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MOMENT ASYMPTOTICS FOR THE CONTINUOUS PARABOLIC ANDERSON MODEL

BY JÜRGEN GÄRTNER AND WOLFGANG KÖNIG

Technische Universität Berlin

We consider the parabolic Anderson problem $\partial_t u = \kappa\Delta u + \xi(x)u$ on $\mathbb{R}_+ \times \mathbb{R}^d$ with initial condition $u(0,x) = 1$. Here $\xi(\cdot)$ is a random shift-invariant potential having high δ-like peaks on small islands. We express the second-order asymptotics of the pth moment ($p \in [1,\infty)$) of $u(t,0)$ as $t \to \infty$ in terms of a variational formula involving an asymptotic description of the rescaled shapes of these peaks via their cumulant generating function. This includes Gaussian potentials and high Poisson clouds.

0. Introduction and main result.

0.1. *The continuous parabolic Anderson problem.* We consider the parabolic Anderson problem

$$\begin{aligned} \partial_t u(t,x) &= \kappa\Delta u(t,x) + \xi(x)u(t,x), \qquad && (t,x) \in (0,\infty)\times\mathbb{R}^d, \\ u(0,x) &= 1, && x \in \mathbb{R}^d, \end{aligned} \tag{0.1}$$

where $\kappa > 0$ is a diffusion constant and $\xi = \{\xi(x); x \in \mathbb{R}^d\}$ denotes a random potential satisfying some asymptotic properties to be stated below. We write $\langle \cdot \rangle$ for expectation w.r.t. ξ. The objective of the present paper is the asymptotic analysis of $\langle u(t,0)^p\rangle$ as $t \to \infty$ for any $p \in [1,\infty)$. This may also be considered as a contribution to a deeper understanding of the intermittent (i.e., spatially highly irregular) behavior of the random fields $u(t,\cdot)$ as $t \to \infty$.

0.2. *The potential.* We assume that the random potential $\xi = \{\xi(x); x \in \mathbb{R}^d\}$ is shift-invariant and, for simplicity, Hölder continuous. We impose the existence of all positive exponential moments:

$$H(t) = \log\left\langle e^{t\xi(0)}\right\rangle < \infty \quad \text{for all } t > 0. \tag{0.2}$$

Under these assumptions, almost surely, problem (0.1) has a minimal nonnegative solution u which will be considered throughout. This solution is shift-invariant and admits a Feynman-Kac representation. Using this, it is not difficult to see that assumption (0.2) guarantees the existence of all moments $\langle u(t,0)^p\rangle$, $p \in [1,\infty)$, for all $t \geq 0$. The cumulant generating function H will play an important role in the sequel.

Our intuitive assumption is that the potential ξ has high δ-like peaks which are located far from each other. The basic idea is that then, asymptotically as

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$t \to \infty$, the essential contribution to the solution $u(t, \cdot)$ comes from higher and higher and more and more δ-like peaks of ξ.

A mathematical formulation of the presence of such high exceedances with low spatial frequency may be given in terms of a large deviation principle for the properly rescaled shapes of these peaks. We shall do this by making a requirement on the asymptotic behavior of the corresponding cumulant generating function rather than directly stating a large deviation property.

In order to make this precise, we introduce a positive scale function $\alpha(t)$ which tends to zero as $t \to \infty$. We are going to determine the decay rate of the probability for the occurrence of a δ-like peak having height of order $H(t)/t$ on an island having a diameter of order $\alpha(t)$.

To this end, denote by

$$\xi_t(x) = \alpha^2(t)\left(\xi(\alpha(t)x) - \frac{H(t)}{t}\right), \qquad x \in \mathbb{R}^d, \tag{0.3}$$

$t > 0$, the scaled, normalized version of the potential.

We denote by $\mathscr{P}_c(\mathbb{R}^d)$ the set of probability measures on $\mathbb{R}^d$ having compact support. For any compact set $K \subset \mathbb{R}^d$, let $\mathscr{P}(K)$ be the set of those $\mu \in \mathscr{P}_c(\mathbb{R}^d)$ for which $\operatorname{supp}\mu \subset K$. We write

$$(\mu, f) = \int_{\mathbb{R}^d} f(x)\,\mu(dx) \tag{0.4}$$

for any $\mu \in \mathscr{P}_c(\mathbb{R}^d)$ and all μ-integrable functions $f: \mathbb{R}^d \to \mathbb{R}$.

We will assume that the rescaled cumulant generating functions J_t: $\mathscr{P}_c(\mathbb{R}^d) \to [0, \infty)$, $t > 0$, defined by

$$J_t(\mu) = -\frac{1}{\beta(t)} \log\left\langle e^{\beta(t)(\mu, \xi_t)}\right\rangle, \tag{0.5}$$

converge in some sense as $t \to \infty$, where the 'large deviation scale' $\beta(t)$ is given by

$$\beta(t) = \frac{t}{\alpha^2(t)}. \tag{0.6}$$

Note that $J_t(\mu) \geq 0$. Indeed, by Jensen's inequality and Fubini's theorem,

$$\left\langle e^{\beta(t)(\mu, \xi_t)}\right\rangle = \frac{\left\langle e^{t(\mu, \xi(\alpha(t)\cdot))}\right\rangle}{\left\langle e^{t\xi(0)}\right\rangle} \leq \frac{\left\langle (\mu, e^{t\xi(\alpha(t)\cdot)})\right\rangle}{\left\langle e^{t\xi(0)}\right\rangle} = 1. \tag{0.7}$$

Hölder's inequality shows that J_t is a concave function. From the shift-invariance of the potential ξ_t it follows that $J_t(\mu)$ is invariant under spatial shifts of μ.

We are now ready to formulate our main assumption.

ASSUMPTION (J). *As $t \to \infty$, J_t converges to a functional $J: \mathscr{P}_c(\mathbb{R}^d) \to [0, \infty)$ uniformly on $\mathscr{P}(K)$ for each compact set $K \subset \mathbb{R}^d$.*

Let us phrase this differently, using an appropriate notion of convergence in $\mathscr{P}_c(\mathbb{R}^d)$. We will say that $\lim_{t\to\infty}\mu_t = \mu$ *in* $\mathscr{P}_c(\mathbb{R}^d)$ if there exists a compact set $K \subset \mathbb{R}^d$ such that $\mu_t \in \mathscr{P}(K)$ for all $t \geq 0$ and $\mu_t \to \mu$ weakly. Then Assumption (J) is equivalent to

$$\lim_{t\to\infty}\mu_t = \mu \text{ in } \mathscr{P}_c(\mathbb{R}^d) \quad \Rightarrow \quad \lim_{t\to\infty} J_t(\mu_t) = J(\mu). \tag{0.8}$$

Since the functionals J_t are nonnegative, concave, and continuous on $\mathscr{P}_c(\mathbb{R}^d)$, the same is true for the limiting functional J.

For a certain part of our results we also need to assume the following.

ASSUMPTION (H). *For all* $\varepsilon > 0$,

$$H\left(t + e^{-\varepsilon\beta(t)}\right) - H(t) \leq O\left(e^{\varepsilon\beta(t)}\right) \qquad \text{as } t \to \infty. \tag{0.9}$$

This is a rather weak regularity assumption which is satisfied in particular if $\limsup_{t\to\infty} t^{-1}\log H'(t) < \infty$.

0.3. *Main result.* We introduce the Donsker-Varadhan functional $\mathscr{S}_d$: $\mathscr{P}_c(\mathbb{R}^d) \to [0,\infty]$ given by

$$\mathscr{S}_d(\mu) = \begin{cases} \left\| (-\Delta)^{\frac{1}{2}} \sqrt{\dfrac{d\mu}{dx}} \right\|_2^2, & \text{if } \mu \ll dx \text{ and } \sqrt{\frac{d\mu}{dx}} \in \mathscr{D}((-\Delta)^{\frac{1}{2}}), \\ \infty, & \text{else,} \end{cases} \tag{0.10}$$

where $-\Delta$ is considered as nonnegative definite self-adjoint operator in $L^2(\mathbb{R}^d)$ and $\mathscr{D}((-\Delta)^{1/2})$ denotes the domain of its square root. Note that $\mathscr{S}_d(\mu)$ is nothing but the Dirichlet form of the Laplacian at $(d\mu/dx)^{1/2}$.

The following quantity turns out to determine the second-order part of the exponential growth of the moments:

$$\chi = \chi(\kappa, d) = \inf\{\kappa\mathscr{S}_d(\mu) + J(\mu) : \mu \in \mathscr{P}_c(\mathbb{R}^d)\} \geq 0. \tag{0.11}$$

THEOREM 1. *Fix* $p \in [1,\infty)$ *arbitrarily and suppose that Assumption* (J) *is satisfied.*

(i) *As* $t \to \infty$,

$$\langle u(t,0)^p \rangle \geq \exp\{H(pt) - \beta(pt)(\chi + o(1))\}. \tag{0.12}$$

(ii) *If* $p = 1$ *or, in addition, Assumption* (H) *is satisfied, then, as* $t \to \infty$,

$$\langle u(t,0)^p \rangle \leq \exp\{H(pt) - \beta(pt)(\chi + o(1))\}. \tag{0.13}$$

0.4. *Discussion.*

0.4.1. *Dependence on covariance.* Let us briefly consider the important particular case when J depends on the covariance only.

To this end, we introduce the mean vector and covariance matrix of $\mu \in \mathscr{P}_c(\mathbb{R}^d)$ given by

$$m(\mu)=\int_{\mathbb{R}^d} x\mu(dx) \quad \text{and} \quad \text{cov}(\mu)=\int_{\mathbb{R}^d}(x-m(\mu))(x-m(\mu))^{\mathrm{T}}\,\mu(dx), \tag{0.14}$$

respectively. Let $\mathscr{M}_d$ (resp. $\mathscr{M}_d^+$) denote the set of symmetric nonnegative (resp. positive) definite $d \times d$–matrices.

If J is a function of the covariance matrix only, that is,

$$J(\mu) = j(\text{cov}(\mu)), \qquad \mu \in \mathscr{P}_c(\mathbb{R}^d), \tag{0.15}$$

for some function $j\colon \mathscr{M}_d \to [0,\infty)$, then χ takes the form

$$\chi = \inf\left\{\frac{\kappa}{4}\,\text{tr}\,(\Gamma^{-1}) + j(\Gamma)\colon \Gamma \in \mathscr{M}_d^+\right\}, \tag{0.16}$$

where $\text{tr}\,(A)$ stands for the trace of a quadratic matrix A. The proof will be given in Section A.1 in the Appendix.

In two of our examples presented in Section 4 we will use (0.16) to identify χ explicitly.

0.4.2. *Positivity of* χ. If $\chi = 0$, then Theorem 1 states that the second-order term of $\langle u(t,0)^p\rangle$ is $e^{o(\beta(pt))}$, which does not provide much information. But this occurs only if the functional J is trivial:

$$\chi = 0 \quad \Leftrightarrow \quad J \equiv 0. \tag{0.17}$$

The proof of this fact will be given in Appendix 4.2. If the field ξ is bounded from above almost surely then the assumption $\alpha(t) \to 0$ always forces J to vanish identically. Our focus is on potentials which are unbounded from above. Loosely speaking, J will be nondegenerate for a suited scale function α, if the field ξ has 'relevant' high δ-like peaks which are located far from each other. For a more detailed explanation we refer to the heuristics below.

0.4.3. *Heuristic derivation of Theorem* 1. We now reveal the mechanism behind the asymptotics in Theorem 1. Here we will choose a more analytically oriented view while our proof given in the subsequent sections will merely rely on probabilistic arguments.

Assumption (J) suggests that the scaled fields ξ_t satisfy a large deviation principle with scale $\beta(t)$ and rate function $I\colon \mathscr{F} \to [0,\infty]$ given by the Legendre transform of $-J$:

$$I(\varphi) = \sup_{\mu}\,[(\mu,\varphi) + J(\mu)]\,. \tag{0.18}$$

Thereby $\mathscr{F}$ is a suitable space of functions $\mathbb{R}^d \to \mathbb{R}$ decaying to $-\infty$ at infinity. In other words, the number $e^{-\beta(t)I(\varphi)}$ is the relative spatial frequency of peaks with shape close to φ for the scaled field ξ_t (asymptotically as $t \to \infty$). This is the place where the geometry of the 'high δ-like peaks' of the potential enters the picture.

Next, we approximate $u(t,\cdot)$ by the solution $\widetilde{u}(t,\cdot)$ of the corresponding initial-boundary value problem with Dirichlet boundary condition in a box whose length $R(pt)$ depends on t and goes to infinity only slightly faster than $pt/\alpha(pt) = e^{o(\beta(pt))}$. We obtain

$$\langle u(t,0)^p\rangle \sim \langle \widetilde{u}(t,0)^p\rangle. \tag{0.19}$$

A Fourier expansion of $\widetilde{u}(t,\cdot)$ with respect to the (random) eigenfunctions of the operator $\kappa\Delta+\xi$ shows that a good approximation for $\widetilde{u}(t,0)$ is provided in terms of its principal (i.e., largest) eigenvalue $\widetilde{\lambda}(\xi)$ in the mentioned box with Dirichlet boundary condition:

$$\widetilde{u}(t,0) \approx e^{t\widetilde{\lambda}(\xi)}. \tag{0.20}$$

This implies

$$\langle \widetilde{u}(t,0)^p\rangle \approx \left\langle e^{pt\widetilde{\lambda}(\xi)}\right\rangle. \tag{0.21}$$

Because of (0.3) and the scaling properties of the Laplacian, the principal eigenvalue scales like $t\widetilde{\lambda}(\xi) = \beta(t)\widetilde{\lambda}(\xi_t) + H(t)$. Hence,

$$\langle e^{pt\widetilde{\lambda}(\xi)}\rangle = e^{H(pt)}\left\langle e^{\beta(pt)\widetilde{\lambda}(\xi_{pt})}\right\rangle. \tag{0.22}$$

Since those local peaks of ξ_{pt} that give the main contribution to the expectation on the r.h.s. of (0.22) will be shown to be located far from each other, $\widetilde{\lambda}(\xi_{pt})$ is close to the maximum of the local principal eigenvalues corresponding to the single peaks. Since their number will turn out to be $e^{o(\beta(pt))}$, and because of the shift-invariance, we find that

$$\left\langle e^{\beta(pt)\widetilde{\lambda}(\xi_{pt})}\right\rangle \approx \langle e^{\beta(pt)\lambda_{\mathrm{f}}(\xi_{pt})}\rangle. \tag{0.23}$$

Here $\lambda_{\mathrm{f}}(\varphi)$ denotes the principal eigenvalue of $\kappa\Delta+\varphi$ in some large, but *fixed* box with Dirichlet boundary condition.

Now an application of the Laplace-Varadhan method with respect to the above large deviation principle yields for the local shapes of ξ_{pt}

$$\left\langle e^{\beta(pt)\lambda_{\mathrm{f}}(\xi_{pt})}\right\rangle \approx e^{-\beta(pt)\chi} \tag{0.24}$$

with

$$\chi = \inf_{\varphi}[I(\varphi) - \lambda_{\mathrm{f}}(\varphi)]. \tag{0.25}$$

Putting together the steps (0.19) and (0.21–0.24), we arrive at the assertion of Theorem 1:

$$\langle u(t,0)^p\rangle \approx e^{H(pt)-\beta(pt)\chi}. \tag{0.26}$$

It remains to check that (0.25) coincides with (0.11). Using the variational representation

$$\lambda_{\mathrm{f}}(\varphi) = \sup_{\mu}[(\mu, \varphi) - \kappa \mathscr{S}_d(\mu)] \tag{0.27}$$

of the principal eigenvalue, we indeed obtain

$$\chi = \inf_{\mu} \left\{ \kappa \mathscr{S}_d(\mu) + \inf_{\varphi}[I(\varphi) - (\mu, \varphi)] \right\} = \inf_{\mu} \{ \kappa \mathscr{S}_d(\mu) + J(\mu) \}. \tag{0.28}$$

The above considerations make it plausible that asymptotically the shapes of the peaks of ξ_{pt} that are relevant for the pth moment are given by the solutions φ of the variational problem (0.25). Moreover, the shapes of the δ-like high peaks of $u(t, \cdot)$ that contribute most to $\langle u(t, 0)^p \rangle$ are correspondingly rescaled time-dependent multiples of the positive eigenfunctions corresponding to the maximizing potential shapes φ. In particular, the relevant shapes of the (δ-like) peaks of $\xi(\cdot)$ and $u(t, \cdot)$ are asymptotically deterministic. Their frequency and geometry essentially determine the second-order asymptotics of the moments.

Thus, the proof of Theorem 1 basically consists of the four main steps (0.19), (0.22), (0.23) and (0.24):

- (i) making the space finite (but still time-dependent);
- (ii) using a Fourier expansion and scaling properties;
- (iii) removing the time-dependence of the box ("compactification");
- (iv) applying large deviation arguments.

Our rigorous proof given below will follow these steps, but some of them will be carried out in a more probabilistic setting. For example, large deviation arguments will be applied in a 'dual' setting, i.e., for the occupation times measures of Brownian motion in the Feynman-Kac formula rather than for local peaks of the field ξ.

0.4.4. *General remarks.* For a general discussion of intermittency and related references we refer to Carmona and Molchanov (1994) and the lectures by Molchanov (1994) and also to the monograph by Sznitman (1998) in which 'negative' Poisson clouds are treated thoroughly.

For Gaussian and Poisson fields, rough logarithmic asymptotics for the moments and the almost sure behavior of $u(t, 0)$ have been derived by Carmona and Molchanov (1995). In a forthcoming paper, Gärtner, König and Molchanov (2000) investigate the second-order term of the almost sure asymptotics.

For the *spatially discrete* Anderson model with i.i.d. potential $\{\xi(x); x \in \mathbb{Z}^d\}$, the second-order asymptotics of the moments $\langle u(t, 0)^p \rangle$ with $p \in \mathbb{N}$ has been investigated by Gärtner and Molchanov (1998). The more subtle asymptotics of the correlation $\langle u(t, 0)u(t, x) \rangle / \langle u(t, 0)^2 \rangle$ for $\xi(x)$ having a double exponential tail has been treated by Gärtner and den Hollander (1999). For several reasons, the approach used in these papers is not directly applicable to our context. For example, no scaling of the size of peaks appears; due to spatial discreteness, δ-like peaks are concentrated on single lattice sites. Secondly, the method of compactification used there fails to work for correlated fields. Also, our difficulties to handle higher moments do not appear in the discrete case.

We mention that our emphasis is not mainly on computing concrete asymptotic formulas for Gaussian and Poisson potentials [which can be done by simpler and more direct calculations like in Pastur and Figotin (1992)], but on revealing the mechanism how high peaks of the potential contribute to the asymptotic behavior of the moments.

0.4.5. *Outline of the paper.* In Section 1 we put down some notation and collect preliminary facts about the initial-boundary value problem in finite boxes and about the Feynman-Kac formula. Sections 2 and 3 are devoted to the proof of parts (i) and (ii) of Theorem 1, respectively. The important examples of a Gaussian and a compound Poisson field are presented in Section 4. The Appendix contains the proofs of (0.16) and (0.17).

1. Preparations for the proof of Theorem 1. The aim of this section is to introduce notation and to collect some simple and well-known facts.

1.1. *Initial-boundary value problem.* We are going to introduce the solution of the parabolic partial differential equation in the first line of (0.1) in a finite box with zero boundary condition, for general potential. Given $r > 0$, let $Q_r = (-r, r)^d$ be the centered cube with side length $2r$.

Let $V: \mathbb{R}^d \to \mathbb{R}$ be an arbitrary Hölder continuous potential. For $r > 0$, let u_r^V be the solution of the initial-boundary value problem for the operator $\kappa\Delta + V$ in the box Q_r with zero boundary condition and initial datum identically equal to one, that is,

$$\begin{aligned} \partial_t u_r^V(t,x) &= \kappa\Delta u_r^V(t,x) + V(x)u_r^V(t,x), && (t,x) \in (0,\infty)\times Q_r,\\ u_r^V(0,x) &= 1, && x \in Q_r,\\ u_r^V(t,x) &= 0, && (t,x)\in(0,\infty)\times\partial Q_r. \end{aligned} \tag{1.1}$$

We trivially extend $u_r^V(\cdot,\cdot)$ to a function on $[0,\infty)\times\mathbb{R}^d$. In order to stress the dependence on the potential we shall write in the sequel u^ξ instead of u for the solution of (0.1). Note that a.s. for $0 < r < R$ (picking $V = \xi$),

$$u_r^\xi(t,x) \le u_R^\xi(t,x) \le u^\xi(t,x), \qquad (t,x)\in(0,\infty)\times Q_r. \tag{1.2}$$

Taking into account (0.3), the reader easily verifies the following scaling relation which is valid for each $R > 0$ and $t > 0$:

$$u^\xi_{R\alpha(t)}(s,x) = e^{sH(t)/t} u_R^{\xi_t}\left(\frac{s}{\alpha^2(t)}, \frac{x}{\alpha(t)}\right), \qquad (s,x)\in(0,\infty)\times Q_{R\alpha(t)}. \tag{1.3}$$

Let $\lambda_1 > \lambda_2 \ge \lambda_3 \ge \cdots$ be the eigenvalues of the operator $\kappa\Delta + V$ in $L^2(Q_r)$ with zero boundary condition. We also write $\lambda_k = \lambda_k^V(Q_r)$ for the kth eigenvalue to emphasize its dependence on the potential V and the box Q_r. For each $R > 0$, the eigenvalues have the scaling property

$$t\lambda_k^\xi(Q_{R\alpha(t)}) = \beta(t)\lambda_k^{\xi_t}(Q_R) + H(t), \qquad t > 0,\ k\in\mathbb{N}. \tag{1.4}$$

Let $(\mathrm{e}_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2(Q_r)$ consisting of corresponding eigenfunctions $\mathrm{e}_k = \mathrm{e}_k^V(Q_r)$.

We also need the fundamental solution $p_r^V(t,\cdot,y)$ of the initial-boundary value problem, that is, the solution of (1.1) with the initial condition $u_r^V(0,\cdot) = \mathbb{1}$ replaced by $p_r^V(0,\cdot,y) = \delta_y(\cdot)$ for each $y \in Q_r$. We have the Fourier expansion

$$p_r^V(t,x,y) = \sum_{k=1}^{\infty} \exp(t\lambda_k)\mathrm{e}_k(x)\,\mathrm{e}_k(y). \tag{1.5}$$

According to Mercer's theorem [see, e.g., Joergens (1982), Theorem III, 8.11], this series converges uniformly in $x, y \in Q_r$. In particular, we also have the Fourier expansion

$$u_r^V(t,\cdot) = \sum_{k=1}^{\infty} \exp(t\lambda_k)(\mathrm{e}_k, \mathbb{1})_r \mathrm{e}_k(\cdot), \tag{1.6}$$

where we write $(\cdot,\cdot)_r$ for the inner product in $L^2(Q_r)$.

1.2. *Feynman-Kac formula.* We are going to express the solutions u^ξ of (0.1) and u_r^V of the initial-boundary value problem (1.1) in terms of the Feynman-Kac formula. Still we assume that $V:\mathbb{R}^d \to \mathbb{R}$ is some Hölder continuous potential.

Let $\{W_t\}_{t\geq 0}$ be Brownian motion in $\mathbb{R}^d$ with generator $\kappa\Delta$. Denote the underlying probability and expectation by $\mathbb{P}_x$ resp. $\mathbb{E}_x$ when $W_0 = x \in \mathbb{R}^d$. Then we have the Feynman-Kac formula

$$u^\xi(t,x) = \mathbb{E}_x \exp\left\{\int_0^t \xi(W_u)\,du\right\}, \qquad (t,x) \in (0,\infty)\times\mathbb{R}^d. \tag{1.7}$$

Next, introduce the normalized occupation time measure of Brownian motion by

$$L_t(dx) = \frac{1}{t}\int_0^t \mathbb{1}\{W_u \in dx\}\,du, \qquad t \in (0,\infty). \tag{1.8}$$

Note that L_t is a random element of $\mathscr{P}_c(\mathbb{R}^d)$. With this notation (1.7) takes the form

$$u^\xi(t,x) = \mathbb{E}_x \exp(t(L_t,\xi)), \qquad (t,x) \in (0,\infty)\times\mathbb{R}^d. \tag{1.9}$$

In order to represent the solution u_r^V of (1.1) in terms of Brownian motion we introduce the stopping time of the first exit from Q_r:

$$\tau_r = \inf\{t \geq 0 : W_t \notin Q_r\}. \tag{1.10}$$

Then, for all $r > 0$ and $(t,x) \in (0,\infty)\times\mathbb{R}^d$, we have

$$\begin{aligned} u_r^V(t,x) &= \mathbb{E}_x \exp\left\{\int_0^t V(W_u)\,du\right\}\mathbb{1}\{\tau_r > t\} \\ &= \mathbb{E}_x \exp(t(L_t,V))\,\mathbb{1}\{\operatorname{supp} L_t \subset Q_r\}. \end{aligned} \tag{1.11}$$

The analogous Feynman-Kac representation for the fundamental solution $p_r^V(t, x, y)$ is in terms of Brownian bridge instead of free Brownian motion:

$$p_r^V(t, x, y) = \mathbb{E}_x \exp(t(L_t, V)) \mathbb{1}\{\operatorname{supp} L_t \subset Q_r\}\, \delta_y(W_t). \tag{1.12}$$

2. Proof of Theorem 1(i): The lower bound. Suppose that Assumption (J) is satisfied, and let $p \in [1, \infty)$ be arbitrary. In this section we prove (0.12) by carrying out the four steps listed at the end of subsection 0.4.3. These steps are handled at the following places: the simple steps (i) and (iii) in (2.1), step (ii) in (2.4), and step (iv) below (2.4).

As before, we write u^ξ for the solution of the Anderson problem (0.1). Given $R > 0$, let $I_{pt} = Q_{R\alpha(pt)}$ be the centered cube with side length $2R\alpha(pt)$, and let $|I_{pt}|$ denote its Lebesgue measure. We use the shift-invariance of $u^\xi(t, \cdot)$, Jensen's inequality and the bound (1.2) to estimate, for large t,

$$\begin{aligned} \langle u^\xi(t,0)^p \rangle &= \left\langle \frac{1}{|I_{pt}|} \int_{I_{pt}} u^\xi(t,x)^p \, dx \right\rangle \\ &\geq \left\langle \left(\frac{1}{|I_{pt}|} \int_{I_{pt}} u^\xi(t,x) \, dx \right)^p \right\rangle \geq \left\langle (u^\xi_{R\alpha(pt)}(t,\cdot), \mathbb{1})^p \right\rangle. \end{aligned} \tag{2.1}$$

Here $(\cdot, \cdot)$ is the inner product in $L^2(\mathbb{R}^d)$ (recall that $u^\xi_{R\alpha(pt)}(t, \cdot) = 0$ outside of I_{pt}).

The following lemma reduces the proof to the case $p = 1$. In order not to interrupt the flow of the argument, we defer its proof to the end of this section.

LEMMA 1. *For every $R > 0$ and any $p \geq 1$, as $t \to \infty$,*

$$\left\langle (u^\xi_{R\alpha(pt)}(t,\cdot), \mathbb{1})^p \right\rangle \geq e^{o(\beta(pt))} \left\langle (u^\xi_{R\alpha(pt)}(pt,\cdot), \mathbb{1}) \right\rangle. \tag{2.2}$$

Assertions (i) [resp. (ii)] of the next lemma are derived by applying large deviation arguments for the occupation times measures of Brownian motion resp. Brownian bridge. This lemma will be crucially needed in the proof of the preceding Lemma 1 as well as for the proofs of the lower and upper bounds in Theorem 1. Recall that χ is given by (0.11).

LEMMA 2. *For every $R > 0$, there is a number $\chi_R > 0$ such that, as $t \to \infty$,*

$$\begin{aligned} &\text{(i)} \qquad \left\langle (u^\xi_{R\alpha(t)}(t,\cdot), \mathbb{1}) \right\rangle = \exp\{H(t) - \beta(t)(\chi_R + o(1))\}, \\ &\text{(ii)} \qquad \left\langle \sum_{k=1}^\infty e^{t\lambda_k^\xi(I_t)} \right\rangle \leq \exp\{H(t) - \beta(t)(\chi_R + o(1))\}. \end{aligned} \tag{2.3}$$

Furthermore, $\lim_{R\to\infty} \chi_R = \chi$.

Now Theorem 1(i) is proved by an application of the Lemmas 1 and 2 (i) to the r.h.s. of (2.1). Indeed, combine (2.1) with (2.2) and Lemma 2 (i) for pt instead of t. Then let $R \to \infty$ to obtain (0.12).

PROOF OF LEMMA 2(i). Recall the notations (0.2) – (0.6). Use the scaling relation (1.3) for $s = t$, the Feynman-Kac representation (1.11), Fubini's theorem, and the definition (0.5) of J_t to see that

$$\begin{aligned}
&\Big\langle (u^{\xi}_{R\alpha(t)}(t,\cdot), \mathbb{1})\Big\rangle e^{-H(t)} e^{o(\beta(t))} \\
&\quad = \Big\langle (u^{\xi_t}_{R}(\beta(t),\cdot), \mathbb{1})\Big\rangle \\
&\quad = \int_{Q_R} dx\, \mathbb{E}_x \big\langle \exp(\beta(t)(L_{\beta(t)}, \xi_t))\big\rangle\, \mathbb{1}\{\mathrm{supp} L_{\beta(t)} \subset Q_R\} \\
&\quad = \int_{Q_R} dx\, \mathbb{E}_x \exp(-\beta(t) J_t(L_{\beta(t)}))\, \mathbb{1}\{\mathrm{supp} L_{\beta(t)} \subset Q_R\}.
\end{aligned} \tag{2.4}$$

Now observe that $(L_{\beta(t)})_{t>0}$ satisfies a weak large deviation principle w.r.t. the uniform initial distribution having rate function $\kappa\mathscr{S}_d$ and scale $\beta(t)$ (cf., e.g., Deuschel and Stroock (1989), Chapter 4). Therefore and because of Assumption (J), Varadhan's lemma implies that the

$$\text{r.h.s. of (2.4)} = \exp(-\beta(t)(\chi_R + o(1))) \qquad \text{as} \quad t \to \infty, \tag{2.5}$$

where

$$\chi_R = \inf\{\kappa\mathscr{S}_d(\mu) + J(\mu) \colon \mu \in \mathscr{P}_{\mathrm{c}}(\mathbb{R}^d),\ \mathrm{supp}\mu \subset Q_R\}. \tag{2.6}$$

Obviously, this quantity tends to χ as $R \to \infty$. □

PROOF OF LEMMA 2(ii). Recall that $I_t = Q_{R\alpha(t)}$. We use the scaling relation (1.4), the Fourier expansion (1.5), and the Feynman-Kac representation (1.12) [and, as in (2.4), Fubini's theorem and (0.5)] to obtain

$$\begin{aligned}
&\Big\langle \sum_{k=1}^{\infty} \exp(t\lambda^{\xi}_k(I_t))\Big\rangle e^{-H(t)} \\
&\quad = \Big\langle \sum_{k=1}^{\infty} \exp(\beta(t)\lambda^{\xi_t}_k(Q_R))\Big\rangle \\
&\quad = \Big\langle \int_{Q_R} p^{\xi_t}_R(\beta(t), x, x)\, dx\Big\rangle \\
&\quad = \int_{Q_R} dx\, \mathbb{E}_x \exp(-\beta(t) J_t(L_{\beta(t)}))\, \mathbb{1}\{\mathrm{supp} L_{\beta(t)} \subset Q_R\}\, \delta_x(W_{\beta(t)}).
\end{aligned} \tag{2.7}$$

Now we estimate the r.h.s. of (2.7) from above as follows. We fix some small number $\delta > 0$ and estimate

$$\mathbb{1}\{\mathrm{supp} L_{\beta(t)} \subset Q_R\} \le \mathbb{1}\{\tau_R > \beta(t) - \delta\}. \tag{2.8}$$

Next, observe that $L_{\beta(t)}$ is a convex combination of $L_{\beta(t)-\delta}$ and L_δ, time-shifted by $\beta(t) - \delta$. Hence, we may use the concavity and nonnegativity of J_t to get

$$-\beta(t) J_t(L_{\beta(t)}) \le -(\beta(t) - \delta) J_t(L_{\beta(t)-\delta}). \tag{2.9}$$

Then applying the Markov property at time $\beta(t)-\delta$, we find that the r.h.s. of (2.7) is less than or equal to

$$\int_{Q_R} dx\,\mathbb{E}_x \exp(-(\beta(t)-\delta)J_t(L_{\beta(t)-\delta}))\mathbb{1}\{\tau_R > \beta(t)-\delta\} p_\delta(W_{\beta(t)-\delta}, x). \tag{2.10}$$

Here p_δ is the transition kernel for our Brownian motion. Note that $p_\delta(W_{\beta(t)-\delta}, x) \le (4\pi\kappa\delta)^{-d/2}$. Now we may conclude as below (2.4) to complete the proof. □

PROOF OF LEMMA 1. Abbreviate $r = R\alpha(pt)$ for a while. We write for short λ_k and e_k instead of $\lambda_k^\xi(Q_r)$ and $\mathrm{e}_k^\xi(Q_r)$ for the kth eigenvalue and corresponding eigenfunction of $\kappa\Delta+\xi$ in $L^2(Q_r)$ with zero boundary condition. Use the Fourier expansion (1.6), the simple estimate

$$\left(\sum_{k=1}^n x_k\right)^p \ge \sum_{k=1}^n x_k^p, \qquad x_1,\dots,x_n \ge 0, \tag{2.11}$$

and Jensen's inequality to obtain

$$\begin{aligned}
\langle (u_r^\xi(t,\cdot), \mathbb{1})_r^p\rangle &= \left\langle \left(\sum_{k=1}^\infty \exp(t\lambda_k)(\mathrm{e}_k, \mathbb{1})_r^2\right)^p\right\rangle \\
&\ge \frac{\left\langle \sum_{k=1}^\infty \exp(pt\lambda_k)(\mathrm{e}_k, \mathbb{1})_r^{2p}\right\rangle}{\langle \sum_{k=1}^\infty \exp(pt\lambda_k)\rangle} \left\langle \sum_{k=1}^\infty \exp(pt\lambda_k)\right\rangle \\
&\ge \left(\frac{\left\langle \sum_{k=1}^\infty \exp(pt\lambda_k)(\mathrm{e}_k, \mathbb{1})_r^{2}\right\rangle}{\langle \sum_{k=1}^\infty \exp(pt\lambda_k)\rangle}\right)^p \\
&\quad \times \left\langle \sum_{k=1}^\infty \exp(pt\lambda_k)(\mathrm{e}_k, \mathbb{1})_r^2\right\rangle \|\mathbb{1}\|_r^{-2} \\
&= e^{o(\beta(pt))}\left(\frac{\langle (u_r^\xi(pt,\cdot), \mathbb{1})\rangle}{\langle \sum_{k=1}^\infty \exp(pt\lambda_k)\rangle}\right)^p \langle (u_r^\xi(pt,\cdot), \mathbb{1})\rangle,
\end{aligned} \tag{2.12}$$

where in the third line we have used that $(\mathrm{e}_k, \mathbb{1})_r^2 \le \|\mathbb{1}\|_r^2$, and in the last line we have again used the Fourier expansion (1.6).

Lemma 2 shows that the quotient on the r.h.s. of (2.12) is not smaller than $e^{o(\beta(pt))}$, which completes the proof. □

3. Proof of Theorem 1(ii): The upper bound. In Section 3.2, we prove (0.13), basically following the program outlined in Section 0.4.3. The main building stone for the proof of step (iii) (the "compactification") may be of independent interest and is isolated in subsection 3.1: We there estimate the principal Dirichlet eigenvalue in a large box from above against the maximum of the principal eigenvalues in small subboxes. Some technical lemma will be proved in Section 3.3.

3.1. *Eigenvalue estimates.* In the following, for any bounded domain $D \subset \mathbb{R}^d$ and any Hölder continuous potential $V: \mathbb{R}^d \to \mathbb{R}$, we write $\lambda^V(D)$ to denote the principal eigenvalue of the operator $\kappa\Delta + V$ in $L^2(D)$ with zero boundary condition. Recall that $Q_r = (-r, r)^d$ for $r > 0$.

The objective of this subsection is to derive, for any $0 < r \ll R$, an upper bound for $\lambda^V(Q_R)$ in terms of the maximum of the eigenvalues $\lambda^V(2rk + Q_r)$ in the small subboxes $2rk + Q_r$ of Q_R with $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$ and lattice norm $|k| = \max\{|k_1|, \dots, |k_d|\} \le R/r$. In order to do this properly, we need to let the small boxes overlap each other slightly, and we need to lower the potential V in the overlapping area, which will be a neighborhood of the grid $(2r\mathbb{Z}^d + \partial Q_r) \cap Q_R$. We turn to the precise formulation.

PROPOSITION 1. *For every $r \ge 2$, there is a smooth function $\Phi_r: \mathbb{R}^d \to [0, \infty)$ whose support is contained in the one-neighborhood of the grid $2r\mathbb{Z}^d + \partial Q_r$ such that for all $R > r$ and all Hölder continuous potentials $V: \mathbb{R}^d \to \mathbb{R}$,*

$$\lambda^{V-\Phi_r}(Q_R) \le \max_{k \in \mathbb{Z}^d, |k| < R/r+1} \lambda^V(2rk + Q_{r+1}). \tag{3.1}$$

Moreover, Φ_r can be chosen periodic in each coordinate with period $2r$ and such that

$$\int_{Q_r} \Phi_r(x)\, dx \le \frac{K}{r} |Q_r| \tag{3.2}$$

for some $K \in (0, \infty)$ which does not depend on r.

PROOF. Fix $r \ge 2$. The idea of the proof is to construct a partition of the one,

$$\sum_{k \in \mathbb{Z}^d} \eta_k^2(x) = 1, \qquad x \in \mathbb{R}^d, \tag{3.3}$$

where $\eta_k(x) = \eta(2rk + x)$ and $\eta: \mathbb{R}^d \to [0, 1]$ is a smooth function with $\eta \equiv 1$ on Q_{r-1} and $\mathrm{supp}\,\eta \subset Q_{r+1}$. (The details are given at the end of the proof.)

Then we may take

$$\Phi_r(x) = \kappa \sum_{k \in \mathbb{Z}^d} |\nabla \eta_k(x)|^2, \qquad x \in \mathbb{R}^d. \tag{3.4}$$

To obtain the eigenvalue estimate (3.1), we apply, for any bounded domain $D \subset \mathbb{R}^d$, the Rayleigh-Ritz formula

$$\lambda^V(D) = \sup_{\|\psi\|=1} G^V(\psi), \tag{3.5}$$

where

$$G^V(\psi) = \int_{\mathbb{R}^d} \left\{ -\kappa |\nabla \psi(x)|^2 + V(x) \psi^2(x) \right\} dx, \tag{3.6}$$

and the supremum in (3.5) is taken over all smooth functions ψ with $\mathrm{supp}\,\psi \subset D$ and L^2-norm $\|\psi\| = 1$.

Now fix some $R > r$. For simplicity, we assume that $R/r \in \mathbb{N}$. Let ψ be an arbitrary smooth function with $\operatorname{supp}\psi \subset Q_R$ and $\|\psi\| = 1$ and define

$$\psi_k = \eta_k \psi, \qquad k \in \mathbb{Z}^d. \tag{3.7}$$

Note that ψ_k is smooth with $\operatorname{supp}\psi_k \subset 2kr + Q_{r+1}$ and $\sum_{k\in\mathbb{Z}^d} \|\psi_k\|^2 = 1$. Then one has the identity

$$G^{V-\Phi_r}(\psi) = \sum_{k\in\mathbb{Z}^d} \|\psi_k\|^2 G^V\left(\frac{\psi_k}{\|\psi_k\|}\right). \tag{3.8}$$

Indeed, using (3.3) and (3.4), one sees that

$$\begin{aligned} \kappa \sum_{k\in\mathbb{Z}^d} |\nabla\psi_k|^2 &= \kappa \sum_{k\in\mathbb{Z}^d} \left\{\psi^2 |\nabla\eta_k|^2 + \tfrac{1}{2} \nabla\eta_k^2 \cdot \nabla\psi^2 + \eta_k^2 |\nabla\psi|^2\right\} \\ &= \psi^2 \Phi_r + \kappa|\nabla\psi|^2. \end{aligned} \tag{3.9}$$

Therefore, one obtains

$$\begin{aligned} \sum_{k\in\mathbb{Z}^d} \|\psi_k\|^2 G^V\left(\frac{\psi_k}{\|\psi_k\|}\right) &= \sum_{k\in\mathbb{Z}^d} G^V(\psi_k) \\ &= \int \left\{-\kappa \sum_{k\in\mathbb{Z}^d} |\nabla\psi_k|^2 + V \sum_{k\in\mathbb{Z}^d} \psi_k^2\right\} dx \\ &= \int \left\{-\kappa|\nabla\psi|^2 + (V - \Phi_r)\psi^2\right\} dx \\ &= G^{V-\Phi_r}(\psi), \end{aligned} \tag{3.10}$$

which yields (3.8).

Since ψ_k is smooth with $\operatorname{supp}\psi_k \subset 2rk + Q_{r+1}$, the Rayleigh-Ritz formula (3.5) yields that $G^V(\psi_k/\|\psi_k\|) \le \lambda^V(2rk + Q_{r+1})$ (provided that $\|\psi_k\| \ne 0$ which may only happen for $|k| \le R/r$). Estimating each of these eigenvalues by their maximum and taking into account that $\sum_{k\in\mathbb{Z}^d} \|\psi_k\|^2 = 1$, we find that the r.h.s. of (3.8) does not exceed the r.h.s. of (3.1). Now also passing to the supremum over ψ on the l.h.s. of (3.8), we arrive at the desired inequality (3.1).

It only remains to construct the function η with the properties required at the beginning of the proof and such that (3.2) holds. One easily checks that the ansatz

$$\eta(x) = \prod_{i=1}^d \zeta(x_i), \qquad x = (x_1, \ldots, x_d) \in \mathbb{R}^d, \tag{3.11}$$

reduces the construction of η to the case $d = 1$ (with η replaced by ζ).

It is not too hard to find a function $\varphi: \mathbb{R} \to [0, 1]$ such that $\sqrt{\varphi}$ and $\sqrt{1-\varphi}$ are smooth, $\varphi \equiv 0$ on $(-\infty, -1]$ and $\varphi \equiv 1$ on $[1, \infty)$, and $\varphi(-x) = 1 - \varphi(x)$ for all $x \in \mathbb{R}$. Then we may take

$$\zeta(x) = \sqrt{\varphi(r + x)(1 - \varphi(-r + x))}, \qquad x \in \mathbb{R}. \tag{3.12}$$

Indeed, ζ is smooth and

$$\sum_{k\in\mathbb{Z}} \zeta_k^2(x) = 1, \qquad x \in \mathbb{R}, \tag{3.13}$$

for $\zeta_k(x) = \zeta(x + 2rk)$.

It is obvious from the construction of ζ and from (3.11) and (3.4) that Φ_r is periodic in each coordinate with period $2r$, and that Φ_r vanishes outside of the one-neighborhood of the grid $2r\mathbb{Z}^d + \partial Q_r$. Since φ does not depend on r, the shape of Φ_r in this one-neighborhood is independent of r, hence $\int_{Q_r} \Phi_r \, dx$ is proportional to the surface area of Q_r as $r \to \infty$, that is, (3.2) holds. □

3.2. *Strategy of the proof of Theorem* 1(ii). For the remainder of this section, we fix $p \in [1, \infty)$ and suppose that Assumption (J) holds.

The steps (i) – (iv) (see the end of Section 0.4.3) will be carried out at the following places: (i) in Lemma 3 below, (ii) in (3.30) and (3.32), (iii) in (3.29) [on base of Proposition 1], (iv) below (3.32) [on base of Lemma 2 (ii)].

Our first lemma states that asymptotically we may replace the solution u^ξ of (0.1) by the solution of the corresponding initial-boundary value problem in some (time-dependent) box. We fix some positive function $R(t)$ such that $R(t) = e^{o(\beta(t))}$ and $R(t)\alpha(t)/t \to \infty$ as $t \to \infty$.

LEMMA 3. *As* $t \to \infty$,

$$\left\langle u^\xi(t,0)^p \right\rangle \le e^{o(\beta(pt))} \left\langle u^\xi_{R(pt)}(t,0)^p \right\rangle. \tag{3.14}$$

PROOF. Use the Feynman-Kac formulas (1.7) and (1.11) to see that (3.14) will follow from

$$\lim_{t\to\infty} \frac{\left\langle \left(\mathbb{E}_0 \exp\left\{ \int_0^t \xi(W_u)\, du \right\} \mathbb{1}\{\tau_{R(pt)} \le t\} \right)^p \right\rangle}{\langle u^\xi(t,0)^p \rangle} = 0. \tag{3.15}$$

Indeed, let $\|\cdot\|_p$ denote the L^p–norm w.r.t. $\langle\cdot\rangle$. Then (1.2), the triangle inequality, and (3.15) yield

$$\begin{aligned} 0 &\le \|u^\xi(t,0)\|_p - \|u^\xi_{R(pt)}(t,0)\|_p \\ &\le \|u^\xi(t,0) - u^\xi_{R(pt)}(t,0)\|_p = o(\|u^\xi(t,0)\|_p). \end{aligned} \tag{3.16}$$

Hence, $\|u^\xi_{R(pt)}(t,0)\|_p \sim \|u^\xi(t,0)\|_p$, and this implies (3.14).

First, we estimate the numerator of the l.h.s. of (3.15). Applying Jensen's inequality twice (first to $\mathbb{P}_0$ and then to the normalized Lebesgue measure on

$[0,t]$) and using Fubini's theorem, we find that

$$
\begin{aligned}
&\left\langle \left(\mathbb{E}_0 \exp\left\{ \int_0^t \xi(W_u)\,du \right\} \mathbb{1}\{\tau_{R(pt)} \le t\} \right)^p \right\rangle \\
&\qquad \le \left\langle \frac{1}{t} \int_0^t du\, \mathbb{E}_0 \exp(pt\xi(W_u)) \mathbb{1}\{\tau_{R(pt)} \le t\} \right\rangle \\
&\qquad = \frac{1}{t} \int_0^t du\, \mathbb{E}_0 \left\langle \exp(pt\xi(W_u)) \right\rangle \mathbb{1}\{\tau_{R(pt)} \le t\} \qquad (3.17)\\
&\qquad = e^{H(pt)} \mathbb{P}_0(\tau_{R(pt)} \le t) \\
&\qquad \le \exp\left\{ H(pt) - (1+o(1)) R(pt)^2/(4\kappa t) \right\},
\end{aligned}
$$

where the last estimate holds for large t and is implied by the reflection principle.

Secondly, using Theorem 1 (i), we estimate the denominator on the l.h.s. of (3.15) as follows:

$$
\left\langle u^\xi(t,0)^p \right\rangle \ge \exp(H(pt) - \beta(pt)(\chi + o(1))). \qquad (3.18)
$$

Substituting (3.17) and (3.18) in (3.15), we get

$$
\text{l.h.s. of (3.15)} \le \limsup_{t\to\infty} \exp\left\{ -\frac{(1+o(1))R(pt)^2}{4\kappa t} + \beta(pt)(\chi + o(1)) \right\}. \qquad (3.19)
$$

This is zero since $R(t)^2/t = \beta(t)(R(t)\alpha(t)/t)^2$ tends to infinity faster than $\beta(t)$. □

The main ingredient in our proof of Theorem 1 (ii) is the following which settles basically the case $p = 1$.

PROPOSITION 2. *As* $t \to \infty$,

$$
\left\langle (u^\xi_{R(t)}(t,\cdot), \mathbb{1}) \right\rangle \le \exp\{H(t) - \beta(t)(\chi + o(1))\}. \qquad (3.20)
$$

The following lemma gives an upper estimate of the pth moment of $u^\xi_{R(pt)}(t,0)$ in terms of the first moment of an integral of $u^\xi_{R(pt)}(pt,\cdot)$. This lemma basically reduces the case $p \ge 1$ to the case $p = 1$.

LEMMA 4. *If Assumption* (H) *is satisfied, then we have, as* $t \to \infty$,

$$
\left\langle u^\xi_{R(pt)}(t,0)^p \right\rangle \le e^{o(\beta(pt))} \left[\left\langle (u^\xi_{R(pt)}(pt,\cdot), \mathbb{1}) \right\rangle + e^{H(pt) - \beta(pt)\chi} \right]. \qquad (3.21)
$$

Since its proof is rather technical, we defer it to subsection 3.3.

Together with Lemma 3 and Lemma 4, Proposition 2 implies Theorem 1 (ii). Indeed, if Assumption (H) is satisfied, substitute (3.21) in (3.14) and apply (3.20) for pt instead of t to arrive at (0.13). In the case $p = 1$, use the shift-invariance of the field ξ, the statement of Lemma 3 for the respective quantities shifted by $x \in Q_{R(t)}$, and integrate over x to obtain

$$\langle u^\xi(t,0)\rangle = (2R(t))^{-d}\langle (u^\xi(t,\cdot),\mathbb{1})_{R(t)}\rangle \le e^{o(\beta(t))}\Big\langle (u^\xi_{2R(t)}(t,\cdot),\mathbb{1})_{R(t)}\Big\rangle, \tag{3.22}$$

where we also have used that the solution in the box $x + Q_{R(t)}$ does not exceed the one in $Q_{2R(t)}$. Now Proposition 2, applied for $2R(t)$ instead of $R(t)$, also implies (0.13).

In the remainder of this subsection, we prove Proposition 2.

PROOF OF PROPOSITION 2. For $r > 0$, we choose a smooth function $\Phi_r : \mathbb{R}^d \to [0,\infty)$ as in Proposition 1. In particular, $K \in (0,\infty)$ is chosen according to (3.2).

For any $\theta \in \mathbb{R}^d$ define a shifted version of Φ_r by

$$\Phi_r^\theta(x) = \Phi_r(x-\theta), \qquad x \in \mathbb{R}^d. \tag{3.23}$$

Let $V : \mathbb{R}^d \to \mathbb{R}$ be some Hölder continuous potential.

STEP 1. *For any $\beta > 0$, $y \in \mathbb{R}^d$, and $2 \le r < R$,*

$$u_R^V(\beta,y) \le \frac{\exp(K\frac{\beta}{r})}{(2r)^d}\int_{Q_r} u_R^{V-\Phi_r^\theta}(\beta,y)\,d\theta. \tag{3.24}$$

PROOF. Since, by periodicity of Φ_r, the map $x \mapsto \int_{Q_r}\Phi_r^\theta(x)\,d\theta$ is constant, we may calculate, with the help of (3.2),

$$\int_{Q_r}(L_\beta,\Phi_r^\theta)\,d\theta = \int_{Q_r}\Phi_r^\theta(0)\,d\theta = \int_{Q_r}\Phi_r(x)\,dx \le \frac{K}{r}|Q_r|. \tag{3.25}$$

Recall (1.11), use (3.25) and apply Jensen's inequality to estimate

$$\begin{aligned} u_R^V(\beta,y) &= \mathbb{E}_y \exp(\beta(L_\beta,V))\mathbb{1}\{\tau_R > \beta\} \\ &\le \exp\Big(K\tfrac{\beta}{r}\Big)\mathbb{E}_y \exp\Big\{-\beta|Q_r|^{-1}\int_{Q_r}(L_\beta,\Phi_r^\theta)\,d\theta\Big\} \\ &\qquad \times \exp(\beta(L_\beta,V))\mathbb{1}\{\tau_R > \beta\} \\ &\le \frac{\exp(K\frac{\beta}{r})}{|Q_r|}\int_{Q_r} d\theta\,\mathbb{E}_y \exp(\beta(L_\beta,V-\Phi_r^\theta))\mathbb{1}\{\tau_R > \beta\}. \end{aligned} \tag{3.26}$$

Now use (1.11) once more to arrive at (3.24). □

For the next three steps, recall the notation from Section 1.1, resp., the beginning of Section 3.1.

STEP 2. *For all $\beta, R > 0$,*

$$(u_R^V(\beta,\cdot),\mathbb{1}) \le (2R)^d \exp(\beta\lambda^V(Q_R)). \tag{3.27}$$

PROOF. Use the Fourier expansion (1.6) and Parseval's equality for the box Q_R to obtain

$$
\begin{aligned}
(u_R^V(\beta,\cdot),\mathbb{1}) &= \big(u_R^V(\beta,\cdot),\mathbb{1}\big)_R = \sum_{k=1}^{\infty}\exp(\beta\lambda_k)(\mathrm{e}_k,\mathbb{1})_R^2 \\
&\le \exp(\beta\lambda_1)\sum_{k=1}^{\infty}(\mathrm{e}_k,\mathbb{1})_R^2 \\
&= \exp(\beta\lambda^V(Q_R))\|\,\mathbb{1}\,\|_R^2 = \exp(\beta\lambda^V(Q_R))(2R)^d. \qquad \square
\end{aligned}
\tag{3.28}
$$

STEP 3. *For any* $\beta > 0$ and $2 \le r < R$,

$$
\begin{aligned}
(u_R^V(\beta,\cdot),\mathbb{1}) &\le \left(\frac{R}{r}\right)^d \exp\Big(K\frac{\beta}{r}\Big) \\
&\quad\times \int_{Q_r} \exp\Big\{\beta \max_{k\in\mathbb{Z}^d,\,|k|<R/r+2} \lambda^V(\theta+2rk+Q_{r+1})\Big\}\,d\theta.
\end{aligned}
\tag{3.29}
$$

PROOF. Integrate (3.24) over $y \in Q_R$ and apply (3.27) for $V - \Phi_r^\theta$ instead of V to obtain that the l.h.s. of (3.29) is less than or equal to

$$
\begin{aligned}
&\frac{\exp(K\frac{\beta}{r})}{(2r)^d}\int_{Q_r}\int_{Q_R} u_R^{V-\Phi_r^\theta}(\beta,y)\,dy\,d\theta \\
&\quad\le \left(\frac{R}{r}\right)^d \exp\left(K\frac{\beta}{r}\right)\int_{Q_r}\exp\left\{\beta\lambda^{V-\Phi_r^\theta}(Q_R)\right\}d\theta.
\end{aligned}
$$

Now apply (3.1) to arrive at (3.29). $\square$

STEP 4. *Conclusion.*

PROOF. Use the scaling relation (1.3) for $s = t$ and recall (0.2)–(0.6) to see that

$$
\Big\langle (u_{R(t)}^{\xi}(t,\cdot),\mathbb{1})\Big\rangle e^{-H(t)} = e^{o(\beta(t))}\Big\langle (u_{\widetilde{R}(t)}^{\xi_t}(\beta(t),\cdot),\mathbb{1})\Big\rangle, \tag{3.30}
$$

where $\widetilde{R}(t) = R(t)/\alpha(t)$.

An application of Step 3 for $V = \xi_t$, $\beta = \beta(t)$ and $R = \widetilde{R}(t)$ yields, for any $r \ge 2$,

$$
\begin{aligned}
&\Big\langle (u_{\widetilde{R}(t)}^{\xi_t}(\beta(t),\cdot),\mathbb{1})\Big\rangle \\
&\quad\le \exp(o(\beta(t)))\exp\Big(K\tfrac{\beta(t)}{r}\Big) \\
&\qquad\times\Big\langle \int_{Q_r}\exp\Big\{\beta(t)\max_{k\in\mathbb{Z}^d,|k|<\widetilde{R}(t)/r+2}\lambda^{\xi_t}(\theta+2rk+Q_{r+1})\Big\}\,d\theta\Big\rangle \\
&\quad\le \exp(o(\beta(t)))\exp\left(K\frac{\beta(t)}{r}\right) \\
&\qquad\times\sum_{k\in\mathbb{Z}^d,|k|<\widetilde{R}(t)/r+2}\int_{Q_r}\big\langle \exp(\beta(t)\lambda^{\xi_t}(\theta+2rk+Q_{r+1}))\big\rangle\,d\theta.
\end{aligned}
\tag{3.31}
$$

Use the shift-invariance of the potential and the additivity of the map $V \mapsto \lambda^V(\theta + 2rk + Q_{r+1})$ for constant functions to see that the expectation in the last line of (3.31) is independent of k and θ. Thus, from (3.31), we have, for any $r \geq 2$,

$$\begin{aligned}
&\left\langle (u^{\xi_t}_{\widetilde{R}(t)}(\beta(t), \cdot), \mathbb{1})\right\rangle \\
&\quad \leq \exp(o(\beta(t))) \exp\left(K\frac{\beta(t)}{r}\right) \left\langle \exp(\beta(t)\lambda^{\xi_t}(Q_{r+1}))\right\rangle \\
&\quad \leq \exp(o(\beta(t))) \exp\left(K\frac{\beta(t)}{r}\right) \left\langle \sum_{k=1}^{\infty} \exp(\beta(t)\lambda^{\xi_t}_k(Q_{r+1}))\right\rangle \\
&\quad \leq \exp(o(\beta(t))) \exp\left(K\frac{\beta(t)}{r}\right) \exp(-H(t)) \\
&\qquad \times \left\langle \sum_{k=1}^{\infty} \exp(t\lambda^{\xi}_k(Q_{(r+1)\alpha(t)}))\right\rangle,
\end{aligned} \tag{3.32}$$

where we have used the scaling relation (1.4). Now use Lemma 2 (ii) to conclude that

$$\limsup_{t\to\infty} \frac{1}{\beta(t)} \log\left\langle (u^{\xi_t}_{\widetilde{R}(t)}(\beta(t), \cdot), \mathbb{1})\right\rangle \leq \frac{K}{r} - \chi_{r+1} \overset{r\to\infty}{\longrightarrow} -\chi. \tag{3.33}$$

In view of (3.30), this completes the proof of Proposition 2. □

3.3. *Proof of Lemma* 4: *Reduction to* $p = 1$. We proceed in three steps.

STEP 1. *For all functions* $\delta = \delta(t) \in (0, t)$, $\gamma = \gamma(t) > 0$, $\eta = \eta(t) > 0$, *we have, as* $t \to \infty$,

$$\begin{aligned}
&\left\langle u^{\xi}_{R(pt)}(t, 0)^p\right\rangle \\
&\quad \leq e^{o(\beta(pt))} \left[\frac{e^{\gamma p}}{\delta^{pd/2}} \left\langle (u^{\xi}_{R(pt)}(t-\delta, \cdot), \mathbb{1})^p\right\rangle + e^{-p\gamma\eta + H(pt+p\eta\delta)}\right].
\end{aligned} \tag{3.34}$$

PROOF. Use the Feynman-Kac formula (1.11), split the Brownian expectation into the parts where $\int_0^\delta \xi(W_s)\,ds \leq \gamma$ and $> \gamma$, use the Markov property at time δ, and Chebyshev's and Jensen's inequality to estimate

$$\begin{aligned}
u^{\xi}_{R(pt)}(t, 0) &= \mathbb{E}_0 \exp\left\{\int_0^t \xi(W_s)\,ds\right\} \mathbb{1}\{\tau_{R(pt)} > t\} \\
&\leq e^{\gamma}\mathbb{E}_0 \mathbb{1}\{\tau_{R(pt)} > \delta\}\mathbb{E}_{W_\delta} \exp\left\{\int_0^{t-\delta} \xi(\widetilde{W}_s)\,ds\right\} \\
&\qquad \times \mathbb{1}\{\widetilde{\tau}_{R(pt)} > t-\delta\} \\
&\qquad + e^{-\eta\gamma}\mathbb{E}_0 \exp\left\{\int_0^t (1 + \eta\,\mathbb{1}_{[0,\delta]}(s))\xi(W_s)\,ds\right\} \\
&\leq e^{\gamma} \int_{\mathbb{R}^d} dz\, p_\delta(z) u^{\xi}_{R(pt)}(t-\delta, z) \\
&\qquad + e^{-\eta\gamma} \int_0^t ds\, \frac{1 + \eta\,\mathbb{1}_{[0,\delta]}(s)}{t+\eta\delta}\, \mathbb{E}_0 \exp\{(t+\eta\delta)\xi(W_s)\}.
\end{aligned} \tag{3.35}$$

Here p_δ denotes the centered Gaussian density with variance $2\delta\kappa$.

Since $\max_{z\in\mathbb{R}^d} p_\delta(z) \le (2\delta\kappa)^{-d/2}$, the first summand on the r.h.s. of (3.35) does not exceed $e^\gamma(2\delta\kappa)^{-d/2}(u^\xi_{R(pt)}(t-\delta,\cdot),\mathbb{1})$. Now take the pth moment, use the inequality $(a+b)^p \le 2^{p-1}(a^p+b^p)$ for $a,b\ge 0$, use Jensen's inequality twice and Fubini's theorem, recall (0.2) and summarize to arrive at the claim. □

STEP 2. *For all functions $\delta=\delta(t)\in(0,t)$, we have, as $t\to\infty$,*

$$\Big\langle (u^\xi_{R(pt)}(t-\delta,\cdot),\mathbb{1})^p\Big\rangle \le e^{o(\beta(pt))}\Big\langle (u^\xi_{R(pt)}(pt,\cdot),\mathbb{1})\Big\rangle. \tag{3.36}$$

PROOF. We write λ_k and e_k for the kth eigenvalue and corresponding eigenfunction of the operator $\kappa\Delta+\xi$ on $Q_{R(pt)}$ with zero boundary condition. Use the Fourier representation (1.6) to estimate

$$\begin{aligned}(u^\xi_{R(pt)}(t-\delta,\cdot),\mathbb{1}) &= \sum_{k=1}^\infty \exp((t-\delta)\lambda_k)(\mathrm{e}_k,\mathbb{1})^2_{R(pt)}\\ &\le \sum_{k:\lambda_k\ge 0} \exp(t\lambda_k)(\mathrm{e}_k,\mathbb{1})^2_{R(pt)} + \sum_{k:\lambda_k<0}(\mathrm{e}_k,\mathbb{1})^2_{R(pt)}\\ &\le \sum_{k=1}^\infty \exp(t\lambda_k)(\mathrm{e}_k,\mathbb{1})^2_{R(pt)} + \|\mathbb{1}\|^2_{R(pt)}.\end{aligned} \tag{3.37}$$

We estimate the first summand on the r.h.s. of (3.37) by using Jensen's inequality w.r.t. the probabilities $(\mathrm{e}_k,\mathbb{1})^2_{R(pt)}/\|\mathbb{1}\|^2_{R(pt)}$, $k\in\mathbb{N}$, to obtain

$$\begin{aligned}\left(\sum_{k=1}^\infty \exp(t\lambda_k)(\mathrm{e}_k,\mathbb{1})^2_{R(pt)}\right)^p &\le \|\mathbb{1}\|^{2(p-1)}_{R(pt)}\sum_{k=1}^\infty \exp(pt\lambda_k)(\mathrm{e}_k,\mathbb{1})^2_{R(pt)}\\ &= \exp(o(\beta(pt)))(u^\xi_{R(pt)}(pt,\cdot),\mathbb{1}).\end{aligned} \tag{3.38}$$

Substituting this in (3.37), we arrive at

$$(u^\xi_{R(pt)}(t-\delta,\cdot),\mathbb{1})^p \le e^{o(\beta(pt))}\Big[\Big(u^\xi_{R(pt)}(pt,\cdot),\mathbb{1}\Big)+1\Big]. \tag{3.39}$$

Note that the assertion (3.36) remains unchanged by adding an arbitrary constant to the potential ξ. Choosing this constant sufficiently large, the reader verifies (using the Feynman-Kac formula, e.g.) that $\langle (u^\xi_{R(pt)}(pt,\cdot),\mathbb{1})\rangle$ tends to infinity as $t\to\infty$. Thus, taking expectations in (3.39), we arrive at (3.36). □

STEP 3. *Conclusion.*

PROOF. In Step 1, we leave $\varepsilon>0$ fixed and make the following choices:

$$\gamma=\varepsilon\beta(pt),\qquad \delta=e^{-2\varepsilon\beta(pt)},\qquad \eta=\frac{1}{p}e^{\varepsilon\beta(pt)}. \tag{3.40}$$

According to Assumption (H), we may choose some $K>0$ such that

$$H(pt+e^{-\varepsilon\beta(pt)})\le H(pt)+Ke^{\varepsilon\beta(pt)},\qquad t>0. \tag{3.41}$$

Substituting this and (3.36) in (3.34) and letting $\varepsilon\to 0$, we obtain the claim. □

4. Examples.

4.1. *Gaussian potential.* Let $\{\xi(x); x \in \mathbb{R}^d\}$ be a Hölder continuous, shift-invariant centered Gaussian field. Assume that the covariance function $B(x) = \langle \xi(0)\xi(x)\rangle$ is twice continuously differentiable in some neighborhood of 0. Abbreviate

$$\sigma^2 = B(0) \qquad \text{and} \qquad \Sigma^2 = -B''(0). \tag{4.1}$$

Thus, $H(t) = t^2\sigma^2/2$. Note that Assumption (H) is satisfied. We assume that the maximum of the covariance function at zero is strict, i.e. we may assume that $\Sigma \in \mathscr{M}_d^+$. We are going to calculate the function J_t. Fix $\mu \in \mathscr{P}_c(\mathbb{R}^d)$. Then

$$\begin{aligned} J_t(\mu) &= -\frac{\alpha^2(t)}{t} \log\left[e^{-H(t)}\left\langle e^{t(\mu,\xi(\alpha(t)\cdot))}\right\rangle\right] \\ &= \frac{t\alpha^2(t)}{2}\sigma^2 - \frac{t\alpha^2(t)}{2}\left\langle\left(\int \xi(\alpha(t)x)\,\mu(dx)\right)^2\right\rangle \\ &= \frac{t\alpha^2(t)}{2}\int\int (B(0) - B(\alpha(t)(x-y)))\,\mu(dx)\mu(dy). \end{aligned} \tag{4.2}$$

Now expand $B(\cdot)$ into a Taylor series around zero and substitute it in the last line of (4.2) to obtain

$$\begin{aligned} J_t(\mu) = -\frac{t\alpha^4(t)}{4}\int_0^1 d\theta\, 2(1-\theta)\int\int &(x-y)^{\mathrm{T}} B''(\alpha(t)\theta(x-y)) \\ &\times (x-y)\,\mu(dx)\mu(dy). \end{aligned} \tag{4.3}$$

Now we see that Assumption (J) is satisfied for $\alpha(t) = t^{-1/4}$ (yielding $\beta(t) = t^{3/2}$). Indeed, assume that $(\mu_t)_{t>0}$ is a family of probability measures on $\mathbb{R}^d$ whose supports are contained in some fixed compact set $K \subset \mathbb{R}^d$ and which converge weakly toward some $\mu \in \mathscr{P}_c(\mathbb{R}^d)$. Since $B''(\cdot)$ is continuous at zero, the integrand on the right side of (4.3) converges to $(x-y)^{\mathrm{T}}B''(0)(x-y)$ as $t \to \infty$, uniformly in $\theta \in (0,1)$ and in $x, y \in K$. Thus, we obtain that $\lim_{t\to\infty} J_t(\mu_t) = J(\mu)$, where

$$J(\mu) = \tfrac{1}{4}\int\int (x-y)^{\mathrm{T}}\Sigma^2(x-y)\,\mu(dx)\mu(dy) = \tfrac{1}{2}\mathrm{tr}(\Sigma^2 \mathrm{cov}(\mu)). \tag{4.4}$$

In particular, J is a function of the covariance.

Therefore, Theorem 1 is applicable here where, according to (0.16), the quantity χ is given by

$$\chi = \inf_{\Gamma \in \mathscr{M}_d^+} f(\Gamma) \qquad \text{with} \qquad f(\Gamma) = \frac{\kappa}{4}\mathrm{tr}(\Gamma^{-1}) + \frac{1}{2}\mathrm{tr}(\Sigma^2\Gamma). \tag{4.5}$$

Let us compute χ explicitly. One easily checks that $\partial_\varepsilon f(\Gamma + \varepsilon A)|_{\varepsilon=0} = 0$ for all symmetric matrices A if and only if $\Gamma = \sqrt{\kappa/2}\,\Sigma^{-1}$. Moreover, since $\partial_\varepsilon^2 f(\Gamma + \varepsilon A)|_{\varepsilon=0} = \kappa \mathrm{tr}(\Gamma^{-3}A^2)/2 \ge 0$ for all Γ, f is convex on $\mathscr{M}_d^+$. Consequently,

$$\chi = f\left(\sqrt{\frac{\kappa}{2}}\Sigma^{-1}\right) = \sqrt{\frac{\kappa}{2}}\mathrm{tr}\Sigma. \tag{4.6}$$

If, for $d = 1$, the correlation function B behaves like

$$B(x) = \sigma^2 (1 - |x|^\gamma(\lambda + o(1))) \qquad \text{as } x \to 0 \tag{4.7}$$

with some $\lambda, \sigma > 0$ and some $\gamma \in (0, 2]$, then Assumption (J) is satisfied with $\alpha(t) = t^{-\frac{1}{2+\gamma}}$ and $\beta(t) = t^{(4+\gamma)/(2+\gamma)}$ and

$$J(\mu) = \frac{\sigma^2 \lambda}{2} \int_{\mathbb{R}^2} |x - y|^\gamma \, \mu(dx)\mu(dy), \qquad \mu \in \mathscr{P}_c(\mathbb{R}). \tag{4.8}$$

In this case, using scaling properties of $\mathscr{S}_1$, we find that

$$\chi = \kappa^{\frac{\gamma}{2+\gamma}} \left(\frac{\sigma^2\lambda}{2}\right)^{2/(2+\gamma)} \inf\left\{\mathscr{S}_1(\mu) + \int_{\mathbb{R}^2} |x - y|^\gamma \, \mu(dx)\mu(dy) \colon \mu \in \mathscr{P}_c(\mathbb{R})\right\}. \tag{4.9}$$

4.2. *Compound Poisson potential.* Let $\Phi(dx)$ denote the realizations of a compound Poisson process on $\mathbb{R}^d$ with intensity parameter $\lambda > 0$ and mass distribution $F(ds)$ on $(0, \infty)$. That is, we have a Poisson distribution of particles with intensity $\lambda > 0$ and independent masses which are distributed according to F. In particular, for every bounded measurable set $B \subset \mathbb{R}^d$, the distribution of the mass $\Phi(B)$ in B is given by

$$\text{Prob}(\Phi(B) \in ds) = e^{-\lambda|B|} \sum_{k=0}^{\infty} \frac{(\lambda|B|)^k}{k!} F^{*k}(ds), \tag{4.10}$$

where $*k$ means k-fold convolution. For $F = \delta_1$ we obtain the Poisson process.

Furthermore, let $\varphi \colon \mathbb{R}^d \to \mathbb{R}$ be twice continuously differentiable with compact support. We assume that φ has a unique global maximum at the origin with $\varphi(0) > 0$. Abbreviate $\Sigma^2 = -\varphi''(0) \in \mathscr{M}_d^+$.

The field we are studying here is given by

$$\xi(x) = \int_{\mathbb{R}^d} \varphi(y + x) \, \Phi(dy). \tag{4.11}$$

Thus, ξ is the weighted sum of copies of clouds φ around each point of the compound Poisson process. Clearly, $\{\xi(x); x \in \mathbb{R}^d\}$ is stationary.

We are going first to identify the function J_t. We shall make use of the formula

$$\log\left\langle e^{\int \psi(y)\,\Phi(dy)}\right\rangle = \lambda \int_{(0,\infty)} \int_{\mathbb{R}^d} (e^{s\psi(y)} - 1) \, dy \, F(ds) \tag{4.12}$$

for continuous functions $\psi \colon \mathbb{R}^d \to \mathbb{R}$ having compact support. In particular, $H(t) = \lambda \int \int (e^{ts\varphi(y)} - 1) \, dy \, F(ds)$. We require that the support of F has a finite upper boundary which will be denoted by m. Then $H(t)$ is finite for all $t > 0$, and also Assumption (H) is satisfied.

We leave the scale function $\alpha(t) \to 0$ arbitrary for a while and fix $\mu \in \mathscr{P}_c(\mathbb{R}^d)$. Inserting (4.11), we obtain

$$\begin{aligned} &\log \langle \exp(\beta(t)(\mu, \xi_t))\rangle \\ &\quad = \log \frac{\langle \exp(t(\mu, \xi(\alpha(t)\cdot)))\rangle}{\langle \exp(t\xi(0))\rangle} \\ &\quad = \log \left\langle \exp\left(t \int\int \varphi(y+\alpha(t)x)\,\mu(dx)\,\Phi(dy)\right)\right\rangle \\ &\qquad - \log\left\langle \exp\left(t\int \varphi(y)\,\Phi(dy)\right)\right\rangle. \end{aligned} \tag{4.13}$$

Hence, in view of (4.12), J_t is given by

$$\begin{aligned} J_t(\mu) = \lambda \frac{\alpha^2(t)}{t} \int_{(0,\infty)} \int_{\mathbb{R}^d} \\ \times\Big(\exp(ts\varphi(y)) - \exp\Big(ts\int \varphi(y+\alpha(t)x)\,\mu(dx)\Big)\Big)\,dy\,F(ds). \end{aligned} \tag{4.14}$$

Now let $(\mu_t)_{t>0}$ be a family of probability measures on $\mathbb{R}^d$ whose supports are contained in one fixed compact set and which converge weakly toward some $\mu \in \mathscr{P}_c(\mathbb{R}^d)$ as $t \to \infty$. We want to determine some appropriate choice of $\alpha(t)$ such that $J_t(\mu_t)$ converges toward some function $J(\mu)$. Since J_t is shift-invariant, we may assume that the probabilities μ_t have zero expectation. We also assume that $\alpha(t) = o(t^{-1/2})$, which will be justified later.

Using the approximation $e^x - 1 \sim x$ for $x \to 0$ and a Taylor expansion, we deduce from (4.14) that

$$\begin{aligned} J_t(\mu_t) &\sim \lambda\alpha^2(t)\int\int e^{ts\varphi(y)}s\int [\varphi(y) - \varphi(y+\alpha(t)x)]\;\mu_t(dx)\,dy\,F(ds) \\ &\sim \frac{\lambda}{2}\alpha^4(t)\int\int e^{ts\varphi(y)}s\,\mathrm{tr}\,(-\varphi''(y)\mathrm{cov}(\mu_t))\;dy\,F(ds). \end{aligned} \tag{4.15}$$

Now an application of the Laplace method yields

$$\begin{aligned} J_t(\mu_t) \sim \frac{\lambda}{2}\alpha^4(t)(2\pi)^{d/2}t^{-d/2}m^{1-d/2}\det(\Sigma^{-1})\mathrm{tr}(\Sigma^2\mathrm{cov}(\mu)) \\ \times \int_{(0,\infty)} e^{ts\varphi(0)}\,F(ds). \end{aligned} \tag{4.16}$$

If we choose

$$\alpha(t) = t^{\frac{d}{8}}\left(m^{1-d/2}\int_{(0,\infty)} e^{ts\varphi(0)}\,F(ds)\right)^{-\frac{1}{4}}, \tag{4.17}$$

then $J(\mu_t) \to J(\mu)$, where

$$J(\mu) = \frac{\lambda}{2}(2\pi)^{d/2}\det(\Sigma^{-1})\,\mathrm{tr}\left(\Sigma^2\mathrm{cov}(\mu)\right). \tag{4.18}$$

Therefore, Assumption (J) is satisfied. Note that $\alpha(t)$ decays exponentially fast toward zero. Using (0.16) and proceeding as below (4.5), we obtain

$$\begin{aligned}\chi &= \inf\left\{\frac{\kappa}{4}\mathrm{tr}(\Gamma^{-1}) + \lambda\frac{(2\pi)^{\frac{d}{2}}}{2\det(\Sigma)}\mathrm{tr}(\Sigma^2\Gamma)\colon \Gamma \in \mathscr{M}_d^+\right\} \\ &= \left(\kappa\lambda\frac{(2\pi)^{\frac{d}{2}}}{2\det(\Sigma)}\right)^{\frac{1}{2}}\mathrm{tr}(\Sigma).\end{aligned} \tag{4.19}$$

In the pure Poisson case (i.e., $F = \delta_1$), we have

$$H(t) = \lambda\int\left(e^{t\varphi(y)} - 1\right)dy \qquad \text{and} \qquad \beta(t) = e^{\frac{t}{2}\varphi(0)}t^{1-\frac{d}{4}}. \tag{4.20}$$

The second-order term $\chi\beta(t)$ in Theorem 1 depends on the cloud φ via $\varphi(0)$ and $\varphi''(0)$ only. Note that, up to $o(\beta(t))$, the first-order term $H(t)$ depends on all values of φ in $\{y\colon \varphi(y) > \varphi(0)/2\}$.

APPENDIX

A.1. *Proof of* (0.16).

STEP 1. *For any* $\mu \in \mathscr{P}_c(\mathbb{R}^d)$ *and* $\Gamma \in \mathscr{M}_d^+$, *we have* $\kappa\mathscr{S}_d(\mu) \geq \sqrt{\kappa}\mathrm{tr}(\Gamma) - \mathrm{tr}(\Gamma^2 \operatorname{cov}\mu)$.

PROOF. Let C be a regular $d\times d$-matrix such that $\Gamma = 2\sqrt{\kappa}C^{\mathrm{T}}C$, and define $\psi(x) = e^{-|Cx|^2}$ for $x \in \mathbb{R}^d$. Then we have

$$\kappa\Delta\psi + p\psi = \lambda\psi, \tag{A.1}$$

where $p(x) = -|\Gamma x|^2$ and $\lambda = -\sqrt{\kappa}\mathrm{tr}(\Gamma)$.

Recall the notation from the beginning of subsection 3.1. Choose $R > 0$ with $\mathrm{supp}\mu \subset Q_{R-1}$. We are going to use the formula

$$\kappa\mathscr{S}_d(\mu) = \sup\{(\mu, V) - \lambda^V(Q_R) \mid V\colon \overline{Q}_R \to \mathbb{R} \text{ Hölder cont.}\}. \tag{A.2}$$

Let ψ_R^0 denote a positive eigenfunction of $\kappa\Delta$ in $L^2(Q_R)$ with Dirichlet boundary condition. With some smooth function $\eta\colon \mathbb{R}^d \to [0, 1]$ satisfying $\eta = 1$ on Q_{R-1} and $\mathrm{supp}\eta \subset Q_R$, define on Q_R,

$$\psi_R = \eta\psi + (1-\eta)\psi_R^0. \tag{A.3}$$

Then ψ_R is positive on Q_R and satisfies an equation of the form

$$\kappa\Delta\psi_R + p_R\psi_R = \lambda\psi_R \tag{A.4}$$

with Dirichlet boundary condition. Since $\psi_R = \psi_R^0$ close to ∂Q_R, the potential p_R is Hölder continuous in $\overline{Q}_R$. Hence, $\lambda = \lambda^{p_R}(Q_R)$. Since, moreover, $p_R = p$ on Q_{R-1}, (A.2) yields

$$\kappa\mathscr{S}_d(\mu) \geq (\mu, p_R) - \lambda^{p_R}(Q_R) = (\mu, p) - \lambda = \sqrt{\kappa}\mathrm{tr}(\Gamma) - \mathrm{tr}(\Gamma^2\mathrm{cov}\mu). \quad \square \tag{A.5}$$

STEP 2. *Proof of* “ ≥ .”

PROOF. Given $\mu \in \mathscr{P}_c(\mathbb{R}^d)$ with regular covariance matrix, apply Step 1 to $\Gamma = \frac{\sqrt{\kappa}}{2}(\mathrm{cov}\mu)^{-1}$ to obtain

$$\kappa \mathscr{S}_d(\mu) \geq \frac{\kappa}{4}\mathrm{tr}((\mathrm{cov}\mu)^{-1}). \tag{A.6}$$

Note that $\mathscr{S}_d(\mu) = \infty$ if $\mathrm{cov}\mu$ is not regular since, in this case, μ is not absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}^d$. Thus, if $J(\mu) = j(\mathrm{cov}\mu)$ for some function j, then, by inserting (A.6) into (0.11), we obtain

$$\begin{aligned} \chi &= \inf\left\{\kappa \mathscr{S}_d(\mu) + j(\mathrm{cov}\mu): \mu \in \mathscr{P}_c(\mathbb{R}^d), \mathrm{cov}\mu \text{ regular}\right\} \\ &\geq \inf\left\{\frac{\kappa}{4}\mathrm{tr}((\mathrm{cov}\mu)^{-1}) + j(\mathrm{cov}\mu): \mu \in \mathscr{P}_c(\mathbb{R}^d), \mathrm{cov}\mu \in \mathscr{M}_d^+\right\} \\ &= \inf\left\{\frac{\kappa}{4}\mathrm{tr}(\Gamma^{-1}) + j(\Gamma): \Gamma \in \mathscr{M}_d^+\right\}. \end{aligned} \tag{A.7}$$

This ends the proof of “≥.” □

STEP 3. *Proof of* “≤ .”

PROOF. Fix $\Gamma \in \mathscr{M}_d^+$ and a regular matrix C such that $2\Gamma = (C^{\mathrm{T}}C)^{-1}$ and define $\psi(x) = e^{-|Cx|^2}$. For $R > 0$, choose a smooth function $\eta_R: \mathbb{R}^d \to [0, 1]$ as in the end of the proof of Proposition 1. Recall that, in particular, $\mathrm{supp}\eta_R \subset Q_{R+1}$ and $\eta_R = 1$ on Q_{R-1}.

Now define $\psi_R = \eta_R^2 \psi$ and $\mu_R(dx) = \psi_R(x)\,dx/Z_R \in \mathscr{P}_c(Q_{R+1})$, where $Z_R = \int \psi_R(x)\,dx$. Clearly, $\lim_{R\to\infty} Z_R = \int \psi(x)\,dx$ and $\lim_{R\to\infty} \mathrm{cov}(\mu_R) = \Gamma$. If $J(\mu) = j(\mathrm{cov}(\mu))$ for some function j and all $\mu \in \mathscr{P}_c(\mathbb{R}^d)$, then j inherits the continuity from J, and therefore we also have $\lim_{R\to\infty} j(\mathrm{cov}(\mu_R)) = j(\Gamma)$.

Furthermore, from the construction of η_R it is clear that $|\nabla \eta_R|$ and $|\nabla \eta_R^2|$ are uniformly bounded by some constant that does not depend on R and vanish outside of $Q_{R+1} \setminus Q_{R-1}$. Therefore, one also easily verifies that

$$\mathscr{S}_d(\mu_R) = \frac{\int |\nabla \sqrt{\psi_R}|^2\,dx}{Z_R} \to \frac{\int |\nabla \sqrt{\psi}|^2\,dx}{\int \psi\,dx} = \frac{1}{4}\mathrm{tr}(\Gamma^{-1}) \tag{A.8}$$

as $R \to \infty$.

Thus, we obtain from (0.11)

$$\begin{aligned} \chi &= \inf\{\kappa \mathscr{S}_d(\mu) + j(\mathrm{cov}\mu): \mu \in \mathscr{P}_c(\mathbb{R}^d)\} \\ &\leq \lim_{R\to\infty}\{\kappa \mathscr{S}_d(\mu_R) + j(\mathrm{cov}\mu_R)\} \\ &= \frac{\kappa}{4}\mathrm{tr}(\Gamma^{-1}) + j(\Gamma). \end{aligned} \tag{A.9}$$

Passing to the infimum over all $\Gamma \in \mathscr{M}_d^+$, we arrive at “≤” in (0.16). □

A.2. *Proof of* (0.17).

PROOF OF ⇐. This is well-known.

PROOF OF $\Rightarrow$. Let $\chi = 0$.

STEP 1. $J(\nu) = 0$ *for all uniform distributions* ν *on cubes*.

PROOF. We fix $R > 0$ and equip the Sobolev space $H^1 = H^1(Q_R)$ with the norm $\|\cdot\|_{H^1}$ given by $\|f\|_{H^1}^2 = \|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2$. For measures $\mu \in \mathscr{P}(\overline{Q}_R)$ having Lebesgue density p with $\sqrt{p} \in H^1$ we define

$$\tilde{\mathscr{I}}_d(\mu) = \|\nabla\sqrt{p}\|_{L^2}^2. \tag{A.10}$$

Note that, due to boundary effects, $\tilde{\mathscr{I}}_d(\mu)$ does not necessarily coincide with $\mathscr{I}_d(\mu)$.

Every $\mu \in \mathscr{P}(\mathbb{R}^d)$ may be written as a convex combination $\mu = \sum_{x\in\mathbb{Z}^d} \gamma_x \mu^x$ with $\mu^x \in \mathscr{P}(2Rx + \overline{Q}_R)$. If $\nu^x \in \mathscr{P}(\overline{Q}_R)$ is a suitable shift of μ^x and $\mathscr{I}_d(\mu) < \infty$, then

$$\mathscr{I}_d(\mu) = \sum_{x\in\mathbb{Z}^d} \gamma_x \tilde{\mathscr{I}}_d(\nu^x). \tag{A.11}$$

This, together with the concavity and shift-invariance of J, implies that

$$0 = \chi \geq \inf\left\{\kappa\tilde{\mathscr{I}}_d(\nu) + J(\nu) : \nu \in \mathscr{P}(\overline{Q}_R), \sqrt{\frac{d\nu}{dx}} \in H^1\right\}. \tag{A.12}$$

Hence, there is a sequence of measures $\nu_n \in \mathscr{P}(\overline{Q}_R)$ with densities $p_n = d\nu_n/dx$ satisfying $\sqrt{p_n} \in H^1$ such that $\|\nabla\sqrt{p_n}\|_{L^2}^2 = \tilde{\mathscr{I}}_d(\nu_n) \to 0$ and $J(\nu_n) \to 0$ as $n \to \infty$. In particular, the sequence $(\|\sqrt{p_n}\|_{H^1})_n$ is bounded. Since H^1 is compactly embedded in $L^2 = L^2(Q_R)$ and the unit sphere of H^1 is weakly compact, we may assume that there exists $q \in H^1$ such that $\sqrt{p_n} \to q$ in L^2 and weakly in H^1. For all $f \in H^1$, we have

$$|(\nabla q, \nabla f)_{L^2}| = \lim_{n\to\infty} |(\nabla\sqrt{p_n}, \nabla f)_{L^2}| \leq \lim_{n\to\infty} \|\nabla\sqrt{p_n}\|_{L^2}\|\nabla f\|_{L^2} = 0. \tag{A.13}$$

Therefore, q is a.e. constant on Q_R. In particular, $(\nu_n)_n$ converges weakly toward the uniform distribution ν on Q_R. By continuity of J, we have $J(\nu) = \lim_{n\to\infty} J(\nu_n) = 0$, and the assertion follows. □

STEP 2. $J(\mu) = 0$ *for all* $\mu \in \mathscr{P}_c(\mathbb{R}^d)$ *having a bounded density*.

PROOF. Let $\mu \in \mathscr{P}_c(\mathbb{R}^d)$ have a bounded density. Choose $R > 0$ so large that $\operatorname{supp}\mu \subset Q_R$. Then the uniform distribution on Q_R is a non-trivial convex combination of μ and some other probability measure with support in Q_R. The assertion now follows from the concavity and nonnegativity of J and Step 1. □

STEP 3. *Conclusion*. Let $\mu \in \mathscr{P}_c(\mathbb{R}^d)$ and let $\varphi: \mathbb{R}^d \to [0,\infty)$ be a smooth probability density with compact support. Then $\mu_\varphi = \varphi * \mu$ has the bounded density $p_\varphi(x) = \int \varphi(x-y)\,\mu(dy)$ which has a compact support. According to Step 2, $J(\mu_\varphi) = 0$. Since μ_φ is a (continuous) convex combination of translations of μ, the continuity and concavity of J imply that $J(\mu) = 0$. □

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TECHNISCHE UNIVERSITÄT BERLIN
FACHBEREICH MATHEMATIK
STR. DES 17. JUNI 135
D-10623 BERLIN
GERMANY
E-MAIL: koenig@math.tu-berlin.de