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HYDRODYNAMIC LIMITS FOR A TWO-SPECIES REACTION-DIFFUSION PROCESS

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We consider a reaction-diffusion process with two components, on the grid $\mathbb{Z}$. This process had been introduced by Durrett and Levin to describe a two-species interaction. We prove the process admits hydrodynamic limits, first with a technique based on correlation functions, then with the method of relative entropy plus coupling.

1. Introduction. The interacting particle system we study in this paper was introduced by Durrett and Levin in [8] where they compare four approaches to model a biological phenomenon. The latter, called *hawks and doves model*, describes two different species living in interaction in the same region. The approaches considered by Levin and Durrett are discrete or infinitesimal, spatial or not spatial: More precisely, they write the model as a patch model (see [12]), as an interacting particle system (see [4]), as a couple of ordinary differential equations and as a couple of partial differential equations. After comparing the resulting behavior for these four models (that they obtain by heuristic methods), they explain how one could derive PDE from the particle system: By rescaling time and space, a discrete particle system would "converge" to the solution of a partial differential equation, called *hydrodynamic equation*. In this paper, we prove rigorously this assertion. First of all, we give a precise description of the dynamics:

Let $\mathbb{Z}$ be the 1-dimensional integer lattice. Two types of particles, namely the hawks and doves, evolve on $\mathbb{Z}$. We denote by $\eta_t(x)$ and $\zeta_t(x)$ the respective number of hawks and doves at site x at time t. So the configurations η_t and ζ_t give the state of the process at time $t \geq 0$ and the state space is $\chi = \mathbb{N}^{\mathbb{Z}} \times \mathbb{N}^{\mathbb{Z}}$. The dynamics splits into two parts: Diffusion and reaction. The diffusion represents the migrations of individuals in their region. It consists in independent symmetric random walks with nearest neighbor jumps which occur at rate 1, that is, after an exponential waiting time of parameter 1. We denote by L_0 the associated generator. There is an interaction between the two species in the reaction part which describes births and deaths of particles. Deaths are due to overpopulation: Each individual at site $x \in \mathbb{Z}$ at time t dies at rate $\kappa(\eta_t(x) + \zeta_t(x))$, where κ is a positive constant. Let $\mathcal{N}$ be a fixed neighborhood of the origin. For example, $\mathcal{N}$ can be constituted by 0 and its nearest neighbors: $\mathcal{N} = \{x \in \mathbb{Z} : |x| \leq 1\}$. We set

$$\hat{\eta}_t(x) = \sum_{y \in \mathcal{N}} \eta_t(x+y), \qquad \hat{\zeta}_t(x) = \sum_{y \in \mathcal{N}} \zeta_t(x+y),$$

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and $p_t(x) = \hat{\eta}_t(x)/(\hat{\eta}_t(x) + \hat{\zeta}_t(x))$ represents the fraction of hawks in the interaction neighborhood. A hawk creates a new one at x at rate $ap_t(x) + b(1 - p_t(x))$ and a dove creates a new one at rate $cp_t(x) + d(1 - p_t(x))$ where a, b, c, d are positive coefficients. This part of the generator is denoted by L_c. So the infinitesimal generator is given for a cylinder function f by

$$(Lf)(\eta, \zeta) = (L_0 f)(\eta, \zeta) + (L_c f)(\eta, \zeta)$$

where

$$\begin{aligned}(L_0 f)(\eta, \zeta) = {} & \tfrac{1}{2} \sum_{x \in \mathbb{Z}} \eta(x)\Big[f(\eta - e_x + e_{x+1}, \zeta) + f(\eta - e_x + e_{x-1}, \zeta) - 2f(\eta, \zeta)\Big] \\ & + \tfrac{1}{2} \sum_{x \in \mathbb{Z}} \zeta(x)\Big[f(\eta, \zeta - e_x + e_{x+1}) + f(\eta, \zeta - e_x + e_{x-1}) - 2f(\eta, \zeta)\Big],\end{aligned}$$

$$\begin{aligned}(L_c f)(\eta, \zeta) = {} & \sum_{x \in \mathbb{Z}} \Big\{\tau_x \beta_1(\eta, \zeta)\Big[f(\eta + e_x, \zeta) - f(\eta, \zeta)\Big] \\ & \qquad + \tau_x \delta_1(\eta, \zeta)\Big[f(\eta - e_x, \zeta) - f(\eta, \zeta)\Big]\Big\} \\ & + \sum_{x \in \mathbb{Z}} \Big\{\tau_x \beta_2(\eta, \zeta)\Big[f(\eta, \zeta + e_x) - f(\eta, \zeta)\Big] \\ & \qquad + \tau_x \delta_2(\eta, \zeta)\Big[f(\eta, \zeta - e_x) - f(\eta, \zeta)\Big]\Big\}\end{aligned}$$

with $\beta_1(\eta, \zeta) = \eta(0)(ap(0) + b(1 - p(0)))$, $\beta_2(\eta, \zeta) = \zeta(0)(cp(0) + d(1 - p(0)))$, $\delta_1(\eta, \zeta) = \kappa\eta(0)(\eta(0) + \zeta(0))$, $\delta_2(\eta, \zeta) = \kappa\zeta(0)(\eta(0) + \zeta(0))$. Here e_x represents the configuration with only one particle at site x: $e_x(x) = 1$ and $e_x(y) = 0$ when $y \neq x$, and the sum of two configurations is understood coordinatewise. We will sometimes denote by L_1 the first part of L_c and by L_2 the second one.

Chen [5] proved the existence for a large class of reaction-diffusion processes in Theorem 13.8. Its criterium, based on a control of the first moment and a Lipschitz condition to obtain convergence, is satisfied for our process. Besides, the distribution which charges only the couple of configurations with no particle is a stationary distribution. So the *hawks and doves model* admits at least an invariant measure. The main obstacle to study an equilibrium behavior of the process is non-attractivity, due to the dependence on the two species of the birth and death rates.

Nevertheless we have some results: We have that the process dies with probability one when the migrations occur at a small rate, using Neuhauser's proof in [11]. In this case, there is a unique invariant measure, the trivial one. For this, we consider the process (η_t^n, ζ_t^n) whose generator is $L_\mu = \mu L_0 + L_c$, with initial configurations $\eta^n \equiv n$, $\zeta^n \equiv n$, that is, $\eta^n(x) = \zeta^n(x) = n$ for all $x \in \mathbb{Z}$. The coefficient $\mu > 0$ slows down the migrations so that the process behaves more like a pure birth and death process.

PROPOSITION 1.1. *There exists* μ_0 *such that for all* $\mu \leq \mu_0$,

$$\lim_{t\to+\infty} E\Big[\eta_t^n(x)\Big] = \lim_{t\to+\infty} E\Big[\zeta_t^n(x)\Big] = 0 \qquad \textit{for all } x \in \mathbb{Z}$$

Moreover, we observe phase transitions, that is, for large but finite birth rates, the process may die or not and then, it is non-ergodic. Let $\lambda = \min\{a, b, c, d\}$. For λ large enough, the process starting from the configurations (e_0, e_0) survives with positive probability.

PROPOSITION 1.2.

$$\lambda_c := \inf\Big\{\lambda \,:\, P[\eta_t^0 + \zeta_t^0 \not\equiv 0,\ \textit{for all } t \geq 0] > 0\Big\} < +\infty$$

where (η_t^0, ζ_t^0) *is the process with initial distribution* (e_0, e_0).

This proposition is proved for one component processes in the book by Chen [5] by using an oriented percolation and a comparison theorem (see [2]). So we compare a one component chosen process to ours to obtain the result.

We now claim that the behavior of the process in the macroscopic limit is described by a system of reaction-diffusion equations (Eq):

$$\frac{\mathrm{d}}{\mathrm{d}t}\binom{u}{v} = \frac{1}{2}\Delta\binom{u}{v} + \binom{F_1(u,v)}{F_2(u,v)}$$

with initial conditions. The function $F_1(u, v)$ is the expectation of $\beta_1(\eta, \zeta) - \delta_1(\eta, \zeta)$ with respect to the product of Poisson measures with respective parameters u and v and $F_2(u, v)$ is the expectation of $\beta_2(\eta, \zeta) - \delta_2(\eta, \zeta)$. In [8], Durrett and Levin compute them:

$$\text{(Eq)}\qquad \begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = \dfrac{1}{2}\Delta u + u\Big[a\Big(h + (1-h)\dfrac{u}{u+v}\Big) + b(1-h)\dfrac{v}{u+v} - \kappa(1+u+v)\Big], \\[2ex] \dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{1}{2}\Delta v + v\Big[c(1-h)\dfrac{u}{u+v} + d\Big(h + (1-h)\dfrac{v}{u+v}\Big) - \kappa(1+u+v)\Big], \end{cases}$$

where

$$h = h(u, v) = \frac{1 - e^{-|\mathcal{N}|(u+v)}}{|\mathcal{N}|(u+v)}.$$

We obtain existence and regularity of the solution by Theorems 14.2 and 14.4 of Smoller [13]. Moreover, the solution is a couple of uniformly bounded functions, when the initial conditions are bounded. So the system admits invariant regions (see Chapter 14, Section B of [13]) for each values of the coefficients. For example, we can set as in [8]: $a = 0.4$, $b = 0.8$, $c = 0.6$, $d = 0.3$, $\kappa = 0.08$ and $|\mathcal{N}| = 3$. Then

$$\Sigma = \{(u, v) :\ 1 \leq u \leq 4.5;\ 1 \leq v \leq 3\}$$

is an invariant region, that is, if the initial conditions are in Σ, then the solution $(u_t(.), v_t(.))$ is still in Σ. In this example, we see that the solution is a couple of functions, bounded below by a positive real number. But sometimes, it will not be true. Let us consider the coefficients $a = 0.5$, $b = 0.4$, $c = 0.4$, $d = 0.8$, $\kappa = 0.1$. Then starting from $u_0(x) = \varepsilon$ and $v_0(x) = 7$ for all $x \in \mathbb{Z}$ (with $\varepsilon \leq 0.5$), u_t will decrease in time to zero. Thus we can not always assume the solutions to be bounded below by a strictly positive number.

We now state the theorems giving the hydrodynamic limits. The passage from microscopic to macroscopic consists in taking a limit when the distance between particles goes to zero. So we set that the distance between two neighboring sites is $1/N$ and N goes to infinity. Once this done, we have to rescale time: The diffusion part of the generator needs an acceleration by N^2, thus we consider the generator $L_N = N^2 L_0 + L_c$. Starting from Poisson product measures, since the independent random walks occur much more than the births and deaths, we hope that the distributions of η_t and ζ_t will still be approximatively Poisson product measures. Actually we prove that the joint distribution of the number of particles at any finite set of points converges to independent Poisson random variables whose parameters are solution of (Eq): There is propagation of chaos. For this we use in section 2 a technique based on correlation functions, introduced by Boldrighini, De Masi, Pellegrinotti and Presutti in [3] (see [6]), for a one component process with polynomial birth and death rates. We extend it to our two components particle system whom birth and death rates are not polynomial.

We will next give in section 3 another proof of the existence of hydrodynamic limits. We will get a *law of large numbers* which describes the collective behavior of the system, by using the relative entropy method, which was first introduced by Yau for the Ginzburg-Landau model [14] in finite volume. It was modified for classical reaction-diffusion processes still in finite volume by Mourragui [10]. We widen it to processes with two types of particles and with less restricting conditions on the birth and death rates, on the torus $T_N = \mathbb{Z}/N\mathbb{Z}$. Then we deduce the result in infinite volume, comparing a process on the torus with a process on $\mathbb{Z}$.

The result with the first method is much stronger but the second proof does not need the initial measure to be product. Furthermore, the second proof demands the solutions of the PDE system to be greater than a positive real number. We have seen in the examples above that it is not always satisfied. Following the approach of Kipnis, Olla and Varadhan [9], Belbase proved in [1] that two-species reaction-diffusion processes with simplier birth and death rates admit hydrodynamic limits. But the technics he used does not seem to be extended to general rates.

Before stating the first theorem, proved using correlations functions, we need some notations. For the sake of simplicity, we prove that our process admits hydrodynamic limits when the birth rates depend only on the number of each species at one site and then we will extend the proof to the general birth rates defined above. We choose the initial measure μ^N on $\mathbb{N}^{\mathbb{Z}} \times \mathbb{N}^{\mathbb{Z}}$ as a product of Poisson measures $\nu^N_{\rho^1}\times\nu^N_{\rho^2}$ such that, for all $x\in\mathbb{N}$, and $k,\ j\in\mathbb{N}$,

$$\nu^N_{\rho^1(.)}\times\nu^N_{\rho^2(.)}\Big\{\eta(x)=k,\,\zeta(x)=j\Big\}$$
$$=\exp\Big\{-\rho^1(x/N)\Big\}\frac{\rho^1(x/N)^k}{k!}\exp\Big\{-\rho^2(x/N)\Big\}\frac{\rho^2(x/N)^j}{j!}$$

where ρ^1 and ρ^2 are assumed to be positive uniformly bounded C^2 functions. Let ρ_0 be the maximum of their upper bounds. We define $\Omega=\{\xi\in\mathbb{N}^{\mathbb{Z}}:|\xi|=\sum_x\xi(x)<+\infty\}$ and $\Omega^{(n)}(L)=\{\xi\in\Omega:|\xi|=n \text{ and } \xi(x)=0 \text{ if } |x|>NL\}$. Let us introduce Poisson polynomials

$$D_k(n)=\begin{cases}1, & \text{if } k=0,\\ n(n-1)\cdots(n-k+1), & \text{else}\end{cases}$$

and Poisson polynomials for configurations

$$\begin{aligned}D:\Omega\times\mathbb{N}^{\mathbb{Z}}&\longrightarrow\mathbb{R}\\ (\xi,\eta)&\longmapsto\prod_x D_{\xi(x)}(\eta(x)).\end{aligned}$$

THEOREM 1.3. *For all $L\in\mathbb{N}$, $T>0$ and $n_1,n_2\in\mathbb{N}$,*

$$\lim_{N\to\infty}\sup_{\substack{\xi^1\in\Omega^{(n_1)}(L)\\ \xi^2\in\Omega^{(n_2)}(L)}}\sup_{0\le t\le T}\bigg|E^N_{\mu^N}[D(\xi^1,\eta_t)D(\xi^2,\zeta_t)]-\prod_x\rho^1_t\Big(\frac{x}{N}\Big)^{\xi^1(x)}\rho^2_t\Big(\frac{x}{N}\Big)^{\xi^2(x)}\bigg|=0 \tag{1}$$

where (ρ^1_t,ρ^2_t) solves (Eq) with initial conditions $(\rho^1(.),\rho^2(.))$.

We now deal with the second theorem, proved by the relative entropy method. We first assume that the hawks and doves are living on the torus $T_N=\mathbb{Z}/N\mathbb{Z}$. Like above, we make the distance between two neighboring sites converging to zero and we accelerate the diffusion by N^2. So the generator L^N of the process is L_N, restricted to T_N, with the associated semi-group (S^N_t). If the empirical measure $(\pi^N_t(\eta_t),\pi^N_t(\zeta_t))$ is defined by

$$\pi^N_t(\eta_t)=\frac{1}{N}\sum_{x=0}^{N-1}\eta_t(x)\delta_{x/N},\qquad \pi^N_t(\zeta_t)=\frac{1}{N}\sum_{x=0}^{N-1}\zeta_t(x)\delta_{x/N}$$

where $\delta_{x/N}$ is the Dirac measure at x/N, we will prove that it converges in measure when N goes to infinity to a deterministic measure, absolutely continuous with respect to the Lebesgue measure, $(\rho^1(t,u)\mathrm{d}u,\rho^2(t,u)\mathrm{d}u)$ where $(\rho^1(.\,,.),\rho^2(.\,,.))$ is solution of the reaction-diffusion system (Eq). The method consists in studying the entropy of the process with respect to Poisson measures with parameter the expected *good profile*: $(\rho^1(t,\,.),\rho^2(t,\,.))$. So let us define the entropy of a measure μ^N on χ_N with respect to a profile $(\rho_1(.),\rho_2(.))$ by

$$\mathrm{H}\Big[\mu^N\Big|\nu^N_{\rho_1(.)}\times\nu^N_{\rho_2(.)}\Big]=\int\mathrm{Log}\frac{\mathrm{d}\mu^N}{\mathrm{d}(\nu^N_{\rho_1(.)}\times\nu^N_{\rho_2(.)})}\,\mathrm{d}\mu^N$$

THEOREM 1.4. *Assume there exist smooth positive functions* $m_1(.)$, $m_2(.)$ *on the torus* $[0, 1]$, *such that*

$$\limsup_{N\to\infty} \frac{1}{N}\mathrm{H}[\mu^N|\nu^N_{m_1(.)} \times \nu^N_{m_2(.)}] = 0. \tag{2}$$

Then for all functions $G_1(.)$ *and* $G_2(.)$, *continuous on* $[0, 1]$, *all* $\delta > 0$ *and* $t \in [0, T]$, *we have*

$$\lim_{N\to\infty} \mu^N S^N_t \Bigg\{(\eta, \zeta) : \ \Big|\frac{1}{N}\sum_{x=0}^{N-1} \eta(x)G_1(x/N) - \int_0^1 G_1(\theta)\lambda_1(t, \theta)\mathrm{d}\theta\Big| > \delta$$
$$\text{and } \ \Big|\frac{1}{N}\sum_{x=0}^{N-1} \zeta(x)G_2(x/N) - \int_0^1 G_2(\theta)\lambda_2(t, \theta)\mathrm{d}\theta\Big| > \delta\Bigg\} = 0$$

where $(\lambda_1(t, .), \lambda_2(t, .))$ *is the unique smooth solution of the system of equations* (Eq) *with initial conditions* $\lambda_1(0, .) = m_1(.)$ *and* $\lambda_2(0, .) = m_2(.)$.

By a coupling method, we will prove in the last section that two processes, one defined on $\mathbb{Z}$ and the other one defined on T_{CN}, the torus $\{-CN, \ldots, CN\}$, are close when C is large. So we can extend the last theorem to infinite volume. Let $(\tilde{S}^N_t)$ be the semi-group of the process on $\mathbb{Z}$, associated to the generator L_N. The hypothesis on the entropy in the last theorem has no sense anymore, so we have to define the specific entropy of a measure μ with respect to a measure ν on $\mathbb{Z}$:

$$\mathscr{H}_N[\mu|\nu] = 1/N \sum_{n\geq 1} H_n[\mu^n|\nu^n]e^{-\theta n/N}$$

where θ is a fixed positive real number and μ^n and ν^n are the respective restrictions of μ and ν on $\Lambda_n = \{-n, \ldots, n\}$.

THEOREM 1.5. *We consider a sequence of initial distributions* μ^N *such that there exists* $M > 0$ *with* $\mu^N(\eta(x)) \leq M$ *for all* $x \in \mathbb{N}$ *and such that there exist smooth positive functions* $m_1(.)$ *and* $m_2(.)$ *on* $\mathbb{R}$ *satisfying*

$$\limsup_{N\to\infty} \frac{1}{N}\mathscr{H}_N[\mu^N|\nu^N_{m_1(.)}\times \nu^N_{m_2(.)}] = 0.$$

Then for all functions $G_1(.)$ *and* $G_2(.)$, *continuous on* $\mathbb{R}$ *with compact support, all* $\delta > 0$ *and* $t \in [0, T]$, *we have*

$$\lim_{N\to\infty} \mu^N \tilde{S}^N_t \Bigg\{(\eta, \zeta) : \ \Big|\frac{1}{N}\sum_{x\in\mathbb{Z}} \eta(x)G_1(x/N) - \int G_1(\theta)\lambda_1(t, \theta)\mathrm{d}\theta\Big| > \delta$$
$$\text{and } \ \Big|\frac{1}{N}\sum_{x\in\mathbb{Z}} \zeta(x)G_2(x/N) - \int G_2(\theta)\lambda_2(t, \theta)\mathrm{d}\theta\Big| > \delta\Bigg\} = 0$$

where $(\lambda_1(t, .), \lambda_2(t, .))$ *is the unique smooth solution of the system of equations* (Eq) *with initial conditions* $\lambda_1(0, .) = m_1(.)$ *and* $\lambda_2(0, .) = m_2(.)$.

2. Hydrodynamical limits: Proof of Theorem 1.3. The tools of this proof are in [6], but the fact that the birth and death rates are polynomial is basic for the original proof, so the main difficulty here is to control the replacement of our rates by polynomials. Moreover, since there are two species, the calculations are more intricate. We first state a duality relation for the correlation functions. The rest of the proof is divided into three parts. The first one gives a uniform bound for the correlation functions. In the second part, we express the limit of an equation satisfied by the correlation functions when N goes to infinity and finally, we prove that the only solution of this limiting equation is $\prod_x \rho_t^1(x/N)^{\xi^1(x)}\rho_t^2(x/N)^{\xi^2(x)}$.

Let ξ^1 and ξ^2 be in $\Omega^{(n_1)}(L)$ and $\Omega^{(n_2)}(L)$ respectively, with $n = n_1 + n_2$. We set $\xi = (\xi^1, \xi^2)$ and we consider the process (ξ_t, η_t, ζ_t) where $\xi_t = (\xi_t^1, \xi_t^2)$ is the process starting from (ξ^1, ξ^2) with generator $N^2 L_0$, and (η_t, ζ_t) is the process with random initial configurations distributed according to μ^N, and generator L_N. We denote by $\mathscr{E}$ the expectation of (ξ_t, η_t, ζ_t). We now define the correlation functions for the process (η_t, ζ_t) as follows:

$$u_t^N(\xi) = E_{\mu^N}^N[D(\xi^1, \eta_t)D(\xi^2, \zeta_t)] = \mathscr{E}[D(\xi_0^1, \eta_t)D(\xi_0^2, \zeta_t)].$$

By a simple calculation, $[L_0 D(\xi, .)](\eta) = [L_0 D(., \eta)](\xi)$, where $[L_0 D(\xi, .)](\eta)$ means that L_0 acts on the second variable. So if we derive the correlation functions, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathscr{E}[D(\xi_{t-s}^1, \eta_s)D(\xi_{t-s}^2, \zeta_s)] = \mathscr{E}[L_c D(\xi_{t-s}^1, \eta_s)D(\xi_{t-s}^2, \zeta_s)] \tag{3}$$

where the generator L_c acts on (η_t, ζ_t). By definition, $\mathscr{E}[D(\xi_0^1, \eta_t)D(\xi_0^2, \zeta_t)] = u_t^N(\xi)$ and

$$\mathscr{E}[D(\xi_t^1, \eta_0)D(\xi_t^2, \zeta_0)] = E^{\xi}\Big[u_0^N(\xi_t)\Big] = \sum_{\bar\xi = (\bar\xi^1, \bar\xi^2)} P_t^N(\xi \to \bar\xi) u_0^N(\bar\xi)$$

where $P_t^N(\xi \to \bar\xi)$ is the transition probability to go from $\xi = (\xi^1, \xi^2)$ to $\bar\xi = (\bar\xi^1, \bar\xi^2)$ in a time t when the generator is $N^2 L_0$. Therefore we integrate in time equation (3):

$$\begin{aligned} u_t^N(\xi) &= \sum_{\bar\xi} P_t^N(\xi \to \bar\xi) u_0^N(\bar\xi) \\ &\quad + \int_0^t \mathrm{d}s \sum_{\bar\xi = (\bar\xi^1, \bar\xi^2)} P_{t-s}^N(\xi \to \bar\xi) E_{\mu^N}^N[L_c D(\bar\xi^1, \eta_s)D(\bar\xi^2, \zeta_s)]. \end{aligned} \tag{4}$$

2.1. *Uniform bounds on the correlation functions.* We set

$$K_t^N(n) = \sup_{s \le t} \sup_{|\xi| \le n} u_s^N(\xi).$$

To bound it, we have to use intensively relation (4).

We set $d(\xi^1(x)) = \xi^1(x)(a + b + c + d + \kappa\xi^1(x) - \kappa)^2/4\kappa$. Note that d is a non negative non decreasing function of $\xi^1(x)$ and that $L_c D(\xi^1(x)e_x, \eta) \le$

$d(\xi^1(x))D(\xi^1(x)e_x - e_x, \eta)$. The same arguments work on $L_c D(\xi^2(x)e_x, \zeta)$ so that

$$\begin{aligned}
u_t^N(\xi) \le \rho_0^n + \int_0^t \mathrm{d}s \sum_{\bar\xi} P_{t-s}^N(\xi \to \bar\xi) \\
\times \Bigg[\sum_{x:\bar\xi^1(x)\ge 1} d(\bar\xi^1(x)) E_{\mu^N}^N[D(\bar\xi^1 - e_x, \eta_s) D(\bar\xi^2, \zeta_s)] \\
+ \sum_{x:\bar\xi^2(x)\ge 1} d(\bar\xi^2(x)) E_{\mu^N}^N[D(\bar\xi^2 - e_x, \zeta_s) D(\bar\xi^1, \eta_s)]\Bigg] \\
\le \rho_0^n + \int_0^t \mathrm{d}s\; d(n) n \sup_{|\xi|<n} E_{\mu^N}^N[D(\xi^1, \eta_s) D(\xi^2, \zeta_s)] \\
\le \rho_0^n + \int_0^t \mathrm{d}s\; n d(n) K_t^N(n).
\end{aligned}$$

Now just apply Gronwall lemma to obtain

$$K_t^N(n) \le (\rho_0 e^{d(n)t})^n =: c_n. \tag{5}$$

2.2. *Hierarchy of correlation functions.* We write $\xi^1 = \sum_{i=1}^{n_1} e_{x_i}$, $\xi^2 = \sum_{i=n_1+1}^{n} e_{x_i}$ and $r_i = x_i/N$ for $1 \le i \le n$. Then we set $\underline{r} = (r_1, \dots, r_{n_1}; r_{n_1+1}, \dots, r_n)$ and $u_t^N(\xi) = \gamma_t^N(\underline{r})$. By linear interpolation, we define $\gamma_t^N(\underline{r})$ for all $\underline{r} \in \mathbb{R}^n$. From (5), $\gamma_t^N(\underline{r})$ is uniformly bounded for all n and $T > 0$ fixed. Furthermore, it is equicontinuous (see the details in [3] and [5]). So we can extract a subsequence (N_k) going to infinity, such that $\gamma_t^{N_k}(\underline{r})$ converges for all $n \ge 1$, $\underline{r}$ and $t \ge 0$ and the convergence is uniform on compact sets. Let $\gamma_t(\underline{r})$ be the limit of $\gamma_t^N(\underline{r})$ when N goes to infinity along the subsequence (N_k). We define $A_n = \{\xi = (\xi^1, \xi^2) \ : \ x_i \ne x_j, \ \forall\ 0 \le i \ne j \le n\}$ and we rewrite equation (4) as

$$\begin{aligned}
u_t^N(\xi) = \sum_{\bar\xi} P_t^N(\xi \to \bar\xi) u_0^N(\bar\xi) \\
+ \int_0^t \mathrm{d}s \sum_{\bar\xi \notin A_n} P_{t-s}^N(\xi \to \bar\xi) E_{\mu^N}^N[L_c D(\bar\xi^1, \eta_s) D(\bar\xi^2, \zeta_s)] \\
+ \int_0^t \mathrm{d}s \sum_{\bar\xi \in A_n} P_{t-s}^N(\xi \to \bar\xi) E_{\mu^N}^N[L_c D(\bar\xi^1, \eta_s) D(\bar\xi^2, \zeta_s)].
\end{aligned} \tag{6}$$

For the first term of the right hand side of (6), we recognize the product of n transition probabilities p_t^N of simple symmetric random walks on $\mathbb{Z}$, accelerated by N^2. By the central limit theorem, when N goes to infinity, it converges weakly to a normal distribution. Therefore

$$\begin{aligned}
\sum_{\bar\xi} P_t^N(\xi \to \bar\xi) u_0^N(\bar\xi) \\
= \sum_{y_1,\dots,y_n} \prod_{i=1}^{n_1} p_t^N(x_i \to y_i) \rho^1(y_i/N) \prod_{i=n_1+1}^{n} p_t^N(x_i \to y_i) \rho^2(y_i/N)
\end{aligned}$$

$$= \prod_{i=1}^{n_1}\Big[\sum_y p_t^N(x_i \to y)\rho^1(y/N)\Big]\prod_{i=n_1+1}^{n}\Big[\sum_y p_t^N(x_i \to y)\rho^2(y/N)\Big]$$

$$\to \prod_{i=1}^{n_1}\Big(\int G_t(r_i - r)\rho^1(r)\mathrm{d}r\Big)\prod_{i=n_1+1}^{n}\Big(\int G_t(r_i - r)\rho^2(r)\mathrm{d}r\Big)$$

with

$$G_t(r_i - r) = \frac{1}{\sqrt{2\pi t}}\exp\{-(r-r_i)^2/2t\}.$$

Consider now the second term of (6), where $\bar{\xi} \notin A_n$. It is well known that $p_t^N(u \to v) \le c/N\sqrt{t}$ where c is a positive constant. Then

$$\begin{aligned}
&\sum_{\bar{\xi}} P_t^N(\xi \to \bar{\xi})\mathbb{1}_{\{\bar{\xi}\notin A_n\}}\\
&\quad = \sum_{\exists i,j:\, \bar{x}_i = \bar{x}_j} P_t^N(\xi \to \bar{\xi})\\
&\quad \le \frac{n(n-1)}{2}\sum_{\bar{x}_2,\dots,\bar{x}_n} p_t^N(x_1 \to \bar{x}_2)p_t^N(x_2 \to \bar{x}_2)\cdots p_t^N(x_n \to \bar{x}_n)\\
&\quad \le \frac{n(n-1)}{2}\frac{c}{\sqrt{N^2 t}}.
\end{aligned} \tag{7}$$

By (5), it is easy to show that $|E_{\mu^N}^N[L_c D(\bar{\xi}^1, \eta_s)D(\bar{\xi}^2, \zeta_s)]|$ is bounded, therefore the second term of the r.h.s. of (6) vanishes when N goes to infinity. Then when N_k goes to infinity, (6) becomes

$$\begin{aligned}
\gamma_t(r_1,\dots,r_n) = {} & \prod_{i=1}^{n_1}\Big(\int G_t(r_i - r)\rho^1(r)\mathrm{d}r\Big)\prod_{i=n_1+1}^{n}\Big(\int G_t(r_i - r)\rho^2(r)\mathrm{d}r\Big)\\
& + \int_0^t \mathrm{d}s \lim_{N_k\to\infty}\sum_{\bar{\xi}\in A_n} P_{t-s}^N(\xi \to \bar{\xi})\\
& \qquad \times E_{\mu^{N_k}}^{N_k}\Big[L_c \prod_{i=1}^{n_1}\eta_s(x_i)\prod_{i=n_1+1}^{n}\zeta_s(x_i)\Big].
\end{aligned} \tag{8}$$

For $i = 1, 2$, we introduce the polynomial

$$P_M^i(\eta,\zeta) = \sum_{k=0}^{M}\sum_{l=0}^{M} a_{lk}^i \prod_{l'\ne l}\frac{\eta(0)-l'}{l-l'}\prod_{k'\ne k}\frac{\zeta(0)-k'}{k-k'}$$

with

$$a_{lk}^1 = l\Big(a\frac{l}{l+k} + b\frac{k}{l+k}\Big), \qquad a_{lk}^2 = k\Big(c\frac{l}{l+k} + d\frac{k}{l+k}\Big)$$

and the set $B_M = \{(k_1,k_2) : \ k_1 \le M \text{ and } k_2 \le M\}$. We denote by $\overline{B_M}$ its complementary. P_M^i and B_M had been chosen to have, when $(\eta(x),\zeta(x)) \in B_M$, $\tau_x P_M^1(\eta,\zeta) = \tau_x\beta_1(\eta,\zeta)$. So we write $\tau_x\beta_1(\eta,\zeta) = \tau_x P_M^1(\eta,\zeta) + (\tau_x\beta_1(\eta,\zeta) - \tau_x P_M^1(\eta,\zeta))\mathbb{1}_{\overline{B_M}}(\eta(x),\zeta(x))$. In the same way, on B_M: $\tau_x P^2_M(\eta,\zeta) = \tau_x\beta_2(\eta,\zeta)$. Since P^i_M and δ_1 are polynomials, we can write them as linear combinations of Poisson polynomials. Then, we shall use the convergence of their expectations. The other terms (which contain $\mathbb{1}_{\overline{B_M}}$) converge to 0 when N_k and then M go to infinity:

$$\begin{aligned}\tau_x\delta_1(\eta,\zeta) = \kappa\eta(x)(\eta(x)+\zeta(x)) &= \kappa D_2(\eta(x)) + \kappa D_1(\eta(x))\\ &\quad + \kappa D_1(\eta(x))D_1(\zeta(x))\\ &:= \sum_{k=1}^{2}\sum_{l=0}^{1}\delta^1_{kl}D_k(\eta(x))D_l(\zeta(x)),\\ \tau_x\delta_2(\eta,\zeta) = \kappa\zeta(x)(\eta(x)+\zeta(x)) &:= \sum_{k=0}^{1}\sum_{l=1}^{2}\delta^2_{kl}D_k(\eta(x))D_l(\zeta(x)),\\ \tau_x P^i_M(\eta,\zeta) &= \sum_{k=0}^{M}\sum_{l=0}^{M}\alpha^i_{kl}D_k[\eta(x)]D_l[\zeta(x)]\end{aligned}$$

where

$$\alpha^i_{kl} = \sum_{x=0}^{k}\sum_{y=0}^{l}\frac{(-1)^{k+l-x-y}a^i_{xy}}{x!y!(k-x)!(l-y)!}.$$

So when $\xi = (\xi^1,\xi^2)\in A_N$,

$$\begin{aligned}&E^{N_k}_{\mu^{N_k}}\Big[L_c\prod_{i=1}^{n_1}\eta_s(x_i)\prod_{i=n_1+1}^{n}\zeta_s(x_i)\Big]\\ &\quad= \sum_{i=1}^{n_1}E^{N_k}_{\mu^{N_k}}\Big[D(\xi^1-e_{x_i},\eta_s)D(\xi^2,\zeta_s)\Big(\tau_{x_i}P^1_M(\eta_s,\zeta_s)-\tau_{x_i}\delta_1(\eta_s,\zeta_s)\Big)\Big]\\ &\qquad+\sum_{i=n_1+1}^{n}E^{N_k}_{\mu^{N_k}}\Big[D(\xi^1,\eta_s)D(\xi^2-e_{x_i},\zeta_s)\Big(\tau_{x_i}P^2_M(\eta_s,\zeta_s)-\tau_{x_i}\delta_2(\eta_s,\zeta_s)\Big)\Big]\\ &\qquad+\sum_{i=1}^{n_1}E^{N_k}_{\mu^{N_k}}\Big[D(\xi^1-e_{x_i},\eta_s)D(\xi^2,\zeta_s)\\ &\qquad\qquad\times\Big(\tau_{x_i}\beta_1(\eta_s,\zeta_s)-\tau_{x_i}P^1_M(\eta_s,\zeta_s)\Big)\mathbb{1}_{\overline{B_M}}(\eta_s(x_i),\zeta_s(x_i))\Big]\\ &\qquad+\sum_{i=n_1+1}^{n}E^{N_k}_{\mu^{N_k}}\Big[D(\xi^1,\eta_s)D(\xi^2-e_{x_i},\zeta_s)\\ &\qquad\qquad\times\Big(\tau_{x_i}\beta_2(\eta_s,\zeta_s)-\tau_{x_i}P^2_M(\eta_s,\zeta_s)\Big)\mathbb{1}_{\overline{B_M}}(\eta_s(x_i),\zeta_s(x_i))\Big].\end{aligned}$$

Denote by $I^M_{N_k}(\xi)$ the sum of the two first terms and by $\bar{I}^M_{N_k}(\xi)$ the sum of the two last terms. For instance, we observe that

$$\begin{aligned}&E^{N_k}_{\mu^{N_k}}\Big[D(\xi^1,\eta_s)D(\xi^2-e_{x_i},\zeta_s)\tau_{x_i}P^1_M(\eta_s,\zeta_s)\Big]\\ &\quad= E^{N_k}_{\mu^{N_k}}\left[\sum_{k=0}^{M}\sum_{l=0}^{M}\alpha^1_{kl}D(\xi^1+ke_{x_i},\eta_s)D(\xi^2+le_{x_i},\zeta_s)\right]\to\sum_{k=0}^{M}\sum_{l=0}^{M}\alpha^1_{kl}\gamma_s(\underline{r}^i_{kl})\end{aligned}$$

when N_k goes to infinity, with $\underline{r}^i_{kl} = (r_1, \dots, r_i,\ r_i, \dots, r_{n_1};\ r_{n_1+1}, \dots, r_i,\ r_i, \dots, r_n)$, where r_i is repeated first k times and then l times. Let

$$(\mathscr{A}_M\gamma_s)(\underline{r}) := \lim_{N_k\to\infty} I^M_{N_k}(\xi) = \sum_{i=1}^{n_1}\biggl(\sum_{k=0}^{M}\sum_{l=0}^{M}\alpha^1_{kl}\gamma_s(\underline{r}^i_{kl}) - \sum_{k=1}^{2}\sum_{l=0}^{1}\delta^1_{kl}\gamma_s(\underline{r}^i_{kl})\biggr) + \sum_{i=n_1+1}^{n}\biggl(\sum_{k=0}^{M}\sum_{l=0}^{M}\alpha^2_{kl}\gamma_s(\underline{r}^i_{kl}) - \sum_{k=0}^{1}\sum_{l=1}^{2}\delta^2_{kl}\gamma_s(\underline{r}^i_{kl})\biggr).$$

Now to study $\bar{I}^M_{N_k}(\xi)$ we state the next lemmas.

LEMMA 2.1. *There exists a constant $c > 0$ such that, for every ξ satisfying $|\xi| \le n$ and for all $t \in [0, T]$: $u^N_t(\xi) \le c^n + o(1/N)$, where $o(1/N)$ is a function vanishing when N goes to infinity.*

This upper bound for the correlation functions is more precise than (5). It is obtained as (5) by using (6) instead of (4) and then (7).

LEMMA 2.2. *For all $1 \le i \le n_1$,*

$$\lim_{M\to\infty}\lim_{N_k\to\infty} E^{N_k}_{\mu^{N_k}}\Biggl[\prod_{\substack{j=1\\ j\ne i}}^{n_1}\eta_s(x_j)\prod_{j=n_1+1}^{n}\zeta_s(x_j)\Bigl(\tau_{x_i}\beta_1(\eta_s,\zeta_s) + \tau_{x_i}P^1_M(\eta_s,\zeta_s)\Bigr) \times \mathbb{1}_{\overline{B_M}}(\eta_s(x_i), \zeta_s(x_i))\Biggr] = 0$$

and we have the same limits for $n_1 + 1 \le i \le n$.

Finally, from (8),

$$\gamma_t(r_1, \dots, r_n) = \prod_{i=1}^{n_1}\Bigl(\int G_t(r_i - r)\rho^1(r)\mathrm{d}r\Bigr)\prod_{i=n_1+1}^{n}\Bigl(\int G_t(r_i - r)\rho^2(r)\mathrm{d}r\Bigr) + \int_0^t \mathrm{d}s\int\Biggl[\prod_{i=1}^{n} G_{t-s}(r_i - r_i{}')\mathrm{d}r'_i\Biggr](\mathscr{A}_M\gamma_s)(\underline{r'}) + o(M^{-1}). \tag{9}$$

PROOF OF LEMMA 2.2. We have to prove that the limit when N_k then M go to infinity of

$$E^{N_k}_{\mu^{N_k}}\Biggl[\prod_{\substack{j=1\\ j\ne i}}^{n_1}\eta_s(x_j)\prod_{j=n_1+1}^{n}\zeta_s(x_j)\Bigl(\tau_{x_i}\beta_1(\eta_s,\zeta_s) + \tau_{x_i}P^1_M(\eta_s,\zeta_s)\Bigr)\mathbb{1}_{\overline{B_M}}(\eta_s(x_i), \zeta_s(x_i))\Biggr]$$

is 0. We will here only show that the term which contains P^1_M vanishes in the case $\eta_s(x_i) > M$ and $\zeta_s(x_i) \le M$. It is then easy to deduce the rest of the proof. For all site x,

$$\tau_x P^1_M(\eta,\zeta)\mathbb{1}_{\{\eta(x)>M,\ \zeta(x)\le M\}} = \sum_{k=0}^{M}\sum_{l=0}^{M} l\Bigl(a\frac{l}{l+k} + b\frac{k}{l+k}\Bigr)\prod_{\substack{l'=0\\ l'\ne l}}^{M}\frac{\eta(x) - l'}{l - l'}\prod_{\substack{k'=0\\ k'\ne k}}^{M}\frac{\zeta(x) - k'}{k - k'}\mathbb{1}_{\{\eta(x)>M,\,\zeta(x)\le M\}}$$

$$\leq \sum_{l=0}^{M} l \prod_{\substack{l'=0\\ l'\neq l}}^{M} \frac{\eta(x)-l'}{l-l'} \mathbb{1}_{\{\eta(x)>M\}} \leq \frac{M}{M!} \sum_{l=0}^{M} C_M^l D_{M+1}(\eta(x)) \mathbb{1}_{\{\eta(x)>M\}}$$

$$\leq \frac{2^M}{(M-1)!} \frac{\eta(x)^M}{M^M} D_{M+1}(\eta(x)).$$

A simple induction gives $X^M D_{M+1}[X] \leq \sum_{j=0}^{M} C_M^j (2M+1)^{M-j} D_{M+1+j}[X]$ and by lemma 2.1, we can bound $E_{\mu^{N_k}}^{N_k}\Big[D_{M+1+j}(\eta_s(x))\Big]$. Then we conclude

$$\lim_{N_k\to\infty} E_{\mu^{N_k}}^{N_k} \left[\prod_{\substack{j=1\\ j\neq i}}^{n_1} \eta_s(x_j) \prod_{j=n_1+1}^{n} \zeta_s(x_j) \tau_{x_i} P_M^1(\eta_s, \zeta_s) \mathbb{1}_{\overline{B_M}}(\eta_s(x_i), \zeta_s(x_i)) \right]$$

$$\leq \frac{2^M}{(M-1)!M^M} \sum_{j=0}^{M} C_M^j (2M+1)^{M-j} c^{n+M+j} \leq c^n \frac{6^M}{(M-1)!}$$

for M big enough. And this last expression vanishes when M goes to infinity. □

2.3. *Solution of the hierarchy.* We set

$$\tilde{\gamma}_t(r_1, \ldots, r_n) = \prod_{i=1}^{n_1} \rho_t^1(r_i) \prod_{i=n_1+1}^{n} \rho_t^2(r_i).$$

We will prove that any γ_t satisfying (9) and $\tilde{\gamma}_t$ are equal. Hence all converging subsequences $(u_t^{N_k})$ converge to the same limit, $\tilde{\gamma}_t$. First we define a semi-norm on $\mathscr{U}$, the set of all sequences $v = (v^{(n)}(r_1, \ldots, r_n))_n$ such that $v^{(n)}(r_1, \ldots, r_n)$ is uniformly bounded, for each $n \geq 1$:

$$\|v\|_n = \sup_{1\leq i\leq n} \sup_{r_1,\ldots,r_i} |v^{(i)}(r_1, \ldots, r_i)|.$$

We will prove that $\|\gamma_t - \tilde{\gamma}_t\|_n = 0$, for all $t \leq T$ and $n \geq 1$. Recall that (ρ_t^1, ρ_t^2) solves (Eq), and then integrate with respect to the semi-group with generator $\sum_{j=1}^{n}(1/2)(\partial^2/\partial r_j^2)$:

$$\begin{aligned}
\tilde{\gamma}_t(r_1, \ldots, r_n) = {} & \prod_{i=1}^{n_1} \left(\int G_t(r_i - r)\rho^1(r)\mathrm{d}r\right) \prod_{i=n_1+1}^{n} \left(\int G_t(r_i - r)\rho^2(r)\mathrm{d}r\right) \\
& + \int_0^t \mathrm{d}s \int \left[\prod_{i=1}^{n} G_{t-s}(r_i - r_i')\mathrm{d}r_i'\right] \\
& \times \Bigg[\sum_{i=1}^{n_1} \prod_{j\neq i} \rho_s^1(r_j') \prod_{j} \rho_s^2(r_j') \int \beta_1(\eta, \zeta) \\
& -\delta_1(\eta, \zeta)\mathrm{d}\nu_{\rho_s^1(r_i')} \times \nu_{\rho_s^2(r_i')}(\eta, \zeta) \\
& + \sum_{i=n_1+1}^{n} \prod_{j} \rho_s^1(r_j') \prod_{j\neq i} \rho_s^2(r_j') \int \beta_2(\eta, \zeta) - \delta_2(\eta, \zeta)\mathrm{d}\nu_{\rho_s^1(r_i')} \\
& \times \nu_{\rho_s^2(r_i')}(\eta, \zeta) \Bigg].
\end{aligned}$$

Besides, the name of Poisson polynomials comes from: $\int D_k(X)\mathrm{d}\nu_\rho(X) = \rho^k$. We deduce with the same arguments than before,

$$\tilde{\gamma}_t(\underline{r}) = \prod_{i=1}^{n_1}\Big(\int G_t(r_i - r)\rho^1(r)\mathrm{d}r\Big)\prod_{i=n_1+1}^{n}\Big(\int G_t(r_i - r)\rho^2(r)\mathrm{d}r\Big)$$
$$+\int_0^t \mathrm{d}s\int\Big[\prod_{i=1}^{n} G_{t-s}(r_i - r_i')\mathrm{d}r_i'\Big]\Big[(\mathscr{A}_M\tilde{\gamma}_s)(\underline{r}') + \bar{J}^M(\underline{r})\Big]$$

where $\bar{J}^M(\underline{r})$ vanishes when M goes to infinity. We set

$$(\mathscr{G}_t v^{(n)})(r_1, \dots, r_n) = \int \mathrm{d}r_1' \cdots \int \mathrm{d}r_n' \prod_{i=1}^{n} G_t(r_i - r_i')v^{(n)}(r_1', \dots, r_n').$$

Using the last equation together with (9), we obtain

$$(\gamma_t - \tilde{\gamma}_t)(r_1, \dots, r_n) = \int_0^t \mathrm{d}s\ \mathscr{G}_{t-s}\mathscr{A}_M(\gamma_s - \tilde{\gamma}_s)(r_1', \dots, r_n') + o(M^{-1}) \tag{10}$$

Since G_t is a density, $\|\mathscr{G}_t\gamma\|_n \leq \|\gamma\|_n$. By lemma 2.2,

$$\|\gamma_t - \tilde{\gamma}_t\|_n = \Big\|\int_0^t \mathrm{d}s\ \mathscr{G}_{t-s}\mathscr{A}_M(\gamma_s - \tilde{\gamma}_s)\Big\|_n + o(M^{-1})$$
$$\leq \int_0^t \mathrm{d}s\|\mathscr{A}_M(\gamma_s - \tilde{\gamma}_s)\|_n + o(M^{-1}).$$

For any integer K we denote by $\mathscr{A}_M^{(K)}\gamma$ the quantity $\mathscr{A}_M \cdots \mathscr{A}_M\gamma$, where $\mathscr{A}_M$ is repeated K times.

LEMMA 2.3. *There exists a constant* $c_0 > 0$ *such that* $\lim_{M\to\infty}\|\mathscr{A}_M^{(K)}\gamma\|_n \leq n\, c_0^n K!(3\kappa n + 9e^{4c_0})^K$, *when* $\|\gamma\|_n \leq c^n$.

PROOF. The proof is technical so we omit it. The explicit form of the coefficients α_{kl}^1 and α_{kl}^2 is intensively used:

$$\|\gamma_t - \tilde{\gamma}_t\|_n \leq \int_0^t \mathrm{d}s_1 \cdots \int_0^{s_{K-1}} \mathrm{d}s_K\ n\, c_0^n K!(3\kappa n + 9e^{4c_0})^K \leq c_0^n[t(3\kappa n + 9e^{4c_0})]^K,$$

and this last expression goes to 0 when K goes to infinity provided $t \leq \tau$ with τ such that $(3\kappa n + 9e^{4c_0})\tau < 1$. Therefore we have proved $\gamma_t = \tilde{\gamma}_t$ for all $t \leq \tau$. For higher values of t, just note that

$$(\gamma_t - \tilde{\gamma}_t)(r_1, \dots, r_n) = \int_0^t \mathrm{d}s\int \mathscr{G}_{t-s}\mathscr{A}_M(\gamma_s - \tilde{\gamma}_s)(r_1, \dots, r_n) + o(M^{-1})$$
$$= \int_\tau^t \mathrm{d}s\int \mathscr{G}_{t-s}\mathscr{A}_M(\gamma_s - \tilde{\gamma}_s)(r_1, \dots, r_n) + o(M^{-1}).$$

We use the same arguments than before and that $\gamma_\tau = \tilde{\gamma}_\tau$. So we show $\gamma_t = \tilde{\gamma}_t$ for $t \leq 2\tau$ and so on until T.

2.4. *Extension to the rates defined in the introduction.* The proof was done for notational simplicity for birth and death rates depending only on the number of particles at one site. To consider rates depending on the density of particles in a neighborhood of the site, we just have to introduce another multipolynomial corresponding to P^i_M, denoted by Q^i_M. For instance, assume $\mathscr{N} = \{-1, 0, 1\}$, then Q^i_M has 6 variables (instead of 2 for P^i_M), that is, $\eta(x_i - 1)$, $\eta(x_i)$, $\eta(x_i + 1)$, $\zeta(x_i - 1)$, $\zeta(x_i)$, $\zeta(x_i + 1)$. For $1 \le i \le n_1$,

$$Q^i_M(\eta, \zeta) = \sum_{l_1, l_2, l_3, k_1, k_2, k_3 = 0}^{M} l_1 \left(a \frac{\sum_i l_i}{\sum_i l_i + k_i} + b \frac{\sum_i k_i}{\sum_i l_i + k_i} \right) p_{l_1}(\eta(x_i)) p_{l_2}(\eta(x_i))$$
$$\times p_{l_3}(\eta(x_i)) p_{k_1}(\zeta(x_i)) p_{k_2}(\zeta(x_i)) p_{k_3}(\zeta(x_i))$$

where

$$p_l(\eta(x)) = \prod_{\substack{l'=0 \\ l' \neq l}}^{M} \frac{\eta(x) - l'}{l - l'}.$$

Like before, Q^i_M coincides with $\tau_{x_i}\beta_1$ on B'_M, the set of configurations with less than M hawks and M doves in the neighborhood of x_i. The terms with $\overline{B'_M}$ vanish when M goes to infinity and we obtain like above an operator $\mathscr{A}'_M$ such that

$$\|\gamma_t - \tilde{\gamma}_t\|_n = \int_0^t \mathrm{d}s \mathscr{G}_{t-s} \mathscr{A}'_M (\gamma_s - \tilde{\gamma}_s) + o(M^{-1}).$$

And the same proof suits.

3. Relative entropy method.

3.1. *Proof of Theorem* 1.4. The proof is related to the work of Mourragui [10]. The two important steps of the proof are the *one-block estimate* and the computation of the entropy production.

We denote by $\mathscr{F}(\chi_N)$ the set of functions $h(\eta, \zeta)$ which depend on η and ζ only through the values of $\eta(y)$ and $\zeta(y)$ for $y \in \mathscr{N}$. For instance, the birth and death rates are in $\mathscr{F}(\chi_N)$. And $\mathscr{F}_b(\chi_N)$ is the set of bounded functions of $\mathscr{F}(\chi_N)$. If a function $h \in \mathscr{F}(\chi_N)$ is integrable with respect to $\nu^N_{a_1} \times \nu^N_{a_2}$, we set

$$\tilde{h}(a_1, a_2) = \int h(\eta, \zeta) \mathrm{d}(\nu^N_{a_1} \times \nu^N_{a_2})(\eta, \zeta).$$

We will use intensively the *entropy inequality*: *for all bounded function U and all $\alpha > 0$,*

$$\int U \mathrm{d}\mu^N \le \frac{1}{\alpha} \mathrm{Log} \int \exp(\alpha U) \mathrm{d}\left(\nu^N_{\rho_1(.)} \times \nu^N_{\rho_2(.)}\right) + \frac{1}{\alpha} \mathrm{H}\left[\mu | \nu^N_{\rho_1(.)} \times \nu^N_{\rho_2(.)}\right]. \tag{11}$$

Let P^μ_N be the law of the process when μ^N is the initial distribution on χ_N, and E^μ_N the associated expectation. We denote by f^N and f^N_t the densities of μ^N and $\mu^N_t = \mu^N S^N_t$ with respect to $(\nu^N_\rho \times \nu^N_\rho)$. We want to prove:

$\lim_{N\to\infty}\mu_t^N(A_{N,t}^{G_1,G_2,\delta})=0$ where $A_{N,t}^{G_1,G_2,\delta}$ is the set considered in the theorem. The entropy inequality (11) allows us to write

$$\mu_t^N(A_{N,t}^{G_1,G_2,\delta})\le\frac{\frac{1}{N}\mathrm{Log}2+\frac{1}{N}\mathrm{H}[\mu_t^N|\nu_{\lambda_1(t,.)}^N\times\nu_{\lambda_2(t,.)}^N]}{\frac{1}{N}\mathrm{Log}\Big[1+\nu_{\lambda_1(t,.)}^N\times\nu_{\lambda_2(t,.)}^N(A_{N,t}^{G_1,G_2,\delta})^{-1}\Big]}.$$

Using large deviations (see Varadhan's theorem in [7]), we have for all profiles $\rho_1(.)$ and $\rho_2(.)$,

$$\lim_{N\to\infty}\frac{1}{N}\mathrm{Log}\left(\nu_{\rho_1(.)}^N\times\nu_{\rho_2(.)}^N\left\{A_{N,t}^{G_1,G_2,\delta}\right\}\right)<0.$$

So we are left to prove that

$$\lim_{N\to\infty}\frac{1}{N}\mathrm{H}[\mu_t^N|\nu_{\lambda_1(t,.)}^N\times\nu_{\lambda_2(t,.)}^N]=0.$$

Here are the two propositions that make up the whole.

PROPOSITION 3.1 (One block estimate). *Let $h\in\mathscr{F}_b(\chi_N)$. We set*

$$\eta^k(x)=\frac{1}{2k+1}\sum_{|x-y|\le k}\eta(y)\qquad and\qquad \zeta^k(x)=\frac{1}{2k+1}\sum_{|x-y|\le k}\zeta(y)$$

Then we obtain the following estimate:

$$\lim_{k\to\infty}\lim_{N\to\infty}E_N^{\mu}\left\{\frac{1}{N}\sum_{x=0}^{N-1}\int_0^T\left|\frac{1}{2k-1}\sum_{|x-y|\le k-1}\Big(\tau_x h(\eta_s,\zeta_s)-\tilde{h}(\eta_s^k(x),\zeta_s^k(x))\Big)\right|\mathrm{d}s\right\}$$
$$=0.$$

PROOF. This proposition allows us to replace a local function h by a function of the mean number of particles in a box of size $2k+1$, that is, by a function of $(\eta^k(0),\zeta^k(0))$. Its proof is now classical, so we omit it (the reader can refer to [10]). □

PROPOSITION 3.2. *Under hypothesis* (2), *for each $t\in[0,T]$, there exists a function A_N^t which converges to zero when N goes to infinity such that*

$$\frac{1}{N}\mathrm{H}\left[\mu_t^N|\nu_{\lambda_1(t,.)}^N\times\nu_{\lambda_2(t,.)}^N\right]\le A_N^t+\frac{C}{N}\int_0^t\mathrm{H}\left[\mu_s^N|\nu_{\lambda_1(s,.)}^N\times\nu_{\lambda_2(s,.)}^N\right]\,\mathrm{d}s.$$

As an immediate corollary, by Gronwall lemma,

$$\lim_{N\to\infty}\frac{1}{N}\mathrm{H}[\mu_t^N|\nu_{\lambda_1(t,.)}^N\times\nu_{\lambda_2(t,.)}^N]=0.$$

PROOF OF PROPOSITION 3.2. First we introduce some notation. The density ψ_t^N of the measure $(\nu^N_{\lambda_1(t,.)}\times\nu^N_{\lambda_2(t,.)})$ with respect to $(\nu^N_\rho\times\nu^N_\rho)$ is

$$\psi_t^N(\eta,\zeta)=\prod_{x=0}^{N-1}\exp\Big(\eta(x)\mathrm{Log}\frac{\lambda_1(t,\frac{x}{N})}{\rho}+\rho-\lambda_1(t,x/N))\Big)$$
$$\times\exp\Big(\zeta(x)\mathrm{Log}\frac{\lambda_2(t,\frac{x}{N})}{\rho}+\rho-\lambda_2(t,x/N))\Big).$$

For each function $h\in\mathscr{F}(\chi_N)$, for all x_1, x_2, y_1, y_2 in $\mathbb{R}^+$, we set

$$(\Gamma h)(x_1,x_2,y_1,y_2)$$
$$=\tilde h(x_1,x_2)-\tilde h(y_1,y_2)-\frac{\mathrm{d}\tilde h}{\mathrm{d}x_1}(y_1,y_2)[x_1-y_1]-\frac{\mathrm{d}\tilde h}{\mathrm{d}x_2}(y_1,y_2)[x_2-y_2].$$

We will often use the following change of variable stated for each cylinder function g:

$$\int\eta(y)g(\eta-e_y+e_x)\mathrm{d}\nu_\rho^N(\eta)=\int\eta(x)g(\eta)\mathrm{d}\nu_\rho^N(\eta). \tag{12}$$

The key of the proof is the derivation of the relative entropy, using that the density f_t^N is the solution of the equation $\partial_t f_t^N=L^{N*}f_t^N$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}\Big[\mu_t^N|\nu^N_{\lambda_1(t,.)}\times\nu^N_{\lambda_2(t,.)}\Big]&=\frac{\mathrm{d}}{\mathrm{d}t}\int f_t^N\mathrm{Log}\frac{f_t^N}{\psi_t^N}\mathrm{d}(\nu_\rho^N\times\nu_\rho^N)\\&=N^2\int f_t^N\left(L_0^N\mathrm{Log}\frac{f_t^N}{\psi_t^N}\right)\mathrm{d}(\nu_\rho^N\times\nu_\rho^N)\\&\quad+\int f_t^N\left(L_1^N\mathrm{Log}\frac{f_t^N}{\psi_t^N}\right)\mathrm{d}(\nu_\rho^N\times\nu_\rho^N)\\&\quad+\int f_t^N\left(L_2^N\mathrm{Log}\frac{f_t^N}{\psi_t^N}\right)\mathrm{d}(\nu_\rho^N\times\nu_\rho^N)\\&\quad-\int\frac{f_t^N}{\psi_t^N}\frac{\mathrm{d}}{\mathrm{d}t}\psi_t^N\mathrm{d}(\nu_\rho^N\times\nu_\rho^N)\\&=I_1+I_2+I_3-I_4.\end{aligned}\tag{13}$$

For I_1, we apply the inequality $x\mathrm{Log}(y/x)\le y-x$ for $x, y>0$ and the fact that L_0^N is selfadjoint:

$$\begin{aligned}I_1&\le N^2\int\frac{f_t^N}{\psi_t^N}L_0^N\psi_t^N\mathrm{d}(\nu_\rho^N\times\nu_\rho^N)\\&=\frac{N^2}{2}\int\sum_{x=1}^{N-1}\bigg\{\Big(\frac{\eta(x)}{\lambda_1(t,x/N)}-1\Big)\Big[\lambda_1(t,\frac{x+1}{N})+\lambda_1(t,\frac{x-1}{N})-2\lambda_1(t,\frac{x}{N})\Big]\\&\qquad+\Big(\frac{\zeta(x)}{\lambda_2(t,x/N)}-1\Big)\Big[\lambda_2(t,\frac{x+1}{N})+\lambda_2(t,\frac{x-1}{N})-2\lambda_2(t,\frac{x}{N})\Big]\bigg\}\\&\qquad\times\mathrm{d}\mu_t^N(\eta,\zeta)\end{aligned}$$

by the explicit expression for ψ_t^N given above. Observe that a Taylor–Young expansion gives

$$\lim_{N\to\infty} \frac{N^2}{2}\Big(\lambda_1(t, x+1/N)+\lambda_1(t, x-1/N)-2\lambda_1(t, x/N)\Big)-\frac{\partial^2}{\partial\theta^2}\lambda_1(t, x/N)=0.$$

We now deal with the second term of (13):

$$\begin{aligned}
I_2 &= \int \frac{f_t^N}{\psi_t^N} L_1^N \mathrm{Log}\frac{f_t^N}{\psi_t^N}\mathrm{d}(\nu^N_{\lambda_1(t,.)}\times\nu^N_{\lambda_2(t,.)})\\
&= \sum_{x=0}^{N-1}\int \frac{f_t^N(\eta, \zeta)}{\psi_t^N(\eta, \zeta)}\Bigg[\tau_x\beta_1(\eta, \zeta)\mathrm{Log}\left(\frac{f_t^N(\eta+e_x, \zeta)}{\psi_t^N(\eta+e_x, \zeta)}\frac{\psi_t^N(\eta, \zeta)}{f_t^N(\eta, \zeta)}\right)\\
&\qquad\qquad +\tau_x\delta_1(\eta, \zeta)\mathrm{Log}\left(\frac{f_t^N(\eta-e_x, \zeta)}{\psi_t^N(\eta-e_x, \zeta)}\frac{\psi_t^N(\eta, \zeta)}{f_t^N(\eta, \zeta)}\right)\Bigg]\\
&\qquad \times\mathrm{d}\left(\nu^N_{\lambda_1(t,.)}\times\nu^N_{\lambda_2(t,.)}\right)(\eta, \zeta).
\end{aligned}$$

Since the birth and death rates are not bounded, we have to truncate them with indicator functions of sets like $A_M = \{\eta(x)\le M, \zeta(x)\le M\}$. To control the terms with $\overline{A_M}$, we have the next lemma.

LEMMA 3.3. *Let $\varphi \in \mathscr{F}(\chi_N)$ such that*

$$\lim_{\eta(0)\to\infty}\frac{\varphi(\eta, \zeta)}{\delta_1(\eta, \zeta)}=0$$

uniformly in the others variables. Then

$$\lim_{M\to\infty}\limsup_{N\to\infty}\frac{1}{N}\sum_{x=0}^{N-1}\int_0^T\int\tau_x\varphi(\eta, \zeta)\mathbb{1}_{\{\eta(x)>M\}}\mathrm{d}\mu_s^N(\eta, \zeta)\mathrm{d}s=0.$$

So we set

$$\begin{aligned}
&\varphi_1(\eta, \zeta)=\eta(0)\beta_1(\eta-e_0, \zeta), \quad \phi_1(\eta, \zeta)=\frac{1}{\eta(0)+1}\delta_1(\eta+e_0, \zeta),\\
&\beta_{1,M}(\eta, \zeta)=\beta_1(\eta, \zeta)\mathbb{1}_{\{\eta(0)\le M\}}, \quad \varphi_{1,M}(\eta, \zeta)=\varphi_1(\eta, \zeta)\mathbb{1}_{\{\eta(0)\le M+1\}},\\
&\delta_{1,M}(\eta, \zeta)=\delta_1(\eta, \zeta)\mathbb{1}_{\{\eta(0)\le M, \zeta(0)\le M\}},\\
&\qquad\qquad \phi_{1,M}(\eta, \zeta)=\phi_1(\eta, \zeta)\mathbb{1}_{\{\eta(0)\le M-1, \zeta(0)\le M-1\}}.
\end{aligned}$$

And we have similar definitions for φ_2, ϕ_2, etc. It follows that

$$
\begin{aligned}
I_2 = \sum_{x=0}^{N-1} \int \frac{f_t^N(\eta,\zeta)}{\psi_t^N(\eta,\zeta)} \Bigg[& \tau_x\beta_{1,M}(\eta,\zeta)\mathrm{Log}\left(\frac{f_t^N(\eta+e_x,\zeta)}{\psi_t^N(\eta+e_x,\zeta)}\frac{\psi_t^N(\eta,\zeta)}{f_t^N(\eta,\zeta)}\right) \\
& +\tau_x\beta_1(\eta,\zeta)\mathrm{Log}\left(\frac{f_t^N(\eta+e_x,\zeta)}{\psi_t^N(\eta+e_x,\zeta)}\frac{\psi_t^N(\eta,\zeta)}{f_t^N(\eta,\zeta)}\frac{1}{\tau_x\beta_1(\eta,\zeta)}\right) \\
& \times \mathbb{1}_{\{\eta(x)>M\}} \\
& +\tau_x\beta_1(\eta,\zeta)\mathrm{Log}\tau_x\beta_1(\eta,\zeta)\mathbb{1}_{\{\eta(x)>M\}} \\
& +\tau_x\delta_1(\eta,\zeta)\mathrm{Log}\left(\frac{f_t^N(\eta-e_x,\zeta)}{\psi_t^N(\eta-e_x,\zeta)}\frac{\psi_t^N(\eta,\zeta)}{f_t^N(\eta,\zeta)}\right)\Bigg]. \\
\times \mathrm{d}\left(\nu^N_{\lambda_1(t,.)}\times\nu^N_{\lambda_2(t,.)}\right)(\eta,\zeta). &
\end{aligned}
\tag{14}
$$

We introduced the third term in order to use Lemma 3.3. Again we use the inequality $x\mathrm{Log}(y/x) \le y - x$ and the variable change (12):

$$
\begin{aligned}
I_2 \le \sum_{x=0}^{N-1} \int \Bigg[& \tau_x\beta_{1,M}(\eta,\zeta)\left(\frac{f_t^N(\eta+e_x,\zeta)}{\psi_t^N(\eta+e_x,\zeta)} - \frac{f_t^N(\eta,\zeta)}{\psi_t^N(\eta,\zeta)}\right) \\
& +\tau_x\delta_1(\eta,\zeta)\left(\frac{f_t^N(\eta-e_x,\zeta)}{\psi_t^N(\eta-e_x,\zeta)} - \frac{f_t^N(\eta,\zeta)}{\psi_t^N(\eta,\zeta)}\right) \\
& +\left(\frac{f_t^N(\eta+e_x,\zeta)}{\psi_t^N(\eta+e_x,\zeta)} - \frac{f_t^N(\eta,\zeta)}{\psi_t^N(\eta,\zeta)}\tau_x\beta_1(\eta,\zeta)\right)\mathbb{1}_{\{\eta(x)>M\}}\Bigg] \\
& \times \mathrm{d}(\nu^N_{\lambda_1(t,.)}\times\nu^N_{\lambda_2(t,.)})(\eta,\zeta) \\
& + \sum_{x=0}^{N-1}\int \tau_x\beta_1(\eta,\zeta)\mathrm{Log}\tau_x\beta_1(\eta,\zeta)\mathbb{1}_{\{\eta(x)>M\}}\mathrm{d}\mu_t^N(\eta,\zeta) \\
\le \sum_{x=0}^{N-1} \int \Bigg[& \frac{\tau_x\varphi_{1,M}(\eta,\zeta)}{\lambda_1(t,x/N)} - \tau_x\beta_{1,M}(\eta,\zeta) + \lambda_1(t,x/N)\tau_x\phi_1(\eta,\zeta) - \tau_x\delta_1(\eta,\zeta) \\
& + \frac{\eta(x)}{\lambda_1(t,x/N)}\mathbb{1}_{\{\eta(x)>M+1\}} + \tau_x\beta_1(\eta,\zeta)\mathrm{Log}\tau_x\beta_1(\eta,\zeta)\mathbb{1}_{\{\eta(x)>M\}}\Bigg] \\
& \times\mathrm{d}\mu_t^N(\eta,\zeta).
\end{aligned}
$$

The third term of (13) has a similar expression. Let us rewrite the fourth term of (13) using the fact that $(\lambda_1(t,.), \lambda_2(t,.))$ is solution of the system (Eq):

$$
\begin{aligned}
I_4 = \sum_{x=0}^{N-1} \int \Bigg[& \left(\frac{\eta(x)}{\lambda_1(t,x/N)} - 1\right)\frac{\mathrm{d}}{\mathrm{d}t}\lambda_1(t,x/N) \\
& +\left(\frac{\zeta(x)}{\lambda_2(t,x/N)} - 1\right)\frac{\mathrm{d}}{\mathrm{d}t}\lambda_2(t,x/N)\Bigg]\mathrm{d}\mu_t^N(\eta,\zeta)
\end{aligned}
$$

$$
\begin{aligned}
= \sum_{x=0}^{N-1} \int \Bigg[& \left(\frac{\eta(x)}{\lambda_1(t, x/N)} - 1 \right) \frac{\partial^2}{\partial \theta^2} \lambda_1(t, x/N) \\
& + \left(\frac{\zeta(x)}{\lambda_2(t, x/N)} - 1 \right) \frac{\partial^2}{\partial \theta^2} \lambda_2(t, x/N) + \left(\frac{\eta(x)}{\lambda_1(t, x/N)} - 1 \right) \\
& \quad \times \Big[\tilde{\beta}_1(\lambda_1(t, x/N), \lambda_2(t, x/N)) - \tilde{\delta}_1(\lambda_1(t, x/N), \lambda_2(t, x/N)) \Big] \\
& + \left(\frac{\zeta(x)}{\lambda_2(t, x/N)} - 1 \right) \\
& \quad \times \Big[\tilde{\beta}_2(\lambda_1(t, x/N), \lambda_2(t, x/N)) - \tilde{\delta}_2(\lambda_1(t, x/N), \lambda_2(t, x/N)) \Big] \Bigg] \\
& \times \mathrm{d}\mu_t^N(\eta, \zeta).
\end{aligned}
$$

Let us now integrate (13). From what precedes, we deduce by removing the negative terms:

$$
\begin{aligned}
& \frac{1}{N} \mathrm{H} \left[\mu_t^N | \nu^N_{\lambda_1(t,.)} \times \nu^N_{\lambda_2(t,.)} \right] \\
& \quad \le \frac{1}{N} \mathrm{H}[\mu^N | \nu_{m_1(.)} \times \nu_{m_2(.)}] + F(M, N, T) + o(1/N) \\
& \qquad + \frac{1}{N} \sum_{x=0}^{N-1} \int_0^t \int \Bigg[\frac{\tau_x \varphi_{1,M}(\eta, \zeta)}{\lambda_1(s, x/N)} - \tau_x \beta_{1,M}(\eta, \zeta) \\
& \qquad\qquad + \lambda_1(s, x/N) \tau_x \phi_{1,M}(\eta, \zeta) - \tau_x \delta_{1,M}(\eta, \zeta) \\
& \qquad\qquad + \frac{\tau_x \varphi_{2,M}(\eta, \zeta)}{\lambda_2(s, x/N)} - \tau_x \beta_{2,M}(\eta, \zeta) \\
& \qquad\qquad + \lambda_2(s, x/N) \tau_x \phi_{2,M}(\eta, \zeta) - \tau_x \delta_{2,M}(\eta, \zeta) \\
& \qquad\qquad - \Big[\tilde{\beta}_{1,M} \left(\lambda_1 \left(s, \frac{x}{N} \right), \lambda_2 \left(s, \frac{x}{N} \right) \right) \\
& \qquad\qquad\quad - \tilde{\delta}_{1,M} \left(\lambda_1 \left(s, \frac{x}{N} \right), \lambda_2 \left(s, \frac{x}{N} \right) \right) \Big] \Big[\frac{\eta(x)}{\lambda_1(s, x/N)} - 1 \Big] \\
& \qquad\qquad - \Big[\tilde{\beta}_{2,M} \left(\lambda_1 \left(s, \frac{x}{N} \right), \lambda_2 \left(s, \frac{x}{N} \right) \right) \\
& \qquad\qquad\quad - \tilde{\delta}_{2,M} \left(\lambda_1 \left(s, \frac{x}{N} \right), \lambda_2 \left(s, \frac{x}{N} \right) \right) \Big] \\
& \qquad\qquad\qquad \times \Big[\frac{\zeta(x)}{\lambda_2(s, x/N)} - 1 \Big] \Bigg] \mathrm{d}\mu_s^N(\eta, \zeta) \mathrm{d}s
\end{aligned}
$$

where F contains all the terms with $\mathbb{1}_{\{\eta(x)>M\}}$ and $\mathbb{1}_{\{\zeta(x)>M\}}$. From Lemma 3.3, we obtain

$$\lim_{M\to\infty}\limsup_{N\to\infty} F(M,N,T)=0.$$

We shall replace the local functions $\varphi_{1,M}$, $\beta_{1,M}$, $\eta(.)$ and $\zeta(.)$ by functions of the empirical density of the particles in microcospic boxes, precisely in boxes of size $2k+1$, with k going to infinity after N and a function $r_N^t(M,k)$ which goes to zero when N and k go to infinity will appear. This is possible thanks to proposition 3.1. Besides a simple computation shows that

$$\tilde{\varphi}_1(a_1,a_2)=a_1\tilde{\beta}_1(a_1,a_2) \text{ and } \tilde{\phi}_1(a_1,a_2)=\frac{1}{a_1}\tilde{\delta}_1(a_1,a_2)$$

so that

$$\begin{aligned}
&\frac{1}{\lambda_1}(\Gamma\varphi_1)(a_1,a_2,\lambda_1,\lambda_2)-(\Gamma\beta_1)(a_1,a_2,\lambda_1,\lambda_2)\\
&\qquad=\frac{1}{\lambda_1}\tilde{\varphi}_1(a_1,a_2)-\tilde{\beta}_1(a_1,a_2)-\tilde{\beta}_1(\lambda_1,\lambda_2)\Big(\frac{a_1}{\lambda_1}-1\Big),\\
&\lambda_1(\Gamma\phi_1)(a_1,a_2,\lambda_1,\lambda_2)-(\Gamma\delta_1)(a_1,a_2,\lambda_1,\lambda_2)\\
&\qquad=\lambda_1\tilde{\phi}_1(a_1,a_2)-\tilde{\delta}_1(a_1,a_2)-\tilde{\delta}_1(\lambda_1,\lambda_2)\Big(\frac{a_1}{\lambda_1}-1\Big).
\end{aligned}$$

We have similar formulas for φ_2, ϕ_2, δ_2 and β_2. And the following lemma will allow us to bound the terms containing such functions.

LEMMA 3.4. *Let $h\in\mathscr{F}_b(\chi_N)$, $\rho_1(.)$ and $\rho_2(.)$ be two positive bounded functions on $[0,1]$ and J be a continuous function on $\mathbb{R}^2$. There exists $\gamma_0>0$ such that, for all $\gamma\le\gamma_0$,*

$$\begin{aligned}
&\frac{1}{N}\sum_{x=0}^{N-1}\int J(\rho_1(x/N),\rho_2(x/N))(\Gamma h)(\eta^k(x),\zeta^k(x),\rho_1(x/N),\rho_2(x/N))\\
&\qquad\times f_t^N(\eta,\zeta)\mathrm{d}\left(\nu_\rho^N\times\nu_\rho^N\right)(\eta,\zeta)\\
&\quad\le\frac{1}{\gamma N}\mathrm{H}[\mu_t^N|\nu_{\rho_1(.)}^N\times\nu_{\rho_2(.)}^N]+o\Big(\frac{2k+1}{N}\Big)+R_N^t(k,\gamma)
\end{aligned}$$

with $\limsup_{k\to\infty}\limsup_{N\to\infty}R_N^t(k,\gamma)\le 0$.

Then

$$\begin{aligned}
&\frac{1}{N}\mathrm{H}\left[\mu_t^N|\nu_{\lambda_1(t,.)}^N\times\nu_{\lambda_2(t,.)}^N\right]\\
&\quad\le\frac{1}{N}\mathrm{H}\left[\mu_t^N|\nu_{m_1(.)}^N\times\nu_{m_2(.)}\right]+F(M,N,T)+r_N^t(M,k)+o(1/N)\\
&\qquad+R_N^t(k,\gamma)+\frac{8}{N\gamma}\int_0^t\mathrm{H}\left[\mu_s^N|\nu_{\lambda_1(s,.)}^N\times\nu_{\lambda_2(s,.)}^N\right]\mathrm{d}s+o\left(\frac{2k+1}{N}\right)
\end{aligned}$$

for all $\gamma \le \gamma_0$, with $\limsup_{k\to\infty} \limsup_{N\to\infty} r_N^t(M,k) = 0$. Hence

$$\limsup_{M\to\infty} \limsup_{k\to\infty} \limsup_{N\to\infty} \frac{1}{N} \mathrm{H}\left[\mu_t^N | \nu^N_{\lambda_1(t,.)} \times \nu^N_{\lambda_2(t,.)}\right] - \frac{8}{N\gamma} \int_0^t \mathrm{H}\left[\mu_s^N / \nu^N_{\lambda_1(s,.)} \times \nu^N_{\lambda_2(s,.)}\right] \mathrm{d}s \le 0.$$

Finally, the Gronwall lemma and hypothesis (2) end the proof:

$$\lim_{N\to\infty} \frac{1}{N} \mathrm{H}\big[\mu_t^N | \nu^N_{\lambda_1(t,.)} \times \nu^N_{\lambda_2(t,.)}\big] = 0. \qquad \square$$

PROOF OF LEMMA 3.3. By the hypothesis on φ, for all $\varepsilon > 0$, there exists $M_1 \in \mathbb{N}$ such that, for all $M \ge M_1$,

$$\varphi(\eta,\zeta)\mathbb{1}_{\{\eta(0)>M\}} \le \frac{\varepsilon}{2}\delta_1(\eta,\zeta)\mathbb{1}_{\{\eta(0)>M\}}.$$

Moreover, since $\delta_1(\eta,\zeta) = \kappa\eta(0)(\eta(0)+\zeta(0))$, for all $\varepsilon > 0$, we have

$$\begin{aligned}\varepsilon\beta_1(\eta,\zeta) \le \varepsilon(a+b)\eta(0) &\le \frac{\varepsilon}{2}\Big[\kappa\eta(0)^2 + \Big(2(a+b) - \kappa\eta(0)\Big)\eta(0)\Big] \\ &\le \frac{\varepsilon}{2}(\delta_1(\eta,\zeta) + 2C).\end{aligned}$$

Then $\varphi(\eta,\zeta)\mathbb{1}_{\{\eta(0)>M\}} \le \varepsilon[\delta_1(\eta,\zeta) - \beta_1(\eta,\zeta) + C] = \varepsilon[-L_1\eta(0) + C]$. We can conclude the proof using the entropy inequality (11). $\square$

PROOF OF LEMMA 3.4. Set

$$g_{\eta,\zeta}(x) = J(\rho_1(x/N), \rho_2(x/N))(\Gamma h)(\eta^k(x), \zeta^k(x), \rho_1(x/N), \rho_2(x/N)).$$

Let $1 \le k \le N$. We split the torus T_N in boxes of size $2k+1$. We obtain

$$\begin{aligned}&\frac{1}{N}\sum_{x=0}^{N-1} \int g_{\eta,\zeta}(x) f_t^N(\eta,\zeta) \mathrm{d}\left(\nu_\rho^N \times \nu_\rho^N\right)(\eta,\zeta) \\ &\quad = \frac{1}{N}\sum_{|y|\le k}\sum_{r=0}^{N_k-1} \int g_{\eta,\zeta}(y + r(2k+1)) f_t^N(\eta,\zeta)\mathrm{d}\left(\nu_\rho^N \times \nu_\rho^N\right)(\eta,\zeta) \\ &\qquad + \frac{1}{N}\sum_{x=(2k+1)N_k-k}^{N-k-1} \int g_{\eta,\zeta}(x) f_t^N(\eta,\zeta)\mathrm{d}\left(\nu_\rho^N \times \nu_\rho^N\right)(\eta,\zeta)\end{aligned} \tag{15}$$

where N_k is the integer part of $N/(2k+1)$. Since J, ρ_1, ρ_2 and h are bounded, from the definition of Γ, $g_{\eta,\zeta}(x) \le K_1 + K_2(\eta^k(x) + \zeta^k(x))$ where K_1 and K_2 are positive constants. Moreover, just like in [10], we have for all $t > 0$ and all $\rho > (a+b+c+d)/\kappa$ the inequality

$$\begin{aligned}\mathrm{H}\left[\mu_t^N | \nu_\rho^N \times \nu_\rho^N\right] &\le \mathrm{H}\left[\mu^N | \nu_\rho^N \times \nu_\rho^N\right] + tCN \\ &\le \mathrm{H}\left[\mu^N | \nu^N_{\rho_1(.)} \times \nu^N_{\rho_2(.)}\right] + C(\rho,\rho_1,\rho_2) + tCN\end{aligned}$$

where C and $C(\rho, \rho_1, \rho_2)$ are positive constants. Gathering all these results and using the entropy inequality, we obtain

$$\begin{aligned}
&\frac{1}{N}\sum_{x=(2k+1)N_k-k}^{N-k-1}\int g_{\eta,\zeta}(x) f_t^N(\eta,\zeta)\mathrm{d}\left(\nu_\rho^N\times\nu_\rho^N\right)(\eta,\zeta)\\
&\quad\le \frac{2k+1}{aN}[aK_1+2\rho(e^{aK_2}-1)]\\
&\quad\quad+\frac{2k+1}{aN}\Big(tCN+C(\rho,\rho_1,\rho_2)+\mathrm{H}\left[\mu^N|\nu^N_{\rho_1(.)}\times\nu^N_{\rho_2(.)}\right]\Big).
\end{aligned}$$

Then when N and a go to infinity, this term vanishes. Furthermore, using the entropy inequality, Hölder inequality and then that $\nu_\rho^N\times\nu_\rho^N$ is a product measure, we have

$$\begin{aligned}
&\frac{1}{N}\sum_{|y|\le k}\sum_{r=0}^{N_k-1}\int g_{\eta,\zeta}(y+r(2k+1)) f_t^N(\eta,\zeta)\mathrm{d}\left(\nu_\rho^N\times\nu_\rho^N\right)(\eta,\zeta)\\
&\quad\le\frac{1}{N\gamma}\mathrm{H}\left[\mu_t^N|\nu^N_{\rho_1(.)}\times\nu^N_{\rho_2(.)}\right]\\
&\quad\quad+\frac{1}{N\gamma}\frac{1}{2k+1}\sum_{x=0}^{N-1}\mathrm{Log}\int\exp\left\{\gamma(2k+1)g_{\eta,\zeta}(x)\right\}\mathrm{d}\left(\nu^N_{\rho_1(.)}\times\nu^N_{\rho_2(.)}\right)(\eta,\zeta)\\
&\quad\quad-\frac{1}{N\gamma}\frac{1}{2k+1}\sum_{x=(2k+1)N_k-k}^{N-k-1}\mathrm{Log}\int\exp\left\{\gamma(2k+1)g_{\eta,\zeta}(x)\right\}\\
&\quad\quad\quad\times\mathrm{d}\left(\nu^N_{\rho_1(.)}\times\nu^N_{\rho_2(.)}\right)(\eta,\zeta).
\end{aligned}$$

The last term is a $o((2k+1)/N)$ (see the above computation). So we just have to study the first one. For each $\gamma>0$,

$$\begin{aligned}
&\limsup_{N\to\infty}\frac{1}{(2k+1)N}\sum_{x=0}^{N-1}\mathrm{Log}\int\exp\Big\{\gamma(2k+1)J(\rho_1(x/N),\rho_2(x/N))\\
&\quad\quad\times(\Gamma h)(\eta^k(x),\zeta^k(x),\rho_1(x/N),\rho_2(x/N))\Big\}\\
&\quad\quad\times\mathrm{d}\left(\nu^N_{\rho_1(.)}\times\nu^N_{\rho_2(.)}\right)(\eta,\zeta)\\
&\quad=\frac{1}{2k+1}\int_S\mathrm{Log}\int\exp\Big\{\gamma(2k+1)J(\rho_1(x),\rho_2(x))\\
&\quad\quad\times(\Gamma h)(\eta^k(0),\zeta^k(0),\rho_1(x),\rho_2(x))\Big\}\\
&\quad\quad\times\mathrm{d}\left(\nu^k_{\rho_1(x)}\times\nu^k_{\rho_2(x)}\right)(\eta,\zeta)\mathrm{d}x.
\end{aligned}$$

We denote by C the upper bound of $J(\rho_1(x), \rho_2(x))$. As in [10], Varadhan's theorem implies that the last expression is bounded above by

$$\int_S \sup_{y_1, y_2 \in \mathbb{R}^+} \left\{ \gamma J(\rho_1(x), \rho_2(x))(\Gamma h)(y_1, y_2, \rho_1(x), \rho_2(x)) - I(y_1, y_2) \right\} \mathrm{d}x$$

where

$$I(y_1, y_2) = y_1 \log \frac{y_1}{\rho_1} + (\rho_1(x) - y_1) + y_2 \mathrm{Log} \frac{y_2}{\rho_2} + (\rho_2(x) - y_2).$$

And this supremum is non positive for γ small enough. Indeed let

$$H(y_1, y_2) = \gamma C(\Gamma h)(y_1, y_2, \rho_1, \rho_2) - y_1 \mathrm{Log} \frac{y_1}{\rho_1} - (\rho_1 - y_1) - y_2 \mathrm{Log} \frac{y_2}{\rho_2} - (\rho_2 - y_2).$$

Let $\delta > 0$ be a fixed small real number. If for example $y_1 \geq \rho_1 + \delta$ and $y_2 \geq \rho_2 + \delta$, it is easy to show that H is non positive when $\gamma \leq (CK_1 + CK_2\delta)^{-1}(\delta - (\rho_1 + \delta)\mathrm{Log}(1 + \delta/\rho_1))$. When $|y_1 - \rho_1| \leq \delta$ and $|y_2 - \rho_2| \leq \delta$,

$$\begin{aligned}
&(\Gamma h)(y_1, y_2, \rho_1, \rho_2) \\
&\quad = \tilde{h}(y_1, y_2) - \tilde{h}(\rho_1, \rho_2) - \frac{\mathrm{d}\tilde{h}}{\mathrm{d}x_1}(\rho_1, \rho_2)[y_1 - \rho_1] - \frac{\mathrm{d}\tilde{h}}{\mathrm{d}x_2}(\rho_1, \rho_2)[y_2 - \rho_2] \\
&\quad \leq 2|y_1 - \rho_1||y_2 - \rho_2| \left| \frac{\mathrm{d}^2\tilde{h}}{\mathrm{d}x_1 \mathrm{d}x_2}(\rho_1, \rho_2) \right| + c|y_1 - \rho_1|^2 + c|y_2 - \rho_2|^2 \\
&\quad \leq \tilde{C}\Big(|y_1 - \rho_1| + |y_2 - \rho_2|\Big)^2
\end{aligned}$$

with $\tilde{C} > 0$. Moreover, we have $y \log \frac{y}{\rho} + \rho - y \sim \rho^{-1}(\rho - y)^2$. It is now easy to deduce that H is non positive for γ small. $\square$

3.2. *Extension to infinite volume*: *Proof of Theorem* 1.5. To widen theorem 1.4 to infinite volume, that is, to the space $\mathbb{Z}$, we make a coupling between two processes: the first one will be on the torus $T_{CN} = \{-CN, \ldots, CN\}$ and the second one on $\mathbb{Z}$. We will prove that, when N goes to infinity and C is large the difference between those two processes is small on the supports of two test-functions G_1 and G_2. Thus the hydrodynamic limits would be proved in infinite volume. First of all, we will describe the chosen coupling. Then we prove an upper bound for the difference between the two processes (the one on T_{CN} and the other one on $\mathbb{Z}$). Finally, we are able to prove theorem 1.5.

Let (η_t^1, ζ_t^1) and (η_t^2, ζ_t^2) be two processes. The second one is on $\mathbb{Z}$ with μ^N as initial distribution, and (η_t^1, ζ_t^1) is on T_{CN} with μ^N restricted to T_{CN} as initial distribution. To couple them, we distinguish between two types of particles: the coupled ones and the non-coupled ones. More precisely, at a site x, the particles belonging to η_t^1 are divided into $\eta_t^*(x)$ and $\eta_t^{1*}(x)$. The particles of $\eta_t^*(x)$ are associated to particles belonging to $\eta_t^2(x)$. These couples of particles move together. All the other particles stay single. At the beginning, $\eta_0^*(x) = \eta_0^1(x) \wedge \eta_0^2(x)$ for all x. We set: $\eta_0^1(x) = \eta_0^*(x) + \eta_0^{1*}(x)$, $\eta_0^2(x) = \eta_0^*(x) + \eta_0^{2*}(x)$

and the same for ζ_0^1 and ζ_0^2. Then, for the diffusion part, the coupled evolution is described at a site x such that $|x| < CN$ by:

- at rate $\eta^*(x)$, two coupled particles go from x to a neighboring site;
- at rate $\eta^{1*}(x) \wedge \eta^{2*}(x)$, two particles (one belonging to η^{1*} and the other to η^{2*}) jump from x to a neighboring site;
- at rate $[\eta^{1*}(x) - \eta^{2*}(x)]^+$, a single particle belonging to η^{1*} jumps to a neighbor;
- at rate $[\eta^{2*}(x) - \eta^{1*}(x)]^+$, a single particle belonging to η^{2*} jumps to a neighbor.

At the site $x = CN$, the particles of the two processes jump outside independently. Those of η^1 arrive at $-CN$ and the others at $CN+1$. The ζ's particles evolve according to the same rules. The diffusion part of the coupled generator $\overline{L_N}$ is denoted by $\overline{L_N^d}$. Concerning the reaction part, we set:

- at rate $\tau_x\beta_1(\eta^1, \zeta^1) \wedge \tau_x\beta_1(\eta^2, \zeta^2)$, two coupled particles are created;
- at rate $[\tau_x\beta_1(\eta^1, \zeta^1) - \tau_x\beta_1(\eta^2, \zeta^2)]^+$, if $\eta^{1*}(x) \geq \eta^{2*}(x)$, a particle for η^1 is created;
- at the same rate, if $\eta^{1*}(x) < \eta^{2*}(x)$, a particle for η^{2*} is removed and two coupled particles are created. So this small part of the generator at x is given by

$$
\begin{aligned}
&\Big\{\tau_x\beta_1(\eta^1, \zeta^1) \wedge \tau_x\beta_1(\eta^2, \zeta^2)\Big[f(\eta^{1*}, \eta^{2*}, \eta^* + e_x) - f(\eta^{1*}, \eta^{2*}, \eta^*)\Big]\\
&\quad \times \big[\tau_x\beta_1(\eta^1, \zeta^1) - \tau_x\beta_1(\eta^2, \zeta^2)\big]^+ \mathbb{1}_{\{\eta^{1*}(x) \geq \eta^{2*}(x)\}}\\
&\quad \times \big[f(\eta^{1*} + e_x, \eta^{2*}, \eta^*) - f(\eta^{1*}, \eta^{2*}, \eta^*)\big]\\
&\quad \times \big[\tau_x\beta_1(\eta^1, \zeta^1) - \tau_x\beta_1(\eta^2, \zeta^2)\big]^+ \mathbb{1}_{\{\eta^{1*}(x) < \eta^{2*}(x)\}}\\
&\quad \times \big[f(\eta^{1*}, \eta^{2*} - e_x, \eta^* + e_x) - f(\eta^{1*}, \eta^{2*}, \eta^*)\big]\Big\}.
\end{aligned}
$$

In a symmetric way, we define the rest of the dynamics. We denote by $\overline{L_N^r}$ the reaction part of the coupled generator and by $\overline{E}_{\mu^N}$ the expectation of the coupled process starting from μ^N. For notational simplicity, we assume that $a + b + c + d \leq 1$. For $x \in T_{CN}$, we compute

$$
\begin{aligned}
&\overline{L_N^r}\Big(\eta^{1*}(x) + \eta^{2*}(x) + \zeta^{1*}(x) + \zeta^{2*}(x)\Big)\\
&\qquad \leq 2\sum_{y \in \mathcal{N}} \Big(\eta^{1*}(x+y) + \eta^{2*}(x+y) + \zeta^{1*}(x+y) + \zeta^{2*}(x+y)\Big).
\end{aligned}
\tag{16}
$$

Moreover, by construction, $|\eta_t^1(x) - \eta_t^2(x)| = |\eta_t^{1*}(x) - \eta_t^{2*}(x)| \leq \eta_t^{1*}(x) + \eta_t^{2*}(x) \leq \eta_t^1(x) + \eta_t^2(x)$. We set $\xi_t^*(x) = \eta_t^{1*}(x) + \eta_t^{2*}(x) + \zeta_t^{1*}(x) + \zeta_t^{2*}(x)$.

Observe that the duality relation (4) obtained in section 2 becomes for $n = 1$

$$
\begin{aligned}
E_{\mu^N}[\eta_t^1(x)] &= \sum_{y\in\mathbb{Z}} P_t^{N,C}(x\to y)\mu^N(\eta^1(y)) \\
&\quad + \int_0^t \mathrm{d}s \sum_{y\in\mathbb{Z}} P_{t-s}^{N,C}(x\to y) E_{\mu^N}[L_c \eta_s^1(y)]
\end{aligned}
\tag{17}
$$

where $P_t^{N,C}(x \to y)$ is the transition probability from x to y for a simple random walk on $\{-CN,\dots,CN\}$, accelerated by N^2. Furthermore $E_{\mu^N}[L_c\eta_s(y)] \le (a+b+c+d)/(2\kappa) := c_0$ since the death rates are quickly larger than the birth rates. Then

$$
E_{\mu^N}[\eta_t^1(x)] \le M + tc_0 \qquad \text{and} \qquad \overline{E}_{\mu^N}[\xi_t^*(x)] \le 4M + 4tc_0 := K_1. \tag{18}
$$

Let $A \in \mathbb{N}$ be fixed. We denote by $P_t^N(x \to y)$ the transition probability on $\mathbb{Z}$. Now we have all the necessary tools to bound above the discrepancy between the two processes in the set Λ_{AN}:

$$
\overline{E}_{\mu^N}\Big[\frac{1}{N}\sum_{x\in\Lambda_{AN}} \eta_t^{1*}(x)\Big] = \frac{1}{N}\sum_{x\in\Lambda_{AN}} \int_0^t \mathrm{d}s \sum_y P_{t-s}^{N,C}(x\to y)\overline{E}_{\mu^N}\Big[\overline{L_N^r}\eta_s^{1*}(y)\Big],
$$

$$
\begin{aligned}
\overline{E}_{\mu^N}\Big[\frac{1}{N}\sum_{x\in\Lambda_{AN}} \eta_t^{2*}(x)\Big] &= \frac{1}{N}\sum_{x\in\Lambda_{AN}}\sum_{|y|>CN} P_t^N(x\to y)\mu^N(\eta(y)) \\
&\quad + \frac{1}{N}\sum_{x\in\Lambda_{AN}} \int_0^t \mathrm{d}s \sum_y P_{t-s}^N(x\to y)\overline{E}_{\mu^N}\Big[\overline{L_N^r}\eta_s^{2*}(y)\Big].
\end{aligned}
$$

Then if we set $\tilde{P}_t^{N,C} = P_t^{N,C} \vee P_t^N$, we obtain using also (16) and (17)

$$
\begin{aligned}
&\overline{E}_{\mu^N}\left[\frac{1}{N}\sum_{x\in\Lambda_{AN}} \xi_t^*(x)\right] \\
&\quad \le \frac{1}{N}\sum_{x\in\Lambda_{AN}}\sum_{|y|>CN} \tilde{P}_t^{N,C}(x\to y)M \\
&\qquad + \frac{1}{N}\sum_{x\in\Lambda_{AN}} \int_0^t \mathrm{d}s \sum_y \tilde{P}_{t-s}^{N,C}(x\to y)\overline{E}_{\mu^N}\Big[\overline{L_N^r}\xi_s^*(y)\Big] \\
&\quad \le \frac{M}{N}\sum_{|y|>CN} P_0[|X_{tN^2}| \ge d(y,\Lambda_{AN})] \\
&\qquad + 2\int_0^t \mathrm{d}s \frac{1}{N}\sum_{y\in\Lambda(A+l)N}\sum_{x\in\Lambda_{AN}} \tilde{P}_{t-s}^{N,C}(x\to y)\overline{E}_{\mu^N}\left[\sum_{z\in\mathcal{N}} \xi_s^*(y+z)\right] \\
&\qquad + 2\int_0^t \mathrm{d}s \frac{1}{N}\sum_{y\notin\Lambda(A+l)N}\sum_{x\in\Lambda_{AN}} \tilde{P}_{t-s}^{N,C}(x\to y)\overline{E}_{\mu^N}\left[\sum_{z\in\mathcal{N}} \xi_s^*(y+z)\right]
\end{aligned}
$$

where (X_t) is a simple symmetric random walk on $\mathbb{Z}$. We denote by I_1, I_2 and I_3 these three terms above. Let $d(.,.)$ be the usual distance on $\mathbb{Z}$. A calculation on random walks and (18) give

$$\begin{aligned} I_1 &\le \frac{2M}{N} \sum_{|y|>CN} \exp\left\{-\frac{d(y,\Lambda_{AN})}{2} \log\left(1+\frac{d(y,\Lambda_{AN})}{2tN^2}\right)\right\} \\ &\le \frac{4M}{N} \sum_{|y|>CN} \exp\left\{-\frac{y-AN}{2} \log\left(1+\frac{2}{N}\right)\right\} \qquad \text{if } C \ge A+4t \\ &\le 4Me^{A}e^{-C} := K_0 e^{-C}, \end{aligned}$$

$$\begin{aligned} I_3 &\le 2\int_0^t \mathrm{d}s \frac{1}{N} \sum_{y\notin\Lambda(A+l)N} P_0[|X_{(t-s)N^2}| \ge d(y,\Lambda_{AN})] \overline{E}_{\mu^N}\left[\sum_{z\in\mathcal{N}} \xi_s^*(y+z)\right] \\ &\le 24\, K_1 \int_0^t \mathrm{d}s \frac{1}{N} \sum_{y>(A+l)N} P_0[X_{tN^2} \ge d(y,\Lambda_{AN})] \\ &\le 24\, K_1 t e^{-l} := K_2 e^{-l} \end{aligned}$$

then

$$\overline{E}_{\mu^N}\left[\frac{1}{N}\sum_{x\in\Lambda_{AN}} \xi_t^*(x)\right] \le K_0 e^{-C} + K_2 e^{-l} + 6\int_0^t \mathrm{d}s \overline{E}_{\mu^N}\left[\frac{1}{N}\sum_{y\in\Lambda_{(A+l)N+1}} \xi_s^*(y)\right].$$

But, in the same way, we have

$$\begin{aligned} \overline{E}_{\mu^N}\left[\frac{1}{N}\sum_{x\in\Lambda_{(A+l)N}} \xi_t^*(x)\right] &\le K_0 e^{-C-l} + K_2 e^{-l} \\ &\quad + 6\int_0^t \mathrm{d}s \overline{E}_{\mu^N}\left[\frac{1}{N}\sum_{y\in\Lambda_{(A+2l)N+1}} \xi_s^*(y)\right]. \end{aligned}$$

Then if we reiterate this calculus,

$$\begin{aligned} &\overline{E}_{\mu^N}\left[\frac{1}{N}\sum_{x\in\Lambda_{AN}} \xi_t^*(x)\right] \\ &\quad \le K_0 e^{-C} \sum_{k=0}^{K} e^{-kl}\frac{(6t)^k}{k!} + K_2 e^{-l} \sum_{k=0}^{K} \frac{(6t)^k}{k!} \\ &\qquad + 6^{k+1}\int_0^t \mathrm{d}s_1 \cdots \int_0^{s_K} \mathrm{d}s_{K+1} \overline{E}_{\mu^N}\left[\frac{1}{N}\sum_{x\in\Lambda_{[A+(K+1)l]N+1}} \xi_{s_{K+1}}^*(x)\right] \\ &\quad \le K_0 e^{-C}\exp(e^{-l}6t) + K_2 e^{6t-l} + 2\frac{(6t)^{K+1}}{(K+1)!}\Big(A+(K+1)l\Big)K_1. \end{aligned}$$

We choose $C = A + (K+1)l$ and we make K then l go to infinity, hence

$$\lim_{l\to\infty}\lim_{K\to\infty} \overline{E}_{\mu^N}\Big[\frac{1}{N}\sum_{x\in\Lambda_{AN}} \xi_t^*(x)\Big] = 0. \tag{19}$$

This result enables us to do the proof of Theorem 1.5.

PROOF OF THEOREM 1.5. We fix $\delta > 0$ and $t \in [0, T]$. We consider two functions G_1 and G_2 on $\mathbb{R}$ with compact support and $A \in \mathbb{N}$ such that the supports of these functions are in $[-A, A]$. To apply theorem 1.4, we need to construct from $m_1(.)$ and $m_2(.)$ two C^2 periodic functions with period C and $C \geq A$. We denote by m_1^C and m_2^C two such functions satisfying: $m_1^C(x) = m_1(x)$ and $m_2^C(x) = m_2(x)$ for all $x \in [-C, C-\varepsilon]$. Let $(\lambda_1^C(t, .), \lambda_2^C(t, .))$ be the solution of the system (Eq) on $[-C, C]$ with initial conditions $(m_1^C(.), m_2^C(.))$. Then for C large enough (see [13], Chapter 11.B)

$$\Big|\int_{\mathbb{R}} G_1(\theta)\Big(\lambda_1^C(t,\theta) - \lambda_1(t,\theta)\Big)\mathrm{d}\theta\Big| \leq \frac{\delta}{4}. \tag{20}$$

This inequality comes from the integral expressions of the solutions: They involve the Gaussian kernel which quickly decreases to 0 when the space coordinate goes to infinity so that the outside of $[-C, C]$ does not influence the solutions in $[-A, A]$ when C is large. Recall that the semi-group associated to the process (η_t^2, ζ_t^2) on $\mathbb{Z}$ is $(\tilde{S}_t^N)$. Then, if we denote by (η_t^1, ζ_t^1) the process on the torus T_{CN} with semi-group (S_t^N), and by $(\bar{S}_t^N)$ the coupled semi-group, we obtain, using (20),

$$\begin{aligned}
&\mu^N \tilde{S}_t^N\left(\eta^2 \;:\; \left|\frac{1}{N}\sum_{x\in\mathbb{Z}}\eta^2(x)G(x/N) - \int G(\theta)\lambda_1(t,\theta)\mathrm{d}\theta\right| > \delta\right)\\
&\quad\leq \mu^N \bar{S}_t^N\left(\eta^1,\, \eta^2 \;:\; \left|\frac{1}{N}\sum_{x\in\mathbb{Z}}\big(\eta^1(x) - \eta^2(x)\big)\,G(x/N)\right|\right.\\
&\qquad\qquad + \left|\frac{1}{N}\sum_{x\in T_{CN}}\eta^1(x)G(x/N) - \int G(\theta)\lambda_1^C(t,\theta)\mathrm{d}\theta\right|\\
&\qquad\qquad\qquad \left.+\Big|\int_{\mathbb{R}} G_1(\theta)\Big(\lambda_1^C(t,\theta) - \lambda_1(t,\theta)\Big)\mathrm{d}\theta\Big| > \delta\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \mu^N \bar{S}_t^N\Big(\eta^1,\, \eta^2 \;:\; \Big|\frac{1}{N}\sum_{x\in\mathbb{Z}}\Big(\eta^1(x) - \eta^2(x)\Big)G(x/N)\Big| > \delta/2\Big)\\
&\quad + \mu^N S_t^N\Big(\eta^1 \;:\; \Big|\frac{1}{N}\sum_{x\in T_{CN}}\eta^1(x)G(x/N) - \int G(\theta)\lambda_1^C(t,\theta)\mathrm{d}\theta\Big|\\
&\qquad\qquad > \delta/2 - \Big|\int_{\mathbb{R}} G_1(\theta)\Big(\lambda_1^C(t,\theta) - \lambda_1(t,\theta)\Big)\mathrm{d}\theta\Big|\Big)\\
&\leq ||G||_\infty \frac{2}{\delta}\overline{E}_{\mu^N}\Big[\frac{1}{N}\sum_{x\in\Lambda_{AN}}|\eta_t^1(x) - \eta_t^2(x)|\Big]\\
&\quad + \mu^N S_t^N\Big(\eta^1 \;:\; \Big|\frac{1}{N}\sum_{x\in T_{CN}}\eta^1(x)G(x/N) - \int G(\theta)\lambda_1^C(t,\theta)\mathrm{d}\theta\Big| > \delta/4\Big).
\end{aligned} \tag{21}$$

We set

$$A_N^C = \left(\eta^1 : \left|\frac{1}{N}\sum_{x\in T_{CN}} \eta^1(x)G(x/N) - \int G(\theta)\lambda_1^C(t,\theta)\mathrm{d}\theta\right| > \delta/4\right)$$

and use the entropy inequality

$$\mu^N S_t^N(A_N^C) \le \frac{\frac{1}{N}\log 2 + \frac{1}{N}H_{CN}\left[\mu_t^N|\nu^N_{\lambda_1^C(t,.)} \times \nu^N_{\lambda_2^C(t,.)}\right]}{\frac{1}{N}\log\left[1+\nu^N_{\lambda_1^C(t,.)} \times \nu^N_{\lambda_2^C(t,.)}(A_N^C)^{-1}\right]}.$$

The denominator of this expression is strictly greater than 0 and from proposition 3.2, we deduce there exists a positive constant α such that

$$\frac{1}{N}H_{CN}\left[\mu_t^N|\nu_{\lambda_1^C(t,.)} \times \nu_{\lambda_2^C(t,.)}\right] \le \frac{\alpha}{N}H_{CN}\left[\mu^N|\nu^N_{m_1^C(.)} \times \nu^N_{m_2^C(.)}\right] + o(1/N).$$

Moreover, using the definition of entropy, we obtain the existence of a constant α' such that

$$\begin{aligned}
&H_{CN}\left[\mu^N|\nu^N_{m_1^C(.)}\times\nu^N_{m_2^C(.)}\right] - H_{CN}\left[\mu^N|\nu^N_{m_1(.)}\times\nu^N_{m_2(.)}\right]\\
&= \int \mathrm{Log}\frac{\mathrm{d}(\nu^N_{m_1(.)}\times\nu^N_{m_2(.)})}{\mathrm{d}(\nu^N_{m_1^C(.)}\times\nu^N_{m_2^C(.)})}\mathrm{d}\mu^N\\
&= \sum_{x=-CN}^{CN} \mu^N[\eta(x)]\mathrm{Log}\frac{m_1(x/N)}{m_1^C(x/N)} + m_1^C(x/N) - m_1(x/N)\\
&\quad + \sum_{x=-CN}^{CN} \mu^N[\zeta(x)]\mathrm{Log}\frac{m_2(x/N)}{m_2^C(x/N)} + m_2^C(x/N) - m_2(x/N)\\
&\le \alpha'\varepsilon N
\end{aligned}$$

because all the terms of the sums are zero except those with $(C-\varepsilon)N \le x \le CN$ and everything is bounded. Besides,

$$\begin{aligned}
&\frac{1}{N}H_{CN}\left[\mu^N|\nu^N_{m_1(.)}\times\nu^N_{m_2(.)}\right]\\
&\quad = \frac{1}{N^2}e^{\theta(C+1)}\sum_{n=CN}^{CN+N} H_{CN}\left[\mu^N|\nu^N_{m_1(.)}\times\nu^N_{m_2(.)}\right]e^{-\theta(C+1)N/N}\\
&\quad \le e^{\theta(C+1)}\frac{1}{N^2}\sum_{n\ge 1} H_n\left[\mu^N|\nu^N_{m_1(.)}\times\nu^N_{m_2(.)}\right]e^{-\theta n/N}\\
&\quad = \frac{1}{N}e^{\theta(C+1)}\mathscr{H}_N\left[\mu^N|\nu^N_{m_1(.)}\times\nu^N_{m_2(.)}\right]
\end{aligned}$$

which goes to zero when N goes to infinity by hypothesis. Therefore

$$\frac{1}{N}H_{CN}\left[\mu_t^N|\nu_{\lambda_1^C(t,.)}\times\nu_{\lambda_2^C(t,.)}\right] \le \alpha'\varepsilon + \frac{1}{N}\alpha e^{\theta C}\mathscr{H}_N\left[\mu^N|\nu^N_{m_1(.)}\times\nu^N_{m_2(.)}\right] + o(1/N).$$

We make N go to infinity and we obtain

$$\lim_{N\to\infty}\mu^N S_t^N\left(\eta^1 \;:\; \left|\frac{1}{N}\sum_{x\in T_{CN}}\eta^1(x)G(x/N)-\int G(\theta)\lambda_1^C(t,\theta)\mathrm{d}\theta\right|>\delta/4\right)\le\alpha''\varepsilon$$

where α'' is a positive constant. Then we come back to (21), using this last estimate and (19),

$$\lim_{C\to\infty}\lim_{N\to\infty}\mu^N\tilde{S}_t^N\left(\eta^2 \;:\; \left|\frac{1}{N}\sum_{x\in\mathbb{Z}}\eta^2(x)G(x/N)-\int G(\theta)\lambda_1(t,\theta)\mathrm{d}\theta\right|>\delta\right)\le\alpha''\varepsilon.$$

So let ε go to zero to obtain the result. □

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