The Annals of Applied Probability
2002, Vol. 12, No. 2, 511–532

DISPERSION RATES UNDER FINITE MODE KOLMOGOROV FLOWS

BY MIKE CRANSTON[1] AND MICHAEL SCHEUTZOW

University of Rochester and Technische Universität Berlin

We consider the growth rate of a collection of passive tracers moving in the plane under the influence of a random, fluctuating, velocity field. The velocity field we consider is a finite mode approximation to Kolmogorov velocity fields, which are commonly used as models for turbulent diffusion. We show that the diameter of the body of passive tracers grows linearly in time under the influence of these velocity fields.

1. Introduction. In this article we consider the growth rate of a body of passive tracers moving under a random velocity field. The topic of passive tracer movement is important in statistical fluid mechanics. It is usual to model, among others, diffusion of a pollutant in the atmosphere or on the ocean's surface, salinity in oceans and diffusion in porous media. The problem we consider is finding the proper growth rate in time of the most remote tracer among a continuum of passive tracers which were originally confined to a bounded region. The study of a finite number of tracers has been carried out by numerical simulation [cf. Carmona, Grishin and Molchanov (1996)]. It was also studied for isotropic flows in Zirbel and Çinlar (1996). Conversely, satellites are used in tracking drifters (buoys) realized on the ocean surface. Our work is influenced by recent results of the authors with David Steinsaltz concerning this problem when the tracers are moving under . In Cranston, Scheutzow and Steinsaltz (2000), we considered stochastic flows on $\mathbf{R}^d$ which solved

$$\phi(t,x) = x + \int_0^t F\big(ds, \phi(s,x)\big),$$

where F is a field of semimartingales, ($F = M + A$, M a martingale, A a bounded variation process) with bounded and regular characteristics. These assumptions imply that the one point motion $t \mapsto \phi(t,x)$, x fixed, is roughly like a Brownian motion with bounded drift. The result of Cranston, Scheutzow and Steinsaltz (2000) is that there is a positive constant C so that for any bounded set $\mathcal{S}$,

$$\limsup_{t\to\infty} \frac{1}{t} \sup_{x\in\mathcal{S}} \|\phi(t,x)\| \le C \quad \text{a.s.}$$

Received August 2000; revised August 2001.

[1]Supported by an NSF grant.

AMS 2000 subject classifications. Primary 60G35; secondary 60G15, 60K40.

Key words and phrases. Kolmogorov velocity fields, stochastic flows, dispersion.

We also gave an example of a flow satisfying the assumptions on F and a compact subset $\mathcal{S}$ for which there exists a $c > 0$ such that

$$\liminf_{t\to\infty} \frac{1}{t} \sup_{x\in\mathcal{S}} \|\phi(t,x)\| \geq c \quad \text{a.s.}$$

In Cranston, Scheutzow and Steinsaltz (1999) we considered this problem under isotropic Brownian flows with a strictly positive top Lyapunov exponent and for more general martingale flows in Scheutzow and Steinsaltz (2001).

Isotropic Brownian flows are solutions of

$$\phi(t,x) = x + \int_0^t W\big(ds, \phi(s,x)\big),$$

where $\{W(\cdot, x) : x \in \mathbf{R}^d\}$ is a field of Brownian motions on $\mathbf{R}^d$ whose correlation tensor

$$s \wedge t b_{ij}(x-y) \equiv E[W_i(s,x)W_j(t,y)]$$

satisfies $U^* b(Uz)U = b(z)$ for every orthogonal matrix U. In this case the one-point motion $t \mapsto \phi(t,x)$, x fixed, is a multiple (independent of x) of Brownian motion on $\mathbf{R}^d$. Thus, the passive tracer at $\phi(t,x)$ at time t will ultimately be no more than a constant times $\sqrt{2t \log\log t}$ from the origin at time t. Whereas the combined results of Cranston, Scheutzow and Steinsaltz (1999, 2000) are that if ϕ is an isotropic Brownian flow with a strictly positive top Lyapunov exponent in dimension $d \geq 2$, then there are positive constants c and C such that for each compact and connected set $\mathcal{S} \in \mathbf{R}^d$ with at least two points,

$$c \leq \inf_{u\in S^{d-1}} \sup_{x\in\mathcal{S}} \liminf_{t\to\infty} \frac{1}{t} \langle \phi(t,x), u\rangle \leq \limsup_{t\to\infty} \frac{1}{t} \sup_{x\in\mathcal{S}} \|\phi(t,0,x)\| \leq C \quad \text{a.s.}$$

That is, from a continuum of passive tracers under this flow, there will be tracers going to infinity in the direction $u \in S^{d-1}$ at a linear rate in time for every direction u. One of the results in Scheutzow and Steinsaltz (2001) is that this property still holds on a set of strictly positive probability for isotropic Brownian flows with a negative top Lyapunov exponent.

Rather than martingale flows we now consider solutions of the random ordinary differential equation $\phi(t,s,x) = x + \int_s^t v(r, \phi(r,s,x)\, dr$, $t > s$, where $\{v(t,x) : t \geq 0,\ x \in \mathbf{R}^2\}$ is an often used approximation to a Gaussian, isotropic, divergence free, time homogenous Markovian vector field process on $\mathbf{R}^2$. This approximation is called a finite mode velocity field. It is Gaussian, volume preserving and symmetric, but not isotropic. Homogenization results [Carmona and Xu (1997), Fannjiang and Komorowski (1999) and — under slightly different conditions — Kesten and Papanicolaou (1979)] have that $\varepsilon(\phi(\cdot/\varepsilon^2, 0, x) - x)$ is convergent in law to a multiple of Brownian motion. Thus, one-point motions $t \mapsto \phi(t,0,x)$ will not wander further than a constant times $\sqrt{2t \log\log t}$ from their starting points for sufficiently large t.

However, when taking the supremum over some continuum set of starting points we find a phenomenon similar to the case of isotropic Brownian flows. Namely, $\frac{1}{t}\sup_{x\in\mathcal{S}}\langle\phi(t,0,x),u\rangle$ will have a positive lim inf almost surely for each $u \in S^1$. This result has been conjectured by Carmona and Sinai [Carmona and Cerou (1999)]. We resolve this for approximations to the velocity fields which have a finite number of modes above. This will be made precise in the next section.

2. Kolmogorov velocity fields. We deal with a finite mode approximation to a common mathematical model for the velocity field of the ocean surface. Let $\{v(t,x): t \geq 0,\ x \in \mathbf{R}^2\}$ be a divergence free, random velocity field on $\mathbf{R}^2$. The divergence free property plays a role analogous to the positivity of the top Lyapunov exponent in our previous work. This assumption guarantees that the image of a set of positive volume will not contract to a point. Thus we will be able at any time to select two points in the image of a set of positive measure, moving under the flow, which are separated by some fixed positive distance. Such points will be sufficiently uncorrelated to give the result. We work in dimension two for simplicity. This is the most physically relevant case and our results should carry over to higher dimensions without much trouble.

Since we shall be primarily interested in the two-dimensional case, there exists a real stream function Ψ such that $v(t,x) = \operatorname{curl}\Psi(t,x)$. Referring to the development in Molchanov (1996), in the case when $\{v(t,x): t \geq 0,\ x \in \mathbf{R}^2\}$ is isotropic, Gaussian, Markov in time, with sufficiently nice correlations, then Ψ has a representation

$$\Psi(t,x) = \int_{\mathbf{R}}\int_{\mathbf{R}^2} \exp\{i\langle k,x\rangle + i\omega t\} Z(dk,d\omega), \tag{1}$$

where Z is a $\mathbf{C}$-valued, Gaussian measure with correlations $E[Z(k+dk,\omega+dw)$ $\overline{Z}(k'+dk,\omega'+d\omega)] = \delta_{k,k'}\delta_{\omega,\omega'}E(k,\omega)dkd\omega$, for some function $E(k,\omega)$.

By a finite mode model we mean an approximation to the stream function at (1) of the form

$$\Psi(t,x) = \sum_{k\in\mathcal{K}} \exp\{i\langle k,x\rangle\} e(k)\xi_t(k) \tag{2}$$

with $\mathcal{K} \subset \mathbf{R}^2\backslash\{0\}$ a finite set and where $e(k)$ are complex constants and the processes $\{\xi_t(k): t \geq 0\}$ are independent complex-valued Ornstein–Uhlenbeck processes for $k \in \mathcal{K}$. Thus, using the fact that $v(t,x) = \operatorname{curl}\Psi(t,x)$, in the finite mode case $\mathcal{K} = \{k_1,\dots,k_N\}$ we have

$$\begin{aligned} v_1(t,x) &= \sum_{i=1}^N k_{i,2}\big[a_i(t)\sin\langle k_i,x\rangle - b_i(t)\cos\langle k_i,x\rangle\big],\\ v_2(t,x) &= \sum_{i=1}^N k_{i,1}\big[-a_i(t)\sin\langle k_i,x\rangle + b_i(t)\cos\langle k_i,x\rangle\big]. \end{aligned} \tag{3}$$

The processes $\{a_i(t):t \geq 0\}$, $\{b_i(t):t \geq 0\}$, $i = 1, \dots, N$ are independent real-valued Ornstein–Uhlenbeck processes. Thus, there are independent N-dimensional Brownian motions $V = (V_1, \dots, V_N)$ and $W = (W_1, \dots, W_N)$ such that

$$\begin{aligned} da_i(t) &= -\alpha_i a_i(t)dt + \sigma_i dV_i(t), \\ db_i(t) &= -\alpha_i b_i(t)dt + \sigma_i dW_i(t), \qquad i = 1, \dots, N \end{aligned} \tag{4}$$

with $\sigma_i > 0$ and $\alpha_i > 0$ for $i = 1, \dots, N$. Unless we make an explicit assumption to the contrary we will assume that the process (a, b) is started at time 0 with its invariant probability measure. We define a filtration by $\mathcal{F}_t = \sigma\{(a(s), b(s)), 0 \leq s \leq t\}$ for $t \geq 0$.

Given $\{v(t, x):t \geq 0, x \in \mathbf{R}^2\}$ satisfying (3) and (4), we define a flow by

$$\phi(t, s, x) = x + \int_s^t v\big(u, \phi(u, s, x)\big)\,du, \qquad t \geq s. \tag{5}$$

Then $\phi(t, s, x)$ is the position of a passive tracer at time t if the tracer was at x at time s. One can expect a substantially different dispersive behavior depending on the dimension of the span of the set $\mathcal{K}$ [cf. Carmona, Grishin and Molchanov (1996)]. We start with the nondegenerate case span $\mathcal{K} = \mathbf{R}^2$ below and treat the easier *shear-flow* case dim(span $\mathcal{K}$) = 1 in Section 4. In the following a random compact subset $\mathcal{S}$ will be called *measurable* if the map $\omega \mapsto \mathcal{S}(\omega)$ is measurable with respect to the Borel σ-algebra generated by the Hausdorff distance on the set of nonempty compact subsets of $\mathbf{R}^2$.

THEOREM. *Suppose v satisfies* (3) *and* (4) *and ϕ is defined by* (5). *Assume* span $\mathcal{K} = \mathbf{R}^2$. *Then there are positive constants c^*, C^* such that for any compact and connected $\mathcal{F}_0$-measurable set $\mathcal{S} \subset \mathbf{R}^2$ with* $\mathrm{vol}(\mathcal{S}) > 0$ *a.s.*,

$$c^* \leq \inf_{u \in S^1} \liminf_{t \to \infty} \frac{1}{t} \sup_{x \in \mathcal{S}} \langle \phi(t, 0, x), u \rangle \leq \limsup_{t \to \infty} \frac{1}{t} \sup_{x \in \mathcal{S}} \|\phi(t, 0, x)\| \leq C^* \quad a.s.$$

The theorem obviously implies the following corollary, which provides a positive answer to the conjecture of Carmona and Sinai stated at the end of the first section for finite mode Kolmogorov flows.

COROLLARY. *Under the assumptions of the theorem we have*

$$2c^* \leq \liminf_{t \to \infty} \frac{1}{t} \operatorname{diam}\big(\phi(t, 0, S)\big) \leq \liminf_{t \to \infty} \frac{1}{t} \operatorname{diam}\big(\phi(t, 0, S)\big) \leq 2C^* \quad a.s.$$

3. Proof of the theorem. We start with the upper bound which is very easy to establish. Let $\kappa := \sum_{j=1}^N \|k_j\|$. Using (3) and (5),

$$\sup_{x \in \mathcal{S}} \|\phi(t, 0, x)\| \leq \sup_{x \in \mathcal{S}} \|x\| + \kappa \int_0^t \sum_{j=1}^N \big(|a_j(s)| + |b_j(s)|\big)\,ds.$$

Thus

$$\liminf_{t\to\infty} \frac{1}{t} \sup_{x\in\mathcal{S}} \|\phi(t,0,x)\| \le \kappa \limsup_{t\to\infty} \frac{1}{t} \int_0^t \sum_{j=1}^N \bigl(|a_j(s)| + |b_j(s)|\bigr)\, ds$$
$$= \frac{2\kappa}{\pi^{1/2}} \sum_{j=1}^N \frac{\sigma_j}{\alpha_j^{1/2}} =: C^* \quad \text{a.s.}$$

by Birkhoff's ergodic theorem. This proves the upper bound.

The middle inequality in the theorem is an obvious consequence of the Cauchy–Schwarz inequality, so it remains to prove the lower bound. Throughout we will call a function g of $k_1, \dots, k_N$ (and possibly additional variables) *rotation-invariant* if $g(k_1, \dots, k_N) = g(Rk_1, \dots, Rk_N)$ for every $R \in SO(2)$. Below we will show that there exists some strictly positive rotation-invariant $c^* = c^*(k_1, \dots, k_N, \alpha_1, \dots, \alpha_N, \sigma_1, \dots, \sigma_N)$ such that

$$c^* \le \liminf_{t\to\infty} \frac{1}{t} \sup_{x\in\mathcal{S}} \langle \phi(t,0,x), e_1\rangle \quad \text{a.s.,} \tag{6}$$

where e_1 denotes the first coordinate direction. Since c^* is rotation-invariant, it follows that for every $u \in S^1$,

$$c^* \le \liminf_{t\to\infty} \frac{1}{t} \sup_{x\in\mathcal{S}} \langle \phi(t,0,x), u\rangle \quad \text{a.s.} \tag{7}$$

In order to show that the exceptional set can be chosen independently of u, let $S^* \subset S^1$ be countable and dense and $\mathcal{N}$ a null set such that (7) (without the qualification a.s.) holds for all $u \in S^*$ and $\limsup_{t\to\infty} \frac{1}{t} \sup_{x\in\mathcal{S}} \|\phi(t,0,x)\| \le C^*$ whenever $\omega \notin \mathcal{N}$. Fix $u \in S^1$, $\varepsilon > 0$ and $u^* \in S^*$ such that $\|u - u^*\| \le \varepsilon$. Then for $\omega \notin \mathcal{N}$ we have

$$\sup_{x\in\mathcal{S}} \langle \phi(t,0,x), u\rangle \ge \sup_{x\in\mathcal{S}} \langle \phi(t,0,x), u^*\rangle - \varepsilon \sup_{x\in\mathcal{S}} \|\phi(t,0,x)\|.$$

Dividing by t, taking $\liminf$ and letting ε go to 0, (7) and therefore the lower bound in the theorem follow.

It remains to prove (6) for some rotation-invariant $c^* > 0$.

Given $\varepsilon > 0$, we define a sequence of almost surely finite stopping times $\{\tau_n(\varepsilon)\}_{n\ge 1}$. Sometimes we will suppress the dependence on ε by writing τ_n instead of $\tau_n(\varepsilon)$. Set

$$\tau_0 = 0$$

and put for $n \ge 1$,

$$\tau_n = \inf\left\{ t \ge \tau_{n-1} + 1 : \sum_{i=1}^N \bigl(a_i^2(t) + b_i^2(t)\bigr) \le \varepsilon^2 \right\}. \tag{8}$$

For $\beta > 0$ we define

$$\lambda(\varepsilon) = \sup_{n\geq 1} \operatorname{ess\,sup}_{\omega} \sup_{Y\in\mathcal{F}_{\tau_n}} \big|E\big[\phi_1(\tau_{n+1}, \tau_n, Y) - Y_1 \big| \mathcal{F}_{\tau_n}\big]\big|, \tag{9}$$

$$\gamma(\varepsilon, \beta) = \inf_{n\geq 1} \operatorname{ess\,inf}_{\omega} \inf_{\substack{Y,Z\in\mathcal{F}_{\tau_n}\\ \|Y-Z\|=\beta}} E\big[\big(\phi_1(\tau_{n+1}, \tau_n, Y) - \phi_1(\tau_{n+1}, \tau_n, Z)\big)^+ \big| \mathcal{F}_{\tau_n}\big]. \tag{10}$$

The idea of the proof is that at the times τ_n the first component of ϕ has a martingale-like property in the sense that $\lambda(\varepsilon)$ is small when ε is small. In fact, in Proposition 1 we show that $\lim_{\varepsilon\to 0} \lambda(\varepsilon) = 0$. On the other hand, we will in Proposition 2 show $\gamma(\varepsilon, \beta)$ is bounded from zero as ε tends to zero. Thus, if $Y_1 = Y_1 \vee Z_1$,

$$\begin{aligned} &E\big(\phi_1(\tau_{n+1}, \tau_n, Y) \vee \phi_1(\tau_{n+1}, \tau_n, Z) \big| \mathcal{F}_{\tau_n}\big) \\ &\quad = E\Big(\phi_1(\tau_{n+1}, \tau_n, Y) + \big(\phi_1(\tau_{n+1}, \tau_n, Z) - \phi_1(\tau_{n+1}, \tau_n, Y)\big)^+ \big| \mathcal{F}_{\tau_n}\Big) \\ &\quad \geq Y_1 \vee Z_1 + \gamma(\varepsilon, \beta) - \lambda(\varepsilon), \end{aligned} \tag{11}$$

so the maximum on the left-hand side exhibits a substantial submartingale property when ε is small enough. This permits a competition-selection procedure which delivers the desired result. This technique was used in Cranston, Scheutzow and Steinsaltz (1999) and Scheutzow and Steinsaltz (2000).

In order to show $\lim_{\varepsilon\to 0} \lambda(\varepsilon) = 0$ we will use a coupling which we now outline. Noting that

$$\phi_1(t, 0, x) = x_1 + \int_0^t v_1\big(u, \phi(u, 0, x)\big)\, du, \tag{12}$$

we write the field v from the point of view of an observer at $\phi(t, 0, z)$ (which is the usual approach to prove homogenization). Define this as

$$D(t, x) = v\big(t, \phi(t, 0, z) + x\big). \tag{13}$$

Then,

$$D_1(t, x) = \sum_{i=1}^N k_{i,2}\big[a_i(t) \sin\langle k_i, x + \phi(t, 0, z)\rangle - b_i(t) \cos\langle k_i, x + \phi(t, 0, z)\rangle\big],$$

$$D_2(t, x) = \sum_{i=1}^N k_{i,1}\big[-a_i(t) \sin\langle k_i, x + \phi(t, 0, z)\rangle + b_i(t) \cos\langle k_i, x + \phi(t, 0, z)\rangle\big].$$

Using the angle addition formulas for sin and cos gives

$$\begin{aligned} D_1(t, x) &= \sum_{i=1}^N k_{i,2}\big(\mathcal{A}_i(t) \sin\langle k_i, x\rangle + \mathcal{B}_i(t) \cos\langle k_i, x\rangle\big), \\ D_2(t, x) &= \sum_{i=1}^N k_{i,1}\big(-\mathcal{A}_i(t) \sin\langle k_i, x\rangle - \mathcal{B}_i(t) \cos\langle k_i, x\rangle\big), \end{aligned} \tag{14}$$

where

$$\mathcal{A}_i(t) = a_i(t)\cos\langle k_i, \phi(t,0,z)\rangle + b_i(t)\sin\langle k_i, \phi(t,0,z)\rangle,$$
$$\mathcal{B}_i(t) = a_i(t)\sin\langle k_i, \phi(t,0,z)\rangle - b_i(t)\cos\langle k_i, \phi(t,0,z)\rangle. \tag{15}$$

From (4) and (15) it follows that the sdes for $\mathcal{A}_i$ and $\mathcal{B}_i$ are

$$\begin{aligned} d\mathcal{A}_i(t) &= -\alpha_i \mathcal{A}_i(t)\,dt - \mathcal{B}_i(t)\big\langle k_i, v\big(t, \phi(t,0,z)\big)\big\rangle\,dt \\ &\qquad + \sigma_i\big[\cos\langle k_i, \phi(t,0,z)\rangle dV_i(t) + \sin\langle k_i, \phi(t,0,z)\rangle\, dW_i(t)\big], \\ d\mathcal{B}_i(t) &= -\alpha_i \mathcal{B}_i(t)\,dt + \mathcal{A}_i(t)\big\langle k_i, v\big(t, \phi(t,0,z)\big)\big\rangle\,dt \\ &\qquad + \sigma_i\big[\sin\langle k_i, \phi(t,0,z)\rangle\, dV_i(t) - \cos\langle k_i, \phi(t,0,z)\rangle\, dW_i(t)\big]. \end{aligned} \tag{16}$$

The first thing to notice about (16) is that the quadratic variations are given by

$$\begin{aligned} d\mathcal{A}_i(t)\,d\mathcal{A}_j(t) &= \sigma_i\sigma_j\delta_{ij}\,dt, \\ d\mathcal{A}_i(t)\,d\mathcal{B}_j(t) &= 0, \\ d\mathcal{B}_i(t)\,d\mathcal{B}_j(t) &= \sigma_i\sigma_j\delta_{ij}\,dt. \end{aligned} \tag{17}$$

The second important observation on (16) is that using (3) one gets

$$\big\langle k_i, v\big(t, \phi(t,0,z)\big)\big\rangle = \sum_{j=1}^{N}(k_{i,1}k_{j,2} - k_{i,2}k_{j,1})\mathcal{B}_j(t).$$

Thus, for $U = (U_1, \dots, U_N)$, $\tilde{U} = (\tilde{U}_1, \dots, \tilde{U}_N)$ independent Brownian motions on $\mathbf{R}^N$,

$$\begin{aligned} d\mathcal{A}_i(t) &= -\alpha_i \mathcal{A}_i(t)\,dt - \left(\mathcal{B}_i(t)\sum_{j=1}^{N}(k_{i,1}k_{j,2} - k_{i,2}k_{j,1})\mathcal{B}_j(t)\right)dt + \sigma_i\,dU_i(t), \\ d\mathcal{B}_i(t) &= -\alpha_i \mathcal{B}_i(t)\,dt + \left(\mathcal{A}_i(t)\sum_{j=1}^{N}(k_{i,1}k_{j,2} - k_{i,2}k_{j,1})\mathcal{B}_j(t)\right)dt + \sigma_i\,d\tilde{U}_i(t). \end{aligned} \tag{18}$$

Now assuming $z = 0$,

$$\dot{\phi}_1(t,0,0) = D_1(t,0) = \sum_{i=1}^{N} k_{i,2}\mathcal{B}_i(t) \equiv \mathcal{C}(t). \tag{19}$$

(15) implies $\mathcal{A}_i(t)^2 + \mathcal{B}_i(t)^2 = a_i(t)^2 + b_i(t)^2$ for every i and hence, denoting $\xi(t) = (\mathcal{A}(t), \mathcal{B}(t))$, we have

$$\|\xi(t)\|^2 = \sum_{i=1}^{N}\big(\mathcal{A}_i(t)^2 + \mathcal{B}_i(t)^2\big) = \sum_{i=1}^{N}\big(a_i(t)^2 + b_i(t)^2\big), \tag{20}$$

so τ_n coincides with

$$\inf\{t \ge \tau_{n-1} + 1 : \|\xi(t)\| \le \varepsilon\}.$$

Returning to λ, in terms of the above, by the strong Markov property,

$$\begin{aligned}\lambda(\varepsilon) &= \sup_{\|x\|\le\varepsilon} \sup_{y\in\mathbf{R}^2} \bigl|E\bigl[\phi_1(\tau_1, 0, y) - y_1 | \xi(0) = x\bigr]\bigr| \\ &= \sup_{\|x\|\le\varepsilon} \bigl|E\bigl[\phi_1(\tau_1, 0, 0) | \xi(0) = x\bigr]\bigr|,\end{aligned} \tag{21}$$

using translation invariance. For later reference we define the function $\tilde{\lambda}(\varepsilon) = \tilde{\lambda}(\varepsilon, \alpha_1, \dots, \alpha_N, \sigma_1, \dots, \sigma_N, k_1, \dots, k_N)$ by

$$\tilde{\lambda}(\varepsilon) := \sup_{R\in SO(2)} \sup_{0<\delta\le\varepsilon} \sup_{\|x\|\le\varepsilon} \bigl|E\bigl[\phi_1\bigl(\tau_1(\delta), 0, 0\bigr) | \xi(0) = x\bigr]\bigr| \ge \lambda(\varepsilon), \tag{22}$$

where the first sup means that we take the supremum over all $R \in SO(2)$ of the flow with $k_1, \dots, k_N$ replaced by $Rk_1, \dots, Rk_N$ (but with the same α_i's and σ_i's). By symmetry,

$$E\bigl[\phi_1(\tau_1, 0, 0) | \xi(0) = 0\bigr] = 0. \tag{23}$$

So we need to show this will not change much if $\xi(0)$ is changed just a little. Since we have (19) we can do this by coupling two copies ξ, ξ' with $\xi(0) = 0$, $\xi'(0) = x$ and $\|x\| \le \varepsilon$, where $\xi = (\mathcal{A}, \mathcal{B})$, $\xi' = (\mathcal{A}', \mathcal{B}')$ are solutions of (18).

We can rewrite (18) as

$$\begin{aligned} d\xi_t &= -\mathbb{A}\xi_t\,dt + c(\xi_t)\,dt + \Sigma\,dB_t, \\ \xi_0 &= 0 \end{aligned} \tag{24}$$

with

$$B = (U, \tilde{U}) \text{ a } 2N\text{-dimensional Brownian motion},$$

$$\mathbb{A} = \operatorname{diag}(\alpha_1, \dots, \alpha_N, \alpha_1, \dots, \alpha_N),$$

$$\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_N, \sigma_1, \dots, \sigma_N),$$

$$c_i(x_1, \dots, x_{2N}) = -\sum_{j=1}^{N} (k_{i,1}k_{j,2} - k_{i,2}k_{j,1}) x_{i+N} x_{j+N}, \qquad 1 \le i \le N,$$

$$c_i(x_1, \dots, x_{2N}) = \sum_{j=1}^{N} (k_{i-N,1}k_{j,2} - k_{i-N,2}k_{j,1}) x_{i-N} x_{j+N}, \qquad N+1 \le i \le 2N.$$

We now specify a choice of another $2N$-dimensional Brownian motion B' which will be used to create a copy ξ' coupled to the original ξ, satisfying

$$\begin{aligned} d\xi'_t &= -\mathbb{A}\xi'_t\,dt + c(\xi'_t)\,dt + \Sigma\,dB'_t, \\ \xi'_0 &= x, \end{aligned} \tag{25}$$

where $x \in \mathbf{R}^{2N}$ has $\|x\| \le \varepsilon$.

Assuming for the moment that we have B' and therefore also ξ', define the coupling time

$$T = \inf\{t > 0 : \xi_t = \xi'_t\} \tag{26}$$

and set $\xi'_t = \xi_t$ for $t \ge T$. This is done by taking $B'_{T+t} - B'_T = B_{T+t} - B_T$ for $t \ge 0$. Our coupling is a coupling by reflection introduced in Lindvall and Rogers (1986). Clearly (the law of) T depends on x but we will suppress this dependence. Define

$$\begin{aligned} \eta_t &= \xi_t - \xi'_t, \\ u_t &= \frac{\Sigma^{-1}\eta_t}{\|\Sigma^{-1}\eta_t\|}, \\ H_t &= I - 2u_t u_t^T, \\ \beta_t &= -\mathbb{A}\eta_t + c(\xi_t) - c(\xi'_t) \end{aligned} \tag{27}$$

and

$$dB'_t = H_t\, dB_t. \tag{28}$$

Note (28) and (25) allow the creation of the $2N$-dimensional Brownian motion B'_t as the process ξ'_t evolves which is then fed back into (25) to create more of ξ'_t. In the following we will assume that (ξ_t, ξ'_t), $t \ge 0$ is the coordinate process on the space $\Omega = C([0,\infty), \mathbf{R}^{4N})$ and θ is the time shift defined by $\theta_t(\omega)(s) = \omega(t+s)$ for $s, t \ge 0$.

For the purpose of estimating $P(T \ge t \mid (\xi_0, \xi'_0) = (0, x))$ we will need to couple ξ and ξ' with a different Brownian motion than at (28) if either gets too large before the coupling time T. To make this precise, put

$$\begin{aligned} \alpha &= \min_{1 \le i \le N} \alpha_i, \\ K &= \max_{1 \le i, j \le N} \big(|k_{i,2}k_{j,1} - k_{i,1}k_{j,2}| \vee 1\big) \end{aligned} \tag{29}$$

and then set

$$S = \inf\left\{t > 0 : \sqrt{\|\xi_t\|^2 + \|\xi'_t\|^2} \ge \frac{\alpha}{4NK}\right\}. \tag{30}$$

On the set $\{T > t \ge S\}$ take B' so that $B'_{S+t} - B'_S$ is a $2N$-dimensional Brownian motion independent of $B_{S+t} - B_S$ and define ξ'_t as in (25) with this new B', that is,

$$\xi'_t = \xi'_S - \int_S^t \mathbb{A}\xi'_u\, du + \int_S^t c(\xi'_u)\, du + \Sigma(B'_t - B'_S).$$

Then define

$$R = \inf\left\{t > S : \sqrt{\|\xi_t\|^2 + \|\xi_t'\|^2} = \frac{\alpha}{c_0 N K}\right\}, \tag{31}$$

where $c_0 > 4$ is a constant to be named later. On $\{T > t \geq R\}$ revert to the B' at (28) until $S_1 \equiv S \circ \theta_R + R$ when we switch back to B' with increments independent of B provided T has not occurred yet. This B' is used in (25) until $R_1 = R \circ \theta_{S_1} + S_1$, when once again dB_t' is given by (28) (provided $R_1 + t < T$) until $S_2 = S \circ \theta_{R_1} + R_1$, and so on. We remark that T can only occur in the time intervals when B' is given by (28). With this coupling in place we are now able to prove the following.

PROPOSITION 1.

$$\lim_{\varepsilon \to 0} \tilde{\lambda}(\varepsilon) = 0.$$

PROOF. Assume $0 < \varepsilon \leq 1$. From (19) and (22),

$$\tilde{\lambda}(\varepsilon) = \sup_{R \in SO(2)} \sup_{0<\delta\leq\varepsilon} \sup_{\|x\|\leq\varepsilon} \left| E\left[\int_0^{\tau_1(\delta)} \mathcal{C}(s)\, ds \,\middle|\, \xi(0) = x\right]\right|.$$

Adopt the notation $\xi(s,x)$ and $\mathcal{C}(s,x)$ to denote the initial conditions $\xi(0,x) = x$ and $\mathcal{C}(0,x) = x \in \mathbf{R}^{2N}$. Take $\{(\xi(s,x), \xi(s,0) : s \geq 0\}$ to be the coupled pair outlined above.

Then, using (23),

$$\begin{aligned}
\tilde{\lambda}(\varepsilon) &= \sup_{R} \sup_{0<\delta\leq\varepsilon} \sup_{\|x\|\leq\varepsilon} \left| E\left[\int_0^{\tau_1(\delta)} [\mathcal{C}(s,x) - \mathcal{C}(s,0)]\, ds\right]\right| \\
&= \sup_{R} \sup_{0<\delta\leq\varepsilon} \sup_{\|x\|\leq\varepsilon} \left| E\left[\int_0^{T\wedge\tau_1(\delta)} [\mathcal{C}(s,x) - \mathcal{C}(s,0)]\, ds\right]\right| \\
&\leq \sup_{R} \sup_{\|x\|\leq\varepsilon} \left(E\left[\int_0^T |\mathcal{C}(s,x)|\, ds\right] + E\left[\int_0^T |\mathcal{C}(s,0)|\, ds\right]\right).
\end{aligned} \tag{32}$$

Also, by Cauchy–Schwarz,

$$E\left[\int_0^T |\mathcal{C}(s,x)|\, ds\right] \leq \int_0^\infty \sqrt{P(T \geq s)} \sqrt{E[\mathcal{C}(s,x)^2]}\, ds. \tag{33}$$

By (4), (19) and (20), $\sup_{R \in SO(2)} E[\mathcal{C}(s,x)^2] \leq c$ for all $s \geq 0$ and all $\|x\| \leq 1$ for some positive constant c.

Thus, by (32) and (33),

$$\tilde{\lambda}(\varepsilon) \leq 2\sqrt{c} \int_0^\infty \sup_{\|x\|\leq\varepsilon} \sqrt{P(T \geq s)}\, ds. \tag{34}$$

Using the dominated convergence theorem, the proof of Proposition 1 will be complete once we have proved the lemma. $\square$

LEMMA 1. *There are positive constants C and δ depending on α, K, N and $\sigma = \max_{1\le i\le N} \sigma_i$, such that for $\|x\| \le 1$ and*

$$T = \inf\{s > 0 : \xi(s, x) = \xi(s, 0)\},$$

$$P(T \ge t) \le \Big(C\sqrt{\|x\|}(e^{\delta t} - 1)^{-1/2}\Big) \wedge 1.$$

PROOF. Fix $x \in \mathbf{R}^{2N}$ such that $0 \le \|x\| \le 1$. Using the notation of (27), Itô's formula gives

$$d(\|\eta_t\|) = 2\Big\langle \frac{\eta_t}{\|\eta_t\|}, \Sigma u_t u_t^T \, dB_t\Big\rangle + \Big\langle \frac{\eta_t}{\|\eta_t\|}, \beta_t\Big\rangle dt, \qquad 0 \le t \le S \wedge T. \tag{35}$$

Notice that

$$\bigg(2\Big\langle \frac{\eta_t}{\|\eta_t\|}, \Sigma u_t u_t^T \, dB_t\Big\rangle\bigg)^2 = 4\frac{\|\eta_t\|^2}{\|\Sigma^{-1}\eta_t\|^2}\, dt. \tag{36}$$

Recalling the definition of S in (30) and c in (24) observe that

$$\big\|c\big(\xi(t, 0)\big) - c\big(\xi(t, x)\big)\big\| \le \frac{\alpha}{2}\big\|\xi(t, 0) - \xi(t, x)\big\|, \qquad 0 \le t \le S. \tag{37}$$

Consequently, if we define $dM_t = 2\langle \frac{\eta_t}{\|\eta_t\|}\rangle, \Sigma u_t u_t^T \, dB_t\rangle$ and ρ_t by

$$\begin{aligned} d\rho_t &= dM_t - \frac{\alpha}{2}\rho_t \, dt, \\ \rho_0 &= \|\eta_0\| = \|x\| \end{aligned} \tag{38}$$

by an elementary comparison theorem [see Ikeda and Watanabe (1981)],

$$\rho_t \ge \|\eta_t\| \qquad \text{for } 0 \le t \le S \wedge T \text{ a.s.} \tag{39}$$

The solution of (38) is given by

$$\rho_t = \exp\Big(-\frac{\alpha}{2}t\Big)\Big(\|x\| + \int_0^t \exp\Big(\frac{\alpha}{2}s\Big)\, dM_s\Big). \tag{40}$$

Writing $\sigma_0(\rho) = \inf\{t > 0 : \rho_t = 0\}$ observe that $\sigma_0(\rho) \ge T$ on $\{T \le S\}$ and so

$$\begin{aligned} P(T \ge t,\ T \le S) &\le P\big(\sigma_0(\rho) \ge t\big) \\ &\le P\Big(\|x\| + \int_0^r \exp\Big(\frac{\alpha}{2}u\Big)\, dM_u \ge 0,\ \forall\, r \le t\Big), \\ &= P\Big(-\int_0^r \exp\Big(\frac{\alpha}{2}u\Big)\, dM_u \le \|x\|;\ \forall\, r \le t\Big) \\ &\le P\Big(b_r \le \|x\|;\ \forall\, r \le \frac{4\sigma^2}{\alpha}(e^{\alpha t} - 1)\Big), \end{aligned} \tag{41}$$

where b is the one-dimensional Brownian motion

$$b_r = \int_0^{\tau_r} e^{\alpha u/2}\, dM_u$$

with

$$\tau_r = \inf\left\{t : 4\int_0^t e^{\alpha u} \frac{\|\eta_u\|^2}{\|\Sigma^{-1}\eta_u\|^2}\, du = r\right\}.$$

Therefore,

$$P(T \geq t,\ T \leq S) \leq \int_0^{\frac{\|x\|\sqrt{\alpha}}{2\sigma\sqrt{\exp(\alpha t)-1}}} \frac{2}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy \tag{42}$$
$$\leq \frac{\|x\|\sqrt{\alpha}}{\sqrt{2\pi}\,\sigma\sqrt{\exp(\alpha t)-1}}.$$

Next we estimate $P(T \geq t,\ T > S)$. First,

$$P(T \geq t,\ T > S) \leq \sqrt{P(T \geq t)}\sqrt{P(T > S)}. \tag{43}$$

By the arguments in Cranston [(1991), Theorem 1, inequality (0.2)],

$$P(T > S) \leq \frac{C(\alpha,\sigma)}{2NK}\|x\|. \tag{44}$$

To handle $P(T \geq t)$, recalling the stopping times R and S of (30) and (31) we create a sequence of stopping times γ_n by

$$\begin{aligned} \gamma_0 &= 0,\\ \gamma_{2n+1} &= S \circ \theta_{\gamma_{2n}} + \gamma_{2n},\\ \gamma_{2n} &= R \circ \theta_{\gamma_{2n-1}} + \gamma_{2n-1}. \end{aligned} \tag{45}$$

Henceforth we adopt the notation $P(\cdot|(z,w))$ to denote the initial condition of the diffusions ξ and ξ' from (24) and (25). Now the existence of a $\delta > 0$ for c_0 large enough, depending on $C(\alpha,\sigma)$ in (44) such that

$$\inf_{\sqrt{\|z\|^2+\|w\|^2}=\frac{\alpha}{c_0KN}} P\big(T \leq S|(z,w)\big) = \delta \tag{46}$$

follows from Cranston [(1991), Theorem 1, inequality (0.2)].

From (20) and the definition of the stopping times R and S it follows easily that there exist positive numbers A and ϵ_0 such that

$$P(\gamma_{2n+2} - \gamma_{2n} \geq s|\mathcal{F}_{\gamma_{2n}}) \leq Ae^{-\epsilon_0 s} \tag{47}$$

for all $n \geq 0$ and $s \geq 0$. From (46) we get for $n \geq 1$,

$$P(T \leq \gamma_{2n+1}|\mathcal{F}_{\gamma_{2n}}) \geq \delta \qquad \text{on } \{T > \gamma_{2n}\}. \tag{48}$$

Hence for any $\lambda > 0, n \in \mathbf{N}$ and $0 < \varepsilon < \varepsilon_0$ we get

$$\begin{aligned} P(T \geq \lambda n) &\leq P(\gamma_{2n+2} \geq \lambda n) + P(T \geq \lambda n, \gamma_{2n+2} < \lambda n) \\ &\leq E e^{\varepsilon \gamma_{2n+2}} e^{-\varepsilon \lambda n} + P(T > \gamma_{2n+2}) \\ &\leq e^{-\varepsilon \lambda n} \left(\frac{A\varepsilon_0}{\varepsilon_0 - \varepsilon} \right)^{n+1} + (1-\delta)^n. \end{aligned} \tag{49}$$

Choosing λ sufficiently large, it follows that there exist $a, c > 0$ such that

$$P(T \geq t) = P\big(T \geq t | (0, x)\big) \leq a e^{-ct}. \tag{50}$$

Combining (42), (43), (44) and (50) we have

$$\begin{aligned} P\big(T \geq t | (0,x)\big) &\leq P\big(T \geq t,\ T \leq S | (0,x)\big) + P\big(T \geq t,\ T > S | (0,x)\big) \\ &\leq \frac{\|x\| \sqrt{\alpha}}{\sqrt{2\pi}\sigma} \big(\exp(\alpha t) - 1\big)^{-1/2} + \sqrt{\frac{C(\alpha, \sigma)}{2NK} \cdot \|x\|} \sqrt{a e^{-ct}}, \end{aligned}$$

which proves Lemma 1 and also completes the proof of Proposition 1. $\square$

We now turn our attention to proving $\tilde{\gamma}(\varepsilon, \beta) := \inf_{R \in SO(2)} \gamma(\varepsilon, \beta)$ remains bounded from 0 as ε tends to 0. For this purpose define, given $\varepsilon > 0$, $x, y \in \mathbf{R}^2$,

$$A(\varepsilon, x, y) = \big\{ w : \|\xi(1)\| \leq \varepsilon, \phi_1(1, 0, x) \geq \phi_1(1, 0, y) + 5 \big\}. \tag{51}$$

Further we define

$$\kappa^* = \max\big\{ \|k\| : k \in \mathcal{K} \big\}.$$

Then we have the following.

LEMMA 2. *Given any* $0 < \beta < \frac{\pi}{2\kappa^*}$ *there is an* $\varepsilon_0 > 0$, *such that for any* $0 < \varepsilon < \varepsilon_0$, *there is a rotation-invariant* $\rho = \rho(\varepsilon, \beta, \alpha_1, \dots, \alpha_N, \sigma_1, \dots, \sigma_N, k_1, \dots, k_N) > 0$ *such that*

$$P_{\xi_0}\big(A(\varepsilon, x, y)\big) \geq \rho$$

holds for any $x, y \in \mathbf{R}^2$ *with* $\|x - y\| = \beta$ *and whenever* $\|\xi_0\| \leq \varepsilon$.

PROOF. For a vector $r = (r_1, r_2) \in \mathbf{R}^2$ we write $r^\perp = (r_2, -r_1)$. Since $\operatorname{span} \mathcal{K} = \mathbf{R}^2$, there exists i_0 and j_0 such that $\langle k_{i_0}, k_{j_0}^\perp \rangle \neq 0$. Relabelling if necessary we can assume that $i_0 = 1$ and $j_0 = 2$.

Observe that for any $u \in S^1$ we have

$$\begin{aligned} |\langle k_1, k_2^\perp \rangle| &= \big| \langle k_1, u \rangle \langle k_2^\perp, u \rangle + \langle k_1, u^\perp \rangle \langle k_2^\perp, u^\perp \rangle \big| \\ &\leq (\|k_1\| + \|k_2\|) \, (|\langle k_1, u \rangle| \vee |\langle k_2, u \rangle|) \end{aligned}$$

and hence

$$|\langle k_1, u\rangle| \vee |\langle k_2, u\rangle| \geq \frac{|\langle k_1, k_2^\perp\rangle|}{\|k_1\| + \|k_2\|} =: \Delta > 0. \tag{52}$$

Exchanging k_1 and k_2 if necessary we can assume that $|\langle k_2, e_2\rangle| \geq \Delta$. Now let $x, y \in \mathbf{R}^2$ satisfy $\|x - y\| = \beta$. Without loss of generality we will assume that $y = 0$.

Define

$$\tilde{\alpha}_1 := \begin{cases} 0, & \text{if } |\langle k_1, x\rangle| < \Delta\beta, \\ \dfrac{\pi - 2\langle x, k_2\rangle}{\langle k_1^\perp, k_2\rangle \sin\langle k_1, x\rangle}, & \text{otherwise,} \end{cases}$$

$$\tilde{\alpha}_2 := \begin{cases} \left(12 + \dfrac{\pi}{\kappa^*}\right) \dfrac{\operatorname{sign}(\langle e_1, k_2^\perp\rangle \sin\langle k_2, x\rangle)}{\Delta \sin(\beta\Delta)}, & \text{if } |\langle k_1, x\rangle| < \Delta\beta, \\ \left(12 + \dfrac{\pi}{\kappa^*} + 2\kappa^* \dfrac{\pi}{|\langle k_1^\perp, k_2\rangle|}\right) \dfrac{\operatorname{sign}(\langle e_1, k_2^\perp\rangle)}{\Delta}, & \text{otherwise,} \end{cases}$$

$$\alpha_1(t) := \begin{cases} 8\tilde{\alpha}_1 t, & 0 \leq t \leq 1/4, \\ 4\tilde{\alpha}_1(1 - 2t), & 1/4 \leq t \leq 1/2, \\ 0, & 1/2 \leq t \leq 1, \end{cases}$$

and

$$\alpha_2(t) := \begin{cases} 0, & 0 \leq t \leq 1/2, \\ 4\tilde{\alpha}_2(2t - 1), & 1/2 \leq t \leq 3/4, \\ 8\tilde{\alpha}_2(1 - t), & 3/4 \leq t \leq 1, \end{cases}$$

and let $\varphi : [0, \infty) \to \mathbf{R}^2$ solve

$$\begin{aligned} \dot{\varphi}(t) &= k_1^\perp \alpha_1(t) \sin\langle k_1, \varphi(t)\rangle + k_2^\perp \alpha_2(t) \sin\langle k_2, \varphi(t)\rangle, \\ \varphi(0) &= x. \end{aligned} \tag{53}$$

Observing that $\langle k_1, \dot{\varphi}(t)\rangle = 0$ for $0 \leq t < 1/2$, we get

$$\varphi\left(\frac{1}{2}\right) = \begin{cases} x, & \text{if } |\langle k_1, x\rangle| < \Delta\beta, \\ x + \dfrac{1}{2} k_1^\perp \dfrac{\pi - 2\langle x, k_2\rangle}{\langle k_1^\perp, k_2\rangle}, & \text{otherwise.} \end{cases} \tag{54}$$

Observing that by (52) $|\langle k_1, x\rangle| < \Delta\beta$ implies $|\langle k_2, x\rangle| \geq \Delta\beta$, that $|\langle k_2, x\rangle| \leq \kappa^*\beta < \frac{\pi}{2}$, that $\langle k_2, \varphi(1/2)\rangle = \pi/2$ if $|\langle k_1, x\rangle| \geq \Delta\beta$ and that $\langle k_2, \dot{\varphi}(t)\rangle = 0$ for $1/2 \leq t < 1$ we get

$$\varphi_1(1) = \begin{cases} x_1 + \dfrac{\langle e_1, k_2^\perp\rangle}{2}\sin\langle k_2, x\rangle\tilde{\alpha}_2 & \\ \qquad \geq -\beta + \dfrac{\Delta}{2}\sin(\Delta\beta)|\tilde{\alpha}_2|, & \text{if } |\langle k_1, x\rangle| < \Delta\beta, \\ x_1 + \dfrac{\langle e_1, k_1^\perp\rangle}{2}\dfrac{\pi - 2\langle x, k_2\rangle}{\langle k_1^\perp, k_2\rangle} + \langle e_1, k_2^\perp\rangle\dfrac{\tilde{\alpha}_2}{2} & \\ \qquad \geq -\beta - \dfrac{\kappa^*\pi}{|\langle k_1^\perp, k_2\rangle|} + \dfrac{\Delta}{2}|\tilde{\alpha}_2|, & \text{otherwise.} \end{cases}$$

In both cases we therefore get

$$\varphi_1(1) \geq 6.$$

Define the functions $\alpha_i(t)$ for $i \geq 3$ and $\beta_i(t)$ for $i \geq 1$ to be identically 0. Then there exists a rotation-invariant $c = c(\beta, k_1, \dots, k_N)$ (not depending on x) such that

$$\int_0^1 \left(\sum_{i=1}^N (\alpha_i'(t)^2 + \beta_i'(t)^2) \right) dt = 32\left(\tilde{\alpha}_1^2 + \tilde{\alpha}_2^2\right) \leq c. \tag{55}$$

By (55) and Girsanov's theorem, given any $\varepsilon > 0$ there is a rotation-invariant $\rho = \rho(\varepsilon, \beta, \alpha_1, \dots, \alpha_N, \sigma_1, \dots, \sigma_N, k_1, \dots, k_N) > 0$ such that for any $\|\xi_0\| \leq \varepsilon$ we have

$$\begin{aligned} P_{\xi_0}\Bigg(\sup_{0\leq t\leq 1}\Bigg(\tfrac{1}{2}(1+t)\Bigg[\sum_{i=1}^N \Big\{\big(a_i(t) - \alpha_i(t)\big)^2 & \\ + \big(b_i(t) - \beta_i(t)\big)^2\Big\}\Bigg]^{1/2}\Bigg) < \varepsilon\Bigg) > \rho. \end{aligned} \tag{56}$$

Select $\varepsilon_0 > 0$ (independently of x) so that if $\varepsilon < \varepsilon_0$ and

$$\sup_{0\leq t\leq 1}\left(\tfrac{1}{2}(1+t)\left[\sum_{i=1}^N \left\{\big(a_i(t) - \alpha_i(t)\big)^2 + \big(b_i(t) - \beta_i(t)\big)^2\right\}\right]^{1/2}\right) < \varepsilon, \tag{57}$$

then with ϕ as defined at (5),

$$\sup_{0\leq t\leq 1} \|\varphi(1) - \phi(t, 0, x)\| < \tfrac{1}{2} \quad \text{and} \quad \sup_{0\leq t\leq 1} \|\phi(t, 0, 0)\| < \tfrac{1}{2}. \tag{58}$$

With $\varepsilon < \varepsilon_0$, (57) implies

$$\phi_1(1, 0, x) \geq \varphi_1(1) - \tfrac{1}{2} \geq 6 - \tfrac{1}{2} - \tfrac{1}{2} + \phi_1(1, 0, 0) = 5 + \phi_1(1, 0, 0). \tag{59}$$

Thus for $\varepsilon \le \varepsilon_0$,

$$A(\varepsilon, x, y) \supseteq \left\{ \sup_{0\le t\le 1} \left(\frac{1}{2}(1+t) \left[\sum_{i=1}^{N} \big(a_i(t) - \alpha_i(t)\big)^2 + \big(b_i(t) - \beta_i(t)\big)^2 \right]^{1/2} \right) < \varepsilon \right\}$$

and Lemma 2 is proved. □

PROPOSITION 2. *Choose* $\varepsilon_1 > 0$ *such that* $\tilde\lambda(\varepsilon_1) \le 1$. *With* β, ε_0 *and* ρ *as in Lemma* 2, *for any* $0 < \varepsilon \le \varepsilon_2 := \varepsilon_0 \wedge \varepsilon_1$,

$$\begin{aligned} \tilde\gamma(\varepsilon, \beta) &= \inf_{R, \|\xi\|\le\varepsilon, \|x-y\|=\beta} E\Big[\big(\phi_1(\tau_1(\varepsilon), 0, x) - \phi_1(\tau_1(\varepsilon), 0, y)\big)^+ \big| \xi(0) = \xi \Big] \\ &\ge 3\rho(\varepsilon_2). \end{aligned}$$

PROOF. Note that Proposition 1 guarantees the existence of such an ε_1. Take any $0 < \varepsilon < \varepsilon_2$, $\|\xi\| \le \varepsilon$, $\xi(0) = \xi$ and $x, y \in \mathbf{R}^2$ such that $\|x - y\| = \beta$. Then by the definition of $\tilde\lambda$ at (22) on the set $A(\varepsilon_2, x, y)$,

$$\begin{aligned} \big|E\big[\phi_1(\tau_1(\varepsilon), 0, x) - \phi_1(1, 0, x) | \mathcal{F}_1\big]\big| &\le \tilde\lambda(\varepsilon_2) \le 1, \\ \big|E\big[\phi_1(\tau_1(\varepsilon), 0, y) - \phi_1(1, 0, y) | \mathcal{F}_1\big]\big| &\le \tilde\lambda(\varepsilon_2) \le 1. \end{aligned} \tag{60}$$

Thus, on $A(\varepsilon_2, x, y)$,

$$\begin{aligned} &E\Big[\Big(\phi_1\big(\tau_1(\varepsilon), 0, x\big) - \phi_1\big(\tau_1(\varepsilon), 0, y\big)\Big)^+ | \mathcal{F}_1\Big] \\ &\quad \ge E\big[\phi_1\big(\tau_1(\varepsilon), 0, x\big) - \phi_1\big(\tau_1(\varepsilon), 0, y\big) | \mathcal{F}_1\big] \\ &\quad \ge E\big[\phi_1\big(\tau_1(\varepsilon), 0, x\big) - \phi_1\big(1, 0, x\big)) | \mathcal{F}_1\big] \\ &\qquad - E\big[\phi_1\big(\tau_1(\varepsilon), 0, y\big) - \phi_1\big(1, 0, y\big) | \mathcal{F}_1\big] + 5 \\ &\quad \ge 3 \qquad \text{by (60)}. \end{aligned} \tag{61}$$

Consequently, from (61) follows that for any $\varepsilon < \varepsilon_2$ on the set $\{\|\xi(0)\| \le \varepsilon\}$, we have

$$\begin{aligned} &E\Big[\big(\phi_1\big(\tau_1(\varepsilon), 0, x\big) - \phi_1\big(\tau_1(\varepsilon), 0, y\big)\big)^+ | \mathcal{F}_0\Big] \\ &\quad \ge E\Big[E\Big[\Big(\phi_1\big(\tau_1(\varepsilon), 0, x\big) - \phi_1\big(\tau_1(\varepsilon), 0, y\big)\Big)^+ \mathbb{1}_{A(\varepsilon_2, x, y)} | \mathcal{F}_1\Big] | \mathcal{F}_0\Big] \\ &\quad \ge 3\rho(\varepsilon_2) \end{aligned}$$

and the proposition is proved. □

PROOF OF THE THEOREM. Let us first fix $\beta \in (0, \frac{\pi}{2\kappa^*})$ and assume that $\mathrm{vol}(\mathcal{S}(\omega)) \ge \pi\beta^2$ for all $\omega \in \Omega$. Since the flow is incompressible,

$$\mathrm{diam}\big(\phi(t, 0, \mathcal{S})\big) \ge 2\beta \qquad \text{for all } t \ge 0.$$

Take $\{\tau_n\}_{n\geq 1}$ to be the stopping times defined at (8) where $\varepsilon = \varepsilon(\beta) \in (0,1)$ is taken so that $\tilde{\gamma}(\varepsilon,\beta) - \tilde{\lambda}(\varepsilon) > 0$; which is possible by Propositions 1 and 2. We now create a sequence $\{(x_i, y_i)\}_{i\geq 1}$ of $\mathcal{F}_{\tau_i}$-measurable $\mathcal{S} \times \mathcal{S}$-valued random variables by a competition-selection procedure. Take $x_1, y_1 \in \mathcal{S}$ to be arbitrary $\mathcal{F}_{\tau_1}$-measurable satisfying $\|\phi(\tau_1, 0, x_1) - \phi(\tau_1, 0, y_1)\| = \beta$. Since $\mathcal{S}$ is connected and $\operatorname{diam}(\phi(\tau_1, 0, \mathcal{S})) \geq 2\beta$, such x_1 and y_1 exist.

Inductively define

$$x_i = \begin{cases} x_{i-1}, & \text{if } \phi_1(\tau_i, 0, x_{i-1}) \geq \phi_1(\tau_i, 0, y_{i-1}), \\ y_{i-1}, & \text{otherwise} \end{cases} \tag{62}$$

and select y_i to be an $\mathcal{S}$-valued $\mathcal{F}_{\tau_i}$-measurable random variable such that

$$\|\phi(\tau_i, 0, x_i) - \phi(\tau_i, 0, y_i)\| = \beta. \tag{63}$$

Since $\operatorname{diam}(\phi(t, 0, S)) \geq 2\beta$, such a y_i always exists.

We now claim that there is a $c > 0$ such that

$$\liminf_{i\to\infty} \frac{\phi_1(\tau_i, 0, x_i)}{\tau_i} \geq \frac{\tilde{\gamma}(\varepsilon,\beta) - \tilde{\lambda}(\varepsilon)}{c} > 0. \tag{64}$$

To verify this, set $Z_i = \phi_1(\tau_i, 0, x_i) - \phi_1(\tau_{i-1}, 0, x_{i-1})$ for $i \geq 2$. Then each Z_i is $\mathcal{F}_{\tau_i}$–measurable and

$$\begin{aligned} E[Z_i | \mathcal{F}_{\tau_{i-1}}] &= E[\phi_1(\tau_i, 0, x_{i-1}) \vee \phi_1(\tau_i, 0, y_{i-1}) - \phi_1(\tau_{i-1}, 0, x_{i-1}) | \mathcal{F}_{\tau_{i-1}}] \\ &= E\big[\big(\phi_1(\tau_i, 0, y_{i-1}) - \phi_1(\tau_i, 0, x_{i-1})\big)^+ | \mathcal{F}_{\tau_{i-1}}\big] \\ &\quad + E[\phi_1(\tau_i, 0, x_{i-1}) - \phi_1(\tau_{i-1}, 0, x_{i-1}) | \mathcal{F}_{\tau_{i-1}}] \\ &\geq \tilde{\gamma}(\varepsilon,\beta) - \tilde{\lambda}(\varepsilon) \qquad (65) \\ &> 0 \qquad \text{by choice of } \varepsilon. \end{aligned}$$

Since $\tau_{i+1} - \tau_i \geq 1$ and $E[(\tau_{i+1} - \tau_i)^2 | \mathcal{F}_{\tau_i}] \leq k$, almost surely for some k and for all i, from the strong law of large numbers for martingales [Hall and Heyde (1980), Theorem 2.18], it follows that

$$\begin{aligned} &\limsup_{i\to\infty} \frac{1}{i} \left\{ \sum_{j=1}^{i-1} \Big[(\tau_{j+1} - \tau_j) - \sqrt{k} \Big] \right\} \\ &\quad \leq \lim_{i\to\infty} \frac{1}{i} \left\{ \sum_{j=1}^{i-1} (\tau_{j+1} - \tau_j - E[\tau_{j-1} - \tau_j | \mathcal{F}_{\tau_i}]) \right\} = 0 \quad \text{a.s.} \end{aligned} \tag{66}$$

Thus,

$$\limsup_{i\to\infty} \frac{\tau_i}{i} \leq \sqrt{k} \quad \text{a.s.} \tag{67}$$

Again by Theorem 2.18 of Hall and Heyde (1980), and Lemma 3 (after this proof),

$$\begin{aligned}\liminf_{i\to\infty}\left[\frac{\phi_1(\tau_i,0,x_i)}{i}-(\tilde{\gamma}-\tilde{\lambda})\right] &= \liminf_{i\to\infty}\frac{\sum_{j=2}^{i}(Z_j-(\tilde{\gamma}-\tilde{\lambda}))+\phi_1(\tau_1,0,x_1)}{i},\\ &\geq 0 \quad \text{a.s.}\end{aligned}$$

whence

$$\liminf_{i\to\infty}\frac{\phi_1(\tau_i,0,x_i)}{\tau_i}\geq\frac{1}{\sqrt{k}}\liminf_{i\to\infty}\frac{\phi_1(\tau_i,0,x_i)}{i}\geq\frac{\tilde{\gamma}(\varepsilon,\beta)-\tilde{\lambda}(\varepsilon)}{\sqrt{k}}\quad\text{a.s.}\tag{68}$$

Observe that

$$\begin{aligned}1 &\leq \liminf_{i\to\infty}\frac{\tau_{i+1}}{\tau_i}\\ &\leq \limsup_{i\to\infty}\frac{\tau_{i+1}}{\tau_i}\\ &= \limsup_{i\to\infty}\frac{\tau_{i+1}-\tau_i}{\tau_i}+1\\ &\leq \limsup_{i\to\infty}\frac{\tau_{i+1}-\tau_i}{i}+1,\end{aligned}$$

where in the last line we have used $\tau_j-\tau_{j-1}\geq 1$ a.s. for all j. Applying the of Borel–Cantelli lemma and the estimate

$$P(\tau_{i+1}-\tau_i\geq is)\leq\frac{E[(\tau_{i+1}-\tau_i)^2]}{s^2i^2}\leq\frac{k}{s^2i^2}$$

for $s>0$ it follows that $P(\tau_{i+1}-\tau_i\geq is\text{ i.o.})=0$. So we have

$$\lim_{i\to\infty}\frac{\tau_{i+1}}{\tau_i}=1\quad\text{a.s.}\tag{69}$$

We can now combine this claim with Lemma 3 to prove

$$\begin{aligned}\liminf_{t\to\infty}\sup_{x\in\mathcal{S}}\frac{\phi_1(t,0,x)}{t} &\geq \liminf_{i\to\infty}\inf_{\tau_i\leq t<\tau_{i+1}}\frac{\phi_1(t,0,x_i)}{t}\\ &= \liminf_{i\to\infty}\left\{\inf_{\tau_i\leq t<\tau_{i+1}}\frac{\phi_1(t,0,x_i)-\phi_1(\tau_i,0,x_i)}{t}\right.\\ &\qquad\left.+\frac{\phi_1(\tau_i,0,x_i)}{t}\right\}\\ &\geq \frac{\tilde{\gamma}(\varepsilon,\beta)-\tilde{\lambda}(\varepsilon)}{\sqrt{k}}=:c^*(\beta)\quad\text{a.s.}\quad\text{by (68) and (69).}\end{aligned}$$

Now take an arbitrary $\mathcal{F}_0$-measurable compact and connected set $\mathcal{S}$ with $\mathrm{vol}(\mathcal{S})>0$ almost surely. Then for any β as above define

$$\tilde{\mathcal{S}}(\omega):=\begin{cases}\mathcal{S}(\omega), & \text{if }\mathrm{vol}(\mathcal{S}(\omega))\geq\pi\beta^2,\\ \bar{B}(0,\beta), & \text{otherwise,}\end{cases}$$

where $\bar{B}(0, \beta)$ denotes the closed ball of radius β centered at 0. The consideration above shows that

$$\liminf_{t\to\infty} \sup_{x\in\tilde{\mathcal{S}}} \frac{\phi_1(t, 0, x)}{t} \geq c^*(\beta)$$

and, letting $\beta \to 0$, it follows that $\operatorname{diam}(\phi(t, 0, \mathcal{S})) \to \infty$ as $t \to \infty$ almost surely.

Now we fix $\beta^* \in (0, \frac{\pi}{2\kappa^*})$ and define $c^* = c^*(\beta^*)$. For $\varepsilon_3 > 0$ there exists some T such that

$$P\Bigl(\inf_{t\geq T} \operatorname{diam}\bigl(\phi(t, 0, \mathcal{S})\bigr) \leq 2\beta^*\Bigr) < \varepsilon_3.$$

Starting the competition-selection procedure at time T with the set $\phi(T, 0, \mathcal{S})$ with respect to β^* and giving up as soon as the diameter of $\phi(t, 0, \mathcal{S})$ falls below $2\beta^*$ after time T, we see that

$$P\Bigl(\liminf_{t\to\infty} \sup_{x\in\mathcal{S}} \frac{\phi(t, 0, \mathcal{S})}{t} \geq c^*\Bigr) \geq 1 - \varepsilon_3.$$

Since $\varepsilon_3 > 0$ was arbitrary, the probability is in fact 1. Observing that c^* is rotation-invariant we have shown (6) and therefore the theorem is proved. $\square$

LEMMA 3. *There is a* $\Gamma > 0$ *such that with* x_i *as defined at* (62),

$$\sup_{i\geq 1} E\Bigl[\sup_{\tau_i\leq t<\tau_{i+1}} \bigl|\phi_1(t, 0, x_i) - \phi_1(\tau_i, 0, x_i)\bigr|^2\Bigr] \leq \Gamma$$

and

$$\sup_{i\geq 1} E Z_i^2 \leq \Gamma.$$

PROOF. Since for $\tau_i \leq t < \tau_{i+1}$,

$$\phi_1(t, 0, x_i) - \phi_1(\tau_i, 0, x_i) = \int_{\tau_i}^{t} v\bigl(s, 0, \phi(s, 0, x_i)\bigr)\, ds,$$

$$\begin{aligned} E\Bigl[&\sup_{\tau_i\leq t<\tau_{i+1}} |\phi_1(t, 0, x_i) - \phi_1(\tau_i, 0, x_i)|^2\Bigr] \\ &\leq E\biggl(\biggl[\int_{\tau_i}^{\tau_{i+1}} \bigl\|v\bigl(s, 0, \phi(s, 0, x_i)\bigr)\bigr\|\, ds\biggr]^2\biggr) \\ &\leq C\ E\biggl(\biggl(\sum_{j=1}^{N} \int_{\tau_i}^{\tau_{i+1}} \bigl(|a_j(s)| + |b_j(s)|\bigr)\, ds\biggr)^2\biggr) \end{aligned}$$

$$
\begin{aligned}
&\leq C \sum_{j=1}^{N} \sum_{k=1}^{N} E \int_0^\infty \int_0^\infty \mathbb{1}_{[0,\tau_{i+1}-\tau_i)}(s) \mathbb{1}_{[0,\tau_{i+1}-\tau_i)}(t) \\
&\qquad \times \big[|a_j(s+\tau_i)||a_k(t+\tau_i)| + 2|a_j(s+\tau_i)||b_k(t+\tau_i)| \\
&\qquad\qquad + |b_j(s+\tau_i)||b_k(t+\tau_i)|\big]\, ds\, dt \\
&\leq C \sum_{j=1}^{N} \sum_{k=1}^{N} \int_0^\infty \int_0^\infty \sqrt{P(\tau_{i+1}-\tau_i \geq s \vee t)} \\
&\qquad \times \Big[\sqrt{E[a_j^2(\tau_i+s) a_k^2(\tau_i+t)]} + 2\sqrt{E[a_j^2(\tau_i+s) b_k^2(\tau_i+t)]} \\
&\qquad\qquad + \sqrt{E[b_j^2(\tau_i+s) b_k^2(\tau_i+t)]}\Big] \\
&\leq C\, N^2 \int_0^\infty \int_0^\infty \sqrt{P(\tau_{i+1}-\tau_i \geq s \vee t)}\, ds\, dt \\
&= \Gamma < \infty
\end{aligned}
$$

for a new constant C (independent of s, t, i), since $\tau_{i+1} - \tau_i$ has moments of all orders.

The proof that $E[Z_i^2] \leq \Gamma$ is almost the same. $\square$

REMARK. The arguments at the end of the proof of the theorem show that it is not necessary to assume that $\mathrm{vol}(\mathcal{S}) > 0$ almost surely. It is sufficient to require that $\liminf_{t\to\infty} \mathrm{diam}(\phi(t, 0, \mathcal{S})) > 0$ almost surely. Since the flow preserves the volume, this property certainly holds for sets with positive volume but one can expect that it also holds for certain other sets. In fact we conjecture that, as in the case of a nondegenerate isotropic Brownian flow, the theorem holds for any compact and connected ($\mathcal{F}_0$-measurable) set $\mathcal{S}$ which contains more than one point almost surely. In any case it is clear that for every such set almost surely either the left inequality in the theorem holds or $\liminf_{t\to\infty} \mathrm{diam}(\phi(t, 0, \mathcal{S})) = 0$.

4. Shear flows. As before we let $\mathcal{K} = \{k_1, \dots, k_N\} \subseteq \mathbf{R}^2 \backslash \{0\}$ be a finite set of modes, but this time we assume that $\dim(\mathrm{span}\, \mathcal{K}) = 1$. The corresponding flow ϕ is then called *shear flow*. We will see below that the rate of expansion of a bounded set $\mathcal{S}$ under a shear flow is strictly sublinear. Without loss of generality we assume that the second component of k is zero for every $k \in \mathcal{K}$. Then the stream function Ψ is of the form

$$\Psi(t, x) = \sum_{i=1}^{N} \big(a_i(t) \cos(r_i x_1) + b_i(t) \sin(r_i x_1)\big),$$

where r_i denotes the first component of k_i and x_1 is the first component of x. Hence by (3),

$$v_1(t,x) = 0,$$
$$v_2(t,x) = \sum_{i=1}^{N} r_i\big(-a_i(t)\sin(r_i x_1) + b_i(t)\cos(r_i x_1)\big),$$

that is,

$$\phi_1(t,0,x) = x_1,$$
$$\phi_2(t,0,x) = x_2 + \sum_{i=1}^{N} r_i\left(-\sin(r_i x_1)\int_0^t a_i(s)\,ds + \cos(r_i x_1)\int_0^t b_i(s)\,ds\right).$$

Since the integral from 0 to t of an Ornstein–Uhlenbeck process is almost surely asymptotically bounded by a (deterministic) constant times $(t\log\log t)^{1/2}$, the same is true for $\sup_{x\in\mathcal{S}}\|\phi(t,0,x)\|$ when $\mathcal{S}$ is a (random) subset of $\mathbf{R}^2$ which is almost surely bounded.

Acknowledgments. We thank Stanislav Molchanov and Tomasz Komorowski for valuable discussions.

REFERENCES

BAXENDALE, P. and HARRIS, T. (1986). Isotropic stochastic flows. *Ann. Probab.* **14** 1155–1179.

CARMONA, R. and CEROU, F. (1999). Transport by incompressible random velocity fields: simulations and mathematical conjectures. In *Stochastic Partial Differential Equations: Six Perspectives* (R. Carmona and B. Rozovskii, eds.) 153–181. Amer. Math. Soc. Providence, RI.

CARMONA, R., GRISHIN, A. and MOLCHANOV, S. (1996). Massively parallel simulations of motions in a Gaussian velocity field. In *Stochastic Modelling in Statistical Oceanography* (R. Adler, P. Müller and B.L. Rozovskii, eds.) 47–68. Birkhäuser, Boston.

CARMONA, R. and XU, L. (1997). Homogenization for time-dependent 2d incompressible Gaussian flows. *Ann. Appl. Probab.* **7** 265–279.

CRANSTON, M. (1991). Gradient estimates on manifolds using coupling. *J. Funct. Anal.* **99** 110–124.

CRANSTON, M., SCHEUTZOW, M. and STEINSALTZ, D. (1999). Linear expansion of isotropic Brownian flows. *Elect. Comm. Probab.* **4** 91–101.

CRANSTON, M., SCHEUTZOW, M. and STEINSALTZ, D. (2000). Linear bounds for stochastic dispersion. *Ann. Probab.* **28** 1852–1869.

FANNJIANG, A. and KOMOROWSKI, T. (1999). Turbulent diffusion in Markovian flows. *Ann. Appl. Probab.* **9** 591–610.

HALL, P. and HEYDE, C. C. (1980). *Martingale Limit Theory and its Application*. Academic Press, New York.

IKEDA, N. and WATANABE, S. (1981). *Stochastic Differential Equations and Diffusion Processes*. North-Holland, Amsterdam.

KESTEN, H. and PAPANICOLAOU, G. (1979). A limit theorem for turbulent diffusion. *Comm. Math. Physics* **65** 97–128.

LE JAN, Y. (1985). On isotropic Brownian motions. *Z. Wahrs. Verw. Gebiete* **70** 609–620.

LINDVALL, T. and ROGERS, L. C. G. (1986). Coupling of multidimensional diffusions by reflection. *Ann. Probab.* **14** 860–872.

MOLCHANOV, S. (1996). Topics in statistical oceanography. In *Stochastic Modelling in Statistical Oceanography* (R. Adler, P. Müller and B. L. Rozovskii, eds.) 343–380. Birkhäuser, Boston.

SCHEUTZOW, M. and STEINSALTZ, D. (2001). Chasing balls through martingale fields. *Ann. Probab.* To appear.

ZIRBEL, C. and ÇINLAR, E. (1996). Dispersion of particle systems by Brownian flows. *Adv. Appl. Probab.* **28** 53–74.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ROCHESTER
ROCHESTER, NEW YORK 14627
E-MAIL: cran@math.rochester.edu

FACHBEREICH 3, MA 7-5
TECHNISCHE UNIV. BERLIN
STRASSE DES 17. JUNI 136
10623 BERLIN
GERMANY
E-MAIL: ms@math.tu-berlin.de