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ON THE EXISTENCE AND CONVERGENCE OF PRICE EQUILIBRIA FOR RANDOM ECONOMIES

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We study an exchange economy comprising n agents, where the excess demands by the agents are random variables. We show that under certain conditions the set of price equilibria is nonempty. We also prove a theorem concerning the convergence of the random price equilibria toward the price equilibria of an associated "expectation economy."

1. Introduction. A central object of study in the analysis of an economic system is the existence of a stabilizing equilibrium price. An equilibrium price is characterized as a price at which the total excess demand equals zero. The classical general existence results are by Arrow and Debreu ([2], [4]).

In the present paper we are concerned with systems called *pure exchange economies*. (For an account, see, e.g., [14] or [15].) In mathematical terms such a system is defined by specifying a subset D of $R^l_+ \doteq \{p = (p^1, \dots, p^l) \in R^l;\ p^1 \geq 0, \dots, p^l \geq 0\}$ and a family of maps

$$z_i\colon D \to R^l, \qquad i = 1, \dots, n,$$

where l and n are given fixed integers. We assume throughout this paper that D is compact and convex, and that its interior $\mathring{D}$ is nonempty. The collection $\mathscr{E} = ((z_i; i = 1, \dots, n), D)$ is called an *exchange economy of size n and with domain D*. The economic interpretation is as follows:

There are $l+1$ commodities traded by n economic agents. The components p^j of the vectors $p \in D$ represent prices for the commodities $j = 1, \dots, l$. The components $z_i^j(p)$ of the vectors $z_i(p) \in R^l$ denote the *individual excess demands* by the agents $i = 1, \dots, n$ for the commodities $j = 1, \dots, l$. The *total excess demand* is defined as the sum

$$z(p) \doteq \sum_{i=1}^n z_i(p) \in R^l.$$

The price of the $l+1$th commodity can be normed in various ways, for example, $p^{l+1} = 1$ or $\sum_{j=1}^{l+1} p^j = 1$, or $\sum_{j=1}^{l+1} (p^j)^2 = 1$. (Under the norming $p^{l+1} = 1$, commodity $l+1$ is commonly referred to as the *numeraire*.) By *Walras's law* the total excess demand for commodity $l+1$ is

$$z^{l+1}(p) = -(p^{l+1})^{-1} p \cdot z(p) = -(p^{l+1})^{-1} \sum_{j=1}^{l} p^j z^j(p). \tag{1.1}$$

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A price $p^* \in D$ is called an *equilibrium price* if the total excess demand at that price is equal to zero:

$$z(p^*) = 0.$$

Note that, then, by (1.1) the excess demand z^{l+1} is zero at p^*, too. The set of all equilibrium prices p^* in D is denoted by π^*. The existence of a price equilibrium can be deduced (under certain boundary conditions for the total excess demand function) using a suitable fixed point theorem (e.g., Brouwer's theorem; cf. Lemma 2.1).

Usually the agents are supposed to be deterministic objects. A more realistic assumption is to allow randomness in their characteristics (initial endowments and utility functions). In this case, the total excess demand is a random vector implying that the price equilibria are random, too. In their pioneering works Hildenbrand [9] and Bhattacharya and Majumdar [3] studied the existence and convergence of price equilibria in a pure exchange economy with random agents. Hildenbrand assumed that the agents are independent and identically distributed, whereas Bhattacharya and Majumdar allowed exchangeability of the joint distribution. An interactive random economic system was studied by Föllmer [7].

The main purpose of the present paper is to prove existence and convergence of price equilibria in random pure exchange economies (Theorems 1 and 3). We do not assume independence or exchangeability. Instead, we formulate conditions in terms of the Laplace transforms of the excess demands. The convergence of the random price equilibria toward the price equilibria of an "expectation economy" is proved using an *entropy estimate for the associated large deviation probabilities* (Theorems 2 and 4). Based on the approach proposed in this paper, the *large deviation theory* and associated *principle of minimum entropy* for random economic systems has been developed [11] and [12]. Finally, we note that the results of this paper could be formulated as well for more general economic systems like economies with production; compare with Remark 2.3 below.

2. Existence of equilibria for random exchange economies. A *random pure exchange economy* (*random economy*, for short) is defined in the same way as a deterministic economy with the exception that the individual excess demands are random variables $\zeta_1(p), \dots, \zeta_n(p)$ depending on the parameter (price) p. (Since we will study sequences of random economies with increasing sizes n in the sequel, we attach the subscript n to their characteristic symbols.) We suppose that all these random variables are defined on some underlying probability space $(\Omega, \mathcal{F}, P)$. Thus in particular, for each $\omega \in \Omega$, the *realization* $\mathcal{E}_n(\omega) \doteq ((\zeta_i(\cdot, \omega);\ i = 1, \dots, n), D)$ is a deterministic pure exchange economy. The (random) total excess demand is denoted by

$$Z_n(p) \doteq \sum_{i=1}^{n} \zeta_i(p).$$

Note that we do not assume that $\zeta_1(p), \ldots, \zeta_n(p)$ ought to be independent or identically distributed.

For a random economy $\mathscr{E}_n$ let $\pi_n^*(\omega)$ denote the (possibly empty) set of equilibrium prices for the realization $\mathscr{E}_n(\omega)$, $\omega \in \Omega$. The set-valued map $\omega \mapsto \pi_n^*(\omega)$ is simply called *the set of equilibrium prices for the random economy* $\mathscr{E}_n$. We will need no measurability assumptions on the map $\pi_n^*(\cdot)$; compare with Remark 2.2. In Theorem 1 we will prove that, under certain conditions, π_n^* is eventually nonempty almost surely; that is, there is a finite (a.s.) random variable $m: \Omega \to N_+ = \{1, 2, \ldots\}$ such that $\pi_n^*(\omega) \neq \varnothing$ for $n \geq m(\omega)$.

Let $c_n: R^l \times D \to R_+$ denote the logarithm of the Laplace transform of the excess demand $Z_n(p)$,

$$c_n(\beta, p) \doteq \log E e^{\beta \cdot Z_n(p)}, \qquad \beta \in R^l,$$

and let

$$c(\beta, p) \doteq \limsup_{n \to \infty} n^{-1} c_n(\beta, p). \tag{2.1}$$

In the large deviation theory, the function c is referred to as the (limiting) *free energy function* of the sequence $(Z_n(p); n = 1, 2, \ldots)$ (see, e.g., [5]). It follows from Hölder's inequality that, for each fixed p, $c(\beta, p)$ is a convex function of the variable β. We shall assume throughout this paper that c is finite at each $\beta \in R^l$ and $p \in D$ and that it defines a twice continuously differentiable map (C^2-map) $c: R^l \times D \to R$. Note that, due to the convexity, the derivative $(\partial^2 c / \partial \beta^2)(\beta, p)$ is positive definite everywhere; compare with condition (3.1).

We have the following result, which comes from the differentiability of the map $c(\cdot, p)$; compare with Lemma 2.2 below:

PROPOSITION 2.1. *There exists* (*a.s.*)

$$z(p) \doteq \lim_{n \to \infty} n^{-1} Z_n(p) \doteq \frac{\partial c}{\partial \beta}(0, p) \quad \textit{for all } p \in D. \tag{2.2}$$

We will call the vector field $z: D \to R^l$ the *expectation economy* and we denote by

$$\pi^* \doteq \{p \in D; z(p) = 0\}$$

its *equilibrium set*. Lemma 2.1 below presents a sufficient condition for π^* being nonempty.

EXAMPLE 2.1. Suppose that, for each $p \in D$, the individual excess demands $\zeta_1(p), \zeta_2(p), \ldots$ form a sequence of independent identically distributed random vectors taking values in R^l. Clearly now

$$n^{-1} c_n(\beta, p) \equiv c(\beta, p) \doteq \log E e^{\beta \cdot \zeta_1(p)}$$

and the partial derivative $z(p) = (\partial c / \partial \beta)((0, p)$ is equal to the expectation of the individual excess demands:

$$z(p) = E \zeta_1(p).$$

The covariance matrix of $\zeta_1(p)$ is equal to the second derivative:

$$\frac{\partial^2 c}{\partial \beta^2}(0; p) = \text{Cov } \zeta_1(p).$$

In the proof of Theorem 1 we will need the following lemma concerning the existence of equilibrium prices for deterministic economies in convex domains. This lemma follows as a corollary of the general theory by Debreu [4]. Since we will need here exactly this particular formulation, we shall for the convenience of the reader equip it with the proof.

Let D be a convex subset of R^l_+ having nonempty interior $\mathring{D}$. For any $p \in \partial D \doteq$ the boundary of D, let $\mathcal{N}(p)$ denote the set of outward unit normals at p, that is, unit vectors pointing outward from D and perpendicular to some tangent plane at p. (Note that in the case of corners there may be several tangent planes and hence several outward unit normals at p.)

LEMMA 2.1. *Suppose that D is a compact convex subset of R^l having nonempty interior $\mathring{D}$. Suppose that z: $D \to R^l$ is a continuous map satisfying the following boundary condition. For all $p \in \partial D$, $u \in \mathcal{N}(p)$:*

$$z(p) \cdot u < |z(p)|. \tag{2.3}$$

Then there is $p^* \in D$ such that

$$z(p^*) = 0.$$

REMARK 2.1. Note that, if (2.3) holds true, then, due to compactness, there is a constant $\delta > 0$ such that

$$z(p) \cdot u \le |z(p)| - \delta \quad \text{for all } p \in \partial D, u \in \mathcal{N}(p). \tag{2.4}$$

PROOF. Suppose first that the boundary ∂D is smooth in the sense that for each point $p \in \partial D$ there exists a unique $u = u(p) \in \mathcal{N}(p)$ and that $z(p) = -u(p)$ at the boundary. Then the map $f(p) \doteq p + \varepsilon z(p)$ maps D into D for $\varepsilon > 0$ small enough. By Brouwer's fixed point theorem this map has a fixed point p^* which is then also a zero point for the map z.

The general case is proved by extending z to the convex set

$$\hat{D} = D \cup \{q \in R^l; q = p + \alpha u,\ p \in \partial D, u \in \mathcal{N}(p), 0 \le \alpha \le 1\}$$

as the function

$$\hat{z}(p + \alpha u) \doteq (1 - \alpha) z(p) - \alpha u$$

(cf. [14], page 336). □

For random economies we have:

THEOREM 1. *Suppose that*:

(2.5) *the expectation economy z satisfies the boundary condition* (2.3) *of Lemma* 2.1;

(2.6) *the realizations $\zeta_i(\cdot, \omega)$: $D \to R^l$ of the individual excess demand maps are continuous almost surely*; *and*

(2.7) *the family* $(\zeta_i(\cdot, \omega);\ i \geq 1,\ \omega \in \Omega)$ *is uniformly* (*over* ω, *a.s.*) *equicontinuous on the boundary ∂D, that means*

$|\zeta_i(p) - \zeta_i(q)| \leq \varepsilon(|p-q|)$ *for all* $i \geq 1$, $p, q \in \partial D$, *a.s.*, *where* $\lim_{h\to 0} \varepsilon(h) = 0$.

Then there is a constant $\theta > 0$ *such that*

$$P\{\pi_n^* = \varnothing\} \leq e^{-n\theta} \quad \textit{eventually}, \tag{2.8}$$

that is, *for all n big enough*. *Consequently*,

$$\pi_n^* \neq \varnothing \quad \textit{eventually } (a.s.). \tag{2.9}$$

COROLLARY 2.1. *Suppose that* $\zeta_1(p), \zeta_2(p), \ldots$ *are i.i.d. as in Example* 2.1, *that* (2.5) *holds true and that the excess demand* $\zeta_1(\cdot, \omega)$ *by agent* 1 *satisfies the continuity conditions* (2.6) *and* (2.7). *Then the conclusions of Theorem* 1 *hold true*.

REMARK 2.2. In the formulation of Theorem 1 we do not postulate any measurability properties for the set functions $\omega \mapsto \pi_n^*(\omega)$. The inequality (2.8) ought to be interpreted as concerning the upper probability measure of the set $\{\pi_n^* = \varnothing\} \doteq \{\omega \in \Omega : \pi_n^*(\omega) = \varnothing\}$; that is, it means that there are measurable sets $A_n \in \mathscr{F}$ such that $\{\pi_n^* = \varnothing\} \subset A_n$ and $P(A_n) \leq e^{-n\theta}$ (eventually).

REMARK 2.3. For economic systems more general than exchange economies (e.g., systems with production, cf. [14], Appendix A) we ought to make assumptions (2.6) and (2.7) for the mean total excess demands $n^{-1}Z_n(p)$, $n = 1, 2, \ldots$. This same remark is valid for the other results of this paper.

REMARK 2.4. The case where the individual excess demands form an exhangeable collection of random variables (like in [3]) does not follow as a corollary of the results of this paper. However, this extension would be possible by utilizing the existing large deviation theory for exchangeable random variables (see, e.g., [1], Theorem 5.1]). In the proof of Theorem 1 we need the following large deviation estimate. For a proof, see, for example, [6], Theorem IV.1. (There it is assumed that $c(\beta)$ exists as a limit. However, as is well known, see, e.g., [1], Theorem 2.1, the lim sup suffices.)

LEMMA 2.2. *Let* $(S_n; n = 1, 2, \ldots)$ *be a sequence of* R^l*-valued random variables with free energy function*

$$c(\beta) \doteq \lim_{n\to\infty} \sup n^{-1} \log E e^{\beta\cdot S_n}.$$

Suppose that the derivative

$$z \doteq c'(0) \in R^l$$

exists. Then for any $\varepsilon > 0$ *there exists* $\eta(\varepsilon) > 0$ *such that*

$$P\{|n^{-1}S_n - z| > \varepsilon\} \le e^{-n\eta(\varepsilon)} \quad \textit{eventually}.$$

Hence (*using the Borel–Cantelli lemma*)

$$\lim_{n\to\infty} n^{-1}S_n = z \quad \textit{almost surely.}$$

PROOF OF THEOREM 1. By Lemma 2.2, for any $p \in \partial D$ and $\varepsilon > 0$,

$$P\{|n^{-1}Z_n(p) - z(p)| > \varepsilon\} \le e^{-n\eta(\varepsilon, p)} \quad \text{eventually,}$$

where $\eta(\varepsilon, p) > 0$ is a constant depending on ε and p. Clearly the family $\{n^{-1}Z_n(p)\}$ is equicontinuous, too; that is,

$$\left|n^{-1}Z_n(p) - n^{-1}Z_n(q)\right| \le \varepsilon(|p - q|).$$

By the compactness of ∂D there is a finite subset $\{q_1, \ldots, q_K\} \subset \partial D$ such that for each $p \in \partial D$ there is some q_i such that

$$|n^{-1}Z_n(p) - n^{-1}Z_n(q_i)| \le \frac{\varepsilon}{3} \quad \text{for all } n \text{ (a.s.).}$$

Since by Lemma 2.2,

$$n^{-1}Z_n(p) \to z(p) \quad \text{for all } p \in \partial D \text{ almost surely,}$$

it follows that $|z(p) - z(q_i)| \le \varepsilon/3$ as well. Consequently,

$$\begin{aligned} &P\Big\{\sup_{p\in\partial D}\left|n^{-1}Z_n(p) - z(p)\right| > \varepsilon\Big\} \\ &\quad\le P\Big\{\max_i\left|n^{-1}Z_n(q_i) - z(q_i)\right| > \frac{\varepsilon}{3}\Big\} \\ &\quad\le e^{-n\eta(\varepsilon)} \quad \text{eventually,} \end{aligned} \tag{2.10}$$

where $\eta(\varepsilon) \doteq \frac{1}{2}\min_i \eta(\varepsilon/3, q_i)$.

Let δ be as in (2.4), that is,

$$\inf_{\substack{p\in\partial D \\ u\in\mathscr{N}(p)}} \left\{|z(p)| - z(p)\cdot u\right\} \ge \delta. \tag{2.11}$$

We apply Lemma 2.1 to the random vector field $n^{-1}Z_n(\cdot)$ and obtain

$$P\{\pi_n^* \neq \varnothing\} \geq P\left\{ \inf_{\substack{p\in\partial D\\ u\in\mathscr{N}(p)}} \left\{ \left|n^{-1}Z_n(p)\right| - n^{-1}Z_n(p)\cdot u \right\} > 0 \right\} \quad \text{(by Lemma 2.1)}$$

$$\geq P\left\{ \sup_{p\in\partial D} \left|n^{-1}Z_n(p) - z(p)\right| \leq \frac{\delta}{3} \right\} \quad \text{[by (2.11)]}$$

$$\geq 1 - e^{-n\eta(\delta/3)} \quad \text{[by (2.10)].}$$

The almost sure convergence follows from the Borel–Cantelli lemma. □

3. Convergence of the random equilibria. Consider a sequence $(\mathscr{E}_n)$ of random economies with expectation economy $z(p) = (\partial c/\partial\beta)(0, p)$ as defined in (2.2). In this section we investigate the convergence of the random equilibria π_n^* toward the equilibrium set π^* of z. We study the convergence in probability as well as almost sure convergence.

Let $C_n \in R^l$ be any sequence of sets. We say that it *converges into a set* $C \subset R^l$ if

$$\varnothing \neq C_n \subset B(C, \varepsilon) \quad \text{eventually, for all } \varepsilon > 0,$$

where $B(C, \varepsilon) \doteq \{y \in R^l; \inf_{x\in C} |y - x| < \varepsilon\}$ denotes the ε-neighborhood of C. We will use the notation $C_n \to C$. In the special case where $C = \{x_0\}$ is a one-point set, we write simply $C_n \to x_0$. This means the same as

$$\varnothing \neq C_n \subset B(x_0, \varepsilon) \quad \text{eventually, for all } \varepsilon > 0.$$

The random equilibria π_n^* are said to *converge into* π^* *almost surely* if

$$\pi_n^*(\omega) \to \pi^* \quad \text{for a.e. } \omega \in \Omega.$$

Associated with the free energy function c defined by (2.1) there is its *convex conjugate function* $\hat{c}$: $R^l \times D \to R_+$ defined as the *Legendre transform* (see [13], Section 26):

$$\hat{c}(v, p) \doteq \sup_{\beta\in R^l} (\beta\cdot v - c(\beta, p)).$$

We denote

$$I(p) \doteq \hat{c}(0, p) = -\inf_{\beta\in R^l} c(\beta, p), \qquad p \in D.$$

The function $I(p)$ turns out to be the *large deviation rate function (entropy function)* for random economies; compare with Theorem 4. We assume in this section that

(3.1) the (positive definite) derivative $(\partial^2 c/\partial\beta^2)(\beta, p)$ is strictly positive definite for all β and p.

It follows from the general properties of convex conjugates ([13], Theorem 26.5) that under the hypothesis (3.1), for each $p \in D$, the *range of the derivative map* $(\partial c/\partial\beta)(\cdot, p)$, that is,

$$\mathscr{R}(p) \doteq \left\{ \frac{\partial c}{\partial \beta}(\beta, p); \beta \in R^l \right\},$$

is an open set in R^l and it equals the interior of the *domain of the convex conjugate map* $\hat{c}(\,\cdot\,, p)$:

$$\mathscr{R}(p) = \big(\hat{\mathscr{D}}(p)\big)^\circ,$$

where

$$\hat{\mathscr{D}}(p) \doteq \big\{ v \in R^l; \hat{c}(v, p) < \infty \big\}.$$

Therefore, if $0 \in \mathscr{R}(p)$, then there is a unique $\beta = \beta(p)$ such that

$$I(p) = -c(\beta(p), p). \tag{3.2}$$

THEOREM 2. *Suppose that*

$$0 \in \mathscr{R}(p) \quad \textit{for all } p \in D. \tag{3.3}$$

Then

$$I \textit{ is a } C^2\textit{-map on } D; \tag{3.4}$$

(3.5) *the equilibrium set* π^* *is equal to the compact set* $\{p \in D; I(p) = 0\}$;

(3.6) $\lim_{|v| \to 0} \hat{c}(v, p) = I(p)$ *uniformly over* $p \in D$.

Before presenting the proof, let us briefly look at the case of i.i.d. individual excess demands: If $\zeta_1(p), \zeta_2(p), \ldots$ are i.i.d. random variables as in Example 2.1, then $\mathscr{R}(p)$ is equal to the interior of the convex hull of the support of the distribution of $\zeta_1(p)$:

$$\mathscr{R}(p) = \big(\mathrm{co}(\mathrm{Supp}\ \zeta_1(p))\big)^\circ. \tag{3.7}$$

Thus we have:

COROLLARY 3.1. *Suppose that* $\zeta_1(p), \zeta_2(p), \ldots$ *are i.i.d. random variables such that*

$$0 \in \big(\mathrm{co}(\mathrm{Supp}\ \zeta_1(p))\big)^\circ \quad \textit{for each } p \in D. \tag{3.8}$$

Then the conclusions of Theorem 2 *hold true.*

Now to the proof of Theorem 2:

PROOF OF THEOREM 2. (i) This follows from the implicit function theorem ([10], Theorem XIV, 2.1) and convex duality (cf. the proof of part (iii)) applied to the map $f(\beta, p) \doteq (\partial c/\partial\beta)(\beta, p)$.

(ii) If $z(p) = (\partial/\partial\beta)c(0, p) = 0$, then $0 = \beta(p)$ is the (unique) minimum point of the map $\beta \mapsto c(\beta, p)$. However, then $I(p) = -c(\beta(p), p) = -c(0, p) = 0$. Conversely, $I(p) = 0$ implies that $\inf_\beta c(\beta, p) = 0 = c(0, p)$. It follows that $(\partial c/\partial\beta)(0, p) = 0$, that is, $p \in \pi^*$. By the continuity of I, π^* is closed.

(iii) It follows from the hypothesis (3.1) and from the compactness of D that for each $p \in D$ there is an open ball $B(0, \varepsilon(p)) \subset \mathscr{R}(p)$, where we can assume $\varepsilon(\cdot) > 0$ to be continuous. Let $U \doteq B(0, \varepsilon/2)$, where $\varepsilon \doteq \min_D \varepsilon(p) > 0$. Then by the implicit function theorem and by the convex duality [13] there is a C^1-map $\beta\colon U \times D \to R^l$ such that

$$\frac{\partial c}{\partial\beta}(\beta(v, p), p) \equiv v$$

and

$$\frac{\partial\hat{c}}{\partial v}(v, p) \equiv \beta(v, p) \quad \text{for all } v \in U,\ p \in D.$$

By the mean value theorem we have

$$\left|\hat{c}(v, p) - I(p)\right| \le \sup_{\substack{w\in\overline{U}\\ p\in D}}\left|\beta(w, p)\right||v| \quad \text{for } v \in U,\ p \in D,$$

which proves the assertion. □

Theorem 3 is the main result of this section:

THEOREM 3. *Suppose that conditions* (2.5), (3.1) *and* (3.3) *hold true. Moreover, suppose that*

(3.9) *the family* $(\zeta_i(\cdot, \omega);\ i \ge 1,\ \omega \in \Omega)$ *is equicontinuous* (*a.s.*) *on the whole domain* D,

that is, condition (2.7) *holds true for all* $p,\ q \in D$. *Then for any* $\varepsilon > 0$ *there is* $\theta(\varepsilon) > 0$ *such that*

$$P\left\{\pi_n^* \ne \varnothing \text{ and } \pi_n^* \subset B(\pi^*, \varepsilon)\right\} \ge 1 - e^{-n\theta(\varepsilon)}.$$

Consequently,

$$\pi_n^* \to \pi^* \quad (a.s.)$$

We postpone the proof until after Theorem 4.

COROLLARY 3.2. *Suppose that* $\zeta_1(p), \zeta_2(p), \ldots$ *are i.i.d. and that conditions* (2.5) *and* (3.8) *are satisfied. Moreover, suppose that* $\zeta_1(\cdot, \omega)$ *satisfies condition* (3.9). *Then the conclusions of Theorem* 3 *hold true.*

The proof of Theorem 3 is based on the following upper bound large deviation result for the random equilibria π_n^*. In particular, this result implies that the function I is a large deviation (upper bound) rate function for the sequence $\{\pi_n^*\}$.

THEOREM 4. *Suppose that conditions* (3.1), (3.3) *and* (3.9) *hold true. Then*

$$\lim_{n\to\infty}\sup\ n^{-1}\log P\{\pi_n^* \cap \{I \geq \delta\} \neq \emptyset\} \leq -\delta \quad \textit{for all } \delta > 0.$$

In the proof of Theorem 4 we need the following large deviation upper bound result, originally by J. Gärtner and R. Ellis:

LEMMA 3.1 ([6], [8]). *Let $(S_n; n = 1, 2, \dots)$ be a sequence of R^l-valued random variables with free energy function c as defined in* (2.1). *Then for all compact sets $F \subset R^l$,*

$$\lim_{n\to\infty}\sup\ n^{-1}\log P\{n^{-1}S_n \in F\} \leq -\inf_{v\in F}\hat{c}(v).$$

PROOF OF THEOREM 4. Fix $\delta > 0$ and $\varepsilon > 0$. Let $\{q_1, \dots, q_d\} \subset \{I \geq \delta\} \cap D$ be a finite set such that for each $p \in \{I \geq \delta\}$ there is q_i satisfying

$$\varepsilon(|p - q_i|) \leq \varepsilon,$$

where $\varepsilon(\cdot)$ is the function defined in condition (3.9). Since the family $\{n^{-1}Z_n\}$ satisfies the equicontinuity condition, too, we can estimate as follows:

$$\begin{aligned} P\{\pi_n^* \cap \{I \geq \delta\} \neq \emptyset\} &= P\{Z_n(p) = 0 \text{ for some } p \in \{I \geq \delta\}\} \\ &\leq P\{|n^{-1}Z_n(q_i)| \leq \varepsilon \text{ for some } q_i\} \\ &\leq d\max_i P\{|n^{-1}Z_n(q_i)| \leq \varepsilon\}. \end{aligned}$$

Consequently,

$$\begin{aligned} &\lim_{n\to\infty}\sup\ n^{-1}\log P\{\pi_n^* \cap \{I \geq \delta\} \neq \emptyset\} \\ &\quad\leq \max_i \lim_{n\to\infty}\sup n^{-1}\log P\{|n^{-1}Z_n(q_i)| \leq \varepsilon\} \\ &\quad\leq -\min_i \inf_{|v|\leq\varepsilon}\hat{c}(v, q_i) \quad \text{(by Lemma 3.1)} \\ &\quad\leq -\min_i I(q_i) + o(1) \quad \text{[by (3.6)]}, \\ &\quad\leq -\delta + o(1). \end{aligned}$$

The assertion follows after letting $\varepsilon \to 0$. □

PROOF OF THEOREM 3. Since I is continuous near π^* (Theorem 2) there is, for any $\varepsilon > 0$, a constant $\delta = \delta(\varepsilon) > 0$ such that

$$\{I < \delta\} \subset B(\pi^*, \varepsilon).$$

Theorems 1 and 4 give the estimate

$$\lim_{n\to\infty}\sup\ n^{-1}\log P\{\pi_n^* = \varnothing \text{ or } \pi_n^* \cap (B(\pi^*, \varepsilon))^c \neq \varnothing\} \leq -\min(\theta, \delta(\varepsilon)),$$

which proves Theorem 3. □

An important special case is the case where only one expected equilibrium price in a subdomain $C \subset D$ exists, that is,

$$\pi^* \cap C = \{p^*\}.$$

We shall refer to an equilibrium price p^* as *stable* if C is compact and convex, $p^* \in \mathring{C}$, p^* is the only equilibrium price in C and $z: C \to R^l$ satisfies the hypothesis (2.3) of Lemma 2.1 on the boundary ∂C.

EXAMPLE 3.1. Suppose that p^* is an equilibrium price for the expectation economy $\mathscr{E} = (z, D)$. Clearly, if the derivative $z'(p) = (\partial^2 c/\partial\beta\partial p)(0, p)$ is negative definite at p^*, that is, $z'(p^*)h \cdot h < 0$ for all nonzero $h \in R^l$, then p^* is stable. For the naturality and interpretation from the economic point of view of this negative definiteness condition, see, for example, [15]. We have the following corollary of Theorem 3:

COROLLARY 3.3. *Suppose that p^* is a stable equilibrium in $C \subset D$. Suppose also that conditions* (3.1), (3.3) *and* (3.9) *hold true on C. Then for all $\varepsilon > 0$, for some $\theta(\varepsilon) > 0$,*

$$P\{\pi_n^* \cap C \neq \varnothing \text{ and } \pi_n^* \cap C \subset B(p^*, \varepsilon)\} \geq 1 - e^{-n\theta(\varepsilon)} \quad \text{eventually},$$

and therefore

$$\pi_n^* \cap C \to p^* \quad (a.s.).$$

4. Examples.

4.1. *Bernoulli economy.* The simplest nontrivial example of a stochastic sequence is the sequence generated by independent coin tossings (Bernoulli sequence). The random exchange economy, which one might call a *Bernoulli economy*, is specified by a given decreasing continuous function

$$\alpha: [0, 1] \to [0, 1].$$

We postulate that the excess demands $\zeta_1(p), \zeta_2(p), \ldots,$ where $p \in [0, 1]$, are i.i.d. taking only the two values 1 and -1 with probabilities

$$P\{\zeta_1(p) = 1\} = \alpha(p),$$
$$P\{\zeta_1(p) = -1\} = 1 - \alpha(p).$$

Then clearly

$$z(p) = E\zeta_1(p) = 2\alpha(p) - 1$$

and thus

$$\pi^* = [a^*, b^*], \quad \text{where } a^* = \min\{p : \alpha(p) = \tfrac{1}{2}\} \text{ and } b^* = \max\{p : \alpha(p) = \tfrac{1}{2}\}.$$

The free energy function is

$$c(\beta, p) = \log\bigl(e^{\beta}\alpha(p) + e^{-\beta}(1 - \alpha(p))\bigr),$$

which attains its minimum at the point

$$\beta = \beta(p) = \frac{1}{2}\log\frac{1 - \alpha(p)}{\alpha(p)}.$$

It follows that the entropy function I is given by

$$\begin{aligned} I(p) &= -c(\beta(p), p) \\ &= \log 2\sqrt{\alpha(p)(1 - \alpha(p))}. \end{aligned}$$

Now the realizations $\zeta_i(p)$ are discontinuous. However, as is easy to see, the results of this paper still hold true; thus, for example, by Theorems 3 and 4, for any $p > b^*$ and any constant $\rho > 2\sqrt{\alpha(p)(1 - \alpha(p))}$,

$$P\bigl\{\pi_n^* \cap [p, 1] \neq \varnothing\bigr\} \leq \rho^n \quad \text{eventually}.$$

Similarly, if $p < a^*$, then

$$P\bigl\{\pi_n^* \cap [0, p] \neq \varnothing\bigr\} \leq \rho^n \quad \text{eventually}.$$

Consequently,

$$\pi_n^* \to [a^*, b^*] \quad \text{a.s.}$$

4.2. *Random Cobb–Douglas economy.* A standard example that has relevance from the point of view of economics theory is provided by the so-called Cobb–Douglas economy: An economy $\mathscr{E}$ comprising the exchange of $l+1$ commodities by n agents is called a *Cobb–Douglas economy* if the demand by each individual agent is determined as a result of maximization of a *Cobb–Douglas utility function*

$$u(x) = \prod_{j=1}^{l+1} (x^j)^{a^j}$$

(see, e.g., [15], page 111). Here $a^1, \dots, a^{l+1}$ are nonnegative parameters satisfying $\sum_{j=1}^{l+1} a^j = 1$. Thus the l-vector $a \doteq (a^1, \dots, a^l)$ belongs to the simplex

$$S^l \doteq \left\{x \in R_+^l; \sum_{j=1}^{l} x^j \leq 1\right\}.$$

When dealing with a Cobb–Douglas economy, it is convenient to use the price norming $\sum_{j=1}^{l+1} p^j = 1$; that is, the price vector $p \doteq (p^1, \dots, p^l) \in R_+^l$ belongs to the simplex S^l. In fact, we shall assume that *the domain D is equal to the whole simplex S^l*. Letting $e \doteq (e^j; j = 1, \dots, l+1) \in R^{l+1}$ denote the agent's *initial endowment,* the resulting demand $d^j(p)$ for the commodity j under the price vector p is equal to

$$d^j(p) = (p^j)^{-1} a^j w(p),$$

where $w(p) \doteq \sum_{k=1}^{l+1} p^k e^k = (p, p^{l+1}) \cdot e$ denotes the *initial wealth* of the agent. Thus a^j will be the share of the initial wealth used to purchase commodity j:

$$p^j d^j(p) = a^j w(p).$$

The corresponding excess demand for commodity j is

$$\begin{aligned} \zeta^j(p) &= d^j(p) - e^j \\ &= (p^j)^{-1} a^j w(p) - e^j. \end{aligned}$$

We shall view $\zeta(p) = (\zeta^1(p), \dots, \zeta^l(p))$ as a vector in R^l. Recall that $\zeta^{l+1}(p)$ is automatically determined by Walras' law; compare with (1.1).

If we allow the parameters $a = (a^j)$ and $e = (e^j)$ to be random vectors, we are concerned with a *random Cobb–Douglas economy*. We shall make the following assumption:

(4.1) The parameter vector pairs $(a_1, e_1), \dots, (a_n, e_n)$ associated with the n individual agents are i.i.d. $S^l \times R^{l+1}$-valued random vectors having a common density function $f(a, e)$ such that $f(a, e) > 0$ for all $a \in S^l$, all $e \in R^{l+1}$ with $|e - 1| < \varepsilon$ for some $\varepsilon = \varepsilon(a) > 0$.

A straightforward calculation shows that, under the assumption (4.1), the common density function $f_\zeta(z, p)$ of the excess demands $\zeta_i(p)$, $i = 1, \dots, n$, satisfies the following condition:

(4.2) For any $\delta > 0$, $f_\zeta(z, p) > 0$ for all $|z| < \delta$, all $p \in D$ such that $p^{l+1} = 1 - \sum_1^l p^j \geq \delta$.

It follows that the positive definiteness condition (3.1) as well as the following condition hold true:

$$0 \in \mathscr{R}(p) \quad \text{for all } p \in D \text{ satisfying } p^{l+1} < 1. \tag{4.3}$$

Let

$$\begin{aligned} m^{jk} &\doteq E(a_1^j e_1^k), \\ \mu^k &\doteq E e_1^k, \qquad j, k \in \{1, \dots, l+1\}. \end{aligned}$$

Then the expectation economy $z(p)$ is given by

$$z^j(p) = (p^j)^{-1} \sum_{k=1}^{l+1} m^{jk} p^k - \mu^j, \qquad j = 1, \dots, k+1.$$

Let $\underline{m} \doteq \min_{j,k} m^{jk}$, and $\overline{\mu} \doteq \max_j \mu^j$. Then

$$z^j(p) \geq \underline{m}(p^j)^{-1} - \overline{\mu} \geq 0$$

whenever $p^j \le \delta_0 \doteq (\overline{\mu})^{-1}\underline{m}$ so that $z(p)$ satisfies the boundary condition (2.3) of Lemma 2.1 on any of the domains $D_\delta \doteq \{p \in D;\ p^j \ge \delta$ for all $j = 1, \dots, l+1\}$ with $0 \le \delta \le \delta_0$. This implies that

$$\pi^* \subset D_{\delta_0}.$$

Since clearly the required continuity conditions hold true on any of the domains D_δ with $0 < \delta \le \delta_0$, we have by Theorem 3 that

$$\pi_n^* \to \pi^* \quad \text{(a.s.)}. \tag{4.4}$$

The free energy function c is given by

$$c(\beta; p) = \log E\exp\left(\sum_{j=1}^{l}(p^j)^{-1}\beta^j a^j \sum_{k=1}^{l+1} p^k e^k - \sum_{j=1}^{l}\beta^j e^j\right),$$

so that the entropy function characterizing the rate in (4.4) is given by

$$I(p) = -c(\beta, p),$$

where $\beta = \beta(p)$ is the solution of the equations

$$\frac{\partial c}{\partial \beta^j}(\beta; p) = 0,$$

namely,

$$E\left[\left((p^j)^{-1}a^j - e^j\right)\exp\left(\sum_{j=1}^{l}(p^j)^{-1}\beta^j a^j \sum_{k=1}^{l+1} p^k e^k - \sum_{j=1}^{l}\beta^j e^j\right)\right] = 0,$$

$$j = 1, \dots, l.$$

In [11] the special case of a Cobb–Douglas economy with deterministic parameters a^j (or deterministic initial endowments e^j) is studied. In these cases, the free energy function c and the entropy function I can be calculated more explicitly.

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