

## ETERNAL SOLUTIONS TO SMOLUCHOWSKI'S COAGULATION EQUATION WITH ADDITIVE KERNEL AND THEIR PROBABILISTIC INTERPRETATIONS

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The cornerstone of this work, which is partly motivated by the characterization of the so-called eternal additive coalescents by Aldous and Pitman, is an explicit expression for the general eternal solution to Smoluchowski's coagulation equation with additive kernel. This expression points at certain Lévy processes with no negative jumps and more precisely at a stochastic model for aggregation based on such processes, which has been recently considered by Bertoin and Miermont and is known to bear close relations with the additive coalescence. As an application, we show that the eternal solutions can be obtained from some hydrodynamic limit of the stochastic model. We also present a simple condition that ensures the existence of a smooth density for an eternal solution.

**1. Introduction.** Roughly, Smoluchowski's coagulation equation is meant to describe the evolution in the hydrodynamic limit of a particle system in which particles coagulate pairwise as time passes. Typically, we are given a symmetric function  $K: ]0, \infty[ \times ]0, \infty[ \rightarrow ]0, \infty[$ , called the coagulation kernel, where  $K(x, y)$  specifies the rate at which two particles with respective masses  $x$  and  $y$  coagulate (i.e., they merge as a single particle with mass  $x + y$ ). If we represent the density of particles with mass  $dx$  at time  $t$  by a measure  $\mu_t(dx)$  on  $]0, \infty[$ , the dynamics of the system are thus described by the equation

$$(1) \quad \partial_t \langle \mu_t, f \rangle = \frac{1}{2} \int_{]0, \infty[ \times ]0, \infty[} (f(x+y) - f(x) - f(y)) K(x, y) \mu_t(dx) \mu_t(dy),$$

where  $f: ]0, \infty[ \rightarrow \mathbb{R}$  is a generic continuous function with compact support and the notation  $\langle \mu_t, f \rangle$  stands for the integral of  $f$  with respect to  $\mu_t(dx)$ . We refer to the survey by Aldous [1] for detailed explanations, physical motivations and references.

More precisely, we shall focus on solutions fulfilling the requirement that  $\int_{]0, \infty[} x \mu_t(dx) < \infty$  for all  $t$ . This means that the total mass density is finite at any time, which is a natural physical assumption; see [1]. We shall further impose that (1) holds for every continuous function  $f$  with  $\sup_{0 < x < \infty} |f(x)|/x < \infty$ . If

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Received July 2001; revised October 2001.

AMS 2000 subject classifications. 60G51, 60K35, 82C21.

Key words and phrases. Smoluchowski's coagulation equation, additive coalescence, Lévy process with no positive jumps.

we assume for simplicity unit total mass density at some given time  $t_0$ , that is,  $\int_{]0, \infty[} x \mu_{t_0}(dx) = 1$ , then by taking  $f(x) = x$  in (1) we see that

$$(2) \quad \int_{]0, \infty[} x \mu_t(dx) = 1 \quad \text{for all } t.$$

In other words, the total mass is conserved when the system evolves.

Next, recall that stochastic coalescents are Markovian models for aggregation which are naturally connected to Smoluchowski's coagulation equation (see also [10, 11] for other interesting related random processes). They have been introduced by Marcus [20] and Lushnikov [19]; see also Evans and Pitman [14] for a general construction and [1] for a survey and further references. Informally, in such models, any pair of distinct particles, say with masses  $x$  and  $y$ , merges as a single cluster with mass  $x + y$  at rate  $K(x, y)$ , independently of the other pairs in the system. It is known that under certain conditions, hydrodynamic limits of stochastic coalescents yield solutions to Smoluchowski's coagulation equation; see Norris [23] and the references therein.

We shall consider Smoluchowski's equation (1) for the additive kernel  $K(x, y) = x + y$ , which arises for instance in the study of droplet formation in clouds [15], gravitational clustering in the universe [26] and phase transition for parking [9]. Here we will be interested in its *eternal* solutions, in the sense that the time parameter  $t$  is real (possibly negative). For instance,

$$(3) \quad \mu_t(dx) = \frac{e^{-t}}{\sqrt{2\pi x^3}} \exp\left(-\frac{xe^{-2t}}{2}\right) dx, \quad t \in \mathbb{R}, x > 0$$

is such a solution; see, for example, Table 2, page 8 in [1]. One of our motivations for investigating eternal solutions, rather than solutions starting at time  $t = 0$  from some initial condition  $\mu_0(dx)$ , stems from a recent work of Aldous and Pitman [3]. Indeed, they constructed all the eternal additive coalescents (i.e., stochastic coalescents for the additive kernel with time-parameter  $t \in \mathbb{R}$ ), and it is therefore natural to compare the latter with the eternal solutions of (1).

In this work, we shall first provide an explicit expression for the general eternal solution to Smoluchowski's coagulation equation with additive kernel. Our approach is essentially standard; it relies on the well-known fact that for  $K(x, y) = x + y$ , (1) can be reduced to some simple PDE using Laplace transforms. The upshot is that this expression suggests a connection with certain Lévy processes with no positive jumps, and more precisely points at a simple stochastic model for aggregation of intervals based on such processes. The latter was recently considered by Bertoin [6] and Miermont [22], who already established close links with the additive coalescence. Here, we shall use results on Lévy processes to derive some information on eternal solutions to Smoluchowski's coagulation equation. In particular, we will show that the eternal solutions can be obtained from an elementary hydrodynamic limit of the stochastic model and present a simple condition ensuring the existence of a smooth density.

**2. Eternal solutions.** To summarize some notions discussed in the introduction, we make the following formal definition.

DEFINITION 1. Let  $\mu = (\mu_t(dx), t \in \mathbb{R})$  be a one-parameter family of measures on  $]0, \infty[$  such that (2) holds. We call  $\mu$  an *eternal solution* to Smoluchowski's coagulation equation with additive kernel if  $K(x, y) = x + y$  and (1) is fulfilled for every continuous function  $f: ]0, \infty[ \rightarrow \mathbb{R}$  with  $\sup_{0 < x < \infty} |f(x)|/x < \infty$ . In that case, we simply write  $\mu \in \mathcal{S}_{\text{eternal}}$ .

Our analysis of eternal solutions will heavily rely on the following.

DEFINITION 2. We call  $(\sigma^2, \Lambda)$  a *Lévy pair* and write  $(\sigma^2, \Lambda) \in \mathcal{L}$  if  $\sigma^2 \geq 0$  and  $\Lambda(dx)$  is a measure on  $]0, \infty[$  with

$$\int_{]0, \infty[} (x \wedge x^2) \Lambda(dx) < \infty,$$

such that

$$\sigma^2 > 0 \quad \text{or} \quad \int_{]0, \infty[} x \Lambda(dx) = \infty.$$

For every  $(\sigma^2, \Lambda) \in \mathcal{L}$ , we define for  $q \geq 0$ ,

$$\Psi_{\sigma^2, \Lambda}(q) = \frac{1}{2} \sigma^2 q^2 + \int_{]0, \infty[} (e^{-qx} - 1 + qx) \Lambda(dx).$$

In the sequel, we shall sometimes need the following elementary facts:  $\Psi_{\sigma^2, \Lambda}$  is a strictly convex increasing function on  $[0, \infty[$ ; it has derivative 0 at  $q = 0$  and  $\infty$  at  $q = \infty$ . We are now able to state the central result of this work.

THEOREM 1. (i) Let  $\mu \in \mathcal{S}_{\text{eternal}}$ . Then there exists a unique  $(\sigma^2, \Lambda) \in \mathcal{L}$ , which we call the *Lévy pair* of  $\mu$ , such that

$$\lim_{t \rightarrow -\infty} \mu_t(]e^t x, \infty]) = \Lambda(]x, \infty])$$

for every  $x > 0$  which is not an atom of  $\Lambda$ , and

$$\lim_{x \rightarrow 0} \lim_{t \rightarrow -\infty} \int_0^x y \mu_t(]e^t y, \infty]) dy = \frac{1}{2} \sigma^2,$$

or equivalently in terms of Laplace transform,

$$\lim_{t \rightarrow -\infty} \int_{]0, \infty[} (\exp(-e^{-t} qx) - 1 + e^{-t} qx) \mu_t(dx) = \Psi_{\sigma^2, \Lambda}(q).$$

(ii) Conversely, take any  $(\sigma^2, \Lambda) \in \mathcal{L}$ . There exists a unique  $\mu \in \mathcal{S}_{\text{eternal}}$  with Lévy pair  $(\sigma^2, \Lambda)$ . More precisely, for every  $s > 0$ , let  $\Phi(\cdot, s): [0, \infty[ \rightarrow [0, \infty[$

be the inverse of the bijection  $q \rightarrow \Psi_{\sigma^2, \Lambda}(sq) + q$ . Then  $\mu_t$  is specified by the identity

$$\Phi(q, e^t) = \int_{]0, \infty[} (1 - e^{-qx}) \mu_t(dx), \quad q \geq 0.$$

For instance, if we take  $\sigma^2 = 1$  and  $\Lambda(dx) = 0$ ; that is,  $\Psi_{\sigma^2, \Lambda}(q) = \frac{1}{2}q^2$ , then we get for  $s > 0$  and  $q \geq 0$ ,

$$\Phi(q, s) = \frac{\sqrt{1 + 2qs^2} - 1}{s^2} = \frac{1}{s\sqrt{2\pi}} \int_0^\infty (1 - e^{-qx}) x^{-3/2} \exp\left(-\frac{x}{2s^2}\right) dx.$$

After the substitution  $s = e^t$ , we recover the eternal solution (3).

Before starting to prove Theorem 1, we point out the following interesting consequence of the representation given in part (ii): every eternal solution  $\mu \in \mathcal{S}_{\text{eternal}}$  has infinite cluster density, in the sense that

$$\mu_t(]0, \infty[) = \infty \quad \text{for all } t \in \mathbb{R}.$$

Indeed, we have  $\mu_t(]0, \infty[) = \Phi(\infty, e^t)$ , and since  $\Psi_{\sigma^2, \Lambda}(\infty) = \infty$ , the preceding quantity is also infinite.

Our approach is essentially standard. It is based on the well-known observation that for certain simple kernels, Smoluchowski's equation (1) can be connected via the Laplace transform to some classical PDE's. See McLeod [21] in the case of the multiplicative kernel  $K(x, y) = xy$ , and also Deaconu and Tanré [12] for a recent contribution. In our framework, the Riemann (or free transport) equation will have a crucial role; Lemma 1, which is perhaps already known, is a first step in this direction.

LEMMA 1. *Let  $\mu \in \mathcal{S}_{\text{eternal}}$ . Define for every  $q \geq 0$ ;  $t \in \mathbb{R}$  and  $s > 0$ ,*

$$\ell(q, t) = \int_{]0, \infty[} (1 - e^{-qx}) \mu_t(dx), \quad \Psi(q, s) = \frac{q}{s} - \ell\left(\frac{q}{s}, \ln s\right).$$

*Then  $\Psi$  is a solution to the Riemann equation on  $[0, \infty[ \times ]0, \infty[$ , that is,*

$$\partial_s \Psi + \Psi \partial_q \Psi = 0.$$

*Moreover, for every fixed  $s > 0$ , the function  $q \rightarrow \Psi(q, s)$  is increasing on  $[0, \infty[$  and has derivative zero at  $q = 0$ , and the function  $q \rightarrow q^{-2} \Psi(q, s)$  is monotone decreasing on  $]0, \infty[$ .*

PROOF. Applying (1) with  $f(x) = (1 - e^{-qx})$  yields

$$\begin{aligned} \partial_t \ell(q, t) &= -\frac{1}{2} \int_{]0, \infty[ \times ]0, \infty[} (1 - e^{-qx} - e^{-qy} + e^{-q(x+y)})(x+y) \mu_t(dx) \mu_t(dy) \\ &= - \int_{]0, \infty[ \times ]0, \infty[} (1 - e^{-qx})(1 - e^{-qy}) x \mu_t(dx) \mu_t(dy). \end{aligned}$$

By (2), we arrive at the equation

$$\partial_t \ell(q, t) = (\partial_q \ell(q, t) - 1) \ell(q, t),$$

and it follows from straightforward calculations that  $\Psi$  solves the Riemann equation.

That  $\Psi(\cdot, s)$  is an increasing function with derivative 0 at the origin is immediate. Finally, integrations by parts show that if we introduce the notation

$$(4) \quad M_t(x) := \int_x^\infty \mu_t(]y, \infty[) dy$$

for every  $x > 0$  and  $t \in \mathbb{R}$ , then

$$q - \ell(q, t) = q^2 \int_0^\infty e^{-qx} M_t(x) dx.$$

It follows that

$$(5) \quad \Psi(q, s) = \frac{q^2}{s} \int_0^\infty M_{\ln s}(sx) e^{-qx} dx,$$

and thus  $q^{-2}\Psi(q, s)$  is decreasing in the variable  $q > 0$ .  $\square$

We stress that Lemma 1 also holds (up to an obvious modification) for solutions started at a fixed time from some given initial condition. This yields an expression for such a solution in terms of the Laplace transform of the initial measure, as all that is needed is to solve the Riemann equation started from an increasing initial condition, which is elementary.

The next step concerns the asymptotic behavior of the solution of the Riemann equation. In particular, we observe that the eternal solution  $\mu$  can be recovered from the latter.

LEMMA 2. *Hypotheses and notation are the same as in Lemma 1.*

(i) *For every  $q \geq 0$ ,  $\Psi(q, s)$  has a finite limit as  $s \rightarrow 0+$ , which will be denoted by  $\Psi(q, 0)$ .*

(ii) *For every fixed  $s > 0$ , the function  $q \rightarrow \Psi(sq, 0) + q$  is a bijection on  $[0, \infty[$ , and the inverse bijection is  $\ell(\cdot, \ln s)$ .*

PROOF. (i) Because  $\Psi$  is a positive solution to the Riemann equation with  $\partial_q \Psi \geq 0$ , we have  $\partial_s \Psi \leq 0$ , so when  $s$  decreases to  $0+$ ,  $\Psi(q, s)$  increases to  $\Psi(q, 0) \in [0, \infty]$ . Let us check that  $\Psi(\cdot, 0)$  is finite everywhere. In this direction, it is immediate that for every  $0 < s < 1$  and  $q \geq 0$ ,

$$\Psi(q, 1) = \Psi(q - (1 - s)\Psi(q, 1), s).$$

Indeed,  $\Psi$  can be thought of as the velocity field for a free particle system; that is,  $\Psi(q, r)$  represents the velocity of the particle located at  $q$  at time  $r$ . Since the

velocity of each particle is constant,  $q - (1 - s)\Psi(q, 1)$  is the location at time  $s$  of the particle located at  $q$  at time 1, which yields the identity above. Next, note that the function

$$q \rightarrow q - (1 - s)\Psi(q, 1) = sq + (1 - s)\ell(q, 0)$$

is a bijection on  $[0, \infty[$ ; denote its inverse by  $\varphi(\cdot, s)$  so that

$$\Psi(\varphi(q, s), 1) = \Psi(q, s).$$

When  $s$  decreases to 0,  $sq + (1 - s)\ell(q, 0)$  converges to  $\ell(q, 0)$  and hence  $\varphi(\cdot, s)$  converges to the inverse of  $\ell(\cdot, 0)$ , where the latter is viewed as a bijection from  $[0, \infty[$  to  $[0, \ell(\infty, 0)[$ . This shows that  $\Psi(q, 0) < \infty$  for every  $q < \ell(\infty, 0)$ . Now recall from Lemma 1 that  $q \rightarrow q^{-2}\Psi(q, s)$  is decreasing, so the fact that  $\Psi(\cdot, 0)$  is finite on some neighborhood of 0 implies that it is finite everywhere.

(ii)  $\Psi$  solves the Riemann equation with an increasing boundary condition  $\Psi(\cdot, 0)$ . It is well known that there is a unique solution which is given in terms of  $\Psi(\cdot, 0)$  by the formula in the statement.  $\square$

We now make the connection with Lévy pairs.

LEMMA 3. *There exists a unique Lévy pair  $(\sigma^2, \Lambda) \in \mathcal{L}$  such that*

$$\lim_{t \rightarrow -\infty} \mu_t(\cdot | e^t x, \infty) = \Lambda(\cdot | x, \infty), \quad \lim_{x \rightarrow 0} \lim_{t \rightarrow -\infty} \int_0^x y \mu_t(\cdot | e^t y, \infty) dy = \frac{1}{2}\sigma^2,$$

and we have  $\Psi(\cdot, 0) = \Psi_{\sigma^2, \Lambda}$ .

PROOF. Recall the notation (4). We know from Lemma 2 and (5) that when  $s \rightarrow 0+$ , we have for every  $q > 0$  that

$$s^{-1} \int_0^\infty e^{-qx} M_{\ln s}(sx) dx = q^{-2}\Psi(q, s) \rightarrow q^{-2}\Psi(q, 0) = \int_{[0, \infty[} e^{-qx} \nu(dx),$$

where  $\nu(dx)$  is the measure on  $[0, \infty[$  obtained as the vague limit of  $s^{-1}M_{\ln s}(sx) dx$ . Since for every  $s > 0$ , the indefinite integral  $\int_0^\cdot M_{\ln s}(sx) dx$  is a concave function, so is the distribution function  $\nu([0, \cdot])$ , and therefore  $\nu(dx)$  must be of the form

$$\nu(dx) = \frac{1}{2}\sigma^2 \delta_0(dx) + \bar{\Lambda}(x) dx,$$

where  $\delta_0(dx)$  is the Dirac point mass at 0,  $\sigma^2 \geq 0$  and  $\bar{\Lambda}(x) = \lim_{s \rightarrow 0+} s^{-1} M_{\ln s}(sx)$  a decreasing function. More precisely, as  $M_{\ln s}(s \cdot)$  is convex, so is  $\bar{\Lambda}$ , and integrations by parts show that

$$\Psi(q, 0) = \frac{1}{2}\sigma^2 q^2 + bq + c + \int_{]0, \infty[} (e^{-qx} - 1 + qx) \Lambda(dx)$$

for some  $b, c \geq 0$ , where  $\Lambda(dx)$  is the Stieltjes measure on  $]0, \infty[$  given by the second derivative of  $\bar{\Lambda}$ . It should be plain that we must have  $\int (x \wedge x^2) \Lambda(dx) < \infty$ .

As  $\Psi(0, s) = 0$  for all  $s > 0$ , we have  $\Psi(0, 0) = 0$  and hence  $c = 0$ . Next, recall from Lemma 2(ii) that  $\ell(\cdot, 0)$  can be recovered as the inverse of the bijection  $q \rightarrow \Psi(q, 0) + q$ . The derivative of the latter at  $q = 0$  is  $b + 1$ , so the derivative of its inverse at 0 is  $1/(b + 1)$ . We deduce from (2) that  $1/(b + 1) = 1$ , so  $b = 0$ .

Finally, if we had  $\sigma^2 = 0$  and  $\int x \Lambda(dx) < \infty$ , then as  $q \rightarrow \infty$ ,  $q^{-1} \Psi(q, 0)$  would have a finite limit, say  $\lambda$ , and by inversion, we would have

$$\int_{]0, \infty[} \left( \frac{1 - e^{-qx}}{q} \right) \mu_0(dx) = q^{-1} \ell(q, 0) \rightarrow 1/(1 + \lambda) > 0.$$

This is impossible, because the integrand above tends to 0 and dominated convergence applies, thanks to (2). We conclude that  $(\sigma^2, \Lambda) \in \mathcal{L}$ .  $\square$

The combination of Lemmas 1–3 proves the first part of Theorem 1. To establish the second part (i.e., the converse direction), we pick an arbitrary Lévy pair  $(\sigma^2, \Lambda) \in \mathcal{L}$  and solve a variation of the Riemann equation.

LEMMA 4. *Pick  $(\sigma^2, \Lambda) \in \mathcal{L}$  and define for every  $s > 0$  the function  $\Phi(\cdot, s): ]0, \infty[ \rightarrow ]0, \infty[$  as the inverse of the bijection  $q \rightarrow \Psi_{\sigma^2, \Lambda}(sq) + q$ . Then  $\Phi$  is a solution to the PDE,*

$$s \partial_s \Phi(q, s) = (\partial_q \Phi(q, s) - 1) \Phi(q, s), \quad q \geq 0, s > 0$$

with boundary condition

$$\lim_{s \rightarrow 0+} \left( \frac{q}{s} - \Phi\left(\frac{q}{s}, s\right) \right) = \Psi_{\sigma^2, \Lambda}(q).$$

PROOF. Starting from the identity

$$(6) \quad \Psi_{\sigma^2, \Lambda}(s\Phi(q, s)) + \Phi(q, s) = q,$$

and taking the derivative with respect to the variable  $s$ , we obtain

$$(7) \quad (\Phi(q, s) + s \partial_s \Phi(q, s)) \Psi'_{\sigma^2, \Lambda}(s\Phi(q, s)) + \partial_s \Phi(q, s) = 0.$$

On the other hand, taking the derivative with respect to the variable  $q$  in (6), we get

$$\Psi'_{\sigma^2, \Lambda}(s\Phi(q, s)) = \frac{1}{s} \left( \frac{1}{\partial_q \Phi(q, s)} - 1 \right).$$

Substituting this in (7) yields the stated PDE.

Next, we have again from (6) that

$$s \Psi_{\sigma^2, \Lambda}(s\Phi(q/s, s)) + s\Phi(q/s, s) = q;$$

it follows that  $s\Phi(q/s, s) \leq q$  and letting  $s$  tend to 0 in the identity above, we see that  $\lim_{s \rightarrow 0^+} s\Phi(q/s, s) = q$ . Plugging this in (6), we conclude that

$$\lim_{s \rightarrow 0^+} \left( \frac{q}{s} - \Phi\left(\frac{q}{s}, s\right) \right) = \lim_{s \rightarrow 0^+} \Psi_{\sigma^2, \Lambda}(s\Phi(q/s, s)) = \Psi_{\sigma^2, \Lambda}(q),$$

and the proof is complete.  $\square$

The final step in the proof of Theorem 1(ii) is based on the following key observation. For every  $s > 0$  and  $q \geq 0$ , we have

$$\Psi(q, s) := \Psi_{\sigma^2, \Lambda}(sq) + q = q + \frac{1}{2}(\sigma sq)^2 + \int_{]0, \infty[} (e^{-qsx} - 1 + sqx)\Lambda(dx).$$

The strictly increasing function  $\Psi(\cdot, s)$  is thus given by a Lévy–Khintchine formula; more precisely,  $\Psi(\cdot, s)$  can be viewed as the Laplace exponent of some real-valued Lévy process (i.e., a process with independent and stationary increments) with no positive jumps. See Section VII.1 in [4] for background. We now have all the ingredients to establish the following lemma, and thus to complete the proof of Theorem 1(ii).

LEMMA 5. *In the notation of Lemma 4, for each  $t \in \mathbb{R}$ , there is a unique measure  $\mu_t$  on  $]0, \infty[$  which fulfills (2) and such that*

$$\Phi(q, e^t) = \int_{]0, \infty[} (1 - e^{-qx})\mu_t(dx).$$

Finally, the family  $\mu = (\mu_t(dx), t \in \mathbb{R})$  belongs to  $\mathcal{S}_{\text{eternal}}$  and its Lévy pair is  $(\sigma^2, \Lambda)$ .

PROOF. We know from Theorem VII.1 in [4] that the inverse  $\Phi(\cdot, s)$  of the bijection  $\Psi(\cdot, s)$  is the Laplace exponent of a subordinator (i.e., an increasing Lévy process). By the Lévy–Khintchine formula for subordinators (see Section 1.2 in [5]), we get an expression in the form

$$\Phi(q, s) = dq + \int_{]0, \infty[} (1 - e^{-qsx})\mu_{\ln s}(dx),$$

where  $d \geq 0$  is the so-called drift coefficient and  $\mu_{\ln s}(dx)$  the Lévy measure. Recall that the assumption  $(\sigma^2, \Lambda) \in \mathcal{L}$  ensures that  $\Psi'_{\sigma^2, \Lambda}(\infty) = \infty$ , and hence  $\lim_{q \rightarrow \infty} \Phi(q, s)/q = 0$ . This forces  $d = 0$ . Moreover, we know that  $\Psi'_{\sigma^2, \Lambda}(0) = 0$ , so  $\partial_q \Phi(0, s) = 1$ . This shows that  $\mu_{\ln s}(dx)$  fulfills (2).

Now if we set for  $t \in \mathbb{R}$  and  $q \geq 0$ ,

$$\ell(q, t) = \Phi(q, e^t) = \int_{]0, \infty[} (1 - e^{-qx})\mu_t(dx),$$

then, by Lemma 4,  $\ell$  solves

$$\partial_t \ell(q, t) = (\partial_q \ell(q, t) - 1)\ell(q, t).$$

By (2), we deduce that

$$\partial_t \ell(q, t) = - \int_{]0, \infty[ \times ]0, \infty[} (1 - e^{-qx})(1 - e^{-qy})x \mu_t(dx) \mu_t(dy),$$

an identity that we may also rewrite in the form

$$\partial_t \langle \mu_t, g \rangle = -\frac{1}{2} \int_{]0, \infty[ \times ]0, \infty[} (g(x+y) - g(x) - g(y))(x+y) \mu_t(dx) \mu_t(dy),$$

where  $g(x) = (1 - e^{-qx})$ . By Laplace inversion, this shows that  $(\mu_t(dx), t \in \mathbb{R})$  is a solution to Smoluchowski's equation (1) with additive kernel for every continuous function  $f$  with compact support on  $]0, \infty[$ , and the rest of the proof is routine.  $\square$

Theorem 1 shows in particular that an eternal solution  $\mu \in \mathcal{S}_{\text{eternal}}$  is characterized by the behavior of  $\mu_t(dx)$  as  $t$  tends to  $-\infty$ . It may be interesting to point out that it is also possible to recover  $\mu$  from its asymptotic behavior at  $+\infty$ . In this direction, let us denote for every Lévy pair  $(\sigma^2, \Lambda)$  by  $\Phi_{\sigma^2, \Lambda}$  the inverse of the bijection  $\Psi_{\sigma^2, \Lambda}: [0, \infty[ \rightarrow [0, \infty[$ . The same probabilistic argument as in Lemma 5 shows that there exists a unique measure  $\Pi_{\sigma^2, \Lambda}(dx)$  on  $]0, \infty[$  with  $\int_{]0, \infty[} (1 \wedge x) \Pi_{\sigma^2, \Lambda}(dx) < \infty$ , such that

$$(8) \quad \Phi_{\sigma^2, \Lambda}(q) = \int_{]0, \infty[} (1 - e^{-qx}) \Pi_{\sigma^2, \Lambda}(dx), \quad q \geq 0.$$

We are now able to state the following.

**COROLLARY 1.** *Let  $\mu \in \mathcal{S}_{\text{eternal}}$  be an eternal solution with Lévy pair  $(\sigma^2, \Lambda) \in \mathcal{L}$ . Then the measure  $e^t \mu_t(dx)$  converges vaguely on  $]0, \infty[$  as  $t \rightarrow \infty$  towards  $\Pi_{\sigma^2, \Lambda}(dx)$ . Moreover,  $\mu_t(dx)$  can be recovered from  $\Pi_{\sigma^2, \Lambda}(dx)$ .*

**PROOF.** We use the notation of Lemmas 4 and 5 with  $s = e^t$ . The identity (6) yields

$$\Psi_{\sigma^2, \Lambda}(s\Phi(q, s)) = q - \Phi(q, s),$$

and by the very definition of  $\Phi_{\sigma^2, \Lambda}$ , we obtain

$$s\Phi(q, s) = \Phi_{\sigma^2, \Lambda}(q - \Phi(q, s)).$$

When  $s$  tends to  $\infty$ ,  $\Phi(q, s)$  converges to 0 and hence

$$\lim_{s \rightarrow \infty} s\Phi(q, s) = \Phi_{\sigma^2, \Lambda}(q).$$

This establishes the first part of our statement.

Finally, we can recover  $\Psi_{\sigma^2, \Lambda}$  for the measure  $\Pi_{\sigma^2, \Lambda}(dx)$ , by inverting the bijection  $\Phi_{\sigma^2, \Lambda}$  which is given by (8). Since Theorem 1 shows how to construct the eternal solution from  $\Psi_{\sigma^2, \Lambda}$ , the proof is complete.  $\square$

To conclude this section, we show how eternal solutions can be obtained as limits of solutions started from a fixed time with a given initial measure. In this direction, using for instance the analysis based on the Riemann equation as in Lemma 1, it is easily proved that given a measure  $\mu_0(dx)$  on  $]0, \infty[$  that fulfills (2), we can construct for every  $t > 0$  a unique measure  $\mu_t(dx)$  on  $]0, \infty[$  also satisfying (2), such that the family  $\mu = (\mu_t(dx), t \geq 0)$  solves Smoluchowski's coagulation equation (1) for the additive kernel  $K(x, y) = x + y$  and for every continuous function  $f: ]0, \infty[ \rightarrow \mathbb{R}$  with  $\sup_{0 < x < \infty} |f(x)|/x < \infty$ . Consider now a sequence  $(\mu_0^{(n)}(dx), n \in \mathbb{N})$  of measures on  $]0, \infty[$  fulfilling (2), and for each integer  $n$ , let  $\mu^{(n)} = (\mu_t^{(n)}(dx), t \geq 0)$  be the solution started at time 0 from  $\mu_0^{(n)}(dx)$ . We have the following limit theorem.

PROPOSITION 1. *Assume there exists a sequence of positive real numbers  $(\varepsilon_n, n \in \mathbb{N})$  with limit 0 and a Lévy pair  $(\sigma^2, \Lambda) \in \mathcal{L}$  such that the following holds:*

$$\lim_{n \rightarrow \infty} \mu_0^{(n)}(] \varepsilon_n x, \infty[) = \Lambda(]x, \infty[)$$

whenever  $x > 0$  is not an atom of  $\Lambda$  and

$$\lim_{x \rightarrow 0} \lim_{n \rightarrow \infty} \int_0^x y \mu_0^{(n)}(]y, \infty[) dy = \frac{1}{2} \sigma^2.$$

Then for every  $t \in \mathbb{R}$ ,  $\mu_{t - \ln \varepsilon_n}^{(n)}(dx)$  converges weakly on  $]0, \infty[$  to  $\mu_t(dx)$  where  $\mu = (\mu_t(dx), t \in \mathbb{R}) \in \mathcal{S}_{\text{eternal}}$  is the eternal solution with Lévy pair  $(\sigma^2, \Lambda)$ .

PROOF. We just sketch the argument. Define for every integer  $n$  and  $s \geq 1$ ,

$$\Psi^{(n)}(q, s) = \int_{]0, \infty[} (\exp(-qx/s) - 1 + qx/s) \mu_{\ln s}^{(n)}(dx).$$

The proof of Lemma 1 shows that  $\Psi^{(n)}$  is a solution to the Riemann equation on  $[0, \infty[ \times [1, \infty[$ . It is then immediate that the function

$$(q, s) \rightarrow \Psi^{(n)}(q/\varepsilon_n, s/\varepsilon_n) = \int_{]0, \infty[} (\exp(-qx/s) - 1 + qx/s) \mu_{\ln s - \ln \varepsilon_n}^{(n)}(dx)$$

is again a solution to the Riemann equation; more precisely, this solution starts at time  $\varepsilon_n$  from the condition

$$q \rightarrow \int_{]0, \infty[} (\exp(-qx/\varepsilon_n) - 1 + qx/\varepsilon_n) \mu_0^{(n)}(dx).$$

Translating the hypothesis of the statement in terms of Laplace transform gives that when  $n \rightarrow \infty$ , this initial condition converges to  $\Psi_{\sigma^2, \Lambda}(q)$ . It follows that for every  $s > 0$  and  $q \geq 0$ ,

$$\lim_{n \rightarrow \infty} \Psi^{(n)}(q, s) = \Psi(q, s),$$

where  $\Psi$  is the solution to the Riemann equation on  $[0, \infty[ \times [0, \infty[$  with initial condition  $\Psi_{\sigma^2, \Lambda}$ . Rewriting this for  $s = e^t$  and  $q = \lambda s$  we thus have that for every  $\lambda \geq 0$  and  $t \in \mathbb{R}$ ,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{]0, \infty[} (\exp(-\lambda x) - 1 + \lambda x) \mu_{t - \ln \varepsilon_n}^{(n)}(dx) \\ &= \int_{]0, \infty[} (\exp(-\lambda x) - 1 + \lambda x) \mu_t(dx), \end{aligned}$$

where  $\mu = (\mu_t(dx), t \in \mathbb{R})$  is the eternal solution with Lévy pair  $(\sigma^2, \Lambda)$ .  $\square$

### 3. A probabilistic interpretation and its applications.

3.1. *A stochastic model for aggregation.* The general form of eternal solutions given in Theorem 1, as well as the argument to establish the part (ii) in the preceding section, invite a probabilistic interpretation. Indeed, recall that for an arbitrary Lévy pair  $(\sigma^2, \Lambda) \in \mathcal{L}$ , the formula that defines  $\Psi_{\sigma^2, \Lambda}$  is of the Lévy–Khintchine type. Specifically,  $\Psi_{\sigma^2, \Lambda}$  can be viewed as the Laplace exponent of a centered Lévy processes with no positive jumps, where  $\sigma^2$  is the so-called Brownian coefficient and  $\Lambda(dx)$  the image of the Lévy measure by the map  $x \rightarrow -x$ .

We now introduce a Lévy process with no positive jumps,  $X = (X_r, r \geq 0)$ , such that

$$\mathbb{E}(\exp(qX_r)) = \exp(r\Psi_{\sigma^2, \Lambda}(q)), \quad q \geq 0.$$

Recall that the assumption  $(\sigma^2, \Lambda) \in \mathcal{L}$  ensures that the sample paths of  $X$  have unbounded variation and oscillate (i.e.,  $\sup_{r \geq 0} X_r = \infty$  and  $\inf_{r \geq 0} X_r = -\infty$  a.s.). Similarly, the function  $\Psi(\cdot, s)$ , which plays a key role in the preceding section, can be viewed as another Laplace exponent; more precisely introduce for every  $s > 0$ ,

$$X_r^{(s)} := sX_r + r, \quad r \geq 0,$$

which is again Lévy process with no positive jumps, whose Laplace exponent given by

$$\Psi(q, s) = \Psi_{\sigma^2, \Lambda}(sq) + q, \quad q \geq 0.$$

It is then well known that the first passage process

$$T_x^{(s)} := \inf\{r \geq 0 : X_r^{(s)} > x\}, \quad x \geq 0$$

is a subordinator, that is, an increasing process with independent and stationary increments. More precisely, the inverse  $\Phi(\cdot, s)$  of the bijection  $\Psi(\cdot, s)$  is the Laplace exponent of  $T^{(s)}$ , in the sense that

$$\mathbb{E}(\exp(-qT_x^{(s)})) = \exp(-x\Phi(q, s)), \quad q, x \geq 0.$$

See, for example, Theorem VII.1 in [4]. The Lévy–Khintchine formula for  $\Phi(\cdot, s)$  is given in Lemma 5, and we conclude that the eternal solution  $\mu_t(dx)$  associated with the Lévy pair  $(\sigma^2, \Lambda)$  coincides with the so-called Lévy measure of the subordinator  $T^{(s)}$  for  $s = e^t$ . Recall also that  $T^{(s)}$  has zero drift.

REMARK. Deaconu and Tanré [12] have given a different probabilistic interpretation of some solutions to Smoluchowski’s coagulation equation with additive kernel. More precisely, they express the solution, say  $\mu_t(dx)$ , in terms of the distribution of the total population of a certain subcritical branching process. The connection with the present interpretation follows readily from the classical link between branching processes and Lévy processes with no negative jumps (see Lamperti [17, 18]), and more specifically, between the distribution of the total population of the former and that of first passage times of the latter. The main difference with the present work is that in [12], the offspring distribution of the branching process depends on the time parameter  $t$ , so one needs a one-parameter family of branching processes to represent the entire solution. Here, the same Lévy process enables us to construct the eternal solution at any time, and this is precisely the feature that points at a stochastic model for aggregation, which we now introduce.

As it has been pointed out by Kingman [16], we may use each subordinator  $T^{(s)}$  to construct an interesting random partition of  $[0, \infty[$ . To that end, introduce the closed range

$$\mathcal{T}^{(s)} := \{T_x^{(s)}, x \geq 0\}^{\text{cl}}.$$

Because the subordinator  $T^{(s)}$  has zero drift,  $\mathcal{T}^{(s)}$  is a closed random set with zero Lebesgue measure; see for example, Proposition 1.8 in [5]. The complementary set  $[0, \infty[ \setminus \mathcal{T}^{(s)}$  can be expressed as the union of disjoint open intervals, which we may view as a random partition of  $[0, \infty[$ .

We now make the key observation that

$$(9) \quad \mathcal{T}^{(s)} \subseteq \mathcal{T}^{(s')} \quad \text{for } 0 < s' < s,$$

because an instant at which  $X^{(s)}$  reaches a new maximum is necessarily also an instant at which  $X^{(s')}$  reaches a new maximum. Roughly, (9) means that the random partitions get coarser as the parameter  $s$  increases; and therefore they induce a process in which intervals aggregate. By a simple time-reversal, the latter gives rise to a fragmentation process that has been studied by Bertoin [6] and Schweinsberg [25] in the Brownian case, and by Miermont [22] in the general case, and which is closely related to the additive coalescence, as we shall now explain.

3.2. *Connection with the additive coalescence.* Recall that the additive coalescence is a Markovian dynamic in which configurations are represented by decreasing numerical sequences  $x = (x_1, \dots)$  with  $x_1 + \dots = 1$ , where each  $x_i$  should be viewed as the mass of the  $i$ th heaviest particle. Roughly, the system evolves as follows: every pair  $\{x_i, x_j\}$  with  $i \neq j$  merges into a single particle with mass  $x_i + x_j$  at rate  $K(x_i, x_j) = x_i + x_j$ , independently of the other pairs. We refer to Evans and Pitman [14] for a precise construction (which is delicate when the system has infinitely many particles) and to Aldous and Pitman [2, 3] for a deep study. Following Aldous and Pitman [3], we call eternal additive coalescent a Markov process which is indexed by times  $t \in \mathbb{R}$  and governed by this dynamic.

Resuming the discussion of the preceding section, it should be plain that

$$\mathcal{T}^{(\infty)} := \bigcap_{s>0} \mathcal{T}^{(s)}$$

coincides with the closed range of the first passage process of  $X$ . Let us pick a random interval  $I$  in the decomposition of  $[0, \infty[\setminus \mathcal{T}^{(\infty)}$ , which is measurable with respect to  $\mathcal{T}^{(\infty)}$ . For instance,  $I$  may be the interval component that contains some fixed point, or the leftmost interval with length at least some fixed  $\lambda > 0$ . The portion of the path of  $X$  restricted to  $I$  is an excursion of the Lévy process  $X$  below its supremum, and by excursion theory, its distribution is specified by the so-called excursion measure of  $\text{It}\hat{o}$ . We refer to Chapters IV and VI in [4] for details.

Let us assume for simplicity that the one-dimensional distributions of the Lévy process  $X$  have smooth density; see the forthcoming equation (10) for a simple sufficient condition. This technical hypothesis allows us to work conditionally on the length  $|I|$  of  $I$ , and we may now relate our stochastic model for aggregation to the additive coalescence, stating the following result which is implicit in [22] (see also [6]).

**PROPOSITION 2.** *For every  $s > 0$ , let  $Y_1(s) \geq Y_2(s) \geq \dots$  be the ranked sequence of the interval components of  $I \setminus \mathcal{T}^{(s)}$ . Then conditionally on  $|I| = 1$ , the process  $((Y_1(e^t), Y_2(e^t), \dots), t \in \mathbb{R})$  is an eternal additive coalescent.*

Let us now focus on the so-called *extreme* additive coalescents, which form the so-called entrance boundary of the additive coalescence, since general eternal coalescents can be obtained as mixtures of extreme ones. Aldous and Pitman [3] have shown that there is a bijective correspondence between the laws of extreme eternal additive coalescents and the nonnegative sequences  $(\xi_0, \xi_1, \dots)$  such that  $\xi_1 \geq \xi_2 \geq \dots$ ,  $\sum_1^\infty \xi_i^2 < \infty$  and

$$\xi_0 > 0 \quad \text{or} \quad \sum_1^\infty \xi_i = \infty.$$

Note that the requirements on the sequence  $(\xi_0, \xi_1, \dots)$  have the same flavor as those in the definition of a Lévy pair (in this direction, it is also interesting to

compare Proposition 13 of [3] with the present Proposition 1). Roughly, each such sequence can be used to construct a certain inhomogeneous continuum random tree (cf. [8]), and logging randomly this tree by an independent Poisson point process on its skeleton produces a fragmentation process which is related to an extreme additive coalescent by a simple time reversal. We also refer to [7] for a different construction based on bridges with exchangeable increments and no positive jumps (of course, bridges of Lévy processes provide natural examples of such processes). In this setting,  $\xi_0 \geq 0$  corresponds to the coefficient of the Brownian bridge component and  $\xi_1 \geq \xi_2 \geq \dots$  to the jumps of such a bridge with exchangeable increments.

We stress that only a strict subclass of eternal additive coalescents appears in Proposition 2; in particular the only extreme eternal processes are Brownian (i.e., corresponding to  $\Lambda = 0$ ). We refer to Section 6 in [22] for the identification of the mixing for the nonextreme eternal additive coalescents arising in Proposition 2.

It might also be interesting to underline some major differences between our stochastic model for aggregation and the additive coalescence. The most obvious is that the former deals with a system with infinite total mass (i.e., the length of  $[0, \infty[$ ), whereas we have a unit total mass in the latter. Perhaps more significantly, all the randomness for our stochastic model for aggregation is encoded by the Lévy process  $X$ , in the sense that the dynamics are deterministic if we know the path of  $X$ . At the opposite, the evolution of the additive coalescent requires the continuous introduction of randomness. Typically, for an extreme eternal additive coalescent, randomness is needed not only for the construction of the inhomogeneous continuum random tree, but also for the specification of the Poissonian cut points on the skeleton of the tree.

*3.3. Application to the existence of smooth density.* As a first application, we present a simple criterion that ensures the existence of a smooth density for eternal solutions. Specifically, suppose that  $(\sigma^2, \Lambda)$  is a Lévy pair such that

$$(10) \quad \sigma^2 > 0 \quad \text{or} \quad \liminf_{\varepsilon \rightarrow 0^+} \varepsilon^{-\alpha} \int_{]0, \varepsilon[} x^2 \Lambda(dx) > 0 \quad \text{for some } \alpha < 2.$$

We stress that this is a very mild assumption as it always holds that  $\int_{]0, \varepsilon[} x \Lambda(dx) = \infty$ . Then it is known that the one-dimensional distributions of the Lévy process  $X$  are absolutely continuous with a  $\mathcal{C}^\infty$  density; see, for example, Proposition 28.3 in [24]. More precisely, a slight variation of the argument there shows that there exists a  $\mathcal{C}^\infty$  function

$$\begin{aligned} p: ]0, \infty[ \times \mathbb{R} &\rightarrow [0, \infty[, \\ (r, x) &\rightarrow p_r(x) \end{aligned}$$

such that

$$\mathbb{P}(X_r \in dy) = p_r(y) dy \quad \text{for every } r > 0.$$

**COROLLARY 2.** *Let  $\mu \in \mathcal{S}_{\text{eternal}}$  be an eternal solution with Lévy pair  $(\sigma^2, \Lambda)$ , and assume that (10) is fulfilled. Then for every  $t \in \mathbb{R}$ ,  $\mu_t(dx)$  is absolutely continuous and there is a version of its density,  $n_t(x) = \mu_t(dx)/dx$  such that  $n: (t, x) \rightarrow n_t(x)$  is a  $\mathcal{C}^\infty$  function. More precisely, in the notation introduced above, we have*

$$n_t(x) = e^{-t} x^{-1} p_x(-x e^{-t}), \quad x > 0.$$

**PROOF.** It is immediate that the one-dimensional density of the Lévy process  $X^{(s)}$  are given, in the obvious notation, by

$$p_r^{(s)}(y) = s^{-1} p_r((y-r)/s).$$

It is then well known that the Lévy measure  $\mu_{\ln s}(dx)$  of  $T^{(s)}$ , the first passage process of  $X^{(s)}$ , is absolutely continuous with density

$$\frac{\mu_{\ln s}(dx)}{dx} = x^{-1} p_x^{(s)}(0) = \frac{1}{sx} p_x(-x/s), \quad x > 0.$$

This follows from Exercise VII.2 in [4]; see also Lemma 7 in [22]. This establishes our claim.  $\square$

**REMARK.** The same technique can be applied to establish the regularity of the solution started at some fixed time, say  $t = 0$ , from some given measure,  $\mu_0(dx)$ . More precisely, one can show using similar arguments that if there exists  $\alpha < 2$  such that

$$\liminf_{\varepsilon \rightarrow 0^+} \varepsilon^{-\alpha} \int_{]0, \varepsilon[} x^2 \mu_0(dx) > 0,$$

then the conclusions of Corollary 2 hold for every  $t > 0$ .

For instance, in the Brownian case ( $\sigma^2 = 1$  and  $\Lambda = 0$ ), we recover the eternal solution (3). In the  $\alpha$ -stable case  $\sigma^2 = 0$  and  $\Lambda(dx) = cx^{-\alpha-1}dx$ , where  $1 < \alpha < 2$ , we obtain a different eternal solution in terms of the completely asymmetric  $\alpha$ -stable density, say,

$$\rho_\alpha(y) = \mathbb{P}(X_1 \in dy)/dy.$$

More precisely, the scaling property of stable processes yields the identity

$$p_r(y) = r^{-1/\alpha} \rho_\alpha(yr^{-1/\alpha})$$

and hence

$$n_t(x) = e^{-t} x^{-1-1/\alpha} \rho_\alpha(-x^{1-1/\alpha} e^{-t}).$$

It does not seem easy to check directly that indeed this provides a solution of Smoluchowski's coagulation equation.

We also observe that the formula in Corollary 2 shows that the density of the solution decays exponentially as a function of time  $t$ , and more precisely,

$$n_t(x) \sim x^{-1} p_x(0) e^{-t} \quad \text{as } t \rightarrow \infty.$$

This can be viewed as a refinement of Corollary 1.

3.4. *Eternal solutions as hydrodynamic limits.* Our next purpose is to show that eternal solutions can be obtained from a hydrodynamic limit of the stochastic aggregation models introduced in Section 3.1. In this direction, we recall that Norris [23] established recently a much more general result in the same vein based on the so-called stochastic coalescence of Marcus [20] and Lushnikov [19]. The main interest of the present result is perhaps the simplicity of its proof.

To give a precise statement, denote for every  $r, s > 0$  by  $\lambda_{r,1}^{(s)} \geq \lambda_{r,2}^{(s)} \geq \dots$  the ranked sequence of the lengths of the intervals components of  $[0, r[\setminus \mathcal{T}^{(s)}$ . We may think of the latter as massive particles that aggregate when time  $s$  increases. Let us now introduce the associated (re-weighted) empirical measure

$$\mu_{r,\ln s}(dx) = \frac{1}{r} \sum_{k=1}^{\infty} \delta_{\lambda_{r,k}^{(s)}}(dx),$$

where  $\delta_y(dx)$  stands for the Dirac point mass at  $y$ ; note that the normalization has been chosen such that  $\int_{]0, \infty[} x \mu_{r,t}(dx) = 1$  for all  $r > 0$  and  $t \in \mathbb{R}$ .

COROLLARY 3. *With probability 1,*

$$\lim_{r \rightarrow \infty} \mu_{r,t}(dx) = \mu_t(dx)$$

*in the sense of vague convergence of measures on  $]0, \infty[$ , where  $(\mu_t(dx), t \in \mathbb{R}) = \mu \in \mathcal{S}_{\text{eternal}}$  is the eternal solution with Lévy pair  $(\sigma^2, \Lambda)$ .*

PROOF. This is an immediate application of the Lévy–Itô decomposition (see, e.g., Section 1.1 in [5]). More precisely, the lengths of the interval components of  $]0, \infty[\setminus \mathcal{T}^{(s)}$  correspond to the jumps of the subordinator  $T^{(s)}$ . We know that the jump process

$$(\Delta_y^{(s)} = T_y^{(s)} - T_{y-}^{(s)}, y \geq 0)$$

is a Poisson point process on  $]0, \infty[$  with characteristic measure the Lévy measure of  $T^{(s)}$ , that is,  $\mu_{\ln s}(dx)$ . Taking  $r = T_y^{(s)}$  and letting  $y$  tend to  $\infty$ , we deduce from (a variation of) the Glivenko–Cantelli theorem that  $\mu_{r,\ln s}(dx)$  converges vaguely to  $\mu_{\ln s}(dx)$ . The general case when no restriction on  $r$  is imposed follows immediately by interpolation.  $\square$

**Acknowledgment.** I thank David Aldous for comments about the literature related to this work and for pointing out relevant references.

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