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LIMIT THEOREMS FOR MANDELBROT'S MULTIPLICATIVE CASCADES

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Let $W \geq 0$ be a random variable with $EW = 1$, and let $Z^{(r)}\,(r \geq 2)$ be the limit of a Mandelbrot's martingale, defined as sums of product of independent random weights having the same distribution as W, indexed by nodes of a homogeneous r-ary tree. We study asymptotic properties of $Z^{(r)}$ as $r \to \infty$: we obtain a law of large numbers, a central limit theorem, a result for convergence of moment generating functions and a theorem of large deviations. Some results are extended to the case where the number of branches is a random variable whose distribution depends on a parameter r.

1. Introduction and main results. Let $\mathbb{N}^\star = \{1, 2, \ldots\}$ be the set of positive integers, and let

$$\mathbf{U} = \{\varnothing\} \cup \bigcup_{k=1}^{\infty} (\mathbb{N}^\star)^k$$

be the set of all finite sequences containing the null sequence $\varnothing$. Let $W \geq 0$ be a non-negative random variable with $EW = 1$ and $P(W = 1) \neq 1$, and let $\{W_u : u \in \mathbf{U}\}$ be independent copies of W. For $r = 2, 3, \ldots$, let $Z^{(r)}$ be the Mandelbrot's variable associated with W and parameter r:

$$Z^{(r)} := \lim_{n\to\infty} Y_n^{(r)},$$

where

$$Y_n^{(r)} = \sum_{(i_1,\ldots,i_n)\in\{1,\ldots,r\}^n} \frac{W_{i_1}\cdots W_{i_1\cdots i_n}}{r^n}.$$

It is easily seen that $Z = Z^{(r)}$ satisfies the following distributional equation:

$$\text{(E)} \qquad Z^{(r)} = \frac{1}{r}\sum_{i=1}^{r} W_i Z_i^{(r)},$$

where $Z_i^{(r)}(1 \leq i \leq r)$ are independent random variables having the same distribution as Z, and are also independent of $\{W_i : 1 \leq i \leq r\}$. In terms of Laplace transforms $\phi^{(r)}(t) = E\exp\{tZ^{(r)}\}$, the equation reads

$$\text{(E}'\text{)} \qquad \phi^{(r)}(t) = [E\phi^{(r)}(Wt/r)]^r, \qquad t \leq 0.$$

The model was first introduced by Mandelbrot (1974a, b) to analyse precisely some problems of turbulence, and is referred as Mandelbrot's multiplicative

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cascades. For fixed r, the properties of $Z^{(r)}$ and related subjects have been studied by many authors; see, for example, Kahane and Peyrière (1976), Durrett and Liggett (1983), Guivarc'h (1990) and Holley and Waymire (1992). In particular, by Theorems 1 and 2 of Kahane and Peyrière (1976), we have $EZ^{(r)} = 1$ if $EW \log W < \log r$, and $Z^{(r)} = 0$ almost surely otherwise; in the case where the condition is satisfied, $E[(Z^{(r)})^2] < \infty$ if and only if $E[W^2] < r$. See also Liu (1997a, b), (1998) for more general results and for related topics.

Since in general, it is hopeless to give explicitly the distribution of $Z^{(r)}$, it is desirable to give its asymptotic properties. The purpose of this paper is to give limit theorems for the process $\{Z^{(r)} : r \geq 2\}$ as $r \to \infty$. The following results will be established.

THEOREM 1.1 (A law of large numbers). *If* $EW \log^+ W < \infty$, *then*

$$\lim_{r\to\infty} Z^{(r)} = 1 \quad \textit{in probability}.$$

Notice that $EZ^{(r)} = 1$ for all $r > 1$ sufficiently large. By Sheffé's theorem, we obtain:

COROLLARY 1.1 (Convergence in L^1). *If* $EW \log^+ W < \infty$, *then*

$$\lim_{r\to\infty} Z^{(r)} = 1 \quad \textit{in } L^1.$$

THEOREM 1.2 (A central limit theorem). *If* $E(W^2) < \infty$, *then as* $r \to \infty$,

$$\frac{\sqrt{r}}{\sqrt{EW^2 - 1}}(Z^{(r)} - 1) \textit{ converges in law to the normal law } \mathscr{N}(0, 1).$$

Let $\underline{w} = ess \inf W$ and $\overline{w} = ess \sup W$. Then

$$0 \leq \underline{w} < 1 < \overline{w} \leq +\infty.$$

THEOREM 1.3 (Convergence of moment generating function). *The following assertions are equivalent*:

(i) $\overline{w} < \infty$;
(ii) *for all* $t > 0$,

$$\lim_{r\to\infty} E \exp\{tZ^{(r)}\} = \exp\{t\}.$$

Before stating our results on large deviations, let us recall some elementary properties of the cumulant generating function of W defined by

$$\Lambda(t) := \log Ee^{tW} \leq +\infty \qquad (t \in \mathbb{R}),$$

and its Fenchel-Legendre dual defined by

$$\Lambda^*(x) := \sup_{t\in\mathbb{R}} tx - \Lambda(t) \leq +\infty \qquad (t \in \mathbb{R}).$$

[See Chapter 2 of Dembo and Zeitouni (1998) and the references therein.] The function Λ is convex and continuously differentiable in the interior of its domain of finiteness which contains $(-\infty, 0)$ since W is non-negative, and which is $\mathbb{R}$ if $\overline{w} < \infty$. The function Λ^* is convex and lower semi-continuous; it is decreasing on $(-\infty, 1]$ with $\Lambda^*(x) = \sup_{t\le 0}\ tx - \Lambda(t)$ $(x \le 1)$, and increasing on $[1, \infty)$ with $\Lambda^*(x) = \sup_{t\ge 0}\ tx - \Lambda(t)$ $(x \ge 1)$; it is always non-negative with

$$\Lambda^*(x) \begin{cases} = +\infty, & \text{if } x < \underline{w} \text{ or } x > \overline{w}, \\ = -\log P(W = \underline{w}), & \text{if } x = \underline{w}, \\ = -\log P(W = \overline{w}), & \text{if } x = \overline{w}, \\ \in (0, \infty), & \text{if } \underline{w} < x < 1 \text{ or } 1 < x < \overline{w} < \infty, \\ = 0, & \text{if } x = 1; \end{cases}$$

it is continuous on $[\underline{w}, 1]$ and, similarly, if $\overline{w} < \infty$, then it is also continuous on $[1, \overline{w}]$. (At left endpoints, we mean the continuity from right, and at right endpoints, the continuity from left.)

We are interested in asymptotic behavior of $P(Z^{(r)} \le x)$ and $P(Z^{(r)} \ge x)$ as $r \to \infty$, for all $x > 0$. Notice that by the law of large numbers, we have

$$\lim_{r\to\infty} P(Z^{(r)} \le x) = 1 \ \text{ if } x > 1 \quad \text{ and } \quad \lim_{r\to\infty} P(Z^{(r)} \ge x) = 1 \ \text{ if } x < 1;$$

by the central limit theorem, $\lim_{r\to\infty} P(Z^{(r)} \le 1) = \lim_{r\to\infty} P(Z^{(r)} \ge 1) = 1/2$. Therefore, it suffices to consider the limit behavior of $P(Z^{(r)} \le x)$ for $x \in (0, 1)$, and that of $P(Z^{(r)} \ge x)$ for $x \in (1, \infty)$. (In these cases, the probabilities tend to 0 as r tends to ∞.) We notice that

$$(0, 1) = \bigcup_{k\ge 0}^{\infty} [\underline{w}^{k+1}, \underline{w}^k) \quad \text{if } \underline{w} > 0$$

and

$$(1, \infty) = \bigcup_{k\ge 0}^{\infty} (\overline{w}^k, \overline{w}^{k+1}] \quad \text{if } \overline{w} < \infty.$$

THEOREM 1.4 (Large deviations). *Assume* $EW \log^+ W < \infty$.

(a) *If* $\underline{w} = 0$, *then for any* $x \in (0, 1)$,

$$\lim_{r\to\infty} \frac{-\log P(Z^{(r)} \le x)}{r} = \Lambda^*(x); \tag{1.1}$$

if $\underline{w} > 0$, *then for any* $k \ge 0$ *and any* $x \in [\underline{w}^{k+1}, \underline{w}^k)$,

$$\lim_{r\to\infty} \frac{-\log P(Z^{(r)} \le x)}{r^{k+1}} = \Lambda^*(x\underline{w}^{-k}). \tag{1.2}$$

(b) *If* $\overline{w} < \infty$, *then for any* $k \ge 0$ *and any* $x \in (\overline{w}^k, \overline{w}^{k+1}]$,

$$\lim_{r\to\infty} \frac{-\log P(Z^{(r)} \ge x)}{r^{k+1}} = \Lambda^*(x\overline{w}^{-k}). \tag{1.3}$$

We remark that the limits are strictly positive and finite, except in the case where $x = \underline{w}^k$ $(k = 0, 1, \ldots)$ and $P(W = \underline{w}) = 0$ for (1.2), and in the case where $x = \overline{w}^{k+1}$ $(k = 0, 1, \ldots)$ and $P(W = \overline{w}) = 0$ for (1.3); in these exceptional cases, the exact equivalent of the tail probabilities remains unknown; however we have in the first case,

$$\lim_{r\to\infty} \frac{-\log P(Z^{(r)} \le \underline{w}^{k+1})}{r^{k+1}} = \infty, \quad k = 0, 1, \ldots \tag{1.4a}$$

$$\lim_{r\to\infty} \frac{-\log P(Z^{(r)} \le \underline{w}^{k+1})}{r^{k+2}} = 0, \quad k = 0, 1, \ldots \tag{1.4b}$$

and in the second case,

$$\lim_{r\to\infty} \frac{-\log P(Z^{(r)} \ge \overline{w}^{k+1})}{r^{k+1}} = \infty, \quad k = 0, 1, \ldots \tag{1.5a}$$

$$\lim_{r\to\infty} \frac{-\log P(Z^{(r)} \ge \overline{w}^{k+1})}{r^{k+2}} = 0, \quad k = 0, 1, \ldots. \tag{1.5b}$$

In fact, (1.4a) follows from (1.2) because $\Lambda^*(\underline{w}) = -\log P(W = \underline{w}) = \infty$; (1.4b) also follows from (1.2) because, for all $k \ge 0$ and all $0 < \varepsilon < 1 - \underline{w}$,

$$\limsup_{r\to\infty} \frac{-\log P(Z^{(r)} \le \underline{w}^k)}{r^{k+1}} \le \limsup_{r\to\infty} \frac{-\log P(Z^{(r)} \le (1-\varepsilon)\underline{w}^k)}{r^{k+1}} = \Lambda^*(1-\varepsilon),$$

so that $\limsup_{r\to\infty}(-\log P(Z^{(r)} \le \underline{w}^k))/r^{k+1} = \Lambda^*(1) = 0$ for all $k \ge 0$. A similar argument shows that (1.3) implies (1.5a) and (1.5b). By (1.4b), we see that the formula (1.2) also holds at right endpoints $\underline{w}^k$, $k \ge 0$; similarly, by (1.5b), the formula (1.3) also holds at left endpoints $\overline{w}^k$, $k \ge 0$.

Our theorem gives a "hierarchy" of large deviation principles with different speeds r^k, $k = 1, 2, \ldots$ and corresponding rate functions $\Lambda^*(.\ \overline{w}^{-k+1})$ for the right tail and $\Lambda^*(.\ \underline{w}^{-k+1})$ for the left tail. For the generic interval $[\underline{w}, \overline{w}]$, the same large deviation principle is satisfied by $Z^{(r)} = \frac{1}{r}\sum_{i=1}^r W_i Z_i^{(r)}$ and by $\frac{1}{r}\sum_{i=1}^r W_i$, which is rather natural in view of Theorem 1.1. For the other intervals, we observe a kind of self-similarity.

If $\overline{w} < \infty$, Theorem 1.4 implies that

$$\lim_{r\to\infty} Z^{(r)} = 1 \quad \text{almost surely.}$$

In closing this section, we point out that the problems also arise in the case where the number of branches is a random variable whose distribution, say F_r, depends on a parameter r. However, for simplicity we shall only give an extension of Theorems 1.2 and 1.4(a), which covers the case where F_r is the r-fold convolution of a fixed distribution on $\mathbb{N}^*$: see Section 7. We also mention that the main results of this paper can be extended to the Mandelbrot's measures (of which $Z^{(r)}$ are the masses): see the forthcoming paper Liu and Rouault (1999).

2. Proof of Theorem 1.1: A law of large numbers. Write

$$\varepsilon(t) = \begin{cases} \dfrac{e^{-t} - 1 + t}{t}, & \text{if } t > 0, \\ 0, & \text{if } t = 0. \end{cases} \tag{2.1}$$

Then ε is increasing and continuous on $[0, \infty)$ and satisfies $0 \leq \varepsilon(t) \leq 1$. Write $\Psi_r(t) = \phi_r(-t) = Ee^{-tZ^{(r)}}$ and

$$\Psi_r(t) - 1 + t = t\varepsilon_r(t), \quad t > 0; \quad \varepsilon_r(0) = 0. \tag{2.2}$$

Then

$$\varepsilon_r(t) = EZ^{(r)}\varepsilon(tZ^{(r)}), \quad t \geq 0, \tag{2.3}$$

and

$$E\Psi_r(tWr^{-1}) = 1 - tr^{-1} + tr^{-1}EW\varepsilon_r(tWr^{-1}).$$

By equation (E'), $\log \Psi_r(t) = r \log E\Psi_r(tWr^{-1})$, so that as $r \to \infty$,

$$\log \Psi_r(t) \sim -t\Big(1 - EW\varepsilon_r(tWr^{-1})\Big), \quad t > 0. \tag{2.4}$$

We need to prove that $\Psi_r(t) \to e^{-t}$, and for this we need only to prove that

$$\lim_{r\to\infty} EW\varepsilon_r(tWr^{-1}) = 0, \quad t > 0. \tag{2.5}$$

By the dominated convergence theorem, since $0 \leq \varepsilon_r \leq 1$, it is enough to prove that, for all $x > 0$, $\lim_{r\to\infty} \varepsilon_r(xr^{-1}) = 0$, that is

$$\lim_{r\to\infty} EZ^{(r)}\varepsilon(xr^{-1}Z^{(r)}) = 0. \tag{2.6}$$

Since $EZ^{(r)} = 1$ we have, for all $\eta > 0$,

$$EZ^{(r)}\varepsilon(xr^{-1}Z^{(r)}) \leq \varepsilon(\eta) + EZ^{(r)}1_{\{Z^{(r)} > r\eta x^{-1}\}}. \tag{2.7}$$

Let us prove that for all $a > 0$,

$$\lim_{r\to\infty} EZ^{(r)}1_{\{Z^{(r)} > ar\}} = 0. \tag{2.8}$$

For fixed $r, a > 0$, put

$$h(x) = h_{r,a}(x) = \begin{cases} x(ar)^{-1}, & \text{if } 0 \leq x \leq ar, \\ 1, & \text{if } ar < x. \end{cases}$$

Then h is concave and increasing with $h(0) = 0$ and $0 \leq h \leq 1$. Therefore, by a lemma of Asmussen and Hering [(1983), page 41], if $\gamma_1, \ldots, \gamma_r$ are independent nonnegative random variables, and if $\gamma = \gamma_1 + \cdots + \gamma_r$, then

$$E\gamma h(\gamma) \leq E\sum_{i=1}^{r} \gamma_i h(\gamma_i) + (E\gamma)h(E\gamma). \tag{2.9}$$

Using this inequality for $\gamma_i = W_i Z_i^{(r)} r^{-1}$ and equation (E), we obtain

$$EZh(Z) \leq E\Big(WZ\ h(WZr^{-1})\Big) + h(1), \tag{2.10}$$

where $Z = Z^{(r)}$, W is independent of Z and of $\{W_1, W_2, \ldots\}$. Using (2.9) again for the sum $WZr^{-1} = \sum_1^r WW_i Z_i r^{-2}$, we obtain

$$E\Big(WZh(WZr^{-1})\Big) \le E\Big(WW_1Z\ h(WW_1Zr^{-2})\Big) + h(r^{-1}).$$

Therefore, by (2.10)

$$EZh(Z) \le h(1) + h(r^{-1}) + E\Big(WW_1Z\ h(WW_1Zr^{-2})\Big).$$

Continuing in this way, we obtain for all $k \ge 1$,

$$EZh(Z) \le h(1) + h(r^{-1}) + \cdots + h(r^{-k+1}) + V_k, \tag{2.11}$$

where

$$V_k = E\Big(WW_1 \cdots W_{k-1}Z\ h(WW_1 \cdots W_{k-1}Zr^{-k})\Big), \tag{2.12}$$

Z, W_1, $W_2, \ldots$ being mutually independent. Fix $r > 1$ such that $\log r > EW \log W$. To prove that $\lim_{k\to\infty} V_k = 0$, we use a change of distribution. Let $\tilde{W}$, $\tilde{W}_1, \ldots$ be a sequence of independent random variables with common distribution $P_{\tilde{W}}(dx) = xP_W(dx)$, which are also independent of Z. Then

$$V_k = E\Big(Zh(\tilde{W}\tilde{W}_1 \cdots \tilde{W}_{k-1}Zr^{-k})\Big). \tag{2.13}$$

By the strong law of large numbers, we have almost surely,

$$\begin{aligned} \lim_{k\to\infty} \frac{\log(\tilde{W}r^{-1}) + \cdots + \log(\tilde{W}_{k-1}r^{-1})}{k} &= E\log(\tilde{W}r^{-1}) \\ &= EW\log(Wr^{-1}) < 0. \end{aligned} \tag{2.14}$$

[Remark that by the definition of $\tilde{W}$, $P(\tilde{W} = 0) = 0$.] Therefore, almost surely

$$\lim_{k\to\infty} \tilde{W}\tilde{W}_1 \cdots \tilde{W}_{k-1}r^{-k} = 0.$$

Since h is continuous with $h(0) = 0$ and $0 \le h \le 1$, the dominated convergence theorem yields $V_k \to 0$. Therefore by (2.11), for all $r > \max\{\exp(EW \log W), 1/a\}$,

$$EZh(Z) \le \sum_{k=0}^{\infty} h(r^{-k}) = \frac{1}{a(r-1)}. \tag{2.15}$$

Because $Z = Z^{(r)}$, this gives

$$\lim_{r\to\infty} EZ^{(r)}h(Z^{(r)}) = 0, \quad a > 0. \tag{2.16}$$

Since $1_{\{Z^{(r)} > ar\}} \le h(Z^{(r)})$, (2.16) gives (2.8). Using (2.8) and letting $r \to \infty$ in (2.7), we see that

$$\limsup_{r\to\infty} Z^{(r)}\varepsilon\big(xr^{-1}Z^{(r)}\big) \le \varepsilon(\eta), \ \ \eta > 0. \tag{2.17}$$

Since $\lim_{\eta\to 0} \varepsilon(\eta) = 0$, this gives (2.6), which ends the proof of Theorem 1.1. $\square$

3. Convergence in L^2. The following result will be used in the next section.

THEOREM 3.1. *If $EW^2 < r < \infty$, then*

$$E\left[\left(Z^{(r)}-1\right)^2\right] = \frac{EW^2-1}{r-EW^2}.$$

In particular, as $r \to \infty$,

$$Z^{(r)} \to 1 \quad \text{in } L^2.$$

PROOF. Since the function $f(s) = \log EW^s$ is convex, we have $f(2) - f(1) \geq f'(1)$, which gives $EW \log W \leq \log EW^2$; therefore the condition $EW^2 < r < \infty$ implies $EW \log W < \log r$, so that by Theorems 1 and 2 of Kahane and Peyrière (1976), $EZ^{(r)} = 1$ and $E[(Z^{(r)})^2] < \infty$. For convenience, we write $Z = Z^{(r)}$ and $Z_i = Z_i^{(r)}$. By equation (E), we have consecutively,

$$Z^2 = \frac{1}{r^2}\left[\sum_{1\leq i\leq r} W_i^2 Z_i^2 + \sum_{1\leq i,j\leq r, i\neq j} W_i W_j Z_i Z_j\right],$$

$$\begin{aligned} EZ^2 &= \frac{1}{r^2}\left[rEW^2EZ^2 + \sum_{1\leq i,j\leq r, i\neq j} EW_iEW_jEZ_iEZ_j\right] \\ &= \frac{1}{r}EW^2EZ^2 + \frac{1}{r^2}(r^2-r). \end{aligned}$$

So $E[(Z^{(r)})^2] = E[Z^2] = (r-1)/(r-EW^2)$. Since $E[(Z^{(r)}-1)^2] = E[(Z^{(r)})^2]-1$, this gives the desired conclusions. □

4. Proof of Theorem 1.2: A central limit theorem. Let $r_0 > EW^2$. By Theorem 3.1, for $r \geq r_0$, $EZ^{(r)} = 1$ and $E[(Z^{(r)})^2] = (r-1)/(r-EW^2)$. By equation (E),

$$rZ^{(r)} - r = \sum_{i=1}^{r}(W_iZ_i^{(r)} - 1).$$

Let $S_r\,(r \geq r_0)$ be the above sum, and let $s_r \geq 0$ be defined by

$$s_r^2 = \sum_{i=1}^{r} E[(W_iZ_i^{(r)}-1)^2].$$

We remark that $W_iZ_i^{(r)} - 1\,(i = 1, \ldots, r)$ are independent and identically distributed random variables with $E[W_iZ_i^{(r)} - 1] = 0$, and that

$$s_r^2 = r[E(W^2)E((Z^{(r)})^2) - 1].$$

We shall verify Lindeberg's condition for the sequence $\{S_r : r \geq r_0\}$. For all $\varepsilon > 0$, we have

$$\begin{aligned}
&\sum_{k=1}^{r} \frac{1}{s_r^2} \int_{|W_k Z_k^{(r)} - 1| \geq \varepsilon s_r} [W_k Z_k^{(r)} - 1]^2 dP \\
&\quad = \frac{r}{s_r^2} \int_{|W_1 Z_1^{(r)} - 1| \geq \varepsilon s_r} [W_1 Z_1^{(r)} - 1]^2 dP \\
&\quad = \frac{1}{E(W^2) E((Z^{(r)})^2)) - 1} \int_{A_r} [W_1 Z_1^{(r)} - 1]^2 dP,
\end{aligned}$$

where $A_r = \{|W_1 Z_1^{(r)} - 1| \geq \varepsilon \sqrt{r[E(W^2)E((Z^{(r)})^2) - 1]}\}$. Since

$$[W_1 Z_1^{(r)} - 1]^2 = W_1^2 \, [(Z_1^{(r)})^2 - 1] - 2W_1 \, [Z_1^{(r)} - 1] + (W_1 - 1)^2,$$

and, as $r \to \infty$

$$\begin{aligned}
EW_1^2 |(Z_1^{(r)})^2 - 1| &= EW_1^2 E|(Z^{(r)})^2 - 1| \to 0, \\
E| - 2W_1 [Z_1^{(r)} - 1]| &= 2EW_1 E|Z_1^{(r)} - 1| \to 0, \\
E(W_1 - 1)^2 1_{\{A_r\}} &\to 0
\end{aligned}$$

(the last assertion holds by the dominated convergence theorem, remarking that $1_{\{A_r\}} \to 0$ in L^1 and in probability by Markov's inequality applied to $[W_1 Z_1^{(r)} - 1]^2$), it follows that

$$\lim_{r \to \infty} \sum_{k=1}^{r} \frac{1}{s_r^2} \int_{|W_k Z_k^{(r)} - 1| \geq \varepsilon s_r} [W_k Z_k^{(r)} - 1]^2 dP = 0.$$

So by Lindeberg's theorem, S_r / s_r converges in law to the normal law $\mathscr{N}(0, 1)$. Since $s_r^2 / [r(EW^2 - 1)] \to 1 \, (r \to \infty)$, this implies that, as $r \to \infty$,

$$\frac{\sqrt{r}}{\sqrt{EW^2 - 1}} (Z^{(r)} - 1) \qquad \text{converges in law to } \mathscr{N}(0, 1). \qquad \square$$

5. Proof of Theorem 1.3: Convergence of moment generating function. We prove the following version of Theorem 1.3. It will be applied in the next section to study large deviations.

THEOREM 5.1. *The following assertions are equivalent*:

(i) $\overline{w} < \infty$;
(ii) *for all* $t > 0$; $\lim_{r \to \infty} Ee^{tZ^{(r)}} = e^t$;
(ii′) *for some* $t > 0$, $\lim_{r \to \infty} Ee^{tZ^{(r)}} = e^t$;
(iii) *for all* $t > 0$, $Ee^{Wt} < \infty$ *and* $\lim_{r \to \infty} Ee^{tWZ^{(r)}} = Ee^{Wt}$;
(iii′) *for some* $t > 0$, $Ee^{Wt} < \infty$ *and* $\lim_{r \to \infty} Ee^{tWZ^{(r)}} = Ee^{Wt}$.

PROOF. The implications (ii) $\Rightarrow$ (ii') and (iii) $\Rightarrow$ (iii') are evident. The implication (ii') $\Rightarrow$ (i) is easy, because (a) if $\overline{w} = \infty$ and $EW \log^+ W < \infty$, then for all $r > 1$ with $EW \log W < \log r$, $Z^{(r)}$ cannot have finite moments of all order [cf. Kahane and Peyrière (1976)], so that $Ee^{tZ^{(r)}} = \infty$ for all $t > 0$; (b) if $EW \log^+ W = \infty$, then $Z^{(r)} = 0$ almost surely for all r. The implication (iii') $\Rightarrow$ (i) follows from the same reason. It remains to prove the implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii).

Assume (i). Put $Y_0^{(r)} = 1$ and, for $n > 0$, let $Y_n^{(r)}$ be defined as in Section 1. Write $\phi_n^{(r)}(t) = Ee^{tY_n^{(r)}}$, $n \geq 0$. Then

$$\phi_{n+1}^{(r)}(t) = [E\phi_n^{(r)}(Wt/r)]^r, \qquad n \geq 0.$$

In the following, we shall use an argument of Rösler [(1992), Theorem 6] to give an upperbound of $\sup_{n\geq 0} \phi_n^{(r)}(t)$, valid for all t in a neighborhood of 0 whose length depends linearly on r. [In the context of Rösler (1992), r is fixed.] Fix $K > 0$, and put

$$g_{r,K}(x) = Ee^{(W-1)x+K(W^2-r)x^2}, \qquad x \geq 0.$$

Then $g_{r,K}(0) = 1$, $g'_{r,K}(0) = 0$ and $g''_{r,K}(0) = E[(W-1)^2] + 2KE[W^2 - r]$. Let $r_K > \max\{2, \overline{w}\}$ be an integer sufficiently large such that $g''_{r_K,K}(0) < 0$. Then there is $\eta_K = \eta(K, r_K) > 0$ small enough such that $g''_{r_K,K}(x) < 0$ for all $x \in [0, \eta_K]$. Consequently $g'_{r_K,K}(x) < g'_{r_K,K}(0) = 0$ and $g_{r_K,K}(x) \leq g_{r_K,K}(0) = 1$ if $0 < x \leq \eta_K$. Note that the function $g_{r,K}(x)$ is decreasing in r, we have

$$g_{r,K}(x) \leq 1 \quad \text{if } r \geq r_K \text{ and } 0 \leq x \leq \eta_K.$$

Therefore, if $r \geq r_K$ and if $\phi_n^{(r)}(t) \leq \exp\{t + Kt^2\}$ for all $0 \leq t \leq r\eta_K$, then for these r and t,

$$\begin{aligned}
\phi_{n+1}^{(r)}(t) &= [E\phi_n^{(r)}(Wt/r)]^r \\
&\leq [E\exp\{Wt/r + K(Wt/r)^2\}]^r \quad \text{(notice that } Wt/r \leq t \leq r\eta_K) \\
&= \exp\{t + Kt^2\}[g_{r,K}(t/r)]^r \\
&\leq \exp\{t + Kt^2\}.
\end{aligned}$$

So by induction on n, we have proved that for all $n \geq 0$,

$$Ee^{tY_n^{(r)}} \leq \exp\{t + Kt^2\} \text{ if } r \geq r_K \text{ and } 0 \leq t \leq r\eta_K.$$

Letting $n \to \infty$ and using Fatou's lemma give

$$Ee^{tZ^{(r)}} \leq \exp\{t + Kt^2\} \text{ if } r \geq r_K \text{ and } 0 \leq t \leq r\eta_K.$$

Now letting $r \to \infty$, we see that

$$\limsup_{r\to\infty} E\exp\{tZ^{(r)}\} \leq \exp\{t + Kt^2\}.$$

Since $t > 0$ and $K > 0$ are arbitrary, letting $K \to 0$ gives

$$\limsup_{r\to\infty} Ee^{tZ^{(r)}} \le e^t, \qquad t > 0.$$

On the other hand, by Jensen's inequality we obtain, for all $t > 0$, $E\exp\{tZ^{(r)}\} \ge \exp\{t\}$, so that,

$$\liminf_{r\to\infty} E\exp\{tZ^{(r)}\} \ge \exp\{t\}.$$

Therefore for all $t > 0$, $\lim_{r\to\infty} E\exp\{tZ^{(r)}\} = \exp\{t\}$, and the assertion (ii) is proved. The assertion (iii) can be obtained in a similar way. This ends the proof of Theorem 5.1. □

6. Proof of Theorem 1.4: Large deviations. We use a version of the Gärtner-Ellis theorem (Dembo and Zeitouni 1998) convenient for the study of right tails and left tails. Theorem 6.1 below is a slight modification of Theorem 1 of Biggins and Bingham [(1993), page759] where the authors give a short sketch of proof. To the sake of completeness and for convenience of readers, we present a proof of Theorem 6.1 in the Appendix.

Let $(\nu_r)_{r\in\mathbb{R}_+}$ be a family of probability distributions on $\mathbb{R}$ and let $\{a_r\}$ be a sequence of positive numbers with $\lim_{r\to\infty} a_r = +\infty$. We assume that for some $t_0 \in [0,\infty]$ and for every $t \in [0, t_0)$, as $r \to \infty$,

$$l_r(t) := \frac{1}{a_r}\log\int \exp\{ta_r\xi\}d\nu_r(\xi) \to l(t) < \infty.$$

We remark that l is convex on $(0, t_0)$ as limit of convex functions, and we do not assume anything outside $[0, t_0)$. Denote the left and right derivative of a convex function g by g'_- and g'_+ respectively; the derivative $g'(t)$ exists if and only if $g'_-(t) = g'_+(t)$. For all $x \in \mathbb{R}$, put

$$l^*(x) = \sup\{ux - l(u); u \in [0, t_0)\}.$$

It can be easily checked that $l^*(x) = tx - l(t)$ if $x \in [l'_-(t), l'_+(t)]$, $t \in (0, t_0)$; in particular, $l^*(l'(t)) = l'(t)t - l(t)$ if l is differentiable at $t \in (0, t_0)$. In the case where l is continuously differentiable on $(0, t_0)$, we have $l'_+(0) = \lim_{t\to 0, u>0} l'(u)$, $l'_-(t_0) = \lim_{u\to t_0, u<t_0} l'(u)$ if $t_0 < \infty$, and we put $l'_-(\infty) = l'(\infty) = \lim_{u\to\infty} l'(u)$ if $t_0 = \infty$.

THEOREM 6.1. (a) *For all* $x > l'_+(0)$,

$$\limsup \frac{1}{a_r}\log\nu_r([x, +\infty)) \le -l^*(x),$$

and this bound is strictly negative.

(b) *If* $x = l'(t)$ *for some* $t \in (0, t_0)$, *then for any* $y < x$,

$$\liminf \frac{1}{a_r}\log\nu_r((y, +\infty)) \ge -l^*(x).$$

(c) *If l is continuously differentiable on $(0, t_0)$, then for all $x \in (l'_+(0), l'_-(t_0))$,*

$$\lim \frac{1}{a_r} \log \nu_r([x, \infty)) = -l^*(x).$$

To apply this theorem we need the asymptotic behavior of

$$\Lambda_k^{(r)}(t) := \frac{1}{r^{k+1}} \log E \exp\{tr^{k+1} Z^{(r)}\}, \tag{6.1}$$

for $r \to \infty$ and $k \geq 0$. It is given by the following proposition. Recall that $\Lambda(t) = \log Ee^{tW}$ by definition.

PROPOSITION 6.1. (a) *If $EW \log^+ W < \infty$, then for any $t \leq 0$,*

$$\lim_{r\to\infty} \Lambda_0^{(r)}(t) = \Lambda(t);$$

if additionally $\underline{w} > 0$, then for any $t \leq 0$ and $k \geq 0$,

$$\lim_{r\to\infty} \Lambda_k^{(r)}(t) = \Lambda(t\underline{w}^k).$$

(b) *If $\overline{w} < \infty$, then for any $t \geq 0$ and $k \geq 0$,*

$$\lim_{r\to\infty} \Lambda_k^{(r)}(t) = \Lambda(t\overline{w}^k).$$

PROOF. From the fundamental equation (E) we have by independence,

$$\Lambda_0^{(r)}(t) = \log E \exp\{tWZ^{(r)}\} \tag{6.2}$$

for any $t \leq 0$; it also holds for any $t \in \mathbb{R}$ if $\overline{w} < \infty$.

(a) From (6.2), Theorem 1.1 and the dominated convergence theorem, we obtain

$$\lim_{r\to\infty} \Lambda_0^{(r)}(t) = \Lambda(t), \quad t \leq 0.$$

When $\underline{w} > 0$, we will prove the remaining result by induction on k. The definition (6.1) may be written as

$$\exp\{r^{k+1}\Lambda_k^{(r)}(t)\} = E \exp\{r^{k+1} t Z^{(r)}\},$$

so that, by inserting an independent extra variable W, we get

$$E \exp r^{k+1}\Lambda_k^{(r)}(tW) = E \exp\{r^{k+1} tWZ^{(r)}\}.$$

Applying the formula (6.2) and the definition (6.1) gives

$$E \exp\{r^{k+1} tWZ^{(r)}\} = \exp \Lambda_0^{(r)}(tr^{k+1}) = \exp\{r^{k+1}\Lambda_{k+1}^{(r)}(t)\};$$

together with the preceding equality, this identity yields the important formula :

$$\Lambda_{k+1}^{(r)}(t) = \frac{1}{r^{k+1}} \log E \exp\{r^{k+1}\Lambda_k^{(r)}(tW)\}. \tag{6.3}$$

Assume that for some $k \geq 0$ and for any $t \leq 0$,

$$\Lambda_k^{(r)}(t) \to \Lambda(t\underline{w}^k).$$

By (6.3), for all $t \leq 0$, $\Lambda_{k+1}^{(r)}(t) \leq \Lambda_k^{(r)}(t\underline{w})$; therefore

$$\limsup_{r\to\infty} \Lambda_{k+1}^{(r)}(t) \leq \Lambda(t\underline{w}^{k+1})$$

by the hypothesis of induction. On the other hand, for any $\varepsilon > 0$ and all $t \leq 0$,

$$\begin{aligned}\Lambda_{k+1}^{(r)}(t) &\geq \frac{1}{r^{k+1}} \log E[1_{W<\underline{w}+\varepsilon} \exp\{r^{k+1}\Lambda_k^{(r)}(tW)\}] \\ &\geq \Lambda_k^{(r)}(t(\underline{w}+\varepsilon)) + \frac{1}{r^{k+1}} \log P(W < \underline{w}+\varepsilon).\end{aligned}$$

Hence

$$\liminf \Lambda_{k+1}^{(r)}(t) \geq \Lambda(t(\underline{w}+\varepsilon)\underline{w}^k).$$

Since $\varepsilon > 0$ is arbitrary, this ends the proof of part (a).

(b) Part (b) can be proved along the same line, but instead of the dominated convergence theorem, we use Theorem 5.1 (iii). □

END OF THE PROOF OF THEOREM 1.4. From the definition of Λ, we have

$$\Lambda'(0) = EW = 1, \quad \Lambda'(-\infty) = \underline{w} \quad \text{and} \quad \Lambda'(+\infty) = \overline{w}.$$

For points in the open intervals, (1.1) and (1.2) follow from Theorem 6.1 where ν_r is the distribution of $-Z^{(r)}$, $t_0 = +\infty$ and $a_r = r$ or r^k, together with Proposition 6.2 a); (1.3) follows from a similar argument using Theorem 6.1 where ν_r is the distribution of $Z^{(r)}$. For the endpoints, we shall only show (1.3) for simplicity. For all $k \geq 0$ and for any $0 < \varepsilon < 1 - 1/\overline{w}$, using the proved result (1.3) for internal points, we obtain

$$\begin{aligned}\limsup \frac{1}{r^{k+1}} \log P(Z^{(r)} \geq \overline{w}^{k+1}) &\leq \limsup \frac{1}{r^{k+1}} \log P(Z^{(r)} \geq \overline{w}^{k+1}(1-\varepsilon)) \\ &= -\Lambda^*(\overline{w}(1-\varepsilon)),\end{aligned}$$

so that

$$\limsup \frac{1}{r^{k+1}} \log P(Z^{(r)} \geq \overline{w}^{k+1}) \leq -\Lambda^*(\overline{w}) = \log P(W = \overline{w}).$$

It remains only to prove that, when $P(W = \overline{w}) > 0$,

$$\liminf \frac{1}{r^{k+1}} \log P(Z^{(r)} \geq \overline{w}^{k+1}) \geq \log P(W = \overline{w}). \tag{6.4}$$

Actually from the fundamental equation (E), we have, for any $k \geq 0$,

$$P(Z^{(r)} \geq \overline{w}^{k+1}) \geq P(W = \overline{w})^r \; P\left(\frac{1}{r}\sum_{k=1}^{r} Z_k^{(r)} \geq \overline{w}^k\right). \tag{6.5}$$

Since by independence

$$P\left(\frac{1}{r}\sum_{k=1}^{r} Z_k^{(r)} \geq \overline{w}^k\right) \geq \left[P(Z^{(r)} \geq \overline{w}^k)\right]^r,$$

a first use of (6.5) shows that $\liminf 1/r^{k+1} \log P(Z^{(r)} \geq \overline{w}^{k+1})$ is non decreasing in k for $k \geq 0$; a second use (for $k = 0$) shows that

$$\begin{aligned} &\liminf \frac{1}{r} \log P(Z^{(r)} \geq \overline{w}) \\ &\quad \geq \log P(W = \overline{w}) + \liminf \frac{1}{r} \log P\left(\frac{1}{r}\sum_{k=1}^{r} Z_k^{(r)} \geq 1\right). \end{aligned} \tag{6.6}$$

It remains only to prove that

$$\liminf \frac{1}{r} \log P\left(\frac{1}{r}\sum_{k=1}^{r} Z_k^{(r)} \geq 1\right) \geq 0. \tag{6.7}$$

Notice that $P(\frac{1}{r}\sum_{k=1}^{r} Z_k^{(r)} \geq 1) = P(U_r \geq 0)$, where

$$U_r := \sum_{k=1}^{r} (Z_k^{(r)} - 1). \tag{6.8}$$

For a lower bound of $P(U_r \geq 0)$, we start from the inequality

$$P(U \geq 0) \geq \frac{(EU^2)^2}{4EU^4}, \tag{6.9}$$

which holds for any random variable U with $EU = 0$ and $EU^4 < \infty$ [Billingsley (1986), Theorem 9.2], and we apply it to $U = U_r$. Again by independence, we have

$$\begin{aligned} EU_r^2 &= rE(Z^{(r)} - 1)^2, \\ EU_r^4 &= rE(Z^{(r)} - 1)^4 + 3r(r-1)[E(Z^{(r)} - 1)^2]^2 \\ &= rE(Z^{(r)} - 1)^4 + 3\left(1 - \frac{1}{r}\right)(EU_r^2)^2. \end{aligned}$$

It is known from Theorem 3.1 that $E(Z^{(r)} - 1)^2 = (EW^2 - 1)/(r - EW^2)$, so that as $r \to \infty$,

$$\lim EU_r^2 = EW^2 - 1.$$

To obtain an upper bound for the fourth moment, for simplicity we use the inequality

$$\frac{(Z^{(r)} - 1)^4}{4!} \leq \exp\{|\, Z^{(r)} - 1 \,|\} \leq \exp\{Z^{(r)} + 1\},$$

although this is not optimal. By Theorem 1.3, this inequality implies

$$\limsup_{r\to\infty} E(Z^{(r)} - 1)^4 < \infty.$$

It is then easy to deduce from (6.9) and the above calculations that

$$\liminf_{r\to\infty} \, rP(U_r \geq 0) > 0,$$

which by (6.8) means

$$\liminf_{r\to\infty} \, rP\left(\frac{1}{r}\sum_{k=1}^{r} Z_k^{(r)} \geq 1\right) > 0.$$

So (6.7) is true, and the proof of Theorem 1.4. is complete. □

7. Extension. Our results may be extended to more general cascades where the number of branches is random. For simplicity, we shall generalize only Theorems 1.2 and 1.4(a).

Let $\{N^{(r)} : r \in \mathbb{N}^*\}$ (or $r \in (0, \infty)$) be a family of random variables with values on $\mathbb{N}^*$, whose distribution are denoted by F_r. We will use the following assumptions:

(A1) For every r the mean $m(r)$ and the variance $\sigma^2(r)$ of $N^{(r)}$ are finite and, as $r \to \infty$, $E \log N^{(r)} \to \infty$.

(A2) As $r \to \infty$, $[N^{(r)} - m(r)]/\sigma(r)$ converges in law to the normal law $\mathscr{N}(0, 1)$.

(A3) As $r \to \infty$,

$$\lambda_r(t) := \frac{1}{m(r)} \log \int e^{tx} dF_r(x) \to \lambda(t)$$

uniformly for $t \leq 0$, with λ continuously differentiable on $(-\infty, 0]$ and $\lambda'_-(0) = 1$.

Notice that in (A1), the condition $E \log N^{(r)} \to \infty$ is satisfied if $N^{(r)} \to \infty$ in probability, and that in (A3), we have $\lambda'_{r-}(0) = 1$ for all r, so it is quite possible that $\lambda'_-(0) = 1$. A typical example for which all the conditions (A1)–(A3) hold is the case where for all $r \in \mathbb{N}^*$, F_r is the r-fold convolution of some distribution F on $\mathbb{N}^*$ with finite variance. The theorems which we shall obtain will be applicable in this case. The classical model is the case where F is the Dirac measure δ_1.

Let $\{N_u^{(r)} : u \in \mathbf{U}\}$ be a family of independent random variables with distributions F_r, and independent of $\{W_u : u \in \mathbf{U}\}$. Put

$$Y_n^{(r)} = \sum \frac{W_{i_1} \cdots W_{i_1 \cdots i_n}}{m(r)^n},$$

where the sum is taken over all $(i_1, \ldots, i_n)$ such that $1 \leq i_1 \leq N^{(r)}, 1 \leq i_2 \leq N_{i_1}^{(r)}, \ldots, 1 \leq i_n \leq N_{i_1 \cdots i_{n-1}}^{(r)}$. For fixed r, the sequence $\{Y_n^{(r)}\}$ is a martingale, and its almost sure limit

$$Z^{(r)} = \lim_{n\to\infty} \, Y_n^{(r)}$$

satisfies the distributional equation

$$(\tilde{\mathrm{E}}) \qquad Z^{(r)} = \frac{1}{m(r)} \sum_{i=1}^{N^{(r)}} W_i Z_i^{(r)},$$

where $Z_i^{(r)}$ are independent random variables with the same distribution as $Z^{(r)}$, and are also independent of $(N^{(r)}, W_1, W_2, \ldots)$. In terms of Laplace transforms $\phi^{(r)}(t) = Ee^{tZ^{(r)}}$, the equation reads

$$(\tilde{\mathrm{E}}') \qquad \phi^{(r)}(t) = E[E\phi^{(r)}(Wt/m(r))]^{N^{(r)}}, \quad t \le 0.$$

It is known [see Theorems 3.1 and 5.1 of Liu (1997a)] that $EZ^{(r)} = 1$ if $E(W \log W) < E(\log N^{(r)})$, and $Z^{(r)} = 0$ almost surely otherwise; when the condition is satisfied, $E[(Z^{(r)})^2] < \infty$ if and only if $E[(N^{(r)})^2] < \infty$ and $EW^2 < m(r)$. Therefore, if $E[W^2] < \infty$ and if (A1) holds, then $EZ^{(r)} = 1$ and $E[(Z^{(r)})^2] < \infty$ for all r large enough, remarking that the condition (A1) implies $m(r) \to \infty$.

The following theorem deals with the convergence in L^2 and the central limit theorem. For a random variable X, we denote by Var X its variance.

THEOREM 7.1. *Assume* (A1) *and* $\sigma_0^2 := EW^2 - 1 < \infty$.

(i) *For all r large enough*,

$$(7.1) \qquad \operatorname{Var} Z^{(r)} = \left[\frac{\sigma_0^2}{m(r)} + \frac{\sigma(r)^2}{m(r)^2}\right]\left[1 - \frac{\sigma_0^2 + 1}{m(r)}\right]^{-1}.$$

Consequently, as $r \to \infty$, $Z^{(r)} \to 1$ in L^2 if and only if

$$(7.2) \qquad \sigma(r)/m(r) \to 0.$$

(ii) *Assume* (A2), (7.2) *and that*

$$(7.3) \qquad \lim_{r\to\infty} \sigma^2(r)/m(r) = a^2$$

for some $a \in [0, \infty)$. Then as $r \to \infty$,

$$\frac{\sqrt{m(r)}}{\sqrt{a^2 + \sigma_0^2}}(Z^{(r)} - 1) \quad \textit{converges in law to the normal law } \mathcal{N}(0, 1).$$

PROOF. (i) Let r be sufficiently large such that $EZ^{(r)} = 1$ and $E[(Z^{(r)})^2] < \infty$. From the fundamental equation $(\tilde{\mathrm{E}})$, by a similar argument as in the proof of Theorem 3.1, we get,

$$(7.4) \qquad E[(Z^{(r)})^2] = \frac{E[N(r)^2] - m(r)}{m(r)^2 - m(r)EW^2},$$

from which (7.1) holds.

(ii) For all real t, put

$$u_r = u_r(t) := \left[E \exp\left(\frac{it}{\sqrt{m(r)}\sigma_0} (WZ^{(r)} - 1) \right) \right]^{m(r)}.$$

From equation ($\tilde{\mathrm{E}}$) and the usual decomposition,

(7.5)
$$E \exp\left(\frac{it\sqrt{m(r)}}{\sigma_0} (Z^{(r)} - 1) \right) = E \left[u_r^{\frac{N^{(r)}}{m(r)}} \exp\left(\frac{it}{\sqrt{m(r)}\sigma_0} (N^{(r)} - m(r)) \right) \right].$$

Since $m(r) \to \infty$ and $Z^{(r)} \to 1$ in L^2, a similar argument as in the proof of Theorem 1.2 shows that for all $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{E(W^2)E((Z^{(r)})^2)) - 1} \int_{\tilde{A}_r} [W_1 Z_1^{(r)} - 1]^2 dP = 0, \tag{7.6}$$

where $\tilde{A}_r = \{ \, |W_1 Z_1^{(r)} - 1| \geq \varepsilon \sqrt{m(r)[E(W^2)E((Z^{(r)})^2) - 1]} \, \}$. We claim that (7.6) implies that for all real t,

$$u_r \to u := \exp\left(-\frac{t^2}{2} \right). \tag{7.7}$$

In fact, if $m(r)$ are integers for all r, then (7.6) simply says that

$$\frac{X_{r,1} + \cdots + X_{r,m(r)}}{s(r)} \quad \text{converges in law to } \mathscr{N}(0,1),$$

where for each r, $\{X_{r,i} : i \geq 1\}$ are independent random variables, each distributed as $W_1 Z_1^{(r)} - 1$, and $s(r)^2 = m(r)E[(W_1 Z_1^{(r)} - 1)^2]$, so that (7.7) follows from (7.6) by Lindeberg's theorem [(7.6) is the Lindeberg condition]. In the general case where $m(r)$ are not necessarily integers, (7.7) also follows from (7.6), using the Fourier method of the proof of Lindeberg's theorem; see Feller (1971), Chapter XV.6, proof of Theorem 1. Note that by (7.2), $N^{(r)}/m(r) \to 1$ in L^2, in probability. So that, by Assumption (A2),

$$\left(\frac{N^{(r)}}{m(r)}, \frac{1}{\sqrt{m(r)}} (N^{(r)} - m(r)) \right) \quad \text{converges in law to } \delta_1 \otimes \mathscr{N}(0, a^2),$$

which implies

$$E u^{\frac{N^{(r)}}{m(r)}} \exp\left\{ \frac{it}{\sqrt{m(r)}\sigma_0} (N^{(r)} - m(r)) \right\} \to u \, \exp\left(-\frac{a^2 t^2}{2\sigma_0^2} \right). \tag{7.8}$$

Let (r_n) be any sequence with $r_n \to \infty$, and let (r'_n) be a subsequence such that $N^{(r'_n)}/m(r'_n) \to 1$ almost surely. By the dominated convergence theorem,

$$u_r^{\frac{N^{(r'_n)}}{m(r'_n)}} - u^{\frac{N^{(r'_n)}}{m(r'_n)}} \to 0 \quad \text{in } L^1.$$

Since the sequence (r_n) is arbitrary, this implies that (for all fixed t)

$$u_r^{\frac{N^{(r)}}{m(r)}} - u^{\frac{N^{(r)}}{m(r)}} \to 0 \quad \text{in } L^1. \tag{7.9}$$

Therefore the difference of the right side of (7.5) and the left side of (7.8) tends to 0, so that by (7.8), the right side of (7.5) tends to

$$u \exp\left(-\frac{a^2t^2}{2\sigma_0^2}\right) = \exp\left(-\frac{(\sigma_0^2+a^2)t^2}{2\sigma_0^2}\right).$$

The last statement means that

$$\frac{\sqrt{m(r)}}{\sigma_0}(Z^{(r)}-1) \quad \text{converges in law to } \mathscr{N}\left(0, \frac{\sigma_0^2+a^2}{\sigma_0^2}\right),$$

which ends the proof of Theorem 7.1. □

We now give a large deviations result. In Section 6, the key tool was the function $\Lambda(t) = \log Ee^{tW}$. Here the same role is played by

$$\tilde{\Lambda}(t) := \lambda(\Lambda(t)).$$

With our assumptions, $\tilde{\Lambda}$ is defined on $(-\infty, 0]$, convex, continuously differentiable on $(-\infty, 0]$ and satisfies $\tilde{\Lambda}(0) = 0$ and $\tilde{\Lambda}'_-(0) = 1$. Define

$$\underline{n} := \lambda'(-\infty) \geq 0 \quad \text{and} \quad \underline{\tilde{w}} := \underline{w}\ \underline{n}. \tag{7.10}$$

Then $\tilde{\Lambda}'(-\infty) = \underline{\tilde{w}}$. Notice that for all $t \leq 0$ and all r, $\lambda_r(t) \geq t$ by Jensen's inequality, so that $\lambda(t) \geq t$. Therefore $\underline{n} = \lim_{t\to-\infty} \lambda(t)/t \leq 1$, $0 \leq \underline{\tilde{w}} < 1$. Let

$$\tilde{\Lambda}^*(x) = \sup\{tx - \tilde{\Lambda}(t); t \leq 0\}.$$

Then $0 \leq \tilde{\Lambda}^*(x) \leq \Lambda^*(x)$ for all $x \in (0, 1]$, and in particular $\tilde{\Lambda}^*(1) = 0$.

THEOREM 7.2. *Assume* (A1), (A3), $EW^2 < \infty$ *and* (7.2).

(a) *If* $\underline{\tilde{w}} = 0$, *then for any* $x \in (0, 1)$

$$\lim_{r\to\infty} \frac{-\log P(Z^{(r)} \leq x)}{m(r)} = \tilde{\Lambda}^*(x); \tag{7.11}$$

(b) *if* $\underline{\tilde{w}} > 0$, *then for any* $k \geq 0$ *and any* $x \in (\underline{\tilde{w}}^{k+1}, \underline{\tilde{w}}^k]$,

$$\lim_{r\to\infty} \frac{-\log P(Z^{(r)} \leq x)}{m(r)^{k+1}} = \underline{n}^k \tilde{\Lambda}^*(x\underline{\tilde{w}}^{-k}). \tag{7.12}$$

PROOF. We follow the same line as in the proof of Theorem 1.4 (a) with similar notation. For $k = 0, 1, \ldots$, put

$$\tilde{\Lambda}_k^{(r)}(t) := \frac{1}{m(r)^{k+1}} \log E \exp\{tm(r)^{k+1}Z^{(r)}\}.$$

It suffices to prove that

$$\lim_{r\to\infty} \tilde{\Lambda}_0^{(r)}(t) = \tilde{\Lambda}(t), \quad t < 0, \tag{7.13}$$

and, if $\underline{w} > 0$, then

$$\lim_{r\to\infty} \tilde{\Lambda}_k^{(r)}(t) = \underline{n}^k \tilde{\Lambda}(t\underline{w}^k), \quad t < 0,\ k = 0, 1, \dots. \tag{7.14}$$

In fact, for internal points, (7.11) and (7.12) follow from (7.13) and (7.14), by Proposition (6.1); for $x = \underline{\tilde{w}}^k > 0$, (7.12) also holds by a similar argument as in the proof of (1.4b).

Now from $(\tilde{\mathrm{E}}')$, we have, for all $t < 0$,

$$\tilde{\Lambda}_0^{(r)}(t) = \lambda_r(s_r) \quad \text{where } s_r = \log E \exp(tWZ^{(r)}). \tag{7.15}$$

By Theorem 7.1(i), $Z^{(r)} \to 1$ in probability, so that $s_r \to \Lambda(t)$ by the dominated convergence theorem; therefore (7.13) follows from (7.15) and the uniform convergence of λ_r.

It remains to prove (7.14). Assume $\underline{\tilde{w}} > 0$. If $k = 0$, then (7.14) reduces to (7.13), so that it holds. Assume that it holds for some $k \geq 0$, we shall prove that it also holds for $k+1$. In fact, again by equation $(\tilde{\mathrm{E}}')$,

$$\tilde{\Lambda}_{k+1}^{(r)}(t) = \frac{1}{m(r)^{k+1}} \lambda_r\big(\log E \exp\{m(r)^{k+1} \tilde{\Lambda}_k^{(r)}(tW)\}\big). \tag{7.16}$$

Therefore, for all $t < 0$,

$$\tilde{\Lambda}_{k+1}^{(r)}(t) \leq \frac{1}{m(r)^{k+1}} \lambda_r(t_r), \tag{7.17}$$

where $t_r = m(r)^{k+1} \tilde{\Lambda}_k^{(r)}(t\underline{w})$. By the induction hypothesis, $\lim \tilde{\Lambda}_k^{(r)}(t\underline{w}) = \underline{n}^k \tilde{\Lambda}(t\underline{w}^{k+1})$. Since $\tilde{\Lambda}$ is convex and satisfies $\tilde{\Lambda}(0) = 0$ and $\tilde{\Lambda}'_-(0) = 1$, we have $\tilde{\Lambda}(s) < 0$ for $s < 0$, and consequently $t_r \to -\infty$ as $r \to +\infty$. By the uniform convergence of $\lambda_r(t)$, we have

$$|\lambda_r(t_r) - \lambda(t_r)| \leq \sup_{t \leq 0} |\lambda_r(t) - \lambda(t)| \to 0,$$

so that

$$\lim \frac{\lambda_r(t_r)}{t_r} = \lim \frac{\lambda(t_r)}{t_r} = \underline{n}. \tag{7.18}$$

Therefore (7.17) implies that

$$\limsup_{r\to\infty} \tilde{\Lambda}_{k+1}^{(r)}(t) \leq \lim \frac{\lambda_r(t_r)}{t_r} \lim \tilde{\Lambda}_k^{(r)}(t\underline{w}) = \underline{n}^{k+1} \tilde{\Lambda}(t\underline{w}^{k+1}).$$

The opposite inequality $\liminf_{r\to\infty} \tilde{\Lambda}_{k+1}^{(r)}(t) \geq \underline{n}^{k+1}\tilde{\Lambda}(t\underline{w}^{k+1})$ follows from a similar argument, remarking that for all $\varepsilon > 0$,

$$\tilde{\Lambda}_{k+1}^{(r)}(t) \geq \frac{1}{r^{k+1}}\lambda_r\big(\log E\big[1_{W<\underline{w}+\varepsilon}\exp\cdots\big]\big)$$
$$\geq \frac{1}{r^{k+1}}\lambda_r(t_r'),$$

where

$$t_r' = \log P(W < \underline{w} + \varepsilon) + r^{k+1}\tilde{\Lambda}_k^{(r)}(t(\underline{w} + \varepsilon)).$$

Hence by induction we have proved that (7.14) holds for all $k = 0, 1, \ldots$ and the proof is complete. □

APPENDIX: PROOF OF THEOREM 6.1

PROOF OF THE UPPER BOUND (a). The classical exponential inequality yields that for any $u \in [0, t_0)$ and any $x \in \mathbb{R}$,

$$\nu_r([x, +\infty)) \leq \int \exp\{ua_r(\xi - x)\}d\nu_r(\xi).$$

Taking logarithm and letting $r \to \infty$, we see that

$$\limsup \frac{1}{a_r}\log \nu_r([x, +\infty)) \leq -[ux - l(u)].$$

Since $u \in [0, t_0)$ is arbitrary, it gives the upper bound $-l^*(x)$. Now, $l'_+(0) = \lim l(u)/u$, so that for $x > l'_+(0)$ there exists $u_0 \in (0, t_0)$ such that $l(u_0)/u_0 < x$. This yields $l^*(x) \geq u_0 x - l(u_0) > 0$, which ends the proof of part (a).

PROOF OF THE LOWER BOUND (b). For $t \in (0, t_0)$, we define the new distributions:

$$d\nu_r^t(\xi) = \exp\{ta_r\xi - a_r l_r(t)\}d\nu_r(\xi).$$

For every y and $z > y$ we have

$$\nu_r((y, +\infty)) = \int_{(y,+\infty)} \exp\{-ta_r\xi + a_r l_r(t)\}d\nu_r^t(\xi)$$
$$\geq \int_{(y,z)} \exp\{-ta_r z + a_r l_r(t)\}d\nu_r^t(\xi),$$

so that

(i) $$\frac{1}{a_r}\log \nu_r((y, +\infty)) \geq -tz + l_r(t) + \frac{1}{a_r}\log \nu_r^t((y, z)).$$

To obtain a lower bound of the last term in the right side of (i), we start with the identity

(ii) $$\nu_r^t((y,z)) = 1 - \nu_r^t((-\infty, y]) - \nu_r^t([z, +\infty)).$$

By the definition of ν_r^t, its normalized cumulant generating function is

$$L_r^t(s) = \frac{1}{a_r}\log\int \exp\{sa_r\xi\}d\nu_r^t(\xi) = l_r(t+s) - l_r(t),$$

so that for every $s \in [-t, t_0 - t)$,

$$L_r^t(s) \to l(t+s) - l(t) =: L^t(s), \qquad r \to \infty.$$

Notice that as $r \to \infty$, the normalized cumulant generating function of $d\nu_r^t(-\xi)$ converges to $L^t(-s)$ for $s \in [0, t)$, whose right derivative at 0 is equal to $-(L^t)'_-(0) = -l'_-(t)$; similarly, the normalized cumulant generating function of $d\nu_r^t(\xi)$ converges to $L^t(s)$ for $s \in [0, t_0 - t)$, whose right derivative at 0 is equal to $(L^t)'_+(0) = l'_+(t)$. Therefore, applying part (a) to the measures $d\nu_r^t(-\xi)$ and to $d\nu_r^t(\xi)$, we see that if $y < l'_-(t)$ and $z > l'_+(t)$, then

$$\limsup \frac{1}{a_r}\log \nu_r^t((-\infty, y]) < 0,$$

$$\limsup \frac{1}{a_r}\log \nu_r^t([z, +\infty)) < 0,$$

so that, from (ii),

$$\liminf \frac{1}{a_r}\log \nu_r^t([y, z]) \geq 0.$$

Consequently, by (i), for all t, y, z satisfying $t_0 > t > 0$ and $y < l'_-(t) \leq l'_+(t) < z$,

$$\liminf \frac{1}{a_r}\log \nu_r((y, +\infty)) \geq -tz + l(t).$$

If l is differentiable at t, then letting $z \to l'(t)$ gives

(iii) $$\liminf \frac{1}{a_r}\log \nu_r((y, +\infty)) \geq -tl'(t) + l(t).$$

Since the right side is just $-l^*(l'(t))$, the proof of part (b) is complete.

PROOF OF PART (c). Fix $x \in (l'_+(0), l'_-(t_0))$. In view of a), it suffices to prove that

(iv) $$\liminf \frac{1}{a_r}\log \nu_r([x, +\infty)) \geq -l^*(x).$$

Since l is continuously differentiable on $[0, \infty)$, and l' non decreasing,

$$\mathscr{X} := \operatorname{int}\{y : y = l'(t), t > 0\} = (l'_+(0), l'_-(t_0)).$$

Therefore by part (b), for all $h > 0$ such that $x + h \in \mathscr{X}$,

$$\liminf \frac{1}{a_r} \log \nu_r([x, +\infty)) \geq \liminf \frac{1}{a_r} \log \nu_r((x, +\infty)) \geq -l^*(x + h).$$

Since l^* is convex, it is continuous on $\mathscr{X}$, so that, letting $h \to 0$ in the last inequality gives (iv), which completes the proof of the theorem. □

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