|b|\big)\left(\int_{-\infty}^{\infty}\frac{\beta(u_\varepsilon(t_1)-\phi)}{(1+x^2)^{k/2}}dx - \int_{-\infty}^{\infty}\frac{\beta(0)}{(1+x^2)^{k/2}}dx\right)$$
$$\leq C|b|\big(||A\phi||_{L^\infty}+|b|\big).$$

Going back to (19) and taking (18) into account, we obtain (15). □

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