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ASYMPTOTIC PROPERTIES OF CERTAIN ANISOTROPIC WALKS IN RANDOM MEDIA

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We discuss a class of anisotropic random walks in a random media on $\mathbb{Z}^d$, $d \geq 1$, which have reversible transition kernels when the environment is fixed. The aim is to derive a strong law of large numbers and a functional central limit theorem for this class of models. The technique of the environment viewed from the particle does not seem to apply well in this setting. Our approach is based on the technique of introducing certain times similar to the regeneration times in the work concerning random walks in i.i.d. random environment by Sznitman and Zerner. With the help of these times we are able to construct an ergodic Markov structure.

1. Introduction. There are many works investigating random motions in random media. The point of view of the "environment viewed from the particle" has played an important role in the progress made so far; see Papanicolaou and Varadhan [13], Kozlov [8], De Masi, Ferrari, Goldstein and Wick [2], Olla [12] and also the lectures of Sznitman [17]. Lawler showed in [9] the central limit theorem for driftless random walks in random environments by using this technique. This technique has mostly been successful when one can find an explicit invariant measure of the Markov chain of the environment viewed from the particle, which is absolutely continuous with respect to the static distribution of the environment, especially when this invariant measure is reversible.

In this article we study a class of anisotropic random walks in random media, which are reversible Markov chains when the environment is fixed, but for which the chain of the environment viewed from the particle has no obvious invariant measure absolutely continuous to the static measure. Paradoxically, we are able to apply a strategy, which has been used in the investigation of a genuinely nonreversible model: the i.i.d. random walks in random environment (cf. Sznitman and Zerner [18] and Sznitman [16]). The principal aim of the present work is to derive a strong law of large numbers with nonvanishing limiting velocity and a functional central limit theorem for the anisotropic random motion in random environment under consideration. Incidentally, let us mention that for the type of models we consider here, the question of the existence of an effective, nonvanishing velocity was asked by Lebowitz and Rost (see [10]) in their investigation of the Einstein relation.

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Let us describe our model in detail. First we denote with $\mathbb{B}^d$ the set of nearest neighbor bonds on $\mathbb{Z}^d$. The random environment is given through i.i.d. nonnegative random variables $\omega(b) \in \mathbb{I} \subset (0, \infty)$, $b \in \mathbb{B}^d$, with common distribution μ. Here $\mathbb{I}$ denotes a compact interval of $(0, \infty)$. A random environment $\omega = (\omega(b))_{b \in \mathbb{B}^d}$ is an element of the product space $\Omega := \mathbb{I}^{\mathbb{B}^d}$ endowed with the canonical product measure $\mathbb{P} = \mu^{\otimes \mathbb{B}^d}$ and the canonical product σ-algebra $\mathscr{A} = (\mathscr{B}(\mathbb{I}))^{\mathbb{B}^d}$, where $\mathscr{B}(\mathbb{I})$ denotes the σ-algebra of Borel subsets of $\mathbb{I}$.

In our model we have a nearest neighbor jump transition kernel $p_\omega(x, x+e)$, that is, $\sum_{|e'|=1} p_\omega(x, x+e') = 1$, where e' denotes unit vectors in $\mathbb{Z}^d$ and $|\cdot|$ the L^1-norm in $\mathbb{R}^d$. Further, we assume that the kernel fulfills the ellipticity condition,

$$p_\omega(x, x+e) \geq \kappa > 0 \qquad \text{for all unit vectors } e \in \mathbb{Z}^d,\ x \in \mathbb{Z}^d,\ \omega \in \Omega, \tag{1.1}$$

and it is reversible; that is, there exists a positive measure $(m_\omega(x))_{x \in \mathbb{Z}^d}$ such that

$$m_\omega(x) p_\omega(x, x+e) = m_\omega(x+e) p_\omega(x+e, x), \tag{1.2}$$

for all $\omega \in \Omega$, $x \in \mathbb{Z}^d$, $|e| = 1$. We also assume that $p_\omega(x, x+e)$ has the form

$$p_\omega(x, x+e) = f\Big((\omega(\{x, x+e'\}))_{|e'|=1}, e \Big), \tag{1.3}$$

for all $x \in \mathbb{Z}^d$ and unit vectors e. This means that the transition kernel $p_\omega(x, x+e)$ depends only on the value of ω for bonds connected to x, in the same way for all $x \in \mathbb{Z}^d$. This is a translation invariance assumption on the jump mechanism.

In addition, we assume there exists a nearest neighbor random walk on $\mathbb{Z}^d$ with jumps distributed according to the law $(q(e))_{|e|=1, e \in \mathbb{Z}^d}$, $q(e) \neq 0$ for all $|e| = 1$, such that

$$\lambda := \frac{1}{2} \Big\| \sum_e \big(e \ \log q(e)\big) \Big\| > 0 \quad \text{and} \quad \ell := \frac{1}{2\lambda} \sum_e e \ \log q(e) \in S^{d-1}, \tag{1.4}$$

with $\|\cdot\|$ denoting the L^2-norm in $\mathbb{R}^d$ and that there exist constants $0 < A < B$, such that

$$A e^{2\lambda \ell . x} \leq m_\omega(x) \leq B e^{2\lambda \ell . x} \qquad \text{for all } \omega \in \Omega, x \in \mathbb{Z}^d, \tag{1.5}$$

where $x . y$ always denotes the standard scalar product of x, $y \in \mathbb{R}^d$ throughout this article.

For instance, if we choose for given $\lambda > 0$ and $\ell \in S^{d-1}$,

$$p_\omega(x, x+e) = \frac{\omega(\{x, x+e\}) e^{\lambda \ell . e}}{\sum_{|e'|=1} \omega(\{x, x+e'\}) e^{\lambda \ell . e'}}, \tag{1.6}$$

then the conditions (1.1), (1.2), (1.3) and (1.5) hold for suitable choices of κ, A and B, provided $q(e) = \frac{e^{\lambda \ell . e}}{\sum_{e'} e^{\lambda \ell . e'}}$ and $m_\omega(x) = e^{2\lambda \ell . x} \sum_e \omega(\{x, x+e\}) e^{\lambda \ell . e} / \sum_{e'} e^{\lambda \ell . e'}$ (the last denominator is simply a matter of normalization).

Actually, (1.6) is a special case of a transition probability with the form

$$p_\omega(x, x+e) = \frac{\omega(\{x, x+e\})q(e)}{\sum_{|e'|=1} \omega(\{x, x+e'\})q(e')}, \tag{1.7}$$

and (1.7) fulfills all the conditions (1.1)–(1.5) for suitable choices of κ, A, B, the reversible measure for (1.7) being now

$$m_\omega(x) = e^{2\lambda\ell . x} \sum_e \omega(\{x, x+e\})q(e),$$

with λ and ℓ from (1.4).

With these assumptions over p_ω, the random walk in the random environment ω is the Markov chain $(X_n)_{n\geq 0}$ on $(\mathbb{Z}^d)^{\mathbb{N}}$, with state space $\mathbb{Z}^d$ and "quenched law" $\mathrm{P}_{x,\omega}$ for $x \in \mathbb{Z}^d$,

$$\begin{aligned} &\mathrm{P}_{x,\omega}[X_{n+1} = X_n + e | X_0, \dots, X_n] \overset{\mathrm{P}_{x,\omega}\text{-a.s.}}{=} p_\omega(X_n, X_n + e), \\ &\mathrm{P}_{x,\omega}[X_0 = x] = 1, \end{aligned} \tag{1.8}$$

where e denotes unit vectors in $\mathbb{Z}^d$. The "annealed law" P_x is then defined as the semidirect product on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$:

$$\mathrm{P}_x := \mathbb{P} \times \mathrm{P}_{x,\omega} \qquad \text{with } x \in \mathbb{Z}^d. \tag{1.9}$$

A degenerate case of the above model is discussed in the physics literature. It corresponds to the anisotropic random walk on the infinite percolation cluster; see pages 136–146 in Havlin and Bunde [6]. In this case the random variable $\omega(b)$ only takes the values 0 or 1. Although random walks an the infinite cluster have been discussed in the isotropic case (cf. [2]), we know of no mathematical reference in the anisotropic situation.

The main goal of this article is to show in Theorem 5.1 that

$$\frac{X_n}{n} \text{ converges } \mathrm{P}_0\text{-a.s. to a deterministic nondegenerate velocity } v.$$

Further, we prove in Theorem 5.3 that the process $B^n_\cdot$,

$$B^n_t = \frac{X_{[tn]} - [tn]v}{\sqrt{n}}, \qquad t \geq 0, \tag{1.10}$$

with $[t]$ denoting the integer part of $t \geq 0$, converges in law under the annealed measure P_0 to a d-dimensional Brownian motion with nondegenerate covariance matrix, as $n \to \infty$.

One special aspect of our work is that our results hold for arbitrarily small anisotropy strength λ. We do not need any Kalikow-like condition as for the i.i.d. random walks in random environment; see [7, 16, 18].

The strategy employed to derive these two theorems is to construct an embedded Markov chain structure under the annealed measure P_0, which has a "small state space" (cf. Corollary 3.6). The times τ_k, $k \geq 1$, defined in (3.12) and (3.26), play a central role here. In essence τ_k is the kth time, when the random walker comes to a new maximum in the direction ℓ and then never comes back below this level. The true definition is in fact more sophisticated (cf. Remark 3.2). The random variables consisting of $\tau_{k+1} - \tau_k$, $X_{\tau_{k+1}} - X_{\tau_k}$ and the value of some bonds connected to X_{τ_k}, $k \geq 1$, build a Markov chain, as shown in Corollary 3.6. In Theorem 3.8 the ergodicity of this Markov chain is shown. Let us mention that the above strategy is in the same spirit as the renewal structure attached to certain regeneration times τ_k for i.i.d. random walks in random environment model in [18] and [16]. However, unlike what happens for the i.i.d. random walks in random environment model, the times τ_k in our model do not yield a renewal structure, but rather lead to a Markov structure with a small state space; see Theorem 3.3 and Corollary 3.6. This comes from the fact that the transition kernel $p_\omega(x, x+e)$ depends on all bonds connected to x; therefore the jump probabilities $p_\omega(x, x+e)$ and $p_\omega(x+e, x+e+e')$ are not independent under $\mathbb{P}$.

Let us explain the organization of this article. In Section 2 we make full use of the ellipticity condition (1.1) and the reversibility assumption (1.2)–(1.5) on $(X_n)_{n\geq 0}$ under the quenched law $\mathrm{P}_{x,\omega}$ to derive a key estimate in Theorem 2.2. In particular, with the help of this estimate we prove that the random walk has a strict positive probability of never coming below its initial level (cf. Corollary 2.3) and at the end of Section 2 we show that $\mathrm{P}_{x,\omega}$-a.s. $(X_n)_{n\geq 0}$ tends to $+\infty$ in the direction ℓ. In Section 3 the times τ_k, $k \geq 1$, are introduced [cf. (3.12) and (3.26)] and the embedded Markov chain $(Y_n)_{n\geq 0}$ under the annealed measure P_0 is constructed in Corollary 3.6. Its ergodicity is then discussed in Theorem 3.8.

In Section 4 we use the key estimate of Theorem 2.2 to derive the integrability properties of X_{τ_1} and τ_1. Our main result is presented in Corollary 4.4. In Section 5, with the help of the embedded Markov chain $(Y_n)_{n\geq 0}$ constructed in Section 3 and the integrability property of τ_1 proved in Corollary 4.4, a strong law of large numbers for $(X_n)_{n\geq 0}$ under the annealed measure P_0 is proved in Theorem 5.1. Further we are able to prove a functional central limit theorem for the process $B^n_\cdot$ in Theorem 5.3.

2. Notation, reversible structure and a key estimate. In this section we use the ellipticity condition (1.1) and the specific reversibility assumption (1.2)–(1.5) on the quenched Markov chain (1.8) to show that the random walk has a positive probability of no-backtracking (cf. Corollary 2.3) and derive transience in direction ℓ (cf. Corollary 2.4). We first provide a uniform lower bound for the generalized principal Dirichlet eigenvalue in Theorem 2.1, which will be useful to prove our key estimate in Theorem 2.2.

Before doing so we introduce some further notations needed throughout this article; c and c_j, $j \in \mathbb{N}$ always stand for positive constants, which depend only on the quantities $(\kappa, d, A, B, q(\cdot))$, which are introduced in (1.1)–(1.5). We denote by

$(\theta_n)_{n\geq 0}$ the canonical shift on $(\mathbb{Z}^d)^{\mathbb{N}}$, and by $\mathscr{F}_n$, $n \geq 0$, the canonical filtration of $(X_n)_{n\geq 0}$, that is, $\mathscr{F}_n = \sigma\{X_0, \dots, X_n\}$ for $n \geq 0$.

The *exit time* T_U for $U \subset \mathbb{Z}^d$ is given by

$$T_U = \inf\{n \geq 0 : X_n \notin U\}, \tag{2.1}$$

and for $u \in \mathbb{R}$ we introduce

$$\begin{aligned} T_u &= \inf\{n \geq 0 : \ell\boldsymbol{.}(X_n - X_0) \geq u\}, \\ \tilde{T}_u &= \inf\{n \geq 0 : \ell\boldsymbol{.}(X_n - X_0) < u\}. \end{aligned} \tag{2.2}$$

Further we shall also need the *first backtracking time* defined through

$$D = \inf\{n \geq 0 : \ell\boldsymbol{.}X_n < \ell\boldsymbol{.}X_0\}. \tag{2.3}$$

2.1. *Principal Dirichlet eigenvalue.* Keeping in mind the reversible structure stated in (1.2)–(1.5), we introduce for each $\omega \in \Omega$ the scalar product on the space of functions $f : \mathbb{Z}^d \to \mathbb{R}$ and its associated norm,

$$(f, g)_{m_\omega} := \sum_{x \in \mathbb{Z}^d} m_\omega(x) f(x) g(x), \qquad \|f\|_{m_\omega} := \sqrt{(f, f)_{m_\omega}}, \tag{2.4}$$

for $f, g : \mathbb{Z}^d \to \mathbb{R}$.

For $\omega \in \Omega$, $U \subset \mathbb{Z}^d$ nonempty, we introduce $\Lambda_\omega(U)$:

$$\Lambda_\omega(U) := \inf\left\{ \frac{\mathcal{E}_{m_\omega}(f, f)}{\sum_x m_\omega(x) f(x)^2} : f \neq 0,\ f|_{U^c} = 0,\ f \in L^2(m_\omega) \right\}, \tag{2.5}$$

with the Dirichlet form

$$\mathcal{E}_{m_\omega}(f, g) = \frac{1}{2} \sum_{x, y} m_\omega(x) p_\omega(x, y) \big(f(x) - f(y)\big)\big(g(x) - g(y)\big), \qquad f, g \in L^2(m_\omega),$$

where for $x, y \in \mathbb{Z}^d$ we use the following convention:

$$p_\omega(x, y) := \begin{cases} p_\omega(x, x + e), & \text{for } y = x + e, \text{ with } |e| = 1, \\ 0, & \text{otherwise,} \end{cases}$$

and by $f|_{U^c}$ we mean the restriction of f to the complement U^c of $U \subset \mathbb{Z}^d$.

With a slight abuse of language, we refer to $\Lambda_\omega(U)$ as the principal Dirichlet eigenvalue attached to U; it is in fact the bottom of the spectrum of the bounded self-adjoint operator $1 - \mathrm{P}_{U,\omega}$ on $L^2(m_\omega)$, where $\mathrm{P}_{U,\omega}$ is defined through

$$\begin{aligned} &\mathrm{P}_{U,\omega} := \mathrm{P}^1_{U,\omega} \qquad \text{provided,} \\ &(\mathrm{P}^n_{U,\omega} f)(x) := \mathrm{E}_{x,\omega}[f(X_n), T_U > n] \qquad \text{for } n \in \mathbb{N},\ f : \mathbb{Z}^d \to \mathbb{R}. \end{aligned} \tag{2.6}$$

The next theorem provides a uniform lower bound for $\Lambda_\omega(U)$.

THEOREM 2.1.

$$\inf_{U,\omega\in\Omega} \Lambda_\omega(U) = \varepsilon > 0, \tag{2.7}$$

where U varies over the collection of nonempty subsets of $\mathbb{Z}^d$.
Consequently,

$$\left\| \mathrm{P}^n_{U,\omega} \right\|_{L^2(m_\omega)} \le e^{-n\gamma} \qquad \textit{with } \gamma = \log \frac{1}{1-\varepsilon}, \tag{2.8}$$

for all $U \subset \mathbb{Z}^d$ and all $\omega \in \Omega$.

PROOF. We begin with the proof of (2.7). The ellipticity condition (1.1) and assumption (1.5) imply that for $x, y \in \mathbb{Z}^d$,

$$m_\omega(x) p_\omega(x, y) \ge A\kappa \tilde{m}(x) q(x, y),$$

with

$$\tilde{m}(x) = e^{2\lambda \ell \cdot x} \quad \text{and} \quad q(x, y) = \begin{cases} q(e), & \text{for } y = x + e, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore $\Lambda_\omega(U) \ge \frac{A\kappa}{B} \tilde{\Lambda}(U)$, with

$$\begin{aligned} &\tilde{\Lambda}(U) \\ &:= \inf\left\{ \frac{\sum_{x,y} \tilde{m}(x) q(x, y)(f(x) - f(y))^2}{2\sum_x \tilde{m}(x) f^2(x)} : f \ne 0,\ f|_{U^c} = 0,\ f \in L^2(\tilde{m}) \right\}. \end{aligned} \tag{2.9}$$

So we only need to provide a positive lower bound in the context of the deterministic random walk with jump probability $(q(e))_{|e|=1}$. Further, because $\Lambda_\omega(\mathbb{Z}^d) = \inf_{U \ne \emptyset} \Lambda_\omega(U)$ and for $f \in L^2(m_\omega)$ we have $\frac{\mathcal{E}_{m_\omega}(f,f)}{(f,f)} = \lim_{U \uparrow \mathbb{Z}^d} \frac{\mathcal{E}_{m_\omega}(f1_U, f1_U)}{(f1_U, f1_U)}$, we see that $\Lambda_\omega(\mathbb{Z}^d) = \inf_{U \ne \emptyset,\ \text{finite}} \Lambda_\omega(U)$, hence we can assume without loss of generality that $\sup\{|\ell \cdot z| : z \in U\} < \infty$.

Let us denote the canonical law of this random walk starting in x by Q_x and its expectation value by $\mathrm{E}^{\mathrm{Q}}_x$. Because $2\lambda\ell \cdot (\sum_{|e|=1} e q(e)) = \sum_{j=1}^d (q(e_j) - q(-e_j))(\log q(e_j) - \log q(-e_j)) > 0$ [recall λ and ℓ are given in (1.4)], we can find $0 < c < 1$ and $\delta > 0$ small enough such that

$$\mathrm{E}^{\mathrm{Q}}_x[e^{-\delta\ell \cdot (X_1 - X_0)}] \le c < 1. \tag{2.10}$$

Defining $\eta = -\log c > 0$, we observe that $\exp\{-\delta\ell \cdot X_n + \eta n\}$ is a Q_x-supermartingale. The stopping theorem implies that

$$\mathrm{E}^{\mathrm{Q}}_x[\exp\{-\delta\ell \cdot (X_{T_U} - x) + \eta T_U\}] \le 1 \qquad \text{for all } x \in U. \tag{2.11}$$

Let $L := \sup\{|\ell \cdot (z - x)| : z \in U\} < \infty$, and since $-\delta\ell \cdot (X_{T_U} - x) \ge -\delta(L + 1)$ we find

$$\sup_{x \in U} \mathrm{E}^{\mathrm{Q}}_x[\exp\{-\delta(L + 1) + \eta T_U\}] \le 1,$$

which implies

$$\sup_{x\in U} \mathrm{E}_x^{\mathrm{Q}}[e^{\eta T_U}] \le e^{\delta\rho} \qquad \text{with } \rho = \sup_{x\in U}\{\ell . x\} - \inf_{x\in U}\{\ell . x\} + 1. \tag{2.12}$$

Notice also

$$\tilde{\Lambda}(U) = 1 - \sup\left\{\frac{(f, \mathrm{Q}_U f)_{\tilde{m}}}{(f, f)_{\tilde{m}}} : f \ne 0,\ f|_{U^c} = 0,\ f \in L^2(\tilde{m})\right\},$$

with the sub-Markov kernel Q_U defined through

$$\begin{aligned} &\mathrm{Q}_U = \mathrm{Q}_U^1 \qquad \text{provided,} \\ &(\mathrm{Q}_U^n f)(x) = \mathrm{E}_x^{\mathrm{Q}}[f(X_n), T_U > n], \qquad n \in \mathbb{N},\ f\colon \mathbb{Z}^d \to \mathbb{R}. \end{aligned} \tag{2.13}$$

We observe also $\mathrm{Q}_U^n = (\mathrm{Q}_U)^n$ and Q_U is a bounded self-adjoint operator on $L^2(\tilde{m})$ with respect to the canonical scalar product $(\cdot,\cdot)_{\tilde{m}}$ attached to $\tilde{m}$.

It now suffices to show that $\|\mathrm{Q}_U\|_{L^2(\tilde{m})} \le e^{-\eta/2}$ to prove (2.7). To show this we observe

$$\begin{aligned} \|\mathrm{Q}_U^n f\|^2_{L^2(\tilde{m})} &= \sum_{x\in U} \tilde{m}(x)(\mathrm{Q}_U^n f)^2(x) \\ &\overset{\text{Jensen}}{\le} (1, \mathrm{Q}_U^n f^2)_{\tilde{m}} = (\mathrm{Q}_U^n 1, f^2)_{\tilde{m}} \\ &\overset{(2.13)}{=} \textstyle\sum_y \tilde{m}(y)\mathrm{Q}_y[T_U > n] f^2(y) \le e^{-\eta n} e^{\delta\rho} \|f\|^2_{L^2(\tilde{m})}, \end{aligned} \tag{2.14}$$

where the Chebychev inequality $\mathrm{Q}_y[T_U > n] \le \mathrm{E}_y^{\mathrm{Q}}[e^{\eta T_U - \eta n}] \overset{(2.12)}{\le} e^{-\eta n} e^{\delta\rho}$ is used in the last step. Taking the nth root, it follows from Theorem VI.6, page 192 in [14], that $\|\mathrm{Q}_U\|_{L^2(\tilde{m})} \le e^{-\eta/2}$, and hence (2.7) follows.

Inequality (2.8) is an immediate consequence of the fact that $\Lambda_\omega(U) = 1 - \|\mathrm{P}_{U,\omega}\|_{L^2(m_\omega)}$ and $\mathrm{P}_{U,\omega}^n = (\mathrm{P}_{U,\omega})^n$. $\square$

2.2. *Key estimate.* Thanks to Theorem 2.1 we can prove the key estimate of this section.

THEOREM 2.2. *There exist constants $c_1 > 0$ and $c_2 > 0$ such that for $m \in \mathbb{N}$,*

$$\sup_{x\in\mathbb{Z}^d,\,\omega\in\Omega} \mathrm{P}_{x,\omega}[\tilde{T}_{-2^m} < T_{2^m}] \le c_1 e^{-c_2 2^m}. \tag{2.15}$$

PROOF. Let $U \subset \mathbb{Z}^d$ be finite, then (2.6) and (2.8) imply that for all $\omega \in \Omega$, $x \in U$,

$$\begin{aligned} m_\omega(x)\mathrm{P}_{x,\omega}[T_U > n] &= (1_{\{x\}}, \mathrm{P}_{U,\omega}^n 1_U)_{L^2(m_\omega)} \\ &\le \|1_{\{x\}}\|_{L^2(m_\omega)} \cdot \|1_U\|_{L^2(m_\omega)} \cdot e^{-\gamma n} \\ &= \sqrt{m_\omega(x)} \cdot \|1_U\|_{L^2(m_\omega)} \cdot e^{-\gamma n}. \end{aligned} \tag{2.16}$$

Using the assumption (1.5), $\mathrm{P}_{x,\omega}[T_U > n]$ can be estimated from above by

$$\begin{aligned}\mathrm{P}_{x,\omega}[T_U > n] &\le \|1_U\|_{L^2(m_\omega)} \cdot e^{-\gamma n}/\sqrt{m_\omega(x)}\\ &\le \frac{1}{\sqrt{A}} e^{-\lambda \ell\boldsymbol{.} x} \|1_U\|_{L^2(m_\omega)} e^{-\gamma n}.\end{aligned} \tag{2.17}$$

Now let U be a box centered at x with width L in the ℓ direction and size L^2 in the directions normal to ℓ, that is, with a rotation R of space $\mathbb{R}^d$ such that $R(e_1) = \ell$:

$$U := \left\{ z \in \mathbb{Z}^d : |(z-x)\boldsymbol{.}\ell| < \frac{L}{2}, \sup_{j\ge 2} |R(e_j)\boldsymbol{.}(z-x)| < \frac{L^2}{2} \right\}. \tag{2.18}$$

With $r_{\max} := \sup\{\ell\boldsymbol{.}z : z \in U\} < \infty$, we see from (1.5) that for $L \ge 1$,

$$\|1_U\|_{L^2(m_\omega)} \le c_3 L^d e^{\lambda r_{\max}}.$$

Thereafter for

$$n \ge \frac{\lambda L}{\gamma} \qquad \text{[recall that } \gamma \text{ is defined in (2.8)]} \tag{2.19}$$

it follows from (2.17) that

$$\begin{aligned}\mathrm{P}_{x,\omega}[T_U > n] &\le \frac{c_3}{\sqrt{A}} e^{-\lambda \ell\boldsymbol{.} x} L^d e^{\lambda r_{\max}} e^{-\gamma n} \le \frac{c_3}{\sqrt{A}} L^d e^{-(\lambda/2)L}\\ &\le c_4 e^{-(\lambda/4)L}.\end{aligned} \tag{2.20}$$

The boundary of U is defined through

$$\partial U = \{ z \notin U : \exists y \in U, |z-y| = 1 \}, \tag{2.21}$$

with $|\cdot|$ denoting the L^1-norm on $\mathbb{R}^d$. Now we divide it into $\partial U = \partial_+ U \cup \partial_- U \cup \partial_0 U$, with

$$\begin{aligned}\partial_+ U &:= \left\{ z \in \partial U : \ell\boldsymbol{.}(z-x) \ge \frac{L}{2} \right\},\\ \partial_- U &:= \left\{ z \in \partial U : \ell\boldsymbol{.}(z-x) \le -\frac{L}{2} \right\},\\ \partial_0 U &:= \partial U \setminus (\partial_+ U \cup \partial_- U)\end{aligned} \tag{2.22}$$

and setting $L = 2^{m+1}$ in the above definition of U, we observe that

$$\mathrm{P}_{x,\omega}[\tilde{T}_{-2^m} < T_{2^m}] \le \mathrm{P}_{x,\omega}\left[T_U > \frac{\lambda L}{\gamma}\right] + \mathrm{P}_{x,\omega}\left[T_U \le \frac{\lambda L}{\gamma}, X_{T_U} \notin \partial_+ U\right]. \tag{2.23}$$

Using (2.20), the first term on the right-hand side of (2.23) can be estimated by

$$\mathrm{P}_{x,\omega}\left[T_U > \frac{\lambda L}{\gamma}\right] \le c_4 e^{-(\lambda/4)L}. \tag{2.24}$$

To estimate the second term, we use Carne's inequality for reversible Markov chains (cf. [1], Theorem 1), (there is a small typo in the paper: x and y are interchanged on the right-hand side of the inequality.):

$$\mathrm{P}_{x,\omega}[X_k = y] \le 2\sqrt{\frac{m_\omega(y)}{m_\omega(x)}}\exp\left\{-\frac{|x-y|^2}{2k}\right\}, \qquad x, y \in \mathbb{Z}^d,\ \omega\in\Omega, \tag{2.25}$$

with $|\cdot|$ denoting the L^1-norm on $\mathbb{R}^d$.

Because $|x-y|^2 \ge \|x-y\|^2$, the second term can now be estimated through

$$\begin{aligned}\mathrm{P}_{x,\omega}\left[T_U \le \frac{\lambda L}{\gamma},\ X_{T_U} \notin \partial_+ U\right] &\le \sum_{k\le \lambda L/\gamma}\ \sum_{y\in\partial_0 U\cup\partial_- U} \mathrm{P}_{x,\omega}[X_k = y]\\ &\le \frac{2\lambda L}{\gamma}\left[\sum_{y\in\partial_0 U}\sqrt{\frac{m_\omega(y)}{m_\omega(x)}}\exp\left(-\frac{\gamma\|x-y\|^2}{2\lambda L}\right)\right.\\ &\qquad\left. + \sum_{y\in\partial_- U}\sqrt{\frac{m_\omega(y)}{m_\omega(x)}}\exp\left(-\frac{\gamma\|x-y\|^2}{2\lambda L}\right)\right].\end{aligned}\tag{2.26}$$

By using (1.5) again the first sum on the right-hand side of (2.26) can be estimated by

$$\begin{aligned}&\sum_{y\in\partial_0 U}\sqrt{\frac{m_\omega(y)}{m_\omega(x)}}\exp\left(-\frac{\gamma\|x-y\|^2}{2\lambda L}\right)\\ &\qquad\le c_5 L^{2d-3}\sqrt{\frac{B}{A}}e^{\lambda L}\exp\left(-\frac{L^4\gamma}{8\lambda L}\right) \le c_6\exp(-c_7 L^3),\end{aligned}\tag{2.27}$$

and the second sum by

$$\begin{aligned}\sum_{y\in\partial_- U}\sqrt{\frac{m_\omega(y)}{m_\omega(x)}}\exp\left(-\frac{\gamma\|x-y\|^2}{2\lambda L}\right) &\le \sum_{y\in\partial_- U}\sqrt{\frac{B}{A}}e^{-c_8\lambda L}e^{-c_9 L}\\ &\le c_{10}L^{2(d-1)}e^{-(c_9+\lambda c_8)L}\\ &\le c_{11}e^{-c_{12}L}.\end{aligned}\tag{2.28}$$

Putting the above inequalities together,

$$\mathrm{P}_{x,\omega}[\tilde{T}_{-2^m} < T_{2^m}] \le c_1 e^{-c_2 2^m} \qquad \text{for all } x\in\mathbb{Z}^d,\ \omega\in\Omega,\ m\ge 0. \tag{2.29}$$ □

2.3. *Transience.* The next corollary of Theorem 2.2 will be useful in Sections 3 and 4.

COROLLARY 2.3. *There exists $c_{13} > 0$ such that for all $x \in \mathbb{Z}^d$ and $\omega\in\Omega$,*

$$\mathrm{P}_{x,\omega}[D = \infty] \ge c_{13} > 0, \tag{2.30}$$

where D is the first backtracking time defined in (2.3).

PROOF. With the notation $U_x^m := \{z \in \mathbb{Z}^d : |\ell\boldsymbol{.}(z-x)| < 2^m\}$, the ellipticity condition (1.1) and the strong Markov property imply that $\mathrm{P}_{y,\omega}[T_{U_x^m} = \infty] = 0$ for all $y \in U_x^m$, $\omega \in \Omega$. Therefore (2.15) implies

$$\inf_{x,\omega} \mathrm{P}_{x,\omega}[\tilde{T}_{-2^m} > T_{2^m}] \geq 1 - c_1 e^{-c_2 2^m}. \tag{2.31}$$

Let $m := \inf\{k \geq 1 : 1 > c_1 e^{-c_2 2^k}\}$; we claim for any $n \geq m+1$, $x \in \mathbb{Z}^d$, $\omega \in \Omega$,

$$\mathrm{P}_{x,\omega}[\tilde{T}_{-2^m} > T_{2^n - 2^m}] \geq \prod_{k=m}^{n-1} (1 - c_1 e^{-c_2 2^k}). \tag{2.32}$$

We show this by induction. The case $n = m+1$ is immediate from (2.31). The step $n \to n+1$ follows easily by the strong Markov property and (2.31):

$$\begin{aligned} &\mathrm{P}_{x,\omega}[\tilde{T}_{-2^m} > T_{2^{n+1} - 2^m}] \\ &\quad \geq \mathrm{E}_{x,\omega}\big[\tilde{T}_{-2^m} > T_{2^n - 2^m}, \mathrm{P}_{X_{T_{2^n - 2^m}},\omega}[\tilde{T}_{-2^n} > T_{2^n}]\big] \\ &\quad \geq \mathrm{P}_{x,\omega}[\tilde{T}_{-2^m} > T_{2^n - 2^m}](1 - c_1 e^{-c_2 2^n}). \end{aligned}$$

From (2.32) it is clear that for all $x \in \mathbb{Z}^d$, $\omega \in \Omega$:

$$\mathrm{P}_{x,\omega}[\tilde{T}_{-2^m} > T_{2^n - 2^m}] \geq \prod_{k \geq m} (1 - c_1 e^{-c_2 2^k}) > 0,$$

and hence

$$\mathrm{P}_{x,\omega}[\tilde{T}_{-2^m} > T_{2^k - 2^m} \text{ for all } k > m] \geq \prod_{k \geq m} (1 - c_1 e^{-c_2 2^k}) > 0.$$

Therefore by using ellipticity condition (1.1) and the strong Markov property again we find that $\mathrm{P}_{x,\omega}$-a.s.,

$$\begin{aligned} \mathrm{P}_{x,\omega}[D = \infty] &\geq \kappa^{c 2^m} \mathrm{E}_{x,\omega}\big[\mathrm{P}_{X_{T_{2^m}},\omega}[\tilde{T}_{-2^m} > T_{2^k - 2^m} \text{ for all } k > m]\big] \\ &\geq \kappa^{c 2^m} \prod_{k \geq m} (1 - c_1 e^{-c_2 2^k}) > 0 \quad \text{ for all } x \in \mathbb{Z}^d,\ \omega \in \Omega. \end{aligned}$$

This completes the proof. $\square$

As an application of the above corollary we prove the transience of X_n in the direction ℓ under the quenched law $\mathrm{P}_{x,\omega}$.

COROLLARY 2.4. *The random walk is transient and* $\mathrm{P}_{x,\omega}[\lim_n \ell\boldsymbol{.}X_n = \infty] = 1$, *for all* $x \in \mathbb{Z}^d$, $\omega \in \Omega$.

PROOF. At first we show

$$P_{x,\omega}\big[\inf_n \ell\centerdot X_n = -\infty\big] = 0 \qquad \text{for all } x \in \mathbb{Z}^d,\ \omega \in \Omega. \tag{2.33}$$

Indeed with

$$D_1 := D \quad \text{and} \quad D_{m+1} := D \circ \theta_{D_m} + D_m, \qquad m \ge 1,$$

we find

$$\begin{aligned}
\sup_{x\in\mathbb{Z}^d} P_{x,\omega}\big[\inf_n \ell\centerdot X_n = -\infty\big] &\le \sup_{x\in\mathbb{Z}^d} P_{x,\omega}[D_m < \infty,\ \forall m] \\
&\le \sup_{x\in\mathbb{Z}^d} E_{x,\omega}\big[D_1 < \infty, P_{X_{D_1},\omega}[D_m < \infty,\ \forall m]\big] \\
&\le \sup_{x\in\mathbb{Z}^d} P_{x,\omega}[D_1 < \infty] \sup_{y\in\mathbb{Z}^d} P_{y,\omega}[D_m < \infty,\ \forall m] \\
&\le (1 - c_{13}) \sup_{y\in\mathbb{Z}^d} P_{y,\omega}[D_m < \infty,\ \forall m],
\end{aligned}$$

where we used (2.30) in the last step. Because $1 - c_{13} < 1$, it follows that $\sup_x P_{x,\omega}[D_m < \infty,\ \forall m] = 0$, and hence (2.33).

Now we claim that for $h > 0$ and $u \in \mathbb{R}$,

$$P_{x,\omega}\text{-a.s.}, \quad \{\ell\centerdot(X_n - x) < u \text{ i.o.}\} \subset \{\ell\centerdot(X_n - x) < u - h \text{ i.o.}\}. \tag{2.34}$$

To verify this, we observe that from the ellipticity condition (1.1) there exists a large enough integer $N > 0$ and $c > 0$, such that

$$P_{x,\omega}[\tilde{T}_{-h} \le N] \ge c \qquad \text{for all } \omega \in \Omega,\ x \in \mathbb{Z}^d. \tag{2.35}$$

Then we define a sequence of auxiliary stopping-times $(\tilde{V}_k)_{k\ge 0}$,

$$\begin{aligned}
&\tilde{V}_0 := 0, \qquad \tilde{V}_1 := \inf\{n \ge 0 : \ell\centerdot(X_n - x) < u\}, \\
&\tilde{V}_{k+1} := \tilde{V}_1 \circ \theta_{\tilde{V}_k + N} + \tilde{V}_k + N \le \infty \qquad \text{for } k \ge 1,
\end{aligned}$$

and let $G_k = \{\tilde{V}_k < \infty\}$, $1_{H_k} = 1_{\{\tilde{T}_{-h} \le N\}} \circ \theta_{\tilde{V}_k}$. We observe that $G_k \in \mathcal{F}_{\tilde{V}_k}$ and $H_k \in \mathcal{F}_{\tilde{V}_{k+1}}$. Using the strong Markov property and (2.35) we find

$$P_{x,\omega}[H_k | \mathcal{F}_{\tilde{V}_k}] \ge c 1_{G_k} \qquad \text{for all } x \in \mathbb{Z}^d,\ \omega \in \Omega,\ k \ge 1. \tag{2.36}$$

Therefore it follows from Borel–Cantelli's second lemma (cf. [4], page 240) that

$$P_{x,\omega}\text{-a.s.}, \quad \sum_{k\ge 1} 1_{H_k} = \infty \quad \text{on } \Big\{\sum_{k\ge 1} 1_{G_k} = \infty\Big\}, \tag{2.37}$$

which implies (2.34).

As an immediate consequence of (2.34) we see that for $u' \in \mathbb{R}$, $P_{x,\omega}$-a.s.,

$$\{\ell\centerdot X_n < u' \text{ f.o.}\} \subset \bigcap_{h\in\mathbb{N}} \{\ell\centerdot X_n < u' + h \text{ f.o.}\} = \{\lim \ell\centerdot X_n = \infty\}. \tag{2.38}$$

Due to (2.33) we have $\mathrm{P}_{x,\omega}[\inf \ell \centerdot X_n > -\infty] = 1$, and since $\{\inf \ell \centerdot X_n > -\infty\} \subset \bigcup_{u' \in \mathbb{Z}} \{\ell \centerdot X_n < u' \quad \text{f.o.}\}$, it follows from (2.38) that

$$\mathrm{P}_{x,\omega}[\lim \ell \centerdot X_n = \infty] = 1. \qquad \square$$

3. Embedded Markov chain and ergodicity. In this section we will define the regeneration times τ_k, $k \geq 1$, introduce the resulting Markov chain under the annealed measure P_0 and then show that this Markov chain has an invariant probability measure, with which the chain is ergodic.

3.1. *The first no-backtracking time* τ_1. First let us introduce some further notations. With $t_x \colon \Omega \to \Omega$, $x \in \mathbb{Z}^d$, we denote the spatial shift operator

$$(t_x\omega)(\{y, z\}) := \omega(\{y + x, z + x\}) \qquad \text{with } \{y, z\} \in \mathbb{B}^d. \tag{3.1}$$

Let us also denote by $\mathscr{E}$ the set of unit vectors in $\mathbb{Z}^d$, which maximize $\{\ell \centerdot e\}$ and fix one such vector from $\mathscr{E}$; call it $\tilde{e}$:

$$\begin{aligned} &\mathscr{E} := \{e \in \mathbb{Z}^d : |e| = 1,\ \ell \centerdot e = \max\{\ell \centerdot e' : e' \in \mathbb{Z}^d,\ |e'| = 1\}\}, \\ &\tilde{e} \in \mathscr{E} \text{ fixed.} \end{aligned} \tag{3.2}$$

With the help of this $\tilde{e}$ we are able to introduce the set of *maximizing bonds* containing the point $x - \tilde{e}$:

$$\mathcal{B}^x := \{b \in \mathbb{B}^d : b = \{x - \tilde{e}, x - \tilde{e} + e\}, e \in \mathscr{E}\} \tag{3.3}$$

and separate $\mathbb{B}^d$ into two subsets, $\mathcal{R}^x$ and $\mathcal{L}^x$ ($\mathcal{R}$ and $\mathcal{L}$, respectively, stand for "right" and "left" of the point $x \in \mathbb{Z}^d$):

$$\begin{aligned} &\mathcal{R}^x := \{\{y, z\} \in \mathbb{B}^d : \max(\ell \centerdot z, \ell \centerdot y) \geq \ell \centerdot x\}, \\ &\mathcal{L}^x := (\mathbb{B}^d \setminus \mathcal{R}^x) \cup \mathcal{B}^x, \end{aligned} \tag{3.4}$$

so that

$$\mathcal{R}^x \cap \mathcal{L}^x = \mathcal{B}^x. \tag{3.5}$$

We depict $\mathcal{L}^x$ and $\mathcal{R}^x$ for $d = 2$ in Figure 1, where solid lines are bonds in $\mathcal{L}^x$, dashed lines are bonds in $\mathcal{R}^x$ and the two thick lines are bonds in $\mathcal{B}^x$.

Further, we introduce two sequences of $(\mathscr{F}_n)_{n\geq 0}$-stopping times S_k, $k \geq 0$ and R_k, $k \geq 1$, and a sequence of successive maxima in the direction $\ell \in \mathbb{R}^d$, M_k, $k \geq 0$ [we recall the definition of D in (2.3)]:

$$\begin{aligned} &S_0 := 0, \qquad M_0 := \ell \centerdot X_0, \\ &S_1 := \inf\{n \geq 2 : X_n - X_{n-1} = \tilde{e}; X_{n-1} - X_{n-2} = \tilde{e}; \\ &\qquad\qquad\qquad \ell \centerdot X_m \leq \ell \centerdot X_{n-2}, \forall m \leq n - 2\}, \\ &R_1 := D \circ \theta_{S_1} + S_1, \\ &M_1 := \sup\{\ell \centerdot X_m : 0 \leq m \leq R_1\} \end{aligned} \tag{3.6}$$

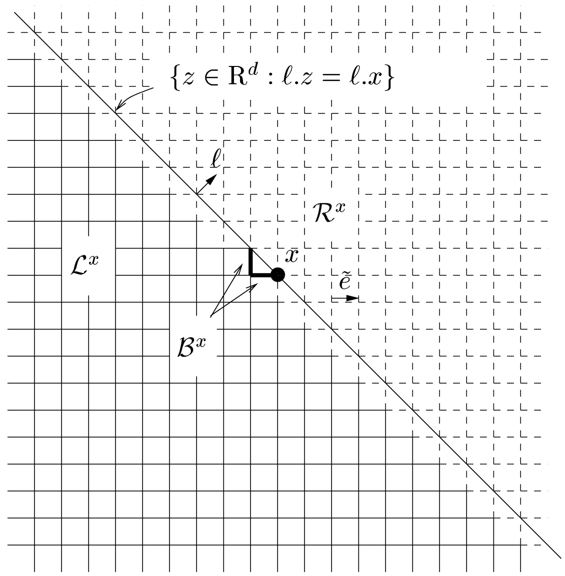

FIG. 1. $\mathcal{L}^x$ *and* $\mathcal{R}^x$.

and inductively for $k \geq 1$,

$$
\begin{aligned}
S_{k+1} &:= \inf\{n \geq R_k : X_n - X_{n-1} = \tilde{e}; X_{n-1} - X_{n-2} = \tilde{e}; \\
&\qquad\qquad \ell.X_m \leq \ell.X_{n-2},\ \forall\, m \leq n-2\}, \\
R_{k+1} &:= D \circ \theta_{S_{k+1}} + S_{k+1}, \\
M_{k+1} &:= \sup\{\ell.X_m : 0 \leq m \leq R_{k+1}\}.
\end{aligned} \tag{3.7}
$$

Clearly we have $0 = S_0 < S_1 \leq R_1 \leq S_2 \leq \cdots \leq \infty$, and the inequalities are strict if the left member is finite.

Now let us introduce

$$K := \inf\{k \geq 1 : S_k < \infty,\ R_k = \infty\}. \tag{3.8}$$

Before defining τ_1 as S_K, we first prove the finiteness of K.

LEMMA 3.1.

$$\mathrm{P}_{x,\omega}[K < \infty] = 1 \qquad \textit{for all } x \in \mathbb{Z}^d,\ \omega \in \Omega. \tag{3.9}$$

PROOF. First we show $\mathrm{P}_{x,\omega}[S_1 < \infty] = 1$, for all $x \in \mathbb{Z}^d$, $\omega \in \Omega$. To this end we introduce a sequence of auxiliary $(\mathcal{F}_n)_{n\geq 0}$-stopping times $\tilde{S}_k$, $k \geq 0$,

$$
\begin{aligned}
\tilde{S}_0 &= 0, \\
\tilde{S}_{k+1} &= \inf\{n \geq \tilde{S}_k + 2 : \ell.X_m \leq \ell.X_n,\ \forall\, m \leq n\}.
\end{aligned}
$$

In words, $\tilde{S}_{k+1}$ is the first time, at least two steps later than $\tilde{S}_k$, when the walk reaches a new maximum.

Because from Corollary 2.4 we have $P_{x,\omega}$-a.s. $\ell.X_n \xrightarrow{n\to\infty} \infty$, it follows that $P_{x,\omega}$-a.s. $\tilde{S}_k < \infty$ and $\tilde{S}_k \xrightarrow{k\to\infty} \infty$, for all $x \in \mathbb{Z}^d$, $\omega \in \Omega$.

We prove now by induction that there exists a constant $c \in (0, 1)$ such that

$$P_{x,\omega}[S_1 > \tilde{S}_k] \le c^k \qquad \text{for all } k, x \in \mathbb{Z}^d,\ \omega \in \Omega, \tag{3.10}$$

which implies by the Borel–Cantelli lemma immediately that

$$P_{x,\omega}[S_1 = \infty] = 0 \qquad \text{for all } x \in \mathbb{Z}^d,\ \omega \in \Omega. \tag{3.11}$$

For $k = 0$, (3.10) is immediate. Assume then (3.10) up to k. Because of (1.1) there exists a $c > 0$ such that $\sup_{y\in\mathbb{Z}^d,\,\omega\in\Omega} P_{y,\omega}[(X_1 - X_0, X_2 - X_1) \neq (\tilde{e}, \tilde{e})\}] \le c < 1$. Using the strong Markov property we get

$$\begin{aligned} P_{x,\omega}[S_1 > \tilde{S}_{k+1}] &\le E_{x,\omega}[S_1 > \tilde{S}_k; (X_{\tilde{S}_k+1} - X_{\tilde{S}_k}, X_{\tilde{S}_k+2} - X_{\tilde{S}_k+1}) \neq (\tilde{e}, \tilde{e})] \\ &= E_{x,\omega}\big[S_1 > \tilde{S}_k, P_{X_{\tilde{S}_k},\omega}[(X_1 - X_0, X_2 - X_1) \neq (\tilde{e}, \tilde{e})]\big] \\ &\le c P_{x,\omega}[S_1 > \tilde{S}_k] \le c^{k+1}. \end{aligned}$$

The claim (3.10) follows.

Now we return to the proof of finiteness of K. By (2.30), $\sup_{y,\omega} P_{y,\omega}[D < \infty] \le 1 - c_{13} < 1$, therefore for $k \ge 1$,

$$\begin{aligned} P_{x,\omega}[R_k < \infty] &= E_{x,\omega}\big[S_k < \infty, P_{X_{S_k},\omega}[D < \infty]\big] \\ &\le (1 - c_{13}) P_{x,\omega}[S_k < \infty] \\ &\le (1 - c_{13}) P_{x,\omega}[R_{k-1} < \infty], \end{aligned}$$

with the convention $R_0 = 0$. By induction it is $P_{x,\omega}[R_k < \infty] \le (1 - c_{13})^k$, for all $x \in \mathbb{Z}^d$, $\omega \in \Omega$, from which we deduce that $P_{x,\omega}$-a.s. $\sum_{k\ge1} 1_{\{R_k<\infty\}} < \infty$, for all $x \in \mathbb{Z}^d$, $\omega \in \Omega$. It is only possible when

$$P_{x,\omega}[K < \infty] = 1 \qquad \text{for all } x \in \mathbb{Z}^d,\ \omega \in \Omega. \qquad \square$$

Now we are ready to define

$$\tau_1 := S_K, \tag{3.12}$$

and certainly we have

$$P_{x,\omega}[\tau_1 < \infty] = 1 \qquad \text{for all } x \in \mathbb{Z}^d,\ \omega \in \Omega. \tag{3.13}$$

Let us give the meaning of τ_1: The random variable τ_1, when finite, is on the one hand the first time n, at which $\ell.X_{n-2}$ reaches a maximum and the next two steps have increment $\tilde{e} \in \mathscr{E}$; that is, $\ell.X_{\tau_1-2} \ge \ell.X_m$ for all $m \le \tau_1 - 2$, and $X_{\tau_1} - X_{\tau_1-1} = \tilde{e}$, $X_{\tau_1-1} - X_{\tau_1-2} = \tilde{e}$. On the other hand it is a time such that after τ_1, $\ell.X_n$ never becomes smaller than $\ell.X_{\tau_1}$.

REMARK 3.2. In the definition of S_k, $k \geq 1$, we chose quite artificially that the random walk $(X_n)_{n\geq 0}$ has increments $\tilde{e}$ in the previous two steps before S_k. Indeed, we can also choose any number of steps larger than two, and this will not affect our later discussion, as the proof of Theorem 4.3 shows.

Loosely speaking, we want to reduce the common dependency of the bonds involved before and after time τ_1 to only finitely many bonds, namely to $\{b \in \mathcal{B}^{X_{\tau_1}}\}$ [recall (3.3) for the definition of $\mathcal{B}^x$]. To achieve this we need that the walker perform at least two steps in the direction $\tilde{e} \in \mathcal{E}$ just before time τ_1. This reduction of dependency is essential to the proof of Theorem 3.3.

Before going to the key result of this section, let us introduce some further notations used in the remainder of this article. Recall the definition of $\mathcal{E}$, $\tilde{e}$ in (3.2) and that $\mathbb{I} \subset \mathbb{R}_+$ is the compact interval given above (1.1). We introduce, for each $x \in \mathbb{Z}^d$,

$$a_x := \big(\omega(\{x-\tilde{e}, x-\tilde{e}+e\})\big)_{e\in\mathcal{E}} = \big(\omega(b)\big)_{b\in\mathcal{B}^x} \in \mathbb{I}^{\mathcal{E}}, \tag{3.14}$$

and for $a \in \mathbb{I}^{\mathcal{E}}$,

$$\mathbb{P}_x^a := \delta_a\Big(\big(\omega(\{x-\tilde{e}, x-\tilde{e}+e\})\big)_{e\in\mathcal{E}}\Big) \otimes \int_{b\in(\mathbb{B}^d\setminus\mathcal{B}^x)} \otimes\, d\mu\big(\omega(b)\big) \tag{3.15}$$

as well as for the annealed measure

$$\mathrm{P}_x^a = \mathbb{P}_x^a \times \mathrm{P}_{x,\omega}. \tag{3.16}$$

We also need the σ-algebra $\mathcal{G}_1$ on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$, describing the history of path and environment involved before τ_1:

$$\mathcal{G}_1 := \sigma\big\{\tau_1, (X_{\tau_1\wedge m})_{m\geq 0}; \{\omega(b) : b \in \mathcal{L}^{X_{\tau_1}}\}\big\}; \tag{3.17}$$

that is, $\mathcal{G}_1$ is generated by the sets

$$\{\tau_1 = m\} \cap \{X_{\tau_1} = x\} \cap A, \tag{3.18}$$

with $m \geq 0$, $x \in \mathbb{Z}^d$, $A \in \sigma\{\omega(b) : b \in \mathcal{L}^x\} \otimes \mathcal{F}_m$ and

$$\{\tau_1 = \infty\} \cap A \qquad \text{with } A \in \mathcal{A} \otimes \mathcal{F}_\infty. \tag{3.19}$$

[Recall $\mathcal{A}$ is defined above (1.1).]

The key step in the study of the embedded Markov chain structure mentioned in Section 1 is now the following.

THEOREM 3.3. *Let f, g, h be bounded and respectively $\sigma\{X_n : n \geq 0\}$-, $\sigma\{\omega(b) : b \in \mathcal{R}^0\}$- and $\mathcal{G}_1$-measurable functions, then for $a \in \mathbb{I}^{\mathcal{E}}$:*

$$\mathrm{E}_0^a[f(X_{\tau_1+\cdot} - X_{\tau_1})\, g \circ t_{X_{\tau_1}} h] = \mathrm{E}_0^a\big[h \mathrm{E}_0^{a_{X_{\tau_1}}}[fg \mid D = \infty]\big], \tag{3.20}$$

where t_x is the spatial shift operator introduced in (3.1).

PROOF. The left-hand side of (3.20) is

$$\mathrm{E}_0^a[f(X_{\tau_1+\cdot}-X_{\tau_1})\, g\circ t_{X_{\tau_1}}\, h]$$

$$(3.21)\qquad \begin{aligned}&=\sum_{k\geq 1, x\in\mathbb{Z}^d}\mathrm{E}_0^a[f(X_{\tau_1+\cdot}-X_{\tau_1})\, g\circ t_{X_{\tau_1}} h, S_k<\infty,\ R_k=\infty,\ X_{S_k}=x]\\ &=\sum_{k,x}\mathbb{E}_0^a\big[\mathrm{E}_{0,\omega}[f(X_{S_k+\cdot}-x)h, S_k<\infty,\ R_k=\infty,\ X_{S_k}=x]g\circ t_x\big].\end{aligned}$$

Observe that on the event $\{\tau_1=S_k\}\cap\{X_{\tau_1}=x\}$, there exists a bounded $\sigma\{\omega(b): b\in\mathcal{L}^x\}\otimes\mathcal{F}_{S_k}$-measurable variable $h_{k,x}$, which coincides with h. Indeed, from the definition of $\mathcal{G}_1$ in (3.18), by applying the monotone class theorem (cf. [4], page 280), on any set $\{\tau_1=m\}\cap\{X_{\tau_1}=x\}$ there exists $\tilde{h}_{m,x}$ which is bounded $\sigma\{\omega(b): b\in\mathcal{L}^x\}\otimes\mathcal{F}_m$-measurable and coincides with h. Now we can define

$$h_{k,x}:=\sum_{m\geq 0}\tilde{h}_{m,x}1_{\{S_k=m\}},$$

so that $h_{k,x}$ is $\sigma\{\omega(b): b\in\mathcal{L}^x\}\otimes\mathcal{F}_{S_k}$-measurable, and coincides with h on $\{\tau_1=S_k\}\cap\{X_{\tau_1}=x\}$.

As a result, the rightmost side of (3.21) equals

$$\sum_{k,x}\mathbb{E}_0^a\big[\mathrm{E}_{0,\omega}[f(X_{S_k+\cdot}-x)h_{k,x}, S_k<\infty,\ D\circ\theta_{S_k}=\infty,\ X_{S_k}=x]g\circ t_x\big];$$

applying the strong Markov property at the stopping time S_k yields

$$\sum_{k,x}\mathbb{E}_0^a\big[\mathrm{E}_{0,\omega}[S_k<\infty,\ X_{S_k}=x, h_{k,x}]\,\mathrm{E}_{x,\omega}[f(X_\cdot-x), D=\infty]g\circ t_x\big].$$

Because by definition of S_k in (3.7), $X_{S_k-1}-X_{S_k-2}=X_{S_k}-X_{S_k-1}=\tilde{e}$ and $\ell.X_m\leq\ell.X_{S_k-2}$ for all $m\leq S_k-2$, and also because $\ell.e\leq\ell.\tilde{e}$ for all unit vectors $e\in\mathbb{Z}^d$, it follows that $\{X_m, X_m+e\}\in\mathcal{L}^{X_{S_k}}$, for all $m\leq S_k-1$. Therefore $\mathrm{E}_{0,\omega}[S_k<\infty, X_{S_k}=x, h_{k,x}]$ is $\sigma\{\omega(b): b\in\mathcal{L}^x\}$-measurable. On the other hand, due to the restriction $D=\infty$, $\mathrm{E}_{x,\omega}[f(X_\cdot-x), D=\infty]\cdot g\circ t_x$ is $\sigma\{\omega(b): b\in\mathcal{R}^x\}$-measurable. Because $\mathcal{L}^x\cap\mathcal{R}^x\neq\varnothing$, these two random variables are not $\mathbb{P}$-independent. Fortunately, by our definition of S_k, we observe the dependence of $\mathrm{E}_{0,\omega}[S_k<\infty, X_{S_k}=x, h_{k,x}]$ and $\mathrm{E}_{x,\omega}[f(X_\cdot-x), D=\infty]\cdot g\circ t_x$ is concentrated on $\{\omega(b): b\in\mathcal{B}^x\}$. [Here we see that it is necessary in the definition of S_k to have the random walk $(X_n)_{n\geq 0}$ going at least two steps in the direction $\tilde{e}\in\mathcal{E}$ before time S_k, otherwise $\mathrm{E}_{0,\omega}[S_k<\infty, X_{S_k}=x, h_{k,x}]$ is not $\sigma\{\omega(b): b\in\mathcal{L}^x\}$-measurable.] Using this fact and Fubini's theorem, the last expression equals

$$\begin{aligned}&\sum_{k,x}\mathbb{E}_0^a\big[\mathrm{E}_{0,\omega}[S_k<\infty,\ X_{S_k}=x, h_{k,x}]\,\mathbb{E}_x^{a_x}\big[\mathrm{E}_{x,\omega}[f(X_\cdot-x), D=\infty]g\circ t_x\big]\big]\\ &\quad=\sum_{k,x}\mathbb{E}_0^a\big[\mathrm{E}_{0,\omega}[S_k<\infty, X_{S_k}=x, h_{k,x}]\,\mathrm{E}_x^{a_x}[f(X_\cdot-x)g\circ t_x, D=\infty]\big].\end{aligned}$$

Using then the translation invariance of $\mathbb{P}$ measure we have $\mathrm{E}_x^{a_x}[f(X_\cdot - x) g \circ t_x, D = \infty] = \mathrm{E}_0^{a_x}[f(X_\cdot - 0) g \circ t_0, D = \infty]$, therefore the rightmost side of the last expression now equals

$$\sum_{k,x} \mathbb{E}_0^a \big[\mathrm{E}_{0,\omega}[S_k < \infty, X_{S_k} = x, h_{k,x}]\, \mathrm{E}_0^{a_x}[fg, D = \infty] \big]$$

$$= \sum_{k,x} \mathbb{E}_0^a \big[\mathrm{E}_{0,\omega}[S_k < \infty, X_{S_k} = x, h_{k,x}]\, \mathrm{P}_0^{a_x}[D = \infty]\, \mathrm{E}_0^{a_x}[fg | D = \infty] \big].$$

This means

$$\begin{aligned} &\mathrm{E}_0^a[f(X_{\tau_1 + \cdot} - X_{\tau_1}) g \circ t_{X_{\tau_1}} h] \\ &\quad = \sum_{k,x} \mathbb{E}_0^a \big[\mathrm{E}_{0,\omega}[S_k < \infty, X_{S_k} = x, h_{k,x}]\, \mathrm{P}_0^{a_x}[D = \infty]\, \mathrm{E}_0^{a_x}[fg | D = \infty] \big]. \end{aligned} \tag{3.22}$$

By taking specially $f = g = 1$, we get from the above equation

$$\mathrm{E}_0^a[h] = \sum_{k,x} \mathbb{E}_0^a \big[\mathrm{E}_{0,\omega}[S_k < \infty,\ X_{S_k} = x, h_{k,x}]\, \mathrm{P}_0^{a_x}[D = \infty] \big]. \tag{3.23}$$

Define now $\varphi(a) := \mathrm{E}_0^a[fg | D = \infty]$, and note that $\varphi(a_x)$ is $\sigma\{\omega(b) : b \in \mathcal{B}^x\}$-measurable, hence $\sigma\{\omega(b) : b \in \mathcal{L}^x\} \otimes \mathcal{F}_{S_k}$-measurable, and thereafter $h_{k,x}\varphi(a_x)$ is $\sigma\{\omega(b) : b \in \mathcal{L}^x\} \otimes \mathcal{F}_{S_k}$-measurable and coincides with the $\mathcal{G}_1$-measurable function $h\varphi(a_{X_{\tau_1}})$ on $\{\tau_1 = S_k\} \cap \{X_{\tau_1} = x\}$.

Substituting h through $h\varphi(a_{X_{\tau_1}})$ in (3.23), we find

$$\begin{aligned} \mathrm{E}_0^a[h\varphi(a_{X_{\tau_1}})] &= \sum_{k,x} \mathbb{E}_0^a \big[\mathrm{E}_{0,\omega}[S_k < \infty,\ X_{S_k} = x, h_{k,x} \cdot \varphi(a_x)]\, \mathrm{P}_0^{a_x}[D = \infty] \big] \\ &= \sum_{k,x} \mathbb{E}_0^a \big[\mathrm{E}_{0,\omega}[S_k < \infty,\ X_{S_k} = x, h_{k,x}]\, \mathrm{P}_0^{a_x}[D = \infty] \\ &\qquad \times \mathrm{E}_0^{a_x}[fg | D = \infty] \big]. \end{aligned}$$

Comparing this with (3.22) yields our claim (3.20). □

REMARK 3.4. Define

$$\psi(X_\cdot, \omega) := (X_{\tau_1 + \cdot} - X_{\tau_1}; t_{X_{\tau_1}}\omega) \in (\mathbb{Z}^d)^{\mathbb{N}} \times \Omega, \tag{3.24}$$

Then equation (3.20) can also be expressed as

$$\mathrm{E}_0^a[(fg) \circ \psi h] = \mathrm{E}_0^a \big[h \mathrm{E}_0^{a_{X_{\tau_1}}}[fg | D = \infty] \big]. \tag{3.25}$$

3.2. *The kth no-backtracking time τ_k and the Markov structure.* Because $\{D = \infty\} = \{D \geq \tau_1\} \in \mathcal{G}_1$, we can define on $\{\tau_1 < \infty\}$ a nondecreasing sequence of random variables inductively, by viewing τ_k, $k \geq 1$, as a function of $X_\cdot$,

$$\tau_{k+1}(X_\cdot) := \tau_1(X_\cdot) + \tau_k(X_{\tau_1 + \cdot} - X_{\tau_1}) \qquad \text{for } k \geq 1, \tag{3.26}$$

and by convention set $\tau_{k+1} = \infty$ on $\{\tau_k = \infty\}$. Because of (3.13) and Theorem 3.3 we observe that P_0-a.s. $\tau_k < \infty$, for all $k \geq 1$. One could ask why we do not use the equivalent formula $\tau_{k+1} = \tau_k(X.) + \tau_1(X_{\tau_k+\cdot} - X_{\tau_k})$ as the definition for τ_{k+1}. The reason will be clear in the proof of Theorem 3.5.

With τ_{k+1}, $k \geq 1$, introduced, we are now ready to introduce σ-algebra $\mathcal{G}_{k+1}$ for $k \geq 1$,

$$\mathcal{G}_{k+1} := \sigma\{\tau_1, \dots, \tau_k, \tau_{k+1};\ (X_{\tau_{k+1}\wedge m})_{m\geq 0};\ \omega(b), b \in \mathcal{L}^{X_{\tau_{k+1}}}\}, \tag{3.27}$$

describing the history of the path and environment involved before time τ_{k+1}.

With $\bar{\mathcal{G}}_k := \sigma\{\tau_1, \dots, \tau_k;\ (X_{\tau_k \wedge m})_{m\geq 0};\ \omega(b), b \in \mathcal{R}^0 \cap \mathcal{L}^{X_{\tau_k}}\}$, which is clearly included in $\mathcal{G}_k$, we also have

$$\mathcal{G}_{k+1} = \sigma\{\mathcal{G}_1 \cup \psi^{-1}(\bar{\mathcal{G}}_k)\} \tag{3.28}$$

with ψ introduced in (3.24).

The main result showing the embedded Markov chain structure comes in the next theorem, displaying the conditional distribution of the joint random variables $((X_{\tau_k+n} - X_{\tau_k})_{n\geq 0}; (\tau_{k+n} - \tau_k)_{n\geq 0}; t_{X_{\tau_k}}\omega(b), b \in \mathcal{R}^{X_{\tau_k}})$ given $\mathcal{G}_k$, $k \geq 1$.

THEOREM 3.5. *Let f, g, h_k be bounded and, respectively, $\sigma\{X_n\colon \geq 0\}$-, $\sigma\{\omega(b)\colon b \in \mathcal{R}^0\}$- and $\mathcal{G}_k$-measurable functions with $k \geq 1$. Then for $a \in \mathbb{I}^{\mathcal{E}}$,*

$$E_0^a[f(X_{\tau_k+\cdot} - X_{\tau_k}) g \circ t_{X_{\tau_k}} h_k] = E_0^a[h_k E_0^{a_{X_{\tau_k}}}[fg | D = \infty]]. \tag{3.29}$$

PROOF. We prove (3.29) by induction. The case $k = 1$ is just Theorem 3.3. For the step k to $k+1$, we observe that in view of (3.28) it is sufficient to show (3.29) for $h_{k+1} = h_1\ h_k \circ \psi$, while h_1 and h_k are bounded and, respectively, $\mathcal{G}_1$ and $\bar{\mathcal{G}}_k$-measurable. For such an h, the left-hand side of (3.29) equals

$$\begin{aligned} &E_0^a[f(X_{\tau_{k+1}+\cdot} - X_{\tau_{k+1}}) g \circ t_{X_{\tau_{k+1}}} h_1 h_k \circ \psi] \\ &\quad = E_0^a[f(X_{\tau_k+\cdot} - X_{\tau_k}) \circ \psi\ (g \circ t_{X_{\tau_k}} \circ \psi)\ (h_k \circ \psi) h_1], \end{aligned}$$

applying now (3.25), the right-hand side of the last expression equals

$$\begin{aligned} &E_0^a[h_1 E_0^{a_{X_{\tau_1}}}[f(X_{\tau_k+\cdot} - X_{\tau_k}) g \circ t_{X_{\tau_k}} h_k | D = \infty]] \\ &\quad = E_0^a[h_1 E_0^{a_{X_{\tau_1}}}[f(X_{\tau_k+\cdot} - X_{\tau_k}) g \circ t_{X_{\tau_k}} h_k, D = \infty]/P_0^{a_{X_{\tau_1}}}[D = \infty]] \end{aligned}$$

and because $h_k 1_{\{D=\infty\}}$ is $\mathcal{G}_k$-measurable, we can use the induction assumption and find

$$\begin{aligned} &= E_0^a[h_1 E_0^{a_{X_{\tau_1}}}[E_0^{a_{X_{\tau_k}}}[fg | D = \infty] h_k, D = \infty]/P_0^{a_{X_{\tau_1}}}[D = \infty]] \\ &= E_0^a[h_1 E_0^{a_{X_{\tau_1}}}[E_0^{a_{X_{\tau_k}}}[fg | D = \infty] h_k | D = \infty]] \\ &= E_0^a[h_1 E_0^{a_{X_{\tau_k}} \circ \psi}[fg | D = \infty] h_k \circ \psi] \end{aligned}$$

$$= \mathrm{E}_0^a\big[h_1 h_k \circ \psi \mathrm{E}_0^{a_{X_{\tau_{k+1}}}}[fg|D=\infty]\big]$$
$$= \mathrm{E}_0^a\big[h_{k+1}\mathrm{E}_0^{a_{X_{\tau_{k+1}}}}[fg|D=\infty]\big],$$

where we applied (3.25) backward in the third line, and this completes the proof. □

As an immediate consequence we get the next corollary.

COROLLARY 3.6. *Let*

$$\Gamma := \mathbb{N} \times \mathbb{Z}^d \times \mathbb{I}^{\mathcal{E}} \tag{3.30}$$

with its canonical product σ-algebra and let $y_k = (j^k, z^k, a^k) \in \Gamma$, $k \geq 0$. *For* $a \in \mathbb{I}^{\mathcal{E}}$ *and* $G \subset \Gamma$ *measurable let also*

$$\tilde{R}(a; G) := \mathrm{P}_0^a[(\tau_1, X_{\tau_1}, a_{X_{\tau_1}}) \in G|D=\infty]. \tag{3.31}$$

Then under P_0 *the* Γ*-valued random variables* (*with convention* $\tau_0 = 0$),

$$Y_k := (J_k, Z_k, A_k) := (\tau_{k+1} - \tau_k, X_{\tau_{k+1}} - X_{\tau_k}, a_{X_{\tau_{k+1}}}), \qquad k \geq 0, \tag{3.32}$$

define a Markov chain on the state space Γ, *which has transition kernel*

$$\mathsf{P}[Y_{k+1} \in G|Y_0 = y_0, \dots, Y_k = y_k] = \tilde{R}(a^k; G), \tag{3.33}$$

and initial distribution

$$\tilde{\Lambda}(G) := \mathrm{P}_0[(\tau_1, X_{\tau_1}, a_{X_{\tau_1}}) \in G]. \tag{3.34}$$

Similarly, *on the state space* $\mathbb{I}^{\mathcal{E}}$, *the random variables*

$$A_k = a_{X_{\tau_{k+1}}}, \qquad k \geq 0, \tag{3.35}$$

also define a Markov chain under P_0. *With* $a \in \mathbb{I}^{\mathcal{E}}$ *and* $B \subset \mathbb{I}^{\mathcal{E}}$ *measurable*, *its transition kernel is*

$$R(a; B) := \mathrm{P}_0^a[a_{X_{\tau_1}} \in B|D=\infty] = \sum_{\substack{j\in\mathbb{N}\\ z\in\mathbb{Z}^d}} \tilde{R}\big(a; (j, z, B)\big), \tag{3.36}$$

and the initial distribution is

$$\Lambda(B) := \mathrm{P}_0[a_{X_{\tau_1}} \in B] = \sum_{\substack{j\in\mathbb{N}\\ z\in\mathbb{Z}^d}} \tilde{\Lambda}\big((j, z, B)\big). \tag{3.37}$$

3.3. *Doeblin condition, invariant measure and ergodicity.* In this section we will show that the transition kernel $\tilde{R}(a;\cdot)$ has an invariant distribution and it is ergodic. First we need the following lemma.

LEMMA 3.7. *There exists a unique probability measure ν on $\mathbb{I}^{\mathcal{E}}$ and two constants $c>0$, $c_{15}>0$ such that for $m \geq 0$,*

$$\sup_{a\in\mathbb{I}^{\mathcal{E}}} \|R^m(a;\cdot)-\nu(\cdot)\|_{\mathrm{var}} \leq ce^{-c_{15}m}, \tag{3.38}$$

where $\|\cdot\|_{\mathrm{var}}$ denotes the variational norm on the space of measures on $\mathbb{I}^{\mathcal{E}}$.

Further, this probability measure ν is invariant with respect to the transition kernel R; that is, $\nu R=\nu$, and the Markov chain $(A_k)_{k\geq 0}$, defined in (3.35) *with transition kernel R and initial distribution ν on the state space $\mathbb{I}^{\mathcal{E}}$ is ergodic.*

Moreover, the initial distribution $\Lambda(\cdot)$ given in (3.37) *is absolutely continuous with respect to $\nu(\cdot)$.*

PROOF. First we show that the kernel $R(a;\cdot)$ satisfies the Doeblin condition (cf. [15], page 178):

$$R(a;B) \geq \kappa^2 c_{13}\big(\otimes_{\mathcal{E}}\mu\big)(B) \qquad \text{for all measurable } B\subset \mathbb{I}^{\mathcal{E}}, \tag{3.39}$$

where we recall that μ is the distribution of $\omega(b)$ on $\mathbb{I}$. Indeed the ellipticity condition (1.1) implies

$$\begin{aligned}
R(a;B) &= \mathrm{P}_0^a[a_{X_{\tau_1}}\in B|D=\infty] = \mathbb{E}_0^a\big[\mathrm{P}_{0,\omega}[a_{X_{\tau_1}}, D=\infty]\big]/\mathrm{P}_0^a[D=\infty]\\
&\geq \mathbb{E}_0^a\big[\mathrm{P}_{0,\omega}[X_1=\tilde{e},\ X_2=2\tilde{e},\ D\circ\theta_2=\infty], a_{X_2}\in B\big]\\
&= \mathbb{E}_0^a\big[\mathrm{P}_{0,\omega}[X_1=\tilde{e},\ X_2=2\tilde{e}]\,\mathrm{P}_{2\tilde{e},\omega}[D=\infty], a_{2\tilde{e}}\in B\big]\\
&\geq \kappa^2\mathbb{E}_0^a\big[\mathrm{P}_{2\tilde{e},\omega}[D=\infty], a_{2\tilde{e}}\in B\big]\\
&\overset{(2.30)}{\geq} \kappa^2 c_{13}\mathbb{P}_0^a[a_{2\tilde{e}}\in B] = \kappa^2 c_{13}(\otimes_{\mathcal{E}}\mu)(B).
\end{aligned}$$

Applying Theorem 6.15 in [11], the Doeblin condition immediately implies that there exists an invariant measure ν and (3.38) holds. (The Doeblin condition implies that the kernel is small and aperiodic in the terminology of [11]; cf. pages 15, 20 and 21.) The uniqueness is a trivial consequence of (3.38).

In view of (3.38) the ergodicity of $(A_n)_{n\geq 0}$ follows from Proposition 2.4 in [15], Chapter 6. To prove that the initial distribution $\Lambda(\cdot)$ is absolutely continuous with respect to the invariant measure $\nu(\cdot)$, we observe that the Doeblin condition (3.39) also implies

$$\nu(B) = \int \nu(da)R(a;B) \geq \kappa^2 c_{13}\int \nu(da)\big(\otimes_{\mathcal{E}}\mu\big)(B) = \kappa^2 c_{13}\big(\otimes_{\mathcal{E}}\mu\big)(B).$$

Therefore $\nu(B)=0$ implies $(\otimes_{\mathcal{E}}\mu)(B)=0$, and hence

$$\Lambda(B) \le \sum_{z\in\mathbb{Z}^d} \mathrm{P}_0[a_z \in B] = \sum_{z\in\mathbb{Z}^d} (\otimes_{\mathcal{E}}\mu)(B) = 0,$$

that is, Λ is absolutely continuous with respect to ν, and this completes the proof. $\square$

With this lemma we can now prove Theorem 3.8.

THEOREM 3.8 (Ergodicity). *$\tilde{\nu} := \nu\tilde{R}$ is the unique invariant distribution for the transition kernel $\tilde{R}$, for which the relation*

$$\sup_{a\in\mathbb{I}^{\mathcal{E}}} \|\tilde{R}^m(a;\cdot) - \tilde{\nu}(\cdot)\|_{\mathrm{var}} \le c_{14}e^{-c_{15}m}, \qquad m \ge 0, \tag{3.40}$$

holds for some $c_{14}>0$. With initial distribution equal $\tilde{\nu}$, the Markov chain $(Y_k)_{k\ge0}$ defined in (3.32) *is ergodic. Moreover, the law of the Markov chain $(Y_{k+1})_{k\ge0}$ under* P_0 *is absolutely continuous with respect to the law of the chain with initial distribution $\tilde{\nu}$.*

PROOF. We observe that for any bounded and measurable function f on $\mathbb{I}^{\mathcal{E}}$ we have $\tilde{R}f = Rf$ and thereafter $\tilde{\nu}\tilde{R} = \nu\tilde{R}\tilde{R} = \nu R\tilde{R} = \nu\tilde{R} = \tilde{\nu}$. This means that $\tilde{\nu}$ is an invariant probability measure with respect to $\tilde{R}$ on Γ. From $R\tilde{R} = \tilde{R}^2$ and (3.38) it follows that $\|\tilde{R}^{m+1}(a;\cdot) - \tilde{\nu}(\cdot)\|_{\mathrm{var}} \le ce^{-c_{15}m}$ for $m \ge 0$, and hence (3.40) with some constant $c_{14} > 0$. Applying again Proposition 2.4 in [15], Chapter 6, the ergodicity of $(Y_k)_{k\ge0}$ with initial distribution $\tilde{\nu}$ follows.

From Corollary 3.6 we know that, the initial distribution of $(Y_{k+1})_{k\ge0}$ under P_0 is $\tilde{\Lambda}\tilde{R}$. From Lemma 3.7, Λ is absolutely continuous with respect to ν, therefore the absolute continuity of the law $(Y_{k+1})_{k\ge0}$ under P_0 with respect to the law with initial distribution $\tilde{\nu}$ follows immediately from the obvious relations $\tilde{\Lambda}\tilde{R} = \Lambda\tilde{R}$ and $\tilde{\nu}\tilde{R} = \nu\tilde{R}$. $\square$

4. Integrability properties of $\ell.X_{\tau_1}$ and τ_1. As a last step of preparation toward the strong law of large numbers and the functional central limit theorem mentioned in Section 1, we will show in this section that for $c>0$ small enough, $\sup_{x,\omega} \mathrm{E}_{x,\omega}[e^{c\tau_1}] < \infty$. The proof will be divided in several auxiliary lemmas.

LEMMA 4.1. *There exists $c_{16} > 0$ such that for all $\omega\in\Omega$, $x\in\mathbb{Z}^d$,*

$$\mathrm{E}_{x,\omega}\big[\exp\{c_{16}\ell.(X_{S_1} - X_0)\}\big] \le 1 + \frac{c_{13}}{4}, \tag{4.1}$$

with c_{13} given in (2.30).

PROOF. At first we define a sequence of auxiliary $(\mathcal{F}_n)_{n\geq 0}$-stopping times [recall the definition of $\tilde{e}$ in (3.2)],

$$N_0 := 0; \qquad N_1 := \inf\{m \geq 0 : \ell\boldsymbol{.}(X_m - X_0) \geq 2\ell\boldsymbol{.}\tilde{e}\};$$
$$N_{k+1} := N_k + N_1 \circ \theta_{N_k} \qquad \text{for } k \geq 1.$$

Observe that for all $k \geq 1$,

$$2\ell\boldsymbol{.}\tilde{e} \leq \ell\boldsymbol{.}(X_{N_k} - X_{N_{k-1}}) \leq 3\ell\boldsymbol{.}\tilde{e} \quad \text{and} \quad N_k - N_{k-1} \geq 2.$$

Therefore we have $\ell\boldsymbol{.}(X_{S_1} - X_0) \leq 3k(\ell\boldsymbol{.}\tilde{e})$ on $\{N_{k-1} < S_1 \leq N_k\}$, and hence

$$\begin{aligned} \mathrm{E}_{x,\omega}[e^{c\ell\boldsymbol{.}(X_{S_1}-X_0)}] &= \sum_{k\geq 1} \mathrm{E}_{x,\omega}[e^{c\ell\boldsymbol{.}(X_{S_1}-X_0)}, N_{k-1} < S_1 \leq N_k] \\ &\leq \sum_{k\geq 1} e^{3ck\ell\boldsymbol{.}\tilde{e}} \mathrm{P}_{x,\omega}[N_{k-1} < S_1 \leq N_k]. \end{aligned} \tag{4.2}$$

Because for all $y \in \mathbb{Z}^d$, $\omega \in \Omega$,

$$\begin{aligned} \mathrm{P}_{y,\omega}[N_{k+1} < S_1] &\leq \mathrm{P}_{y,\omega}[N_k < S_1, (X_{N_k+1} - X_{N_k}, X_{N_k+2} - X_{N_k+1}) \neq (\tilde{e}, \tilde{e})] \\ &\leq (1-\kappa^2)\mathrm{P}_{y,\omega}[N_k < S_1], \end{aligned}$$

where we used the ellipticity condition (1.1) in the last step, the rightmost side of (4.2) can be estimated further by

$$\begin{aligned} \sum_{k\geq 1} e^{3ck\ell\boldsymbol{.}\tilde{e}} \mathrm{P}_{x,\omega}[N_{k-1} < S_1 \leq N_k] &\leq \sum_{k\geq 1} e^{3ck\ell\boldsymbol{.}\tilde{e}} \mathrm{P}_{x,\omega}[N_{k-1} < S_1] \\ &\leq \sum_{k\geq 1} e^{3ck\ell\boldsymbol{.}\tilde{e}} (1-\kappa^2)^{k-1} < \infty, \end{aligned} \tag{4.3}$$

provided c is small enough.

Take now $c_0 > 0$ and $m_0 \in \mathbb{N}$ such that $\sum_{k>m_0} e^{3c_0k\ell\boldsymbol{.}\tilde{e}}(1-\kappa^2)^{k-1} < \frac{c_{13}}{8}$, (4.2) and (4.3) imply that for all $c < c_0$,

$$\begin{aligned} \mathrm{E}_{x,\omega}[e^{c\ell\boldsymbol{.}(X_{S_1}-X_0)}] &\leq \sum_{m\leq m_0} e^{3cm_0\ell\boldsymbol{.}\tilde{e}} \mathrm{P}_{x,\omega}[N_{m-1} < S_1 \leq N_m] + \frac{c_{13}}{8} \\ &\leq e^{3cm_0\ell\boldsymbol{.}\tilde{e}} \mathrm{P}_{x,\omega}[S_1 \leq N_{m_0}] + \frac{c_{13}}{8}. \end{aligned}$$

Thereafter there exists $c_{16} \in (0, c_0)$ small enough such that $e^{3c_{16}m_0\ell\boldsymbol{.}\tilde{e}} < 1 + \frac{c_{13}}{8}$ and that completes our proof. □

Let us introduce the random variable

$$M := \sup\{\ell\boldsymbol{.}(X_n - X_0) : 0 \leq n \leq D\}, \tag{4.4}$$

which is the maximal displacement in the direction ℓ before backtracking. It will turn out that M is a key variable later in studying integrability properties of $\ell\boldsymbol{.}X_{\tau_1}$. Because for all $a \in \mathbb{I}^{\mathcal{E}}$, $\mathrm{P}_0^a[D = \infty] > 0$, we cannot expect $M < \infty$ P_0^a-a.s. Nevertheless we claim the following lemma.

LEMMA 4.2. *There exists some $c_{17} > 0$ small enough such that*

$$\left(1+\frac{c_{13}}{4}\right)\left\{\sup_{\substack{x\in\mathbb{Z}^d\\ \omega\in\Omega}} \mathrm{E}_{x,\omega}[e^{c_{17}M}, D<\infty]\right\} \le 1-\frac{c_{13}}{2}. \tag{4.5}$$

PROOF. At first we show that

$$\mathrm{P}_{x,\omega}[2^m \le M < 2^{m+1},\ D<\infty] \le c_{18}e^{-c_{19}2^{m+1}} \qquad \text{for all } x\in\mathbb{Z}^d,\ \omega\in\Omega. \tag{4.6}$$

Recall the definition (2.18) for the box U centered in x with width L in the direction ℓ and size L^2 in the direction normal to ℓ; also recall (2.22) for its boundary $\partial U = \partial_+ U \cup \partial_- U \cup \partial_0 U$ and setting $L = 2^{m+1}$, we observe that

$$\begin{aligned} &\mathrm{P}_{x,\omega}[2^m \le M < 2^{m+1}, D<\infty] \\ &\quad\le \mathrm{P}_{x,\omega}\left[T_U > \frac{\lambda L}{\gamma}\right] + \mathrm{P}_{x,\omega}\left[T_U \le \frac{\lambda L}{\gamma},\ X_{T_U} \notin \partial_+ U\right] \\ &\qquad + \mathrm{P}_{x,\omega}\left[T_U \le \frac{\lambda L}{\gamma},\ X_{T_U} \in \partial_+ U, \mathrm{P}_{X_{T_U},\omega}[\tilde{T}_{-2^m} < T_{2^m}]\right]. \end{aligned}$$

By (2.24)–(2.29) the first two terms together are $\le c_1 e^{-c_2 2^m}$. To estimate the third term we observe

$$\begin{aligned} &\mathrm{P}_{x,\omega}\left[T_U \le \frac{\lambda L}{\gamma}, X_{T_U} \in \partial_+ U, \mathrm{P}_{X_{T_U},\omega}[\tilde{T}_{-2^m} < T_{2^m}]\right] \\ &\quad\le \sum_{y\in\partial_+ U} \sup_{y\in\partial_+ U} \mathrm{P}_{y,\omega}[\tilde{T}_{-2^m} < T_{2^m}], \end{aligned}$$

and using $\sup_{y,\omega} \mathrm{P}_{y,\omega}[\tilde{T}_{-2^m} < T_{2^m}] \le c_1 e^{-c_2 2^m}$,

$$\le C(d,\ell) L^{2d-2} c_1 e^{-c_2 2^m}.$$

Putting them together the claim (4.6) follows.

With (4.6) in mind we show in the second step that $\sup_{x,\omega} \mathrm{E}_{x,\omega}[e^{cM}, D<\infty] \le 1 - \frac{3c_{13}}{4}$, provided $c > 0$ small enough. This can be seen by the obvious estimate

$$\begin{aligned} &\mathrm{E}_{x,\omega}[e^{cM}, D<\infty] \\ &\quad\le e^{2c}\mathrm{P}_{x,\omega}[0\le M<1, D<\infty] + \sum_{m\ge 0} \mathrm{P}_{x,\omega}[2^m \le M < 2^{m+1}, D<\infty] e^{c2^{m+1}} \\ &\quad\le \mathrm{P}_{x,\omega}[D<\infty] e^{c2^{m_0+1}} + \sum_{m>m_0} \mathrm{P}_{x,\omega}[2^m \le M < 2^{m+1}, D<\infty] e^{c2^{m+1}} \\ &\quad\le \mathrm{P}_{x,\omega}[D<\infty] e^{c2^{m_0+1}} + \sum_{m>m_0} c_{18} e^{(c-c_{19})2^{m+1}}. \end{aligned}$$

Now let $c_0 = \frac{c_{19}}{2}$ and $m_0 \in \mathbb{N}$ be chosen such that $\sum_{m>m_0} c_{18} e^{(c_0-c_{19})2^{m+1}} \le \frac{c_{13}}{8}$, the rightmost side above is less than or equal to

$$(1-c_{13})e^{c2^{m_0+1}} + \frac{c_{13}}{8} \le 1 - \frac{3c_{13}}{4},$$

with $0 < c < c_0$ small enough. Our claim follows immediately. □

With the help of these two lemmas we can now provide the integrability of $\mathrm{E}_{x,\omega}[e^{c\ell\cdot X_{\tau_1}}]$.

THEOREM 4.3. *There exists $c_{20} > 0$ small enough such that*

$$\sup_{\substack{x\in\mathbb{Z}^d\\ \omega\in\Omega}} \mathrm{E}_{x,\omega}[\exp\{c_{20}\ell\cdot(X_{\tau_1} - X_0)\}] < \infty. \tag{4.7}$$

PROOF. Since

$$\begin{aligned}\mathrm{E}_{x,\omega}\big[e^{c\ell\cdot(X_{\tau_1}-X_0)}\big] &= \sum_{k\ge1} \mathrm{E}_{x,\omega}\big[e^{c\ell\cdot(X_{S_k}-X_0)}, S_k<\infty, D\circ\theta_{S_k}=\infty\big]\\ &\le \sum_{k\ge1} \mathrm{E}_{x,\omega}\big[e^{c\ell\cdot(X_{S_k}-X_0)}, S_k<\infty\big],\end{aligned} \tag{4.8}$$

in view of (4.1) it suffices to show that $\sup_{x,\omega} \sum_{k\ge2} \mathrm{E}_{x,\omega}[e^{c\ell\cdot(X_{S_k}-X_0)}, S_k<\infty] < \infty$.

To this end we define another sequence of auxiliary $(\mathscr{F}_n)_{n\ge0}$-stopping times [recall the definition of M_k in (3.7)],

$$V_k := \inf\{n \ge R_k : \ell\cdot X_n \ge M_k\} \qquad \text{for } k \ge 1, \tag{4.9}$$

that is, V_k is the first time after R_k such that the random walker $(X_n)_{n\ge0}$ reaches a maximum in the direction ℓ again.

It is clear that $R_k \le V_k \le S_{k+1}$, and the inequalities are strict if $S_{k+1} < \infty$. We observe that for $k \ge 2$,

$$\begin{aligned}\ell\cdot(X_{S_k} - X_0) &= \ell\cdot X_{S_k} - \ell\cdot X_{V_{k-1}} + \ell\cdot(X_{V_{k-1}} - X_0)\\ &\le \ell\cdot(X_{S_1} - X_0)\circ\theta_{V_{k-1}} + \ell\cdot(X_{V_{k-1}} - X_0),\end{aligned}$$

whence

$$\begin{aligned}\mathrm{E}_{x,\omega}\big[e^{c\ell\cdot(X_{S_k}-X_0)}, S_k<\infty\big] &\le \mathrm{E}_{x,\omega}\big[e^{c\ell\cdot(X_{S_k}-X_0)}, V_{k-1}<\infty\big]\\ &\le \mathrm{E}_{x,\omega}\big[e^{c\ell\cdot(X_{V_{k-1}}-X_0)}, V_{k-1}<\infty, \mathrm{E}_{X_{V_{k-1}},\omega}[e^{c\ell\cdot(X_{S_1}-X_0)}]\big]\\ &\overset{(4.1)}{\le} \mathrm{E}_{x,\omega}\Big[e^{c\ell\cdot(X_{V_{k-1}}-X_0)}\Big(1+\frac{c_{13}}{4}\Big), V_{k-1}<\infty\Big].\end{aligned} \tag{4.10}$$

Further, we observe that

$$\begin{aligned}\ell\boldsymbol{.}(X_{V_{k-1}} - X_0) &= \ell\boldsymbol{.}(X_{V_{k-1}} - X_{S_{k-1}}) + \ell\boldsymbol{.}(X_{S_{k-1}} - X_0)\\ &\le M_{k-1} + 1 - \ell\boldsymbol{.}X_{S_{k-1}} + \ell\boldsymbol{.}(X_{S_{k-1}} - X_0)\\ &= M \circ \theta_{S_{k-1}} + 1 + \ell\boldsymbol{.}(X_{S_{k-1}} - X_0).\end{aligned}$$

Therefore with the strong Markov property the rightmost side of (4.10) can be further estimated by

$$\begin{aligned}&\le e^c \mathrm{E}_{x,\omega}\biggl[\exp\{c(M\circ\theta_{S_{k-1}} + \ell\boldsymbol{.}(X_{S_{k-1}} - X_0))\}\biggl(1+\frac{c_{13}}{4}\biggr), R_{k-1} < \infty\biggr]\\ &= e^c \mathrm{E}_{x,\omega}\biggl[e^{c\ell\boldsymbol{.}(X_{S_{k-1}} - X_0)}, S_{k-1} < \infty, \biggl(1+\frac{c_{13}}{4}\biggr)\mathrm{E}_{X_{S_{k-1}},\omega}[e^{cM}, D < \infty]\biggr],\end{aligned}$$

and this is, by (4.5) and induction,

$$\begin{aligned}&\le e^c\biggl(1-\frac{c_{13}}{2}\biggr)\mathrm{E}_{x,\omega}[e^{c\ell\boldsymbol{.}(X_{S_{k-1}} - X_0)}, S_{k-1} < \infty]\\ &\le \biggl(e^c\biggl(1-\frac{c_{13}}{2}\biggr)\biggr)^k,\end{aligned}$$

provided $0 < c \le c_{17}$.

Therefore we can find $c_{20} \in (0, c_{17})$ small enough such that $e^{c_{20}}(1 - \frac{c_{13}}{2}) < 1$. Therefore,

$$\sum_{k\ge 2} \mathrm{E}_{x,\omega}[e^{c_{20}\ell\boldsymbol{.}(X_{S_k} - X_0)}, S_k < \infty] \le \sum_{k\ge 2}\biggl(e^{c_{20}}\biggl(1-\frac{c_{13}}{2}\biggr)\biggr)^k < \infty.$$

And with (4.8) this completes the proof. □

As a corollary we obtain an estimate on the tail of τ_1 and its integrability properties.

COROLLARY 4.4. *There exists $c_{21} > 0$ and $c_{22} > 0$ such that for $u \in \mathbb{N}$*,

$$\sup_{\substack{x\in\mathbb{Z}^d\\ \omega\in\Omega}} \mathrm{P}_{x,\omega}[\tau_1 > u] \le c_{21}\, e^{-c_{22}u}, \tag{4.11}$$

and consequently,

$$\sup_{\substack{x\in\mathbb{Z}^d\\ \omega\in\Omega}} \mathrm{E}_{x,\omega}[e^{c_{23}\tau_1}] \le c_{24} < \infty, \tag{4.12}$$

for some $c_{23} > 0$ and $c_{24} > 0$.

PROOF. Recall $\gamma = \log \frac{1}{1-\varepsilon}$ from Theorem 2.1 and choose $u \in \mathbb{N}$, $u \geq \frac{2\lambda}{\gamma}$. We denote with U the box defined in (2.18), with center x, width $\frac{\gamma}{2\lambda}u$ in the direction ℓ and size $(\frac{\gamma}{2\lambda}u)^2$ in the direction normal to ℓ.

By Chebychev's inequality and with c_{20} from (4.7) we observe

$$\begin{aligned}
&P_{x,\omega}[\tau_1 > u] \\
&\quad \leq P_{x,\omega}\left[\tau_1 > u, \ell\boldsymbol{.}(X_{\tau_1} - X_0) \leq \frac{\gamma}{4\lambda}u\right] + P_{x,\omega}\left[\ell\boldsymbol{.}(X_{\tau_1} - X_0) > \frac{\gamma}{4\lambda}u\right] \\
&\quad \leq P_{x,\omega}\left[\tau_1 > u, \ell\boldsymbol{.}(X_{\tau_1} - X_0) \leq \frac{\gamma}{4\lambda}u\right] + \exp\left\{-c_{20}\frac{\gamma}{4\lambda}u\right\}E_{x,\omega}[e^{c_{20}\ell\boldsymbol{.}(X_{\tau_1}-X_0)}] \\
&\quad \leq P_{x,\omega}\left[\tau_1 > u, \ell\boldsymbol{.}(X_{\tau_1} - X_0) \leq \frac{\gamma}{4\lambda}u\right] + c_{25}e^{-c_{26}u};
\end{aligned}$$

further we have

$$\begin{aligned}
&P_{x,\omega}\left[\tau_1 > u, \ell\boldsymbol{.}(X_{\tau_1} - X_0) \leq \frac{\gamma}{4\lambda}u\right] \\
&\quad \leq P_{x,\omega}[T_{(\gamma/4\lambda)u} > u] \\
&\quad \leq P_{x,\omega}[T_{(\gamma/4\lambda)u} > T_U] + P_{x,\omega}[T_U = T_{(\gamma/4\lambda)u} > u] \\
&\quad \leq P_{x,\omega}\left[T_U > \frac{u}{2}\right] + P_{x,\omega}\left[T_U \leq \frac{u}{2}, X_{T_U} \notin \partial_+ U\right] \\
&\qquad + P_{x,\omega}[T_U = T_{(\gamma/4\lambda)u} > u].
\end{aligned}$$

Using the same argument as in (2.23)–(2.29), the first two terms on the right-hand side together can be estimated uniformly: for all $x \in \mathbb{Z}^d$, $\omega \in \Omega$ and for all $u \in \mathbb{N}$,

$$P_{x,\omega}\left[T_U > \frac{u}{2}\right] + P_{x,\omega}\left[T_U \leq \frac{u}{2}, X_{T_U} \notin \partial_+ U\right] \leq c_{27}e^{-c_{28}u}, \tag{4.13}$$

and by (2.20) the last term can also be estimated uniformly: for all $x \in \mathbb{Z}^d$, $\omega \in \Omega$ and $u \in \mathbb{N}$, $u \geq \frac{2\lambda}{\gamma}$,

$$P_{x,\omega}[T_U > u] \leq c_4 e^{-(\gamma/8)u}, \tag{4.14}$$

because in our construction of U, $u \geq \frac{\lambda}{\gamma}\frac{\gamma}{2\lambda}u = \frac{u}{2}$, the condition (2.19) is fulfilled.

Altogether we get that for all $u \in \mathbb{N}$, $x \in \mathbb{Z}^d$ and $\omega \in \Omega$,

$$P_{x,\omega}[\tau_1 > u] \leq c_{21}e^{-c_{22}u}, \tag{4.15}$$

our claim (4.11) follows immediately, and finally, (4.12) is an easy consequence of (4.11). $\square$

5. Law of large numbers and central limit theorem. In this section we will provide the main results of this article: first a strong law of large numbers; moreover, we are able to prove a functional central limit theorem. Some parts of the proofs presented in this section are similar to the proofs of [18], Theorem 2.3, page 1864 and [16], Theorem 4.1, pages 130–131.

THEOREM 5.1 (Strong law of large numbers). *Under the assumptions* (1.1)–(1.5) *we have*

$$\text{(5.1)} \qquad \mathrm{P}_0\text{-}a.s.\ \frac{X_n}{n} \xrightarrow{n\to\infty} v = \frac{\mathrm{E}^\Pi[X_{\tau_1}]}{\mathrm{E}^\Pi[\tau_1]} \quad \textit{and} \quad \ell\cdot v > 0,$$

where

$$\text{(5.2)} \quad \Pi[\cdot] := \int \nu(da)\mathrm{P}_0^a[\cdot|D=\infty] \quad \textit{and} \quad \mathrm{E}^\Pi[\cdot] := \int \nu(da)\mathrm{E}_0^a[\cdot|D=\infty].$$

(*We recall that* ν *is the unique invariant distribution on* $\mathbb{I}^{\mathcal{E}}$ *given in Lemma* 3.7.)

PROOF. Let $Y_k = (J_k, Z_k, A_k) = (\tau_{k+1} - \tau_k, X_{\tau_{k+1}} - X_{\tau_k}, a_{X_{\tau_{k+1}}})$, $k \geq 0$, be the random variables on Γ defined in (3.32). We know from Theorem 3.8 that the Markov chain $(Y_k)_{k\geq 0}$ with initial distribution $\tilde{\nu}$ is stationary and ergodic, further, the law of $(Y_{k+1})_{k\geq 0}$ under P_0 is absolutely continuous with respect to the law with initial distribution $\tilde{\nu}$. Therefore from the Birkhoff's ergodic theorem (cf. [4], page 341) it follows that for any $f \in L^1(\Gamma, \tilde{\nu})$, P_0-a.s.,

$$\frac{1}{n}\sum_{k=1}^{n} f(Y_k) \xrightarrow{n\to\infty} \int d\tilde{\nu}\, f.$$

Applying this formula to $f(y) = j$ and $f(y) = z$ for $y = (j, z, a) \in \Gamma$, we find that P_0-a.s.,

$$\text{(5.3)} \quad \begin{aligned} \frac{1}{n-1}(\tau_n - \tau_1) &\xrightarrow{n\to\infty} \int d\tilde{\nu} J_1 = \int \nu(da)\mathrm{E}_0^a[\tau_1|D=\infty] = \mathrm{E}^\Pi[\tau_1] < \infty, \\ \frac{1}{n-1}(X_{\tau_n} - X_{\tau_1}) &\xrightarrow{n\to\infty} \int d\tilde{\nu} Z_1 = \int \nu(da)\mathrm{E}_0^a[X_{\tau_1}|D=\infty] = \mathrm{E}^\Pi[X_{\tau_1}], \end{aligned}$$

where the finiteness follows from (4.12). We also observe that $\ell\cdot v > 0$, because P_0-a.s. $\ell\cdot X_{\tau_1} > 0$ by definition (3.6), (3.7) and (3.12), and $\mathrm{E}^\Pi[|X_{\tau_1}|] \leq \mathrm{E}^\Pi[\tau_1] < \infty$.

From (3.13) we observe that P_0-a.s. $\frac{\tau_1}{n-1} \to 0$, as $n \to \infty$. Therefore (5.3) implies that

$$\text{(5.3*)} \quad \begin{aligned} \frac{1}{n}\tau_n &\xrightarrow{n\to\infty} \int d\tilde{\nu} J_1 = \int \nu(da)\mathrm{E}_0^a[\tau_1|D=\infty] = \mathrm{E}^\Pi[\tau_1], \\ \frac{1}{n}X_{\tau_n} &\xrightarrow{n\to\infty} \int d\tilde{\nu} Z_1 = \int \nu(da)\mathrm{E}_0^a[X_{\tau_1}|D=\infty] = \mathrm{E}^\Pi[X_{\tau_1}]. \end{aligned}$$

Now let us define a nondecreasing sequence k_n, $n \geq 0$, which tends to $+\infty$ P_0-a.s., such that

$$\tau_{k_n} \leq n < \tau_{k_n+1} \qquad \text{(with the convention } \tau_0 = 0\text{).} \tag{5.4}$$

Dividing the above inequality by k_n and using (5.3*), we find that P_0-a.s.,

$$\frac{k_n}{n} \xrightarrow{n\to\infty} \frac{1}{\mathrm{E}^{\Pi}[\tau_1]}. \tag{5.5}$$

Further, we observe that

$$\frac{X_n}{n} = \frac{X_{\tau_{k_n}}}{n} + \frac{X_n - X_{\tau_{k_n}}}{n}, \tag{5.6}$$

then in view of (5.3*) and (5.5), we obtain that P_0-a.s.,

$$\frac{X_{\tau_{k_n}}}{n} = \frac{X_{\tau_{k_n}}}{k_n}\frac{k_n}{n} \xrightarrow{n\to\infty} \frac{\mathrm{E}^{\Pi}[X_{\tau_1}]}{\mathrm{E}^{\Pi}[\tau_1]}, \tag{5.7}$$

and by (5.5) again, that P_0-a.s.,

$$\frac{|X_n - X_{\tau_{k_n}}|}{n} \leq \frac{\tau_{k_n+1} - \tau_{k_n}}{n} = \frac{\tau_{k_n+1}}{k_n+1}\frac{k_n+1}{n} - \frac{\tau_{k_n}}{k_n}\frac{k_n}{n} \xrightarrow{n\to\infty} 0.$$

Combining this with (5.6) and (5.7), we have proved that P_0-a.s. $\frac{X_n}{n} \xrightarrow{n\to\infty} v$, with v given in (5.1). $\square$

We are now able to derive a functional central limit theorem for the process

$$B_t^n = \frac{1}{\sqrt{n}}\big(X_{[tn]} - [tn]v\big), \qquad t \geq 0, \tag{5.8}$$

where $[t]$ denotes the integer part of $t \in \mathbb{R}_+$.

We denote by $D_{\mathbb{R}^d}[0,\infty)$ the set of $\mathbb{R}^d$-valued functions on $[0,\infty)$, which are right-continuous and possess left limits (also called cadlag functions). We endow this set with the Skorohod topology (cf. [5], page 117) and its Borel-σ-algebra, so that $B_\cdot^n$ defines a $D_{\mathbb{R}^d}[0,\infty)$-valued random variable.

To simplify notations let us temporarily denote the law of the Markov chain $(Y_m)_{m\geq 0}$ with invariant distribution $\tilde{\nu}$ by $\mathsf{P}_{\tilde{\nu}}[\cdot]$ and its expectation value by $\mathsf{E}_{\tilde{\nu}}[\cdot]$. Further, we use x^T to denote the transposed vector of $x \in \mathbb{R}^d$.

LEMMA 5.2. *Let* $f(y) := z - jv$ *for* $y = (j, z, a) \in \Gamma$ *and* v *from Theorem* 5.1. *Then*

$$\sup_{a\in\mathbb{I}^{\mathcal{E}}} (\tilde{R}|f|)(a) < \infty, \tag{5.9}$$

where we recall that $|\cdot|$ *denotes the* L^1*-norm on* $\mathbb{R}^d$. *Further the* $\mathbb{R}^d$*-valued random variables*

$$F(a) := \sum_{m=1}^{\infty} (\tilde{R}^m f)(a), \qquad G_n := \sum_{m=1}^{n} f(Y_m), \\ W_n := G_n + F(A_n), \qquad n \geq 1, \tag{5.10}$$

[*with notations from* (3.32) *and* (3.35)] *are well defined*, *and under* P_0, $(W_n)_{n\geq 1}$ *is a* $(\mathcal{H}_n)_{n\geq 1}$*-martingale with respect to* $\mathcal{H}_n := \sigma\{Y_1, \dots, Y_n\}$. *We use the convention* $W_0 := 0$ *and* $\mathcal{H}_0$ *equals the trivial* σ*-algebra*.

Finally, *the partial sum* $\frac{1}{\sqrt{n}} G_{[n\cdot]}$ *converges under* P_0 *on the space* $D_{\mathbb{R}^d}[0,\infty)$ *in law to a* d*-dimensional Brownian motion with covariance matrix* $\mathbf{K}$,

$$\begin{aligned} \mathbf{K} &= \mathsf{E}_{\tilde{\nu}}[(W_2 - W_1)(W_2 - W_1)^T] \\ &= \mathrm{E}^{\Pi}[(X_{\tau_1} - \tau_1 v)(X_{\tau_1} - \tau_1 v)^T] \\ &\quad + \sum_{m=1}^{\infty} \mathrm{E}^{\Pi}\big[(X_{\tau_1} - \tau_1 v)\big(X_{\tau_{m+1}} - X_{\tau_m} - (\tau_{m+1} - \tau_m)v\big)^T\big] \\ &\quad + \sum_{m=1}^{\infty} \mathrm{E}^{\Pi}\big[\big(X_{\tau_{m+1}} - X_{\tau_m} - (\tau_{m+1} - \tau_m)v\big)(X_{\tau_1} - \tau_1 v)^T\big], \end{aligned} \tag{5.11}$$

where the last two terms converge in all matrix norms. [*We recall the definition of* E^{Π} *in* (5.2).]

PROOF. Inequality (5.9) follows immediately from (2.30) and (4.12), because

$$\begin{aligned} \sup_{a \in \mathbb{I}^{\mathcal{E}}} (\tilde{R}|f|)(a) &\leq \sup_a \mathrm{E}_0^a[|X_{\tau_1}| + |v|\tau_1 | D = \infty] \\ &\leq (1 + |v|) \sup_a \mathrm{E}_0^a[\tau_1 | D = \infty] < \infty. \end{aligned}$$

With this, we can now show that

$$\sup_{a \in \mathbb{I}^{\mathcal{E}}} |F(a)| < c_{29} < \infty. \tag{5.12}$$

Indeed, Theorem 5.1 implies that $\tilde{\nu}\tilde{R}f = \tilde{\nu}f = 0$ and hence for $a \in \mathbb{I}^{\mathcal{E}}$, $m \geq 1$,

$$\begin{aligned} |(\tilde{R}^m f)(a)| &\leq \big|\big(\tilde{R}^{m-1} \circ (\tilde{R}f)\big)(a) - \tilde{\nu}\tilde{R}f\big| + |\tilde{\nu}\tilde{R}f| \\ &\leq \|\tilde{R}^{m-1}(a;\cdot) - \tilde{\nu}(\cdot)\|_{\mathrm{var}} \cdot \|\tilde{R}f\|_{L^\infty} \\ &\leq c_{14} e^{-c_{15}(m-1)} \|\tilde{R}f\|_{L^\infty}, \end{aligned} \tag{5.13}$$

where (3.40) is used in the last step, and this with (5.9) proves (5.12).

To show that $(W_n)_{n\ge 1}$ is a $(\mathcal{H}_n)_{n\ge 1}$-martingale, we observe from Corollary 3.6 that for $n \ge 1$,

$$\begin{aligned}\mathrm{E}_0[W_{n+1} - W_n|\mathcal{H}_n] &= \mathrm{E}_0[f(Y_{n+1}) + F(A_{n+1}) - F(A_n)|\mathcal{H}_n]\\ &= (\tilde{R}f)(A_n) + (\tilde{R}F)(A_n) - F(A_n) = 0.\end{aligned}$$

Now we show that under P_0,

$$\frac{1}{\sqrt{n}} W_{[n\cdot]} \xrightarrow{n\to\infty} B(\cdot) \qquad \text{in law on } D_{\mathbb{R}^d}[0,\infty), \tag{5.14}$$

where $B(\cdot)$ is a $\mathbb{R}^d$-valued Brownian motion with covariance matrix $\mathbf{K}$ given by the first line of (5.11). With (5.14) proved, we can replace $W_{[n\cdot]}$ by $G_{[n\cdot]}$ in (5.14), because of (5.12).

To show (5.14), we observe at first that

$$\begin{aligned}\mathrm{E}_0\biggl[\biggl(\frac{1}{\sqrt{n}}\sup_{1\le k\le [nT]}|W_k - W_{k-1}|\biggr)^4\biggr] &\le \frac{1}{n^2}\sum_{1\le k\le [nT]}\mathrm{E}_0[|W_k - W_{k-1}|^4]\\ \le \frac{1}{n^2}\mathrm{E}_0[(|X_{\tau_1} - v\tau_1| + c_{29})^4]&\\ + \frac{[nT]-1}{n^2}\sup_{a\in\mathbb{I}^{\mathcal{E}}}\mathrm{E}_0^a[(|X_{\tau_1} - v\tau_1| + 2c_{29})^4|D=\infty]&\\ \xrightarrow{n\to\infty} 0 \qquad \text{by (4.12) and (2.30)},&\end{aligned}$$

where we used (5.10), (5.12) and Corollary 3.6 in the second and third line.

Second, by Birkhoff's ergodic theorem (cf. [4], page 341) we get from Theorem 3.8 that $\mathsf{P}_{\tilde{\nu}}$-a.s. and hence P_0-a.s.,

$$\sum_{k=1}^{[nt]}\frac{1}{n}(W_{k+1} - W_k)(W_{k+1} - W_k)^T \xrightarrow{n\to\infty} t\mathsf{E}_{\tilde{\nu}}[(W_2 - W_1)(W_2 - W_1)^T]$$

and the same limit holds true for a sum from $k = 0$ to $[nt]$.

Thereafter, (5.14) follows immediately from the martingale central limit theorem (cf. [5], Theorem 1.4 (a), Remark 1.5, pages 339–340). It remains to show the second equality in (5.11). We show at first that the last two terms in (5.11) are well defined; that is, the series converges in any matrix norm. Let $\|\cdot\|$ be an arbitrary matrix norm, then with the notations of (3.32) we have for $m \ge 1$,

$$\begin{aligned}&\|\mathrm{E}^{\Pi}[(X_{\tau_1} - \tau_1 v)(X_{\tau_{m+1}} - X_{\tau_m} - (\tau_{m+1} - \tau_m)v)^T]\|\\ &\quad = \|\mathsf{E}_{\tilde{\nu}}[(Z_0 - J_0 v)(Z_m - J_m v)^T]\| = \|\mathsf{E}_{\tilde{\nu}}[f(Y_0)(\tilde{R}^m f)(A_0)^T]\|\\ &\quad \le c' \sup_a(\tilde{R}^m|f|)(a)\ \mathsf{E}_{\tilde{\nu}}[|Z_0 - J_0 v|],\end{aligned} \tag{5.15}$$

where $c' > 0$ is a dimension dependent constant. Thereafter it follows now from (5.13) that the rightmost side above is

$$\begin{aligned}&\le c' c_{14} e^{-c_{15}(m-1)} \|\tilde{R} f\|_{L^\infty} \cdot \mathsf{E}_{\tilde{\nu}}[|Z_0 - J_0 v|]\\&\le c_{30} e^{-c_{15} m}.\end{aligned}$$

Consequently, the right-hand side of (5.11) converges in any matrix norm.

To verify the second equality, we put in the definition of W_m, $m = 1, 2$,

$$\begin{aligned}\mathbf{K} &= \mathsf{E}_{\tilde{\nu}}[(W_2 - W_1)(W_2 - W_1)^T]\\&= \mathsf{E}_{\tilde{\nu}}[\{f(Y_2) + F(A_2) - F(A_1)\}\{f(Y_2) + F(A_2) - F(A_1)\}^T]\\&= \mathsf{E}_{\tilde{\nu}}[f(Y_2) f(Y_2)^T] + \mathsf{E}_{\tilde{\nu}}[f(Y_2) F(A_2)^T] + \mathsf{E}_{\tilde{\nu}}[F(A_2) f(Y_2)^T]\\&\quad + \mathsf{E}_{\tilde{\nu}}[F(A_2) F(A_2)^T] - \mathsf{E}_{\tilde{\nu}}\big[F(A_1)\big(f(Y_2) + F(A_2)\big)^T\big]\\&\quad + \mathsf{E}_{\tilde{\nu}}[F(A_1) F(A_1)^T] - \mathsf{E}_{\tilde{\nu}}\big[\big(f(Y_2) + F(A_2)\big) F(A_1)^T\big].\end{aligned}$$

Using the fact that $\tilde{\nu}$ is the invariant distribution of the kernel $\tilde{R}$, and applying the Markov property, we see that the second and third line on the right-hand side of the above equation vanish.

Now putting in the definition of F from (5.10), the second equality of (5.11) follows from (5.15) and Corollary 3.6. This completes our proof. □

Thanks to Lemma 5.2, we can now prove the following.

THEOREM 5.3 (Functional central limit theorem). *Under assumption* (1.1)–(1.5), *the* $D_{\mathbb{R}^d}[0, \infty)$*-valued random variable* $B^n_\cdot$ *defined in* (5.8) *converges under* P_0 *in law to a* d*-dimensional Brownian motion with a nondegenerate covariance matrix*

$$\frac{\mathbf{K}}{\mathrm{E}^{\Pi}[\tau_1]}, \tag{5.16}$$

with $\mathbf{K}$ *given in* (5.11) *and* $\mathrm{E}^{\Pi}[\cdot]$ *defined in* (5.2).

PROOF. Let k_n, $n \ge 0$ be the sequence introduced in (5.4). Then (5.5) and Dini's theorem (cf. [3], page 129) imply that P_0-a.s.,

$$\text{for all } T > 0, \quad \sup_{0 \le t \le T} \left| \frac{k_{[tn]}}{n} - \frac{t}{\mathrm{E}^{\Pi}[\tau_1]} \right| \xrightarrow{n \to \infty} 0. \tag{5.17}$$

Further, for the random variables B^n_t and G_n, respectively, defined in (5.8) and (5.10), we observe that P_0-a.s. for any $T > 0$,

$$\sup_{0 \le t \le T} \left| B^n_t - \frac{G_{k_{[tn]}}}{\sqrt{n}} \right| \le (1 + |v|) \sup_{0 \le k \le k_{[nT]}} \frac{\tau_{k+1} - \tau_k}{\sqrt{n}},$$

and

$$\sup_{0\le k\le k_{[nT]}} \frac{\tau_{k+1}-\tau_k}{\sqrt{n}} \xrightarrow{n\to\infty} 0 \qquad \text{in } \mathrm{P}_0\text{-probability}. \tag{5.18}$$

To see (5.18), we observe that thanks to Corollary 3.6, and since $k_n \le n$, for $u > 0$,

$$\begin{aligned}&\mathrm{P}_0\Big[\sup_{0\le k\le k_{[nT]}} \frac{\tau_{k+1}-\tau_k}{\sqrt{n}} > u\Big]\\ &\qquad \le \mathrm{P}_0[\tau_1 > \sqrt{n}u] + nT \sup_{a\in\mathbb{I}^{\mathcal{E}}} \mathrm{P}_0^a[\tau_1 > \sqrt{n}u | D=\infty] \xrightarrow{n\to\infty} 0,\end{aligned}$$

where we used (4.11) and (2.30) in the last step.

Therefore, the Skorohod-distance of $B_\cdot^n$ and $\frac{G_{k_{[n\cdot]}}}{\sqrt{n}}$ (cf. [5], page 117), tends to 0 in P_0-probability, as $n \to \infty$. From this fact, (5.17) and Lemma 5.2 we obtain that, under P_0, $B_\cdot^n$ converges in law to a d-dimensional Brownian motion with covariance matrix $\frac{\mathbf{K}}{\mathrm{E}^{\Pi}[\tau_1]}$.

What remains to prove is the nondegeneracy of $\mathbf{K}$. If $w^T\mathbf{K}w = 0$ for some $w \in \mathbb{R}^d$, it follows from the first line of (5.11) that

$$\mathsf{P}_{\tilde{\nu}}[w\boldsymbol{.}f(Y_2) = w\boldsymbol{.}F(A_1) - w\boldsymbol{.}F(A_2)] = 1,$$

and since from (5.12) we know that F is bounded, we can find some constant $c_{31} > 0$ such that

$$\mathsf{P}_{\tilde{\nu}}[w\boldsymbol{.}f(Y_2) \in (-c_{31}, c_{31})] = 1. \tag{5.19}$$

Because $\tilde{\nu}$ is the invariant distribution of $\tilde{R}$ we obtain [recall the definition of Π in (5.2)]

$$\begin{aligned}1 &= \mathsf{P}_{\tilde{\nu}}[w\boldsymbol{.}f(Y_1) \in (-c_{31}, c_{31})]\\ &= \Pi[(v\boldsymbol{.}w)\tau_1 \in (X_{\tau_1}\boldsymbol{.}w - c_{31}, X_{\tau_1}\boldsymbol{.}w + c_{31})].\end{aligned} \tag{5.20}$$

Now let $r > 2\sqrt{d}$ and $H = \{z \in \mathbb{Z}^d : \ell\boldsymbol{.}z < r + 2\ell\boldsymbol{.}\tilde{e}\}$. Then for all $x \in \partial H$ we can construct a path in H such that $X_0 = 0$, $X_{S_1} = x$. To see this, we first notice that with the argument in [16], page 102, the set $\{z \in \mathbb{Z}^d : 0 \le \ell\boldsymbol{.}z < r\}$ is connected. Therefore there is a path connecting 0 and $x - 2\tilde{e}$, which remains in $\{z \in \mathbb{Z}^d : 0 \le \ell\boldsymbol{.}z < r\}$ except for the last point. By inserting a loop at each step of this path, which goes back to the previous point and then returns to the current position, we can make sure that X_{S_1} does not occur within $\{z \in \mathbb{Z}^d : 0 \le \ell\boldsymbol{.}z < r\}$. Now letting the modified path go two steps in the direction $\tilde{e}$ after it reaches $x - 2\tilde{e}$, we get a path $(X_n)_{n\ge 0}$ with $X_0 = 0$ and $X_{S_1} = x$.

This and (2.30) together imply that for each $x \in \partial H$ there exists $n \in \mathbb{N}$ such that for all $a \in \mathbb{I}^{\mathcal{E}}$,

$$\mathrm{P}_0^a[X_{\tau_1} = x, \tau_1 = S_1 = n, D = \infty] > 0.$$

Using a nearest neighbor loop of length $2k$, $k \in \mathbb{N}$, inserted at the first jump step, we get from the ellipticity condition (1.1) that for all $k \in \mathbb{N}$ and $a \in \mathbb{I}^{\mathcal{E}}$,

$$\mathrm{P}_0^a[X_{\tau_1} = x,\ \tau_1 = S_1 = n + 2k,\ D = \infty] > 0. \tag{5.21}$$

On the other hand it follows from (5.20) and (5.21) that for $x \in \partial H$, there exists $n \in \mathbb{N}$ such that

$$(2k + n)(v\boldsymbol{.}w) \in (x\boldsymbol{.}w - c_{31}, x\boldsymbol{.}w + c_{31}) \qquad \text{for all } k \in \mathbb{N}.$$

This is only possible when

$$v\boldsymbol{.}w = 0. \tag{5.22}$$

Taking now limits points in ∂H, we observe from (5.20) that

$$w\boldsymbol{.}y = 0 \qquad \text{for all } y \perp \ell, \tag{5.23}$$

hence w is colinear to ℓ. But since $v\boldsymbol{.}\ell > 0$, (5.22) implies that $w = 0$, which completes our proof. □

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