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ON THE EQUIVALENCE OF THE TUBE AND EULER CHARACTERISTIC METHODS FOR THE DISTRIBUTION OF THE MAXIMUM OF GAUSSIAN FIELDS OVER PIECEWISE SMOOTH DOMAINS

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Consider a Gaussian random field with a finite Karhunen–Loève expansion of the form $Z(u) = \sum_{i=1}^n u_i z_i$, where z_i, $i = 1, \dots, n$, are independent standard normal variables and $u = (u_1, \dots, u_n)'$ ranges over an index set M, which is a subset of the unit sphere S^{n-1} in R^n. Under a very general assumption that M is a manifold with a piecewise smooth boundary, we prove the validity and the equivalence of two currently available methods for obtaining the asymptotic expansion of the tail probability of the maximum of $Z(u)$. One is the tube method, where the volume of the tube around the index set M is evaluated. The other is the Euler characteristic method, where the expectation for the Euler characteristic of the excursion set is evaluated. General discussion on this equivalence was given in a recent paper by R. J. Adler. In order to show the equivalence we prove a version of the Morse theorem for a manifold with a piecewise smooth boundary.

1. Introduction.

1.1. *Maximum of a Gaussian field.* Let M be a closed subset of the unit sphere S^{n-1} in R^n. We consider a random field $\{Z(u) \mid u = (u_1, \dots, u_n)' \in M\}$ defined by

$$Z(u) = u'z = \sum_{i=1}^n u_i z_i, \tag{1.1}$$

where $z = (z_1, \dots, z_n)'$ is distributed according to the n-dimensional standard multivariate normal distribution $N_n(0, I_n)$. The covariance function is given by

$$r(u, v) = E[Z(u)Z(v)] = u'v.$$

The variance of $Z(u)$ is $r(u, u) = \|u\|^2 = 1$ since $u \in S^{n-1}$. Let $\{X(t) \mid t \in I\}$ be a Gaussian random field such that $E[X(t)] = 0$, $E[X(t)^2] = 1$ and $X(t)$ has a finite Karhunen–Loève expansion,

$$X(t) = \sum_{i=1}^n \phi_i(t) z_i = \phi(t)'z, \qquad t \in I,$$

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where $\phi(t) = (\phi_1(t), \dots, \phi_n(t))'$ and $z = (z_1, \dots, z_n)'$ has the same distribution as above. If we put $u_i = \phi_i(t)$, $i = 1, \dots, n$, and $M = \{\phi(t) \mid t \in I\} \subset S^{n-1}$, then $X(t)$ can be written as $Z(u)$ in (1.1). Therefore (1.1) is the canonical form of centered Gaussian random fields with a finite Karhunen–Loève expansion and constant variance. Many testing problems in multivariate analysis can be formulated in the canonical form [e.g., Kuriki and Takemura (2001)].

In this paper we study the asymptotic behavior of the upper tail probability,

$$P\left(\max_{u \in M} Z(u) \geq x\right) \tag{1.2}$$

as x goes to infinity. As a related random field to (1.1) we define

$$Y(u) = u'y = \sum_{i=1}^{n} u_i y_i, \tag{1.3}$$

where $y = (y_1, \dots, y_n)' = z/\|z\|$ is distributed according to the uniform distribution Unif(S^{n-1}) on the unit sphere S^{n-1}. We also study the upper tail probability,

$$P\left(\max_{u \in M} Y(u) \geq x\right). \tag{1.4}$$

Once formulated in the canonical form (1.1), the upper tail probabilities (1.2) and (1.4) depend on the geometry of the index set M. Although in our setting we are restricted to random fields with a finite Karhunen–Loève expansion, we want to consider a class of index sets M which is as general as possible. This class should include polyhedral regions, (geodesically) convex regions, and manifolds with or without boundaries. In our previous works we studied convex regions [Takemura and Kuriki (1997)] and manifolds without boundary [Kuriki and Takemura (1999, 2001)]. Unifying these cases in this paper we assume that M is a *manifold with a piecewise smooth boundary*. Furthermore we assume that M is *locally convex* in the sense that M is approximated by a convex support cone at each point $u \in M$. The precise definition of these notions and formal assumptions of this paper will be given in Section 1.2.

The convexity of the support cone is essential for the validity of the asymptotic expansion of the upper tail probability (1.2). In our subsequent work [Takemura and Kuriki (2000)] we discuss in detail that the tube method and the Euler characteristic method lead to incorrect asymptotic expansion when the support cone is not convex.

In order to derive the asymptotic expansion of the upper tail probability (1.2) for $Z(u)$, two methods are currently available. One is the *tube method* developed by Sun (1993). She showed that given an expression for the upper probability (1.4) for $Y(u)$ valid for $x \in [x_c, 1]$ ($x_c < 1$ is a constant), the asymptotic expansion of the upper probability (1.2) for $Z(u)$ is obtained automatically from the expression for (1.4). As will be explained in Section 2, the upper probability for $Y(u)$ is exactly the ratio of volume of tube (tubular neighborhood) around M to the volume of the unit sphere S^{n-1}. Therefore the problem is reduced to obtaining the formula for volume of tube (tube formula). The tube formula for a manifold of general dimension without boundary was obtained by Weyl (1939). For a manifold with a piecewise smooth boundary, the tube formula for $\dim M = 1$ was given in Hotelling (1939) and for $\dim M = 2$ it was given in Knowles and Siegmund (1989). When M is a geodesically convex domain with piecewise smooth boundary, Takemura and Kuriki (1997) gave a formula which is essentially equivalent to the tube formula. In this paper we present the tube formula for a manifold with a piecewise smooth boundary of general dimension under the assumption of local convexity.

The other method for obtaining the asymptotic expansion of the tail probability (1.2) is the *Euler characteristic method* developed by Adler (1981) and Worsley (1995a, b). As we will see in Section 3, the Euler characteristic method is applicable in principle to any random field. However, in contrast to the tube method, the Euler characteristic method is a heuristic approach and its validity in a general setting has not been proved. Recently, Adler (2000) using results from Piterbarg (1996) showed that the Euler characteristic method for isotropic Gaussian random fields on a piecewise smooth domain gives the valid asymptotic expansion. In this paper, in the case where the Gaussian field is of the form (1.1) but not assumed to be isotropic, we give a proof that the Euler characteristic method is equivalent to the tube method and hence gives a valid asymptotic expansion. In Kuriki and Takemura (1999) we proved the equivalence for the case of M without a boundary. In order to show the general equivalence, we prepare a version of the Morse theorem for a manifold with a piecewise smooth boundary. Moreover our geometric consideration gives us an alternative proof of Naiman's inequality [Naiman (1986), Johnstone and Siegmund (1989)].

The outline of this paper is as follows. In Section 2 we define the tube on the sphere and give a tube formula for a manifold with a piecewise smooth boundary. We also discuss how to calculate the critical radius of the tube, which is essential for determining the order of the remainder term of the asymptotic expansion. In Section 3 we first explain the Euler characteristic method for the Gaussian random field (1.1). Then we prove the equivalence of the tube method and the Euler characteristic method using a generalized version of the Morse theorem. Furthermore, we give an alternative simplified proof of Naiman's inequality. In Section 4 as an example we discuss the distribution of the maximum of the cosine field, which was examined in Piterbarg (1996).

1.2. *A manifold with a piecewise smooth boundary.* Here we define a manifold with a piecewise smooth boundary and state the formal assumptions of this paper.

A cone K is called *proper* if $K \cap (-K) = \{0\}$.

DEFINITION 1.1. Let M be a topological m-dimensional manifold with a boundary. M is called a manifold with a piecewise smooth boundary of class C^r if each $x \in M$ has a neighborhood $W(x)$ which is C^r-diffeomorphic to the following set $\widetilde{W}$ in R^m:

$$\widetilde{W} = (-\varepsilon, \varepsilon)^d \times (K \cap \varepsilon B_1^{m-d}), \tag{1.5}$$

where $0 \leq d \leq m$, $\varepsilon > 0$, K is a closed proper cone in R^{m-d}, $B_1^{m-d} = \{x \mid \|x\| < 1\} \subset R^{m-d}$ is the open ball in R^{m-d}, and $\times$ denotes the direct product.

In this definition $K = K(x)$ and ε can depend on x. Roughly speaking this definition requires that at each $u \in M$, M is approximated by a support cone and the support cone varies in a piecewise smooth manner as u varies over M. Piecewise smoothness means that the support cone can change discontinuously when the dimension d of the tangent space at u changes. K need not be a polyhedral cone. We studied one important example of a nonpolyhedral cone in Kuriki and Takemura (2000a).

For each $0 \leq d < m$, let ∂M_d denote the set of points x having a neighborhood $W(x)$ which is C^r-diffeomorphic to $\widetilde{W}$ of the form (1.5). By standard argument it can be shown that ∂M_d forms a d-dimensional manifold of class C^r; $\partial M = \bigcup_{d=0}^{m-1} \partial M_d$ forms the boundary of M. For convenience and notational consistency we also write $\partial M_m = M^o$, the interior of M, although ∂ symbol here might be somewhat confusing.

Definition 1.1 is an intrinsic definition and M is not necessarily a submanifold of a Euclidean space. However for our purposes it suffices to consider submanifolds of a Euclidean space and we assume that all manifolds are submanifolds of R^n endowed with the standard inner product $\langle x, y\rangle = x'y$, where $x,\ y$ are considered as an n-dimensional column vector. As a submanifold of R^n the topology on M coincides with the relative topology induced from R^n. Therefore M^o denotes the relative interior of M and ∂M denotes the relative boundary of M in R^n.

Let $x \in \partial M_d \subset R^n$, $0 \leq d \leq m$. Take a local coordinate system $(w_1, \dots, w_m)$ and write $x(w_1, \dots, w_m)$ for points in a neighborhood of $x = x(0, \dots, 0)$ in accordance with (1.5), that is, $W(x)$ in Definition 1.1 is written as $W(x) = \{x(w_1, \dots, w_m) \mid (w_1, \dots, w_m) \in \widetilde{W}\}$. Then

$$\frac{\partial x}{\partial w_j} = \frac{\partial x}{\partial w_j}(0, \dots, 0) \in R^n, \qquad j = 1, \dots, d,$$

form a basis for the tangent space of $T_x(\partial M_d)$ of ∂M_d at $x = x(0, \dots, 0)$. The *support cone* $S_x(M)$ of M at $x = x(0, \dots, 0)$ is defined by

$$\begin{aligned} S_x(M) = {} & T_x(\partial M_d) \\ & \oplus \{w_{d+1} N_{d+1} + \cdots + w_m N_m \mid (w_{d+1}, \dots, w_m) \in K\}, \end{aligned} \tag{1.6}$$

where $\oplus$ is the direct sum of vector spaces and

$$N_j = N_j(x) = \frac{\partial x}{\partial w_j}(0, \ldots, 0), \qquad j = d+1, \ldots, m.$$

The support cone $S_x(M)$ is a cone approximating M at x. Furthermore, we define the *normal cone* $N_x(M)$ of M at x as the dual cone of $S_x(M)$ in R^n,

$$N_x(M) = \{y \in R^n \mid y'z \le 0,\ \forall\, z \in S_x(M)\}.$$

Some examples and figures of these cones are given in Section 2.1. It can be easily shown that we can take the above local coordinate systems in such a way that $\{N_{d+1}(x), \ldots, N_m(x)\}$ form an orthonormal basis of $T_x(M) \cap T_x^{\perp}(\partial M_d)$ for each x and of class C^{r-1} as functions of x. Using this particular local coordinates, $N_x(M)$ is written as

$$\begin{aligned} N_x(M) = {} & T_x(M)^{\perp} \\ & \oplus \{w_{d+1}N_{d+1} + \cdots + w_m N_m \mid (w_{d+1}, \ldots, w_m) \in K^*\}, \end{aligned} \tag{1.7}$$

where $K^* = K^*(x)$ denotes the dual cone of $K(x)$ in R^{m-d}. For the case where M is a convex set the notions of support cone and normal cone given here coincide with the those in Section 2.2 of Schneider (1993). See also Section 2.3 of Takemura and Kuriki (1997).

We now state assumptions of this paper.

ASSUMPTION 1.1. *M is a compact m-dimensional C^2-manifold with a piecewise smooth boundary in the sense of Definition* 1.1.

ASSUMPTION 1.2. *At each point $u \in M$, the support cone $S_u(M)$ of M is convex.*

2. Tube method. In this section we derive the tube formula for tubes around a piecewise smooth $M \subset S^{n-1}$ and the asymptotic expansion of probabilities (1.2) and (1.4) based on the tube formula. For instructive purposes we also discuss the tube formula for tubes in R^n, because the Euclidean case is simpler and helpful for understanding the spherical case.

2.1. *The tube and its critical radius.* Let

$$M_\theta = \{y \in S^{n-1} \mid u'y \ge \cos\theta \text{ for some } u \in M\}.$$

Since y in (1.3) is distributed uniformly on S^{n-1}, the probability (1.4) for $x = \cos\theta$ is written as

$$P\Big(\max_{u \in M} Y(u) \ge \cos\theta\Big) = \frac{1}{\Omega_n}\,\mathrm{Vol}(M_\theta),$$

where $\mathrm{Vol}(\cdot)$ denotes the spherical volume on S^{n-1} and

$$\Omega_n = \mathrm{Vol}(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

is the total volume of S^{n-1}.

Let

$$\mathrm{dist}(u, v) = \cos^{-1}(u'v) \in [0, \pi], \qquad u, v \in S^{n-1},$$

be the distance on the unit sphere S^{n-1}, and let

$$\mathrm{dist}(u, M) = \min_{v \in M} \mathrm{dist}(u, v).$$

Then the set M_θ can be written as

$$M_\theta = \{y \in S^{n-1} \mid \mathrm{dist}(y, M) \le \theta\};$$

that is, M_θ is the set of points whose distance from M are less than or equal to θ. We call M_θ a spherical *tube* around M with radius θ. Therefore the evaluation of the tail probability (1.4) is reduced to the evaluation of the volume of tube M_θ.

Since M is closed, for each $y \in S^{n-1}$ there exists a closest point y_M in M (projection of y onto M) such that

$$\mathrm{dist}(y, y_M) = \mathrm{dist}(y, M). \tag{2.1}$$

Although y_M might not be unique, the distance $\mathrm{dist}(y, M)$ is uniquely determined.

Define a subset of S^{n-1} by

$$C_u(\theta) = M_\theta \cap (u + N_u(M)), \qquad u \in M,$$

where $+$ denotes the vector sum and $N_u(M)$ is the normal cone of M at u, which is the dual cone of $S_u(M)$ in R^n. $C_u(\theta)$ is the cross section of M_θ crossing M at $u \in M$ and consists of points $y \in M_\theta$ such that $u = y_M \in M$. Since each $y \in M_\theta$ belongs to $C_{y_M}(\theta)$, M_θ can be written as the union of cross sections,

$$M_\theta = \bigcup_{u \in M} C_u(\theta).$$

Figure 1 depicts $N_u(M)$ for the case of $\dim M = 2$ and $d = 0, 1, 2$.

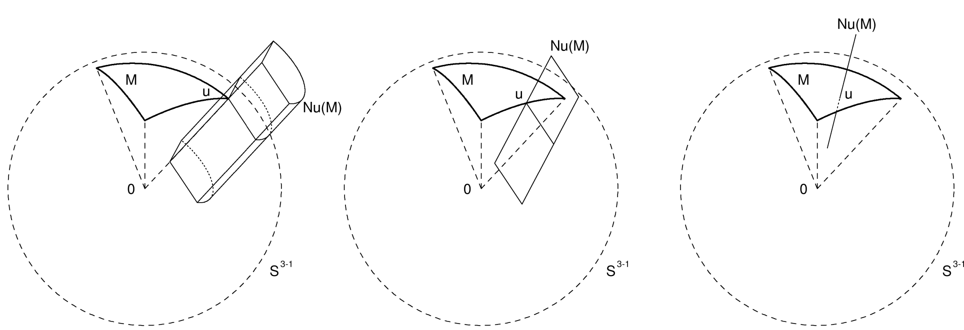

FIG. 1. *Normal cone $N_u(M)$ (left: $d = 0$, center: $d = 1$, right: $d = 2$).*

By Assumption 1.2 at each point $u \in M$, M is locally approximated by the convex support cone $S_u(M)$. Because of this, for each y sufficiently close to M the point y_M is uniquely defined. From the compactness of M it can be shown that there exists $\theta > 0$ such that for every $y \in M_\theta$ the point y_M is unique. The supremum θ_c of such θ is called the *critical radius* of M,

$$\theta_c = \theta_c(M) = \sup\{\theta > 0 \mid y_M \text{ unique for all } y \in M_\theta\}.$$

It is easily shown that θ_c can also be defined by

$$\theta_c = \sup\{\theta > 0 \mid C_u(\theta),\ u \in M,\ \text{are disjoint}\}.$$

Properties of the set of points y with the unique projection y_M onto M are discussed in detail in Section 2.1 of Takemura and Kuriki (2000).

Let

$$K = K(M) = \bigcup_{c \ge 0} cM$$

denote the smallest cone containing M. The critical radius can be computed using the following formula.

LEMMA 2.1. *For $M \subset S^{n-1}$ satisfying Assumptions* 1.1 *and* 1.2,

$$\inf_{u,v \in M} \frac{\|u-v\|^2}{2\|P_v^\perp(u-v)\|} = \begin{cases} \tan\theta_c, & \theta_c < \pi/2, \\ \infty, & \theta_c \ge \pi/2, \end{cases} \tag{2.2}$$

where $P_v^\perp$ is the orthogonal projection in R^n onto the normal cone $N_v(K)$ of K at v.

For the case of a one-dimensional smooth manifold, this result is given in Proposition 4.3 of Johansen and Johnstone (1990). Extension to smooth manifolds of higher dimension is stated in Lemma A.1 of Kuriki and Takemura (2001). We omit the proof of Lemma 2.1, since it is essentially the same as the proof given by Johansen and Johnstone (1990).

It can be proved that $\theta_c \ge \pi/2$ if and only if K $(\neq R^n)$ is convex. If M is a geodesically convex region on S^n, then the critical radius $\theta_c(M)$ may be greater than $\pi/2$. In this case the denominator of the left-hand side of (2.2) is 0 and (2.2) does not give the critical radius.

We now briefly discuss corresponding notions for the tubes in R^n. Let M be a compact m-dimensional submanifold of R^n with piecewise smooth boundary of class C^2 satisfying Assumptions 1.1 and 1.2. Let $x \in R^n$. Since M is closed there exists a closest point $x_M \in M$ from x. The tube M_r around M with radius r is defined by

$$M_r = \{x \in R^n \mid \|x - x_M\| \le r\}.$$

The cross section $C_x(r)$ of M_r at $x \in M$ is defined by

$$C_x(r) = M_r \cap \bigl(x + N_x(M)\bigr),$$

where $+$ denotes the vector sum and $N_x(M)$ is the normal cone of M at x. We see that M_r can be written as the union of cross sections:

$$M_r = \bigcup_{x \in M} C_x(r).$$

From the compactness and the local convexity of M it can be shown that there exists $r > 0$ such that every $x \in M_r$ has unique projection point x_M. The critical radius r_c of M is the supremum of such r:

$$r_c = r_c(M) = \sup\bigl\{r > 0 \mid x_M \text{ unique for all } x \in M_r\bigr\}.$$

In integral geometry literature the critical radius of M is called the *reach* of M [e.g., Federer (1959), Stoyan, Kendall and Mecke (1995)]. The critical radius can be computed using the following formula.

LEMMA 2.2. *Let M be a compact subset of R^n satisfying Assumptions* 1.1 *and* 1.2. *The critical radius r_c of M is given by*

$$r_c = \inf_{x,y \in M} \frac{\|x-y\|^2}{2\|P_y^{\perp}(x-y)\|},$$

where $P_y^{\perp}$ is the orthogonal projection onto the normal cone $N_y(M)$ of M at y.

As in the case of S^{n-1}, the positiveness of the critical radius is assured by Assumption 1.2. Based on this the property of local convexity of Assumption 1.2 is called *positive-reach* in integral geometry literature.

2.2. *Tubal coordinates and Jacobian.* Let $M \subset S^{n-1}$ be piecewise smooth and let y be an interior point of $M_{\theta_c} \cap M^c$. We introduce here the tubal coordinates of M_{θ_c} around y.

Suppose that the projection y_M of y onto M is a relative interior point of a component of d-dimensional boundary ∂M_d of M. Here we assume $d \le n-2$. Let $u = y_M$, $\theta = \cos^{-1}(u'y)$. If $\theta \le \pi/2$, then $u\cos\theta$ is the projection in R^n of y onto $K = K(M)$. Put

$$v = \frac{y - y_M\cos\theta}{\|y - y_M\cos\theta\|} = \frac{y - u\cos\theta}{\|y - u\cos\theta\|} = \frac{y - u\cos\theta}{\sin\theta} \in S^{n-1}.$$

Considering the two-dimensional plane spanned by y and u we see that y is uniquely written as

$$y = u\cos\theta + v\sin\theta, \qquad 0 < \theta < \theta_c,\ \ u \in \partial M_d,\ \ v \in N_u(K) \cap S^{n-1}.$$

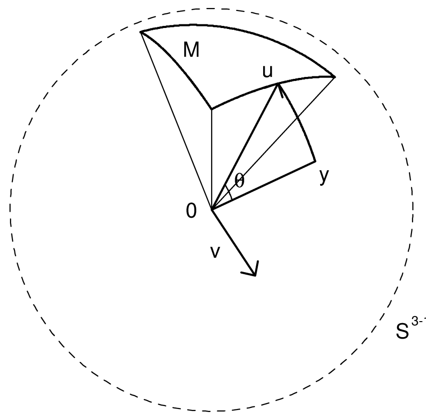

FIG. 2. *Tubal coordinates* ($y = u\cos\theta + v\sin\theta$).

We call the coordinates (θ, u, v) tubal coordinates. Figure 2 depicts the tubal coordinates for the case of $\dim M = 2, d = 1$.

Let $(w_1, \dots, w_d)$ be a local coordinate system of ∂M_d around u and write $u = u(w_1, \dots, w_d)$. The volume element of ∂M_d at u is defined by $du = \sqrt{\det(g_{ij})}\, dw_1 \cdots dw_d$, where $g_{ij} = (\partial u/\partial w_i)'(\partial u/\partial w_j)$ is the metric of ∂M_d at u. Let $H(u, v)$ denote the second fundamental form of M at u in the direction v with the (i, j)th element $-\sum_{k=1}^{d} v'(\partial^2 u)/(\partial w_i \partial w_k) g^{kj}$, where g^{kj} is the (k, j)th element of the inverse matrix of (g_{ij}). Then the Jacobian of the transformation $y \leftrightarrow (\theta, u, v)$ is given as follows.

LEMMA 2.3. *Let dy be the volume element of M_{θ_c} (or S^{n-1}) at y, du be the volume element of ∂M_d at $u = y_M$, dv be the volume element of $S^{n-d-2} = N_u(K(M)) \cap S^{n-1}$ at v. Then for $0 \le d \le n-2$,*

$$dy = \det(\cos\theta I_d + \sin\theta H(u, v)) \sin^{n-d-2}\theta\, d\theta\, du\, dv, \tag{2.3}$$

where for the case $d = 0$ the determinant term equals 1 and du is the unit point mass at u; for the case $d = n-2$, dv is the unit point mass at v.

PROOF. Since the case $d = 0$ is straightforward, assume $d \ge 1$. Introduce a parameter $t > 0$. Let $z = ty \in R^n$ and put $r = t\cos\theta$, $s = t\sin\theta$. Then $z = ru + sv$, which gives a one-to-one correspondence between z and (r, s, u, v). The Jacobian of this transformation is essentially given by Weyl (1939) as

$$dz = \det(rI_d + sH(u, v)) s^{n-d-2}\, dr\, ds\, du\, dv. \tag{2.4}$$

See Appendix A.1 of Kuriki and Takemura (2000b) for a proof. Here note that the Lebesgue measure dz at z is decomposed as

$$dz = t^{n-1}\, dt\, dy, \tag{2.5}$$

where dy is the volume element of S^{n-1} at $y = z/\|z\|$. Note also that

$$dr\,ds = t\,dt\,d\theta. \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.4), and comparing the coefficients of $t^{n-1}\,dt$, we have the lemma. □

Note that for $\theta < \theta_c$ the determinant in (2.3) is nonnegative.

2.3. *Tube formula and tail probabilities.* Here we present the tube formula for the spherical volume of a tube around M. The tube formula of this section unifies the tube formula in the sense of Weyl (1939) and the Steiner formula for the convex sets discussed in Takemura and Kuriki (1997).

Let $u \in \partial M_d$ and let $v \in N_u(K(M))$, $\|v\| = 1$. The lth symmetric function of the principal curvatures of M, that is, the eigenvalues of the second fundamental form $H(u, v)$, is denoted by $\mathrm{tr}_l H(u, v)$. The tube formula $\mathrm{Vol}(M_\theta)$ for M_θ is given as follows.

PROPOSITION 2.1. *For $e = 0, \ldots, m$, let*

$$\begin{aligned} &w_{m+1-e} \\ &\quad = \frac{1}{\Omega_{m+1-e}\Omega_{n-m-1+e}} \sum_{d=m-e}^{m} \int_{\partial M_d} du \int_{N_u(K(M))\cap S^{n-1}} dv\, \mathrm{tr}_{d-m+e} H(u, v), \end{aligned} \tag{2.7}$$

where for each $0 \le d \le m$, du and dv are the volume elements defined in Lemma 2.3. The integral with respect to dv in (2.7) is assumed to be unity when $m = n-1$ and $e = 0$ [i.e., $N_u(K(M)) \cap S^{n-1} = \varnothing$]. Then for $\theta \le \theta_c(M)$ the spherical volume of M_θ is given by

$$\mathrm{Vol}(M_\theta) = \begin{cases} \Omega_n \displaystyle\sum_{e=0}^{m} w_{m+1-e}\bigl(1 - B_{(m+1-e)/2,(n-m-1+e)/2}(\cos^2\theta)\bigr), \\ \hfill 0 \le \theta \le \pi/2, \\ \Omega_n \displaystyle\sum_{e=0}^{m} w_{m+1-e}\bigl(1 + (-1)^{m-e} B_{(m+1-e)/2,(n-m-1+e)/2}(\cos^2\theta)\bigr), \\ \hfill \pi/2 < \theta \le \theta_c, \end{cases} \tag{2.8}$$

where $B_{a,b}(\cdot)$ denotes the cumulative distribution function of beta distribution with parameter (a, b).

PROOF. By virtue of the Jacobian given in Lemma 2.3, the spherical volume of $M_{\theta'}$ for $\theta' \le \theta_c$ is given by

$$\begin{aligned} \mathrm{Vol}(M_{\theta'}) = \sum_{d=0}^{m} \int_0^{\theta'} d\theta \int_{\partial M_d} du \int_{N_u(K)\cap S^{n-1}} dv\, \det\bigl(\cos\theta\, I_d + \sin\theta\, H(u, v)\bigr) \\ \times \sin^{n-d-2}\theta. \end{aligned}$$

Using the expansion formula for the determinant $\det(I_d + A) = \sum_{l=0}^{d} \mathrm{tr}_l\, A$, we obtain the result by straightforward integration. □

COROLLARY 2.1. *For $x \geq \cos\theta_c(M)$,*

$$P\Big(\max_{u\in M} Y(u) \geq x\Big) = \begin{cases} \displaystyle\sum_{e=0}^{m} w_{m+1-e}\big(1 - B_{(m+1-e)/2,(n-m-1+e)/2}(x^2)\big), & 0 \leq x \leq 1, \\ \displaystyle\sum_{e=0}^{m} w_{m+1-e}\big(1 + (-1)^{m-e} B_{(m+1-e)/2,(n-m-1+e)/2}(x^2)\big), & \cos\theta_c \leq x < 0, \end{cases} \tag{2.9}$$

where w_{m+1-e} is given in (2.7).

Note that in (2.8) and (2.9) the second cases are needed only when $\theta_c > \pi/2$.

Now consider the maximum of $Z(u)$. Let $G_k(\cdot)$ and $g_k(\cdot)$ denote the cumulative distribution function and the density function of χ^2 distribution with k degrees of freedom, respectively. Using the independence of $\|z\|$ and $y = z/\|z\| \in S^{n-1}$, we obtain the following result from Corollary 2.1 by integrating out $\|z\|$ [see Sun (1993) and Kuriki and Takemura (2001)].

PROPOSITION 2.2. *If $\theta_c < \pi/2$, then as $x \to \infty$,*

$$P\Big(\max_{u\in M} Z(u) \geq x\Big) = \sum_{e=0}^{m} w_{m+1-e}\big(1 - G_{m+1-e}(x^2)\big) + O\big(g_n\big(x^2(1+\tan^2\theta_c)\big)\big). \tag{2.10}$$

If $\theta_c \geq \pi/2$, then for each $x \geq 0$,

$$P\Big(\max_{u\in M} Z(u) \geq x\Big) = \sum_{e=0}^{m} w_{m+1-e}\big(1 - G_{m+1-e}(x^2)\big). \tag{2.11}$$

Here w_{m+1-e} is given in (2.7).

Note that the remainder term in (2.10) is of the order of $o(1 - G_1(x^2))$.

REMARK 2.1. When $\theta_c(M) \geq \pi/2$, all of the coefficients w_{m+1-e} in (2.11) are nonnegative since $K(M)$ is convex and hence the second fundamental form $H(u, v)$ in (2.7) is nonnegative definite. This distribution is a finite mixture of χ^2 distributions referred to as $\bar{\chi}^2$ (chi-bar-squared) distribution [e.g., Shapiro (1988)].

For the case of Euclidean tubes around piecewise smooth M in R^n satisfying Assumptions 1.1 and 1.2, the tube formula can be directly derived from the expression of the Jacobian in (2.4). Let

$$V_n(M_r) = \sum_{e=0}^{m} \frac{r^{n-m+e}}{n-m+e} \sum_{d=m-e}^{m} \int_{\partial M_d} dy \int_{N_y(M)\cap S^{n-1}} dv \, \mathrm{tr}_{d-m+e} H(y,v), \tag{2.12}$$

where for each $0 \le d \le m$, dy denotes the volume element of ∂M_d and dv denotes the volume element of $S^{n-d-1} = N_y(M) \cap S^{n-1}$. The integral with respect to dv is assumed to be unity when $m = n$ and $e = 0$. Then the following result holds.

PROPOSITION 2.3. *For $r \le r_c(M)$, $V_n(M_r)$ in* (2.12) *is equal to the n-dimensional volume* $\mathrm{Vol}(M_r)$ *of M_r.*

3. Euler characteristic method. In order to approximate tail probabilities of random fields such as (1.2) or (1.4), Adler (1981) and Worsley (1995a, b) have developed a technique based on the Euler characteristic of excursion set. In this paper we call their method the *Euler characteristic method*. We begin with a brief examination of the idea of their method in the case of $Z(u)$ in (1.1). Then we prepare a generalization of the Morse theorem, and prove the equivalence of the tube method and the Euler characteristic method.

3.1. *Excursion set and its expectation.* The excursion set of a random field $\{X(t) \mid t \in I\}$ is a subset of the index set I consisting of $t \in I$ such that $X(t)$ is greater than or equal to a threshold. Hence in our case

$$A(z,x) = \{u \in M \mid u'z \ge x\}$$

is the excursion set for $Z(u) = u'z$. It holds by definition that

$$P\Big(\max_{u\in M} Z(u) \ge x\Big) = P\big(A(z,x) \ne \varnothing\big).$$

Let $\chi(A(z,x))$ denote the Euler characteristic (Euler–Poincaré characteristic) of the excursion set $A(z,x)$. The Euler characteristic method approximates the tail probability (1.2) for large x by

$$P\Big(\max_{u\in M} Z(u) \ge x\Big) \approx E\big[\chi\big(A(z,x)\big)\big]. \tag{3.1}$$

A rationale for the approximation (3.1) is as follows. The Euler characteristic is an integer-valued topological invariant. In particular it takes the values

$$\chi(A(z,x)) = \begin{cases} 1, & A(z,x) \text{ is homotopy equivalent to a point,} \\ 0, & A(z,x) \text{ is empty.} \end{cases}$$

Suppose that the threshold x is large. If $\max_{u\in M} u'z < x$, then $A(z,x)=\varnothing$. Now consider the case $\max_{u\in M} u'z > x$. Note that the maximizing point u^*, that is, $\max_{u\in M} u'z = (u^*)'z$, is uniquely determined with probability 1. Therefore, given $\max_{u\in M} u'z > x$, with a conditional probability nearly equal to 1, $A(z,x)$ will be some neighborhood of u^*, which is homotopic to a point set $\{u^*\}$.

Summarizing the discussions above, it is expected that for large x,

$$I\bigl(A(z,x)\neq\varnothing\bigr)\approx\chi\bigl(A(z,x)\bigr)\quad\text{with a probability nearly equal to 1,}\tag{3.2}$$

where $I(\cdot)$ is the indicator function. By taking the expectation for (3.2), we have $P(A(z,x)\neq\varnothing)\approx E[\chi(A(z,x))]$, and (3.1) follows.

In contrast to the tube method in Section 2, the Euler characteristic method is applicable to any random field. However this method as described above is heuristic; the meaning of the symbol "≈" in (3.1) has to be examined in each case. Recently, [Adler (2000), Theorem 4.5.2] showed that in the case of isotropic Gaussian random field the Euler characteristic method gives the valid asymptotic expansion for (1.2) as x goes to infinity under mild regularity conditions. Adler (2000) proved this by checking that all terms of expansions are the same as a formula obtained earlier by Piterbarg (1996), Theorem 5.1. [See Section 2.5 of Adler (1981) for the definition of isotropic field.] In the following subsections we prove a version of Morse theorem for a manifold with a piecewise smooth boundary and then prove that the Euler characteristic method for the Gaussian random field $Z(u)$ in (1.1) is reduced to the tube method of Section 2. This implies that the Euler characteristic method is valid for the case of $Z(u)$ in (1.1). We also point out that the study of the manifold with a piecewise smooth boundary is essential, because the boundary of an excursion set may well be only piecewise smooth even when the index set M has everywhere smooth boundary.

3.2. *Morse theorem for a manifold with a piecewise smooth boundary.* Here we prepare a generalization of Theorem 10.2 of Morse and Cairns (1969) for M with a piecewise smooth boundary of class C^2. This is needed for our proof of the equivalence of the tube and the Euler characteristic methods. A similar generalization of the Morse theorem was given by Fu (1989) for sets with positive reach. However for the sake of self-contained argument, we give a proof of the generalization to the case of piecewise smooth boundary.

For a real valued function f defined on X, $f_{|X'}$ denotes its restriction to $X'\subset X$. Let f be a real-valued C^2-function defined on some relatively open neighborhood $\widetilde{M}$ of M. As in Morse and Cairns (1969) we assume the following conditions:

1. There is no critical point of f on the relative boundary ∂M of M.
2. For each $0\le d\le m$, $f_{|\partial M_d}$ is nondegenerate (i.e., having nonsingular Hessian) at its critical points.

We call f satisfying these conditions the *Morse function* on M.

Note that f needs to be defined only on $\widetilde{M}$. Therefore we can discuss Morse functions on M intrinsically without reference to R^n. However for our purposes it is convenient to consider M and its Morse function in R^n. Let f be a C^2-function defined on the whole R^n. As a Morse function on M we require that $f_{|\widetilde{M}}$ satisfies the above conditions (1) and (2). Note that the gradient of $f_{|\widetilde{M}}$ at $x \in \widetilde{M}$ coincides with the orthogonal projection of the gradient of f to the tangent space $T_x(\widetilde{M})$ and condition (1) requires that the gradient of f has nonzero $T_x(\widetilde{M})$ component for each $x \in \partial M$.

Let f be a Morse function on M. In the case of M with a smooth $(m-1)$-dimensional boundary, the critical point $x \in \partial M$ of $f_{|\partial M}$ is counted in Theorem 10.2 of Morse and Cairns (1969) if and only if the gradient of f, which is normal to the tangent space $T_x(M)$, is directed into the interior M^o of M. Noting that the normal cone $N_x(M)$ at x is the one-dimensional cone generated by the outward normal vector at x this condition can be expressed as $-\operatorname{grad} f \in N_x(M)$. We use this condition as a criterion for counting critical points on ∂M.

DEFINITION 3.1. Let $0 \le d < m$ and let $x \in \partial M_d$ be a critical point of $f_{|\partial M_d}$. x is extended inward critical point if

$$-\operatorname{grad} f \in N_x(M).$$

Let ν_k, $k = 0, \ldots, m-1$, denote the number of extended inward critical points of index k on ∂M and let μ_k, $k = 0, \ldots, m$, denote the number of critical points on M^o of index k. The augmented type numbers μ'_k, $k = 0, \ldots, m$, of f are

$$\mu_0 + \nu_0, \mu_1 + \nu_1, \ldots, \mu_{m-1} + \nu_{m-1}, \mu_m.$$

Worsley (1995a) shows how the boundary critical points are counted in the Euler characteristic for the case of R^2 and R^3. Definition 3.1 clarifies which critical points are counted in the general dimension.

We are ready to state a generalization of Theorem 10.2 of Morse and Cairns (1969).

PROPOSITION 3.1. *Let M be a compact m-dimensional manifold with a piecewise smooth boundary. The Euler characteristic $\chi(M)$ of M is given by*

$$\chi(M) = \mu'_0 - \mu'_1 + \cdots + (-1)^m \mu'_m.$$

PROOF. We follow the line of argument given in Section 11 of Morse and Cairns (1969). We omit their discussion on "critical arc" because it is basically the same for the case of M with piecewise smooth boundary. The essential point of their argument is to modify f by some function ζ such that the gradient field of $\hat{f} = f + \zeta$ is directed outwards everywhere on the boundary on M. By doing this they shift all inward critical points into the interior of M. This operation reduces their Boundary Condition B to their Boundary Condition A. For our present set-up we need to smoothly approximate ∂M in addition to shifting all extended inward critical points. For doing this we find it easier to shift extended inward critical points outward to the exterior of M (rather than shifting inward).

For our proof it is convenient to use a particular relative open neighborhood of M. Define the open r cross section at $x \in M$ by $C_x^o(r) = x + (N_x(M) \cap r B_1^n)$, where $+$ denote the vector sum and B_1^n denotes the open unit ball in R^n. For $r < r_c(M)$, let

$$\widetilde{M} = \bigcup_{x \in M} C_x^o(r) \cap \bigl(x + T_x(M)\bigr).$$

This $\widetilde{M}$ extends M at each $x \in \partial M$ outward along the tangent space through x [i.e., along $(x + T_x(M))$] in such a way that it is flat in the direction of $N_x(M)$. Without loss of generality we can assume that f is defined on this $\widetilde{M}$. In addition choose a sufficiently small r' and let

$$\overline{M}_{r'} = \bigcup_{x \in M} C_x(r') \cap \bigl(x + T_x(M)\bigr).$$

Although the boundary of $\overline{M}_{r'}$ is only of class C^1, it can be arbitrarily closely approximated by a manifold with a boundary of class C^∞. Note that $\overline{M}_{r'}$ is homotopic to M and hence $\chi(\overline{M}_{r'}) = \chi(M)$. We use the coordinate system in (1.7). Our modifying function ζ is an increasing convex function of $r^2 = \|x - x_M\|^2$ with $\zeta(0) = 0$. Hence $\zeta(x) > 0$ only for $x \notin M$. For $x_M \in \partial M_d$,

$$\zeta(x) = \zeta(w_{d+1}^2 + \cdots + w_m^2).$$

On the cross section $C_x^o(r)$ the gradient field of $\hat{f} = f + \zeta$ is given by

$$\operatorname{grad} \hat{f} = \operatorname{grad} f + 2\zeta'(r^2)(w_{d+1}N_{d+1} + \cdots + w_m N_m),$$
$$(w_{d+1}, \ldots, w_m) \in K^*,$$

in the notation of (1.7). Note that by making $\zeta'(r^2)$, $r^2 > 0$, sufficiently large, we add a strong outward vector field to the gradient field of f. Therefore by appropriate choice of ζ the gradient field of $\hat{f}$ is directed outwards at every $x \in \partial\overline{M}_{r'}$, thus reducing our case to the Boundary Condition A of Morse and Cairns (1969). A more explicit choice of ζ may be described as on page 78 of Morse and Cairns (1969).

Now suppose that $x \in \partial M_d$ is an extended inward critical point of $f_{|\partial M_d}$. Then $-\operatorname{grad} f(x) \in N_x(M)$ and in terms of the basis $\{N_{d+1}, \ldots, N_m\} = \{N_{d+1}(x), \ldots, N_m(x)\}$ we can write

$$-\operatorname{grad} f(x) = a_{d+1}N_{d+1} + \cdots + a_m N_m$$

for some coefficient vector $(a_{d+1}, \ldots, a_m) = (a_{d+1}(x), \ldots, a_m(x)) \in K^* = K^*(x)$.

By setting

$$2\zeta'(r^2)(w_{d+1},\dots,w_m)=(a_{d+1},\dots,a_m), \tag{3.3}$$

we see that the extended inward critical point is shifted outwards and becomes a critical point in the interior of $\overline{M}_{r'}$.

We need to check that the index of the Hessian matrix is not changed by the above shifting. We follow the argument on page 81 of Morse and Cairns (1969). Since ζ depends only on $r^2=w_{d+1}^2+\cdots+w_m^2$, the Hessian matrix of $\hat f$ differs from that of f only in the lower-right $(m-d)\times(m-d)$ submatrix as follows:

$$\left(\frac{\partial^2\hat f}{\partial w_i\partial w_j}\right)=\left(\frac{\partial^2 f}{\partial w_i\partial w_j}\right)+\begin{pmatrix} O & O\\ O & M\end{pmatrix},$$

where

$$M=2\zeta'(r^2)I_{m-d}+4\zeta''(r^2)\begin{pmatrix} w_{d+1}\\ \vdots\\ w_m\end{pmatrix}(w_{d+1},\dots,w_m).$$

Note that the second term on the right-hand side is nonnegative definite, whereas the first term is positive definite being a positive multiple of the identity matrix I_{m-d}. It follows that by letting $\zeta'(r^2)$ be sufficiently large, we can make the index of the Hessian matrix of $\hat f$ equal to the index of the Hessian matrix of f.

It is easy to see that by modification $f\to\hat f$, no critical point appears in the interior of $\overline{M}_{r'}$ other than those given in (3.3). Hence $\hat f$ satisfies the Boundary Condition A of Morse and Cairns (1969) and has type numbers equal to the augmented type numbers of Definition 3.1. This completes the proof of Proposition 3.1. □

3.3. *Equivalence to the tube method.* Here we prove the equivalence of the tube method and the Euler characteristic method first for $Y(u)$ in (1.3) and then for $Z(u)$ in (1.1).

Let

$$A(y,x)=\{u\in S^{n-1}\,|\,u'y\ge x\}\cap M$$

be the excursion set of the random field $Y(u)=u'y$, $y\sim\mathrm{Unif}(S^{n-1})$. In order to evaluate the expectation of the Euler characteristic of $A(y,x)$ we use Proposition 3.1. The following result together with Proposition 2.1 establishes the equivalence of two methods for $Y(u)$ in (1.3).

PROPOSITION 3.2. *Let y be distributed uniformly on S^{n-1}. Then*

$$
\begin{aligned}
E[\chi(A(y,x))] &= \int_{S^{n-1}} \chi(A(y,x))\,dy/\Omega_n \\
&= \begin{cases} \displaystyle\sum_{e=0}^{m} w_{m+1-e}\big(1 - B_{(m+1-e)/2,(n-m-1+e)/2}(x^2)\big), & 0 \le x \le 1, \\ \displaystyle\sum_{e=0}^{m} w_{m+1-e}\big(1 + (-1)^{m-e} B_{(m+1-e)/2,(n-m-1+e)/2}(x^2)\big), & -1 \le x < 0, \end{cases}
\end{aligned} \tag{3.4}
$$

where dy denotes the volume element of S^{n-1} and w_{m+1-e} is given in (2.7).

PROOF. Let $y \in S^{n-1}$. The key idea of the proof is to consider

$$f_y(u) = -u'y$$

as a Morse function. Using the same line of argument as Theorem 6.6 of Milnor (1963), we see that $f_{y|M}$ is a Morse function on M for almost all y. Since the gradient of $f_y(u)$, $u \in R^n$, is $-y$, the gradient of $f_{y|M}$ is given by the $T_u(M)$ component of $-y$. Using this fact it is easily shown that $u \in \partial M$ is an extended inward critical point of $f_{y|M}$ if and only if $y \in N_u(M)$. Similarly, concerning the relative interior M^o of M, $u \in M^o$ is a critical point of $f_{y|M}$ if and only if $y \in N_u(M) = T_u(M)^{\perp}$.

We now consider $\chi(A(y,x))$ using $f_{y|A(y,x)}$. If u is on the relative boundary of $A(y,x)$, then either $-u'y = -x$ or $u \in \partial M$. Suppose that u_0 with $-u_0'y = -x$ is a critical point of $f_{y|A(y,x)}$. Because u_0 is an inner point of some relative neighborhood $\widetilde{M}$ of M and $-u'y$ is increasing as we leave $A(y,x)$ outward at u_0, the gradient of $f_{y|A(y,x)}$ is directed outward on u_0. Hence u_0 is not counted in the Euler characteristic $\chi(A(y,x))$. On the other hand suppose that $u_0 \in \partial M$, $-u'y < -x$, is a critical point of $f_{y|A(y,x)}$. This u_0 is counted in $\chi(A(y,x))$ exactly as it is counted in $\chi(M)$. Also note that if $u_0 \in M^o$, $-u_0'y < -x$, is a critical point of $f_{y|A(y,x)}$, it is counted in $\chi(A(y,x))$ exactly as it is counted in $\chi(M)$. We see that the Euler characteristic $\chi(A(y,x))$ is written as Proposition 3.1, where augmented type numbers are obtained by counting critical points u of $f_{y|M}$ with $-u'y < -x$.

Consider the index of $f_{y|\partial M_d}$ at the critical point $u \in \partial M_d$, $y \in N_u(M)$. Let $H(u,y)$ denote the second fundamental form of ∂M_d at u with respect to the vector y. Then by the same line of argument as stated on page 36 of Milnor (1963), the Hessian matrix of $f_{y|\partial M_d}$ at u is given by $(u'y)I_d + H(u,y)$ and hence the index of the critical point u is the number of negative characteristic roots of $(u'y)I_d + H(u,y)$. In the tubal coordinates, this matrix is written as

$$\cos\theta I_d + H(u, u\cos\theta + v\sin\theta) = \cos\theta I_d + \sin\theta H(u,v),$$

where $\theta = \cos^{-1}(u'y)$ and $v = (y - u\cos\theta)/\sin\theta$. It follows that u is counted in $\chi(M)$ or $\chi(A(y,x))$ with the sign $\operatorname{sgn}\det(\cos\theta I_d + \sin\theta H(u,v))$. That is, we have

$$\chi\bigl(A(y,x)\bigr) = \sum_{u\in\partial M\,:\,N_u(M)\ni y} I(\cos\theta > x)\operatorname{sgn}\det\bigl(\cos\theta I_d + \sin\theta H(u,v)\bigr) \quad \text{a.s.}$$

By Lemma 2.3 the Jacobian of the correspondence between the volume element of S^{n-1} and tubal coordinates (in the sense of unsigned measures) is written as

$$dy = \bigl|\det\bigl(\cos\theta I_d + \sin\theta H(u,v)\bigr)\bigr| \sin^{n-d-2}\theta\, d\theta\, du\, dv,$$

where $|\cdot|$ is the absolute value. [Although Lemma 2.3 treats only the case $y \in M_{\theta_c}$ and $u = y_M$, it can be extended to the case $y \in S^{n-1}$ and $u \in M$ such that $y \in N_u(M)$ by taking the absolute value of determinant.] Since

$$\begin{aligned}\operatorname{sgn}\det&\bigl(\cos\theta I_d + \sin\theta H(u,v)\bigr) \times \bigl|\det\bigl(\cos\theta I_d + \sin\theta H(u,v)\bigr)\bigr| \\ &= \det\bigl(\cos\theta I_d + \sin\theta H(u,v)\bigr),\end{aligned}$$

we have

$$\begin{aligned}&\int_{S^{n-1}} \chi\bigl(A(y,x)\bigr)\,dy \\ &\quad = \sum_{d=0}^{m} \int_0^{\cos^{-1}(x)} d\theta \int_{\partial M_d} du \int_{N_u(K(M))\cap S^{n-1}} dv \det\bigl(\cos\theta I_d + \sin\theta H(u,v)\bigr) \\ &\qquad\qquad \times \sin^{n-d-2}\theta.\end{aligned}$$

As in the proof of Proposition 2.1 this yields (3.4). $\square$

REMARK 3.1. As stated in the proof of Proposition 3.2, $\chi(A(y,\cos\theta))$ is the degree of many-valued map $y \in M_\theta \mapsto u \in M$ such that $y \in N_u(M)$ and the orientation of $N_u(M)$ is taken into account. In this sense the integral of the Euler characteristic $\int_{S^{n-1}} A(y,\cos\theta)\,dy$ for θ greater than the critical radius $\theta_c(M)$ can be regarded as the signed volume of tube.

REMARK 3.2. Let D_0 and D_1 be a pair of domains of S^{n-1}. Suppose D_0 is fixed and D_1 is moving. Let dK_1 denote the kinematic density of D_1, that is, an invariant measure for the group of motions in S^{n-1}. The evaluation of the integral of the following type:

$$\int_{D_0\cap D_1\neq\varnothing} \chi(D_0\cap D_1)\,dK_1$$

is studied as the *kinematic fundamental formula* in integral geometry. The kinematic fundamental formula when both ∂D_0 and ∂D_0 are smooth (of class C^2) is given in Section IV.18.3 of Santaló (1976). Our Proposition 3.2 is a version of the kinematic fundamental formula for $D_0 = M$, $D_1 = \{u \in S^{n-1} \mid u'y \geq x\}$ but $\partial D_0 = \partial M$ is not necessarily smooth.

It is now easy to translate the above equivalence of two methods for $Y(u)$ to the equivalence for $Z(u)$. The expectation of the Euler characteristic for the excursion set $A(z,x) = \{u \in M \mid u'z \geq x\}$ of $Z(u) = u'z$ is given in the following proposition.

PROPOSITION 3.3. *Let z be distributed according to the standard multivariate normal distribution $N_n(0, I_n)$. Then*

$$E[\chi(A(z,x))] = \begin{cases} \displaystyle\sum_{e=0}^{m} w_{m+1-e}\bigl(1 - G_{m+1-e}(x^2)\bigr), & x \geq 0, \\ \displaystyle\sum_{e=0}^{m} w_{m+1-e}\bigl(1 + (-1)^{m-e} G_{m+1-e}(x^2)\bigr), & x < 0. \end{cases}$$

PROOF. Note that $A(z,x) = A(y, x/\|z\|)$ with $y = z/\|z\|$. Since y and $\|z\|$ are independent, the expectation $E[\chi(A(z,x))]$ can be calculated by substituting $x^2 := x^2/\|z\|^2$ in (3.4) and taking the expectation with respect to $\|z\|^2 \sim \chi^2(n)$. $\square$

The above proposition and Proposition 2.2 show that the asymptotic expansion obtained by the tube method and the Euler characteristic method are the same.

In the rest of this section we state various results obtained from the above development.

Consider the special case of $x = -1$ in (3.4). Noting that $A(y, -1) = M$, we have the following corollary.

COROLLARY 3.1.

$$\chi(M) = 2 \sum_{\substack{e=0 \\ m-e:\text{even}}}^{m} w_{m+1-e} = \begin{cases} 2(w_1 + w_3 + \cdots + w_{m+1}), & m \text{ is even}, \\ 2(w_1 + w_3 + \cdots + w_m), & m \text{ is odd}, \end{cases} \tag{3.5}$$

where w_{m+1-e} is given in (2.7).

Corollary 3.1 is an extension of Lemma 3.5 of Kuriki and Takemura (2001). At the end of this subsection we give an alternative derivation of Corollary 3.1 via a version of the Gauss–Bonnet theorem for a positive-reach manifold with a boundary.

REMARK 3.3. Suppose that $K(M)$ is a convex proper cone, which is the case considered in Takemura and Kuriki (1997). Then $\chi(M) = 1$ and Corollary 3.1 yields

$$\frac{1}{2} = \begin{cases} w_1 + w_3 + \cdots + w_{m+1}, & m \text{ is even}, \\ w_1 + w_3 + \cdots + w_m, & m \text{ is odd}. \end{cases}$$

This is exactly Shapiro's conjecture [Shapiro (1987)] on the weights of $\bar{\chi}^2$ distribution. Therefore Corollary 3.1 is a generalization of Shapiro's conjecture.

We now state the equivalence of the tube method and the Euler characteristic method for Euclidean tubes in R^n. For $x \in R^n$ let

$$A(x, r) = \{z \in R^n \mid \|z - x\| \le r\} \cap M$$

denote the intersection of M and the closed ball around x of radius r. The basic relation linking the tube method and the Euler characteristic method is given in the following proposition.

PROPOSITION 3.4. *Let M be a compact m-dimensional submanifold of R^n with a piecewise smooth boundary of class C^2 satisfying Assumptions* 1.1 *and* 1.2 *and let $V_n(M_r)$ be defined by* (2.12). *Then for $r \ge 0$,*

$$V_n(M_r) = \int_{R^n} \chi\bigl(A(x, r)\bigr)\, dx, \tag{3.6}$$

where dx denotes the Lebesgue measure and $\chi(A(x, r))$ denotes the Euler characteristic of $A(x, r)$.

As stated in Remark 3.2, (3.6) is a version of the kinematic fundamental formula for the case of Euclidean space [cf. Section III.15.4 of Santaló (1976)].

The following is a Gauss–Bonnet theorem for a positive-reach manifold with a boundary [Federer (1959), Theorem 5.19, and Section IV.17.2 of Santaló (1976)]. The Euler characteristic of M is given by the coefficient of r^n in the signed tube formula (2.12). The notation is the same as in (2.12).

PROPOSITION 3.5. *The Euler characteristic of M is given by*

$$\chi(M) = \frac{1}{\Omega_n} \sum_{d=0}^{m} \int_{\partial M_d} dy \int_{N_y(M) \cap S^{n-1}} dv \det H(y, v). \tag{3.7}$$

As mentioned above, Proposition 3.5 is equivalent to Corollary 3.1 for $M \subset S^{n-1}$. This can be shown as follows. For a given $y \in M$, $v \in N_y(M) \cap S^{n-1}$ is uniquely written as

$$v = y \cos\theta + w \sin\theta, \qquad w \in N_y\bigl(K(M)\bigr) \cap S^{n-1}, \qquad 0 \le \theta < \pi.$$

Correspondingly, the second fundamental form in (3.7) is written as

$$H(y, v) = \cos\theta I_d + \sin\theta H(y, w).$$

Also for y fixed,

$$dv = \sin^{n-d-2}\theta\, d\theta\, dw,$$

where dw is the volume element of $N_y(K(M)) \cap S^{n-1}$. Therefore we have

$$\chi(M) = \frac{1}{\Omega_n} \sum_{d=0}^{m} \int_0^{\pi} d\theta \int_{\partial M_d} dy \int_{N_u(K(M))\cap S^{n-1}} dw \det\bigl(\cos\theta I_d + \sin\theta H(y, w)\bigr) \times \sin^{n-d-2}\theta.$$

Expanding the determinant and integrating out θ, we see that this is equivalent to (3.5).

3.4. *Alternative proof of Naiman's inequality.* In this subsection we give an alternative proof of Naiman's inequality [Naiman (1986), Johnstone and Siegmund (1989)]. It is based on the following characterization of the critical radius $\theta_c(M)$.

LEMMA 3.1.

$$\theta_c(M) = \sup\bigl\{\theta > 0 \mid I\bigl(A(y, \cos\theta) \neq \varnothing\bigr) = \chi\bigl(A(y, \cos\theta)\bigr) \text{ for all } y\bigr\}. \tag{3.8}$$

PROOF. If $\theta < \theta_c(M)$ each $y \in M_\theta$ has a unique nearest point $y_M \in M$. As in the proof of Proposition 3.2 let $f_y(u) = -u'y$ and let $f_{y|M}$ denote its restriction on M. The index of $f_{y|M}$ at y_M is 1 and this is the only index counted in $\chi(A(y, \cos\theta))$. Therefore $\chi(A(y, \cos\theta)) = I(A(y, \cos\theta) \neq \varnothing)$. On the other hand if $\theta > \theta_c(M)$, it is easy to see that there exists an open set U such that to $y \in U$ correspond two u's such that $u'y > \cos\theta$ and $y \in N_u(M)$. Then $\chi(A(y, \cos\theta))$ is either 0 or 2. This proves (3.8). □

From this lemma we have

$$I\bigl(A(y, \cos\theta) \neq \varnothing\bigr) = \chi\bigl(A(y, \cos\theta)\bigr), \qquad \theta < \theta_c.$$

On the other hand, when $\theta \geq \theta_c$, there is no general relation between $\chi(A(y, \cos\theta))$ and $I(A(y, \cos\theta) \neq \varnothing)$. However in the particular case where $M \subset S^{n-1}$ is one-dimensional and homotopic to the line segment [0, 1], then $\chi(A(y, \cos\theta))$ equals the number of connected components of $A(y, \cos\theta)$, and therefore the inequality

$$I\bigl(A(y, \cos\theta) \neq \varnothing\bigr) \leq \chi\bigl(A(y, \cos\theta)\bigr) \tag{3.9}$$

always holds.

By taking the expectations of both sides of (3.9) with respect to $y \sim \mathrm{Unif}(S^{n-1})$, we have for $0 \leq \theta \leq \pi$ that

$$\begin{aligned} \frac{\mathrm{Vol}(M_\theta)}{\Omega_n} &\leq \frac{1}{\Omega_2 \Omega_{n-2}} \mathrm{Vol}(M)\,\mathrm{Vol}\bigl(S^{(n-2)-1}\bigr)\bigl(1 - B_{1,(n-2)/2}(\cos^2\theta)\bigr) \\ &\quad + \frac{1}{\Omega_1 \Omega_{n-1}} \mathrm{Vol}(\partial M) \frac{\mathrm{Vol}(S^{(n-1)-1})}{2}\bigl(1 \mp B_{1/2,(n-1)/2}(\cos^2\theta)\bigr) \\ &= \frac{1}{2\pi} \mathrm{Vol}(M)\bigl(1 - B_{1,(n-2)/2}(\cos^2\theta)\bigr) + \frac{1}{2}\bigl(1 \mp B_{1/2,(n-1)/2}(\cos^2\theta)\bigr) \end{aligned} \tag{3.10}$$

by Proposition 3.2. Noting that

$$1 - B_{1,(n-2)/2}(x^2) = (1 - x^2)^{(n-2)/2},$$
$$1 \mp B_{1/2,(n-1)/2}(x^2) = \frac{2\Gamma(n/2)}{\sqrt{\pi}\,\Gamma((n-1)/2)} \int_x^1 (1 - x^2)^{(n-3)/2}\, dx,$$

we see that (3.10) is the same as the inequality (3.4) of Johnstone and Siegmund (1989).

Naiman's inequality states that the inequality (3.10) holds even when M is a piecewise C^1-curve. We can show this by taking a sequence of C^2-curves $\{M^i\}_{i=1,2,\dots}$ such that

$$\mathrm{Vol}(M^i) \to \mathrm{Vol}(M), \qquad \mathrm{Vol}\big((M^i)_\theta\big) \to \mathrm{Vol}(M_\theta).$$

4. Maximum of the cosine field: An example. In this section we study the cosine field at some length, because it is the building block for isotropic random fields in the sense of Section 2.5 of Adler (2000) and of basic importance.

4.1. *Cosine field.* The cosine field is defined as

$$Z(t) = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} (z_{2i-1} \cos t_i + z_{2i} \sin t_i)$$

with the index

$$t = (t_1, \dots, t_m) \in [0, T_1] \times \cdots \times [0, T_m], \qquad 0 \le T_i < 2\pi,$$

where $z = (z_1, \dots, z_n)' \sim N_n(0, I_n)$, $n = 2m$. Piterbarg [(1996), Lemmas 5.1, 5.2] derived an asymptotic expansion for the tail probability of the maximum of $Z(t)$. In this section we show that our tube formula gives another derivation of the asymptotic expansion. In addition, we evaluate the remainder term of asymptotic expansion more precisely than Piterbarg (1996) by explicitly evaluating the critical radius.

$Z(t)$ is written as $Z(t) = \phi(t)'z$, where

$$\phi = \phi(t) = (\cos t_1, \sin t_1, \dots, \cos t_m, \sin t_m)'/\sqrt{m} \in S^{2m-1}.$$

$\phi(t)$ is injective and the index set M on S^{2m-1} is $\phi([0, T_1] \times \cdots \times [0, T_m])$.

Denote the partial differential of ϕ with respect to t_i by the subscript i, for example,

$$\phi_i = \frac{\partial}{\partial t_i}\phi = (0, \dots, 0, -\sin t_i, \cos t_i, 0, \dots, 0)'/\sqrt{m},$$
$$\phi_{ij} = \frac{\partial^2}{\partial t_i \partial t_j}\phi = \delta_{ij}(0, \dots, 0, -\cos t_i, -\sin t_i, 0, \dots, 0)'/\sqrt{m},$$

where δ_{ij} is Kronecker's delta.

4.2. *Distribution of maximum.* The d-dimensional boundary ∂M_d consists of 2^{m-d} disjoint components. Note that the phase ψ_i of $(z_{2i-1}, z_{2i}) = (r_i \cos\psi_i, r_i \sin\psi_i)$, $r_i^2 = z_{2i-1}^2 + z_{2i}^2$, is uniformly distributed on $[0, 2\pi)$. Therefore without loss of generality, we may consider a component

$$\{\phi(t) \mid t_{d+1} = \cdots = t_m = 0\},$$

and fix $u = \phi(t)$ as a relative interior point of the component. The metric of ∂M_d at u is given by

$$g_{ij}(t) = \phi_i' \phi_j = (1/m)\delta_{ij}.$$

The support cone $S_u(M)$ at u is the convex cone spanned by the lines $\mathrm{span}\{\phi_i\}$, $i = 1, \ldots, d$, and the rays $\mathrm{cone}\{\phi_i\} = \{c\phi_i \mid c \geq 0\}$, $i = d+1, \ldots, m$. Note that for $i = d+1, \ldots, m$, $\phi_i = e_{2i} = (0, \ldots, 0, 1, 0, \ldots, 0)'$, where 1 is the $2i$th element.

The normal cone $N_u(K(M))$ is the dual cone of $S_u(K(M)) = \mathrm{span}\{u\} \oplus S_u(M)$. It is easily seen that

$$\begin{aligned} &N_u\bigl(K(M)\bigr) \\ &\quad = \bigl\{v = (a_1 \cos t_1, a_1 \sin t_1, \ldots, a_d \cos t_d, a_d \sin t_d, \\ &\qquad a_{d+1}, b_{d+1}, \ldots, a_m, b_m)' \mid a_1 + \cdots + a_m = 0,\ b_{d+1}, \ldots, b_m \leq 0\bigr\}. \end{aligned} \tag{4.1}$$

The squared length of v in (4.1) is $\|v\|^2 = a_1^2 + \cdots + a_m^2 + b_{d+1}^2 + \cdots + b_m^2$.

The second fundamental form of ∂M_d at u with respect v in (4.1) is given by $H(u, v)_{ij} = -v'\phi_{ij}(u) \times m = \sqrt{m} a_i \delta_{ij}$ or

$$H(u, v) = \sqrt{m}\, \mathrm{diag}(a_1, \ldots, a_d).$$

Now we proceed to evaluate the weights w_{m+1-e} of (2.7) for the cosine field. Write $w_{m+1-e} = \sum_{d=m-e}^{m} w_{m+1-e}^{(d)}$, where

$$\begin{aligned} &w_{m+1-e}^{(d)} \\ &\quad = \frac{1}{\Omega_{m+1-e}\Omega_{n-m-1+e}} \int_{\partial M_d} du \int_{N_u(K(M)) \cap S^{n-1}} dv\, \mathrm{tr}_{d-m+e} H(u, v). \end{aligned} \tag{4.2}$$

For convenience write $l = d - m + e$ and

$$J_1 = \int_{N_u(K(M)) \cap S^{n-1}} dv\, \mathrm{tr}_l\, H(u, v).$$

Let $R^2 \sim \chi^2(2m - d - 1)$, and consider the expectation

$$\begin{aligned} J_2 &= E\biggl[\int_{N_u(K(M)) \cap S^{n-1}} dv\, \mathrm{tr}_l\, H(u, Rv)\biggr] \Big/ \Omega_{2m-d-1} \\ &= J_1 \times \frac{E[(\chi_{2m-d-1}^2)^{l/2}]}{\Omega_{2m-d-1}}. \end{aligned} \tag{4.3}$$

Since the degrees of freedom of R^2 is the dimension of the normal cone $\dim N_u(K(M)) = n-1-d = 2m-d-1$, J_2 can be calculated by taking the expectation

$$J_2 = E\big[I(b_{d+1},\dots,b_m \le 0)\,\mathrm{tr}_l\big(\sqrt{m}\,\mathrm{diag}(a_1,\dots,a_d)\big)\big],$$

where

$$(a_1,\dots,a_m) \sim N_m\big(0, I_m - (1/m)1_m 1_m'\big), \qquad 1_m = (1,\dots,1)' \in R^m,$$
$$b_{d+1},\dots,b_m \sim N(0,1),$$

and $(a_1,\dots,a_m), b_{d+1},\dots,b_m$ are mutually independent.

Since $E[a_i a_j] = -1/m$ $(i \ne j)$, we have

$$E[a_1 a_2 \cdots a_k] = \begin{cases} (k-1)!!\,(-1/m)^{k/2}, & \text{for } k \text{ even},\\ 0, & \text{for } k \text{ odd},\end{cases}$$

where $(k-1)!! = (k-1)(k-3)\cdots 3\cdot 1$. Therefore J_2 for l even is

$$J_2 = (1/2)^{m-d} \times \binom{d}{l}(l-1)!!(-1/m)^{l/2} \times m^{l/2} = \frac{d!(-1)^{l/2}}{2^{m-d+l/2}(d-l)!(l/2)!} \tag{4.4}$$

and $J_2 = 0$ for l odd. Combining (4.2), (4.3) and (4.4), and noting that $\int_{\partial M_d} du = 2^{m-d}\Sigma_d$, where

$$\Sigma_d = \sum_{i_1<\cdots<i_d} T_{i_1}\cdots T_{i_d}, \tag{4.5}$$

we get for l even

$$\begin{aligned} w^{(d)}_{m+1-e} = w^{(d)}_{d+1-l} &= \frac{d!(-1)^{l/2}\Gamma((d+1-l)/2)}{2^{l+1}\pi^{(d+1)/2}(d-l)!(l/2)!}\Sigma_d \\ &= \frac{d!(-1)^{l/2}}{2^{d+1}\pi^{d/2}\Gamma((d+2-l)/2)(l/2)!}\Sigma_d \end{aligned}$$

and $w^{(d)}_{d+1-l} = 0$ for l odd.

Let

$$H_k(x) = (-1)^k \frac{d^k}{dx^k} e^{-x^2/2} \Big/ e^{-x^2/2}$$

be the Hermitian polynomial of degree k. Denote the coefficient of x^{k-l} in $H_k(x)$ by $h(k;l)$. It is well known that

$$h(k;l) = \begin{cases} (-1)^{l/2}\dbinom{k}{l}(l-1)!! = (-1)^{l/2}\dfrac{k!}{(k-l)!2^{l/2}(l/2)!}, & \text{for } l \text{ even},\\ 0, & \text{for } l \text{ odd}.\end{cases}$$

Using this we have

$$w^{(d)}_{d+1-l} = h(d;l) \times \frac{2^{(d+1-l)/2}\Gamma((d+1-l)/2)}{2(2\pi)^{(d+1)/2}}\Sigma_d.$$

Multiplying this by $1 - G_{d+1-l}(x^2)$, and taking a summation over $l = 0, \dots, d$ for fixed d we have

$$\begin{aligned}
&\sum_{l=0}^{d} w^{(d)}_{d+1-l}\bigl(1 - G_{d+1-l}(x^2)\bigr) \\
&\qquad = \frac{\Sigma_d}{2(2\pi)^{(d+1)/2}} \sum_{l=0}^{d} h(d;l) \int_{x^2}^{\infty} y^{(d+1-l)/2-1} e^{-y/2}\,dy \\
&\qquad = \frac{\Sigma_d}{(2\pi)^{(d+1)/2}} \sum_{l=0}^{d} h(d;l) \int_{x}^{\infty} z^{d-l} e^{-z^2/2}\,dz \\
&\qquad = \frac{\Sigma_d}{(2\pi)^{(d+1)/2}} \int_{x}^{\infty} H_d(z) e^{-z^2/2}\,dz \\
&\qquad = \frac{\Sigma_d}{(2\pi)^{(d+1)/2}} H_{d-1}(x) e^{-x^2/2}.
\end{aligned}$$

By summing this up over $d = 0, \dots, m$, the asymptotic expansion for the tail probability of $\max_t Z(t)$ is obtained. We will summarize the result at the end of this section as Proposition 4.1.

By Corollary 3.1 we have

$$\begin{aligned}
\chi(M) &= 2 \sum_{d=0}^{m} \sum_{\substack{l=0 \\ d-l:\text{even}}}^{d} w^{(d)}_{d+1-l} \\
&= 2 \sum_{\substack{d=0 \\ d:\text{even}}}^{m} \sum_{\substack{l=0 \\ l:\text{even}}}^{d} \frac{d!}{2^{d+1}\pi^{d/2}(d/2)!}(-1)^{l/2}\binom{d/2}{l/2}\Sigma_d \\
&= 2 \sum_{\substack{d=0 \\ d:\text{even}}}^{m} \frac{1}{2}\delta_{d,0} = 1,
\end{aligned}$$

which was expected since M is homotopic to a point $\phi(0) = (1, 0, \dots, 1, 0)'$.

4.3. *Critical radius.* Here we evaluate the critical radius θ_c by Lemma 2.1. We make the following additional assumption on the index set as is done in Piterbarg (1996):

$$0 \le T_i \le \pi, \qquad i = 1, \dots, m. \tag{4.6}$$

The orthogonal projection matrix onto the space span$\{\phi, \phi_1, \ldots, \phi_d\}$ is written as

$$Q_\phi = \phi\phi' + m \sum_{i=1}^{d} \phi_i \phi_i'.$$

Fix v in the relative interior of $\{\phi(t) \mid t_{d+1} = \cdots = t_m = 0\}$. The orthogonal projection onto the normal cone $N_v(K(M))$ is give by

$$P_v^\perp(w) = (I_{2m} - Q_v)w + \sum_{i=d+1}^{m} e_{2i} \min(0, w_{2i}),$$

where $w = (w_1, \ldots, w_{2m})' \in R^{2m}$.

By Lemma 2.1,

$$\begin{aligned}\tan^2\theta_c &= \inf_{u,v\in M} \frac{\|u-v\|^4}{4\|P_v^\perp(u-v)\|^2} \\ &= \inf_{u,v\in M} \frac{(1-u'v)^2}{\|(I_{2m}-Q_v)(u-v)\|^2 + \sum_{i=d+1}^{m} \min(0, u_{2i}-v_{2i})^2}.\end{aligned}$$

In the expression above we assumed that the infimum is attained when $v = \phi(t)$ with $t_{d+1} = \cdots = t_m = 0$ for a particular value of d. Put $u = \phi(s)$, $s = (s_1, \ldots, s_m)$.

Note that for $i = d+1, \ldots, m$, $v_{2i} = 0$, $u_{2i} = \sin s_i \geq 0$, and hence $\min(0, u_{2i} - v_{2i}) = 0$ by assumption (4.6).

Put

$$u'v = \phi(s)'\phi(t) = \frac{1}{m}\sum_{i=1}^{m} x_i, \qquad x_i = (\cos s_i, \sin s_i)\begin{pmatrix}\cos t_i \\ \sin t_i\end{pmatrix}.$$

Noting

$$\|(I_{2m} - Q_v)(u-v)\|^2 = 1 - u'Q_v u = 1 - \big(\phi(s)'\phi(t)\big)^2 - m\sum_{i=1}^{d}\big(\phi(s)'\phi_i(t)\big)^2$$

and

$$\phi(s)'\phi_i(t) = \frac{1}{m}(\cos s_i, \sin s_i)\begin{pmatrix}-\sin t_i \\ \cos t_i\end{pmatrix} = \pm\frac{1}{m}\sqrt{1-x_i^2},$$

the argument of the infimum is written as

$$\begin{aligned}&\frac{(1-(1/m)\sum_{i=1}^{m} x_i)^2}{1-((1/m)\sum_{i=1}^{m} x_i)^2 - (1/m)\sum_{i=1}^{d}(1-x_i^2)} \\ &\quad = \frac{(\sum_{i=1}^{m} y_i)^2}{m\sum_{i=1}^{d} y_i^2 + 2m\sum_{i=d+1}^{m} y_i - (\sum_{i=1}^{m} y_i)^2},\end{aligned}$$

where we put $y_i = 1 - x_i$. Note that $0 \le y_i \le 2$. By virtue of the inequality

$$\sum y_i^2 \le \Big(\sum y_i\Big)^2$$

(the equality holds iff $y_i = 0$ except for at most one index i), we see

$$\begin{aligned}
&\frac{(\sum_{i=1}^m y_i)^2}{m\sum_{i=1}^d y_i^2 + 2m\sum_{i=d+1}^m y_i - (\sum_{i=1}^m y_i)^2}\\
&\qquad \ge \frac{(\sum_{i=1}^m y_i)^2}{m(\sum_{i=1}^d y_i)^2 + 2m\sum_{i=d+1}^m y_i - (\sum_{i=1}^m y_i)^2}\\
&\qquad \ge \frac{(\sum_{i=1}^d y_i)^2}{m(\sum_{i=1}^d y_i)^2 - (\sum_{i=1}^d y_i)^2}\\
&\qquad = \frac{1}{m-1},
\end{aligned}$$

where the equality of the second inequality holds iff $\sum_{i=d+1}^m y_i = 0$.

This infimum $1/(m-1)$ is attained in the case where

$$y_1 \to +0, \qquad y_2 = \cdots = y_d = y_{d+1} = \cdots = y_m = 0.$$

This is possible when at least T_1 is positive. Since the infimum $1/(m-1)$ is independent of d, we conclude that

$$\tan^2\theta_c = \frac{1}{m-1}$$

when $0 \le T_i \le \pi$ and $\exists\, i,\ T_i > 0$. The case $T_1 = \cdots = T_m = 0$ is trivial.

PROPOSITION 4.1. *Assume that* $0 \le T_i \le \pi$, $i = 1, \dots, m$. *Then*

$$P\Big(\max_t Z(t) \ge x\Big) = \sum_{d=1}^m \frac{\Sigma_d}{(2\pi)^{d/2}} H_{d-1}(x)\varphi(x) + \int_x^\infty \varphi(x)\,dx + O\bigg(g_{2m}\Big(\frac{m}{m-1}x^2\Big)\bigg)$$

as $x \to \infty$, *where* $\varphi(x) = e^{-x^2/2}/\sqrt{2\pi}$ *and* Σ_d *is given in* (4.5).

This gives the same asymptotic expansion as Piterbarg (1996). In addition we have made the remainder term more precise.

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REFERENCES

ADLER, R. J. (1981). *The Geometry of Random Fields*. Wiley, Chichester.

ADLER, R. J. (2000). On excursion sets, tube formulas and maxima of random fields. *Ann. Appl. Probab.* **10** 1–74.

FEDERER, H. (1959). Curvature measures. *Trans. Amer. Math. Soc.* **93** 418–491.

FU, J. H. G. (1989). Curvature measures and generalized Morse theory. *J. Differential Geom.* **30** 619–642.

HOTELLING, H. (1939). Tubes and spheres in n-spaces, and a class of statistical problems. *Amer. J. Math.* **61** 440–460.

JOHANSEN, S. and JOHNSTONE, I. (1990). Hotelling's theorem on the volume of tubes: Some illustrations in simultaneous inference and data analysis. *Ann. Statist.* **18** 652–684.

JOHNSTONE, I. and SIEGMUND, D. (1989). On Hotelling's formula for the volume of tubes and Naiman's inequality. *Ann. Statist.* **17** 184–194.

KNOWLES, M. and SIEGMUND, D. (1989). On Hotelling's approach to testing for a nonlinear parameter in regression. *Internat. Statist. Rev.* **57** 205–220.

KURIKI, S. and TAKEMURA, A. (1999). Distribution of the maximum of Gaussian random field: tube method and Euler characteristic method. *Proc. Inst. Statist. Math.* **47** 199–219 (in Japanese).

KURIKI, S. and TAKEMURA, A. (2000a). Some geometry of the cone of nonnegative definite matrices and weights of associated $\bar{\chi}^2$ distribution. *Ann. Inst. Statist. Math.* **52** 1–14.

KURIKI, S. and TAKEMURA, A. (2000b). Shrinkage estimation towards a closed convex set with a smooth boundary. *J. Multivariate Anal.* **75** 79–111.

KURIKI, S. and TAKEMURA, A. (2001). Tail probabilities of the maxima of multilinear forms and their applications. *Ann. Statist.* **29** 328–371.

MILNOR, J. (1963). *Morse Theory*. Princeton Univ. Press.

MORSE, M. and CAIRNS, S. S. (1969). *Critical Point Theory in Global Analysis and Differential Topology*. Academic Press, New York.

NAIMAN, D. Q. (1986). Conservative confidence bands in curvilinear regression. *Ann. Statist.* **14** 896–906.

PITERBARG, V. I. (1996). *Asymptotic Methods in the Theory of Gaussian Processes and Fields*. Amer. Math. Soc. Providence, RI.

SANTALÓ, L. A. (1976). *Integral Geometry and Geometric Probability*. Addison-Wesley, London.

SCHNEIDER, R. (1993). *Convex Bodies: The Brunn–Minkowski Theory*. Cambridge Univ. Press.

SHAPIRO, A. (1987). A conjecture related to chi-bar-squared distributions. *Amer. Math. Monthly* **94** 46–48.

SHAPIRO, A. (1988). Towards a unified theory of inequality constrained testing in multivariate analysis. *Internat. Statist. Rev.* **56** 49–62.

STOYAN, D., KENDALL, W. S. and MECKE, J. (1995). *Stochastic Geometry and its Applications*, 2nd ed. Wiley, Chichester.

SUN, J. (1993). Tail probabilities of the maxima of Gaussian random fields. *Ann. Probab.* **21** 34–71.

TAKEMURA, A. and KURIKI, S. (1997). Weights of $\bar{\chi}^2$ distribution for smooth or piecewise smooth cone alternatives. *Ann. Statist.* **25** 2368–2387.

TAKEMURA, A. and KURIKI, S. (2000). Tail probability via tube formula when critical radius is zero. Unpublished manuscript.

WEYL, H. (1939). On the volume of tubes. *Amer. J. Math.* **61** 461–472.

WORSLEY, K. J. (1995a). Estimating the number of peaks in a random field using the Hadwiger characteristic of excursion sets, with applications to medical images. *Ann. Statist.* **23** 640–669.

WORSLEY, K. J. (1995b). Boundary corrections for the expected Euler characteristic of excursion sets of random fields, with an application to astrophysics. *Adv. Appl. Probab.* **27** 943–959.

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