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# A REFINEMENT OF THE HUNT–KURTZ THEORY OF LARGE LOSS NETWORKS, WITH AN APPLICATION TO VIRTUAL PARTITIONING

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This paper gives a refinement of the results of Hunt and Kurtz on the dynamical behavior of large loss networks. We introduce a Liapounov function technique which, under the limiting regime of Kelly, enables the unique identification of limiting dynamics in many applications. This technique considerably simplifies much previous work in this area. We further apply it to the study of the dynamical behavior of large single-resource loss systems under virtual partitioning, or dynamic trunk reservation, controls. We identify limiting dynamics under the above regime, describing the behavior of the number of calls of each type in the system. We show that all trajectories of these dynamics converge to a single fixed point, which we identify. We also identify limiting stationary behavior, including call acceptance probabilities.

**1. Introduction.** This paper presents a refinement of the results of Hunt and Kurtz (1994) on the dynamical behavior of large loss networks. It then considers an application to the virtual partitioning control strategy of Mitra and Ziedins (1996) discussed below.

In recent years there has been considerable interest in the dynamical behavior of loss networks. Although such networks typically reach equilibrium very fast, an understanding of the dynamical behavior both permits the investigation of stability issues and is often the only way to establish equilibrium behavior—see Bean, Gibbens and Zachary (1997) and also the results of Section 3 of the present paper. Hunt and Kurtz (1994) prove a functional law of large numbers describing limiting dynamics in a sequence of loss networks in which arrival rates and capacities are allowed to grow in proportion. However, their result does not always identify these dynamics uniquely. In applications it is necessary to resort to further, ad hoc arguments, and these usually only work in problems where the number of resource constraints is small—typically at most two. In Section 2 we give a result which shows how a Liapounov function technique may be used to refine the original Hunt–Kurtz result in such a way as to permit the unique identification of limiting dynamics in considerably more complex problems. While we give a simple application in that section, our main use of the technique is in the study of the virtual partitioning control scheme considered in Section 3.

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*Virtual partitioning* is a mechanism for sharing the capacity of a resource between a number of competing call types, while preventing any call type from overwhelming the others. Here we consider the classical model of a single-resource loss system modified as follows. Calls of each type $r$ are accepted if there is sufficient available capacity to do so, except when those type-$r$ calls already in progress occupy capacity in excess of some suitable *threshold* $C_r$. In that case, any further call of this type is accepted only if the resulting free capacity in the network will be greater than or equal to some suitably chosen *reservation parameter* $t_r$. Here $t_r$ effectively acts as a "trunk reservation" parameter which is operative only when the number of type-$r$ calls already in progress is sufficiently large. In a large network only a modest value of each $t_r$ is required to ensure that the capacity occupied by type-$r$ calls cannot significantly exceed $C_r$ except only when there would otherwise be spare capacity in the system. This result, which is a property of (rapidly achieved) equilibrium behavior, is a consequence of the analysis of Section 3.

Control mechanisms based on the use of reservation parameters have been both used and investigated for a long time—see Kelly (1991) for a review. However, virtual partitioning as a formal scheme was first proposed by Mitra and Ziedins (1996). The idea has since been applied in several contexts—see, for example, Borst and Mitra (1998), Kumaran and Mitra (1998) and Mitra, Reiman and Wang (1997). The policy has properties of fairness and efficiency and is robust under deviations from the engineered load.

In Section 3 we investigate the limiting dynamics of a loss system with virtual partitioning in a sequence of large loss networks as above, under the assumption of a slowly growing reservation parameter. We prove stability and also deduce limiting equilibrium behavior, including call acceptance probabilities.

For general reviews of loss networks, see, in particular, Kelly (1991) and Ross (1995).

**2. Refinement of the Hunt–Kurtz theory.** In this section we give a brief description of the Hunt–Kurtz theory of large loss networks. We generalize the original description to permit consideration of a wider class of control strategies—for example, the virtual partitioning strategy studied in Section 3. We then give a refinement (Theorem 2.1) to this theory which, in many applications, can be used to uniquely identify limiting dynamics.

As discussed in Section 1, the theory is concerned with the dynamical behavior of loss networks with large capacities and arrival rates and with the establishment of a functional law of large numbers for a suitably normalized version of the dynamics. Thus we consider a sequence of networks, indexed by a scale parameter $N$, with a common set of call types indexed in a finite set $R$. For the $N$th member of the sequence, calls of each type $r \in R$ arrive as a Poisson process of rate $\kappa_r(N)$, where

$$\frac{\kappa_r(N)}{N} \to \kappa_r > 0 \qquad \text{as } N \to \infty. \tag{2.1}$$

A call of type $r$ is accepted, or else rejected and lost, as described below. If accepted, it remains in the network for a period of time which is exponentially distributed with mean $\mu_r^{-1}$ independent of $N$. All arrival processes and call holding times are statistically independent. Define $\mathbf{n}^N(t) = (n_r^N(t), r \in R)$, where $n_r^N(t)$ is the number of calls of type $r$ in progress at time $t$.

We now give the rules for call acceptance. Let $A = (A_{jr},\ j \in J,\ r \in R)$, where $J$ is a finite set and each $A_{jr} \in \mathbb{Z}_+$ (the set of nonnegative integers). For each $N$, let $\mathbf{C}(N) = (C_j(N), j \in J)$, where $C_j(N) \in \mathbb{Z}_+$, and suppose that

$$\frac{C_j(N)}{N} \to C_j > 0 \qquad \text{as } N \to \infty. \tag{2.2}$$

Again for each $N$, define the process $\mathbf{m}^N(\cdot) = (m_j^N(\cdot), j \in J)$ by

$$m_j^N(\cdot) = C_j(N) - \sum_{r \in R} A_{jr} n_r^N(\cdot). \tag{2.3}$$

We assume that, for some $J_1 \subseteq J$ and for each $j \in J_1$, the process $m_j^N(\cdot) \in \mathbb{Z}_+$. This corresponds to the interpretation of each such $j$ as indexing a resource of capacity $C_j(N)$ where a call of each type $r$ requires $A_{jr}$ units of this capacity. For each $j \in J_2 = J \setminus J_1$, we have $m_j^N(\cdot) \in \mathbb{Z}$. Thus, for each $j \in J$, define

$$\mathbb{Z}_j = \begin{cases} \mathbb{Z}_+, & \text{if } j \in J_1, \\ \mathbb{Z}, & \text{if } j \in J_2, \end{cases}$$

and let $D = \prod_{j \in J} \mathbb{Z}_j$ be the product set. A call of type $r$ arriving at time $t$ is accepted if and only if $\mathbf{m}^N(t-)$ belongs to some appropriate subset of $D$ (see below), which depends on $r$ but is independent of $N$. The introduction of the set $J_2$ [with appropriately defined $A_{jr}$ and $C_j(N)$ for each $j \in J_2$] permits the consideration of, for example, trunk reservation strategies in which the trunk reservation parameter grows with $N$, as well as the virtual partitioning strategy considered in Section 3.

To obtain the functional law of large numbers of Hunt and Kurtz (1994), we require an additional condition on each of the acceptance sets. Compactify each of the sets $\mathbb{Z}_j$ to $\mathbb{Z}_j^\Delta$, where

$$\mathbb{Z}_j^\Delta = \begin{cases} \mathbb{Z}_+ \cup \{\infty\}, & \text{if } j \in J_1, \\ \{-\infty\} \cup \mathbb{Z} \cup \{\infty\}, & \text{if } j \in J_2, \end{cases}$$

where the topologies are those of the usual one-point compactification of $\mathbb{Z}_+$ for $j \in J_1$ and the analogous two-point compactification of $\mathbb{Z}$ for $j \in J_2$. Let $E = \prod_{j \in J} \mathbb{Z}_j^\Delta$ be given the corresponding product topology. Let $\mathcal{C}$ be the collection of subsets $F$ of $E$ such that the indicator function $I_F$ is continuous. [For $F$ to belong to $\mathcal{C}$ it is necessary and sufficient that there exists some $M \in \mathbb{Z}_+$ with the following property: for $\mathbf{m} \in E$ such that, for some $j$, $m_j = \pm\infty$, we have $I_F(\mathbf{m}) = I_F(\mathbf{m}')$

for all $\mathbf{m}'$ with $m'_k = m_k$, $k \neq j$, $|m'_j| \geq M$ and $\operatorname{sign} m'_j = \operatorname{sign} m_j$.] We regard the acceptance sets as subsets of $E$ [although, for each $N$, the process $\mathbf{m}^N(\cdot)$ is, of course, confined to the space $D \subset E$] and require that, for all $N$, a call of type $r$ arriving at time $t$ is accepted if and only if $\mathbf{m}^N(t-) \in \mathcal{A}_r$, where $\mathcal{A}_r \in \mathcal{C}$. This condition is satisfied for all reasonable control strategies.

Now, for each $N$, define the normalized process

$$\mathbf{x}^N(\cdot) = \mathbf{n}^N(\cdot)/N.$$

Following Hunt and Kurtz (1994), define also the random occupation measure $\nu^N$ on $[0,\infty) \times E$, associated with the process $\mathbf{m}^N(\cdot)$, by

$$\nu^N([0,t] \times \Gamma) = \int_0^t I_{\{\mathbf{m}^N(u) \in \Gamma\}}\, du, \qquad \Gamma \in \mathcal{B}(E) \tag{2.4}$$

[where $\mathcal{B}(E)$ is the Borel $\sigma$-algebra induced by the above topology]. Then $\nu^N \in \mathcal{L}_0(E)$, where $\mathcal{L}_0(E)$ denotes the space of all measures $\gamma$ on $[0,\infty) \times E$ such that $\gamma([0,t] \times E) = t$. Again following Hunt and Kurtz, $\mathcal{L}_0(E)$ is given the topology corresponding to weak convergence of the measures restricted to $[0,t] \times E$ for each $t$.

We are interested in any possible "fluid limit" process $\mathbf{x}(\cdot)$ of the process $\mathbf{x}^N(\cdot)$ [see, e.g., Kelly (1991)]. Note that any such limit necessarily takes values in the space $X = \{\mathbf{x} \in \mathbb{R}_+^R \colon \sum_r A_{jr} x_r \leq C_j \text{ for all } j \in J_1\}$. For each $\mathbf{x} \in X$, let $\mathbf{m}_{\mathbf{x}}(\cdot)$ be the Markov process on $E$ with transition rates given by

$$\mathbf{m} \to \begin{cases} \mathbf{m} - \mathbf{A}_r, & \text{at rate } \kappa_r I_{\{\mathbf{m} \in \mathcal{A}_r\}}, \\ \mathbf{m} + \mathbf{A}_r, & \text{at rate } \mu_r x_r, \end{cases} \tag{2.5}$$

where $\mathbf{A}_r$ denotes the vector $(A_{jr},\ j \in J)$ and $\infty \pm a = \infty$ for any $a \in \mathbb{Z}_+$. Then the process $\mathbf{m}_{\mathbf{x}}(\cdot)$ is reducible, and so does not always have a unique invariant distribution. Lemmas 1 and 2 and Theorem 3 of Hunt and Kurtz (1994) apply without change to the present generalized control strategy. They show that the sequence $\{(\mathbf{x}^N(\cdot),\ \nu^N)\}$ is relatively compact in $D_{\mathbb{R}^R}[0,\infty) \times \mathcal{L}_0(E)$ and that any weakly convergent subsequence has a limit $(\mathbf{x}(\cdot),\ \nu)$ which obeys the relations

$$x_r(t) = x_r(0) + \int_0^t \bigl(\kappa_r \pi_u(\mathcal{A}_r) - \mu_r x_r(u)\bigr)\, du \tag{2.6}$$

and

$$\nu([0,t] \times \Gamma) = \int_0^t \pi_u(\Gamma)\, du, \qquad \Gamma \in \mathcal{B}(E). \tag{2.7}$$

Here, for each $t$, $\pi_t$ is some invariant distribution of the Markov process $\mathbf{m}_{\mathbf{x}(t)}(\cdot)$ and additionally satisfies

$$\pi_t\{\mathbf{m} \colon m_j = \infty\} = 1 \quad \text{if } \sum_{r \in R} A_{jr} x_r(t) < C_j, \qquad j \in J, \tag{2.8}$$

$$\pi_t\{\mathbf{m} \colon m_j = -\infty\} = 1 \quad \text{if } \sum_{r \in R} A_{jr} x_r(t) > C_j, \qquad j \in J_2. \tag{2.9}$$

REMARK 2.1. For a discussion of these results—which involve a separation, in the limit, of the time scales of the processes $\mathbf{x}^N(\cdot)$ and $\mathbf{m}^N(\cdot)$—again see Hunt and Kurtz (1994). Conditions (2.8) and (2.9) are easily seen to hold by noting that, for any $j$, away from the boundary region $\sum_{r\in R} A_{jr}x_r = C_j$ in $X$, the limiting dynamics $\mathbf{x}(\cdot)$ may be deduced without reference to the control $j$.

Depending on the model under study, it may or may not now be the case that there exists a function $\pi'$ on $X$ (each value of which is a probability distribution on $E$) with the property that, for *all* convergent subsequences, we may take $\pi_t$ above to be given by $\pi_t = \pi'_{\mathbf{x}(t)}$. In particular, if there does exist such a function, then we may define a *velocity field* $\mathbf{v} = (v_r,\ r \in R)$ on $X$ by

$$v_r(\mathbf{x}) = \kappa_r \pi'_{\mathbf{x}}(\mathcal{A}_r) - \mu_r x_r, \tag{2.10}$$

so that (2.6) becomes

$$x_r(t) = x_r(0) + \int_0^t v_r\big(\mathbf{x}(u)\big)\,du. \tag{2.11}$$

However, Hunt (1995) gives some, rather pathological, examples in which the function $\pi'$ fails to exist.

There remains now the general problem in applications of establishing the existence of such a function $\pi'$ and of identifying it. For each $t$, the state space of the process $\mathbf{m}_{\mathbf{x}(t)}(\cdot)$ fragments into $2^{J_1} \times 3^{J_2}$ closed components, each of which is usually not further reducible. Each of these components *may* have an associated invariant distribution, and, subject to the restrictions imposed by conditions (2.8) and (2.9), the above results merely require the distribution $\pi_t$ to be some convex combination of these (extreme) invariant distributions.

In previous work—see Hunt and Kurtz (1994), Hunt (1995), Hunt and Laws (1997), Bean, Gibbens and Zachary (1995, 1997) and Alanyali (1999)—the distribution $\pi_t, t \geq 0$, is usually identified by making use of the further observation that the process $\mathbf{x}(\cdot)$ must necessarily remain within the space $X$. This observation, however, only provides sufficient additional information in quite simple examples, where the set $J$ is small. Otherwise, it is necessary to further refine the limiting theory described above to establish additional properties of the distribution $\pi_t$, essentially via some form of tightness argument. We give below, in Theorem 2.1, a very general result (see also the further discussion at the end of this section). Once established, this result considerably simplifies the arguments used in earlier work to identify $\pi_t$. The theorem further permits the identification of $\pi_t$ in more complex models. An example is the virtual partitioning strategy considered in Section 3.

For any function $f$ on $D$, and for each $\mathbf{x} \in \mathbb{R}_+^R$, define the function $d_{\mathbf{x}} f$ on $D$ by

$$\begin{aligned} d_{\mathbf{x}} f(\mathbf{m}) = \sum_r \big[&\kappa_r I_{\{\mathbf{m}\in\mathcal{A}_r\}}\big(f(\mathbf{m} - \mathbf{A}_r) - f(\mathbf{m})\big) \\ &+ \mu_r x_r\big(f(\mathbf{m} + \mathbf{A}_r) - f(\mathbf{m})\big)\big] \end{aligned} \tag{2.12}$$

here $d_{\mathbf{x}}$ is simply the generator of the Markov process $\mathbf{m}_{\mathbf{x}}(\cdot)$ defined by (2.5). Similarly, for each $N \geq 1$, define the function $d_{\mathbf{x}}^N f$ on $D$ as for $d_{\mathbf{x}} f$ in (2.12) but with each $\kappa_r$ replaced by $\kappa_r(N)/N$.

THEOREM 2.1. *Let* $(\mathbf{x}(\cdot), \nu)$ *be the limit, in some convergent subsequence, of the sequence* $\{(\mathbf{x}^N(\cdot), \nu^N)\}$. *Suppose that, associated with some region* $X' \subseteq X$, *there exist a set* $F$ *satisfying* $D \subseteq F \subset E$, *a real-valued function* $f$ *on* $D$, *a set* $K \subseteq J$ *and a constant* $\bar{a} \in \mathbb{R}$ *such that*

$$|f(\mathbf{m} + \mathbf{A}_r) - f(\mathbf{m})| \quad \textit{is bounded in } \mathbf{m} \in D,\ r \in R, \tag{2.13}$$

$$\sum_{r \in R} A_{jr} x_r = C_j \quad \textit{for all } j \in K \textit{ and } \mathbf{x} \in X', \tag{2.14}$$

$$\textit{the function } f \textit{ depends only on those } m_j \textit{ with } j \in K, \tag{2.15}$$

$$G(a) \subseteq F(a) \subseteq F \quad \textit{for some compact set } F(a) \subset E \textit{ for all } a \in \mathbb{R}, \tag{2.16}$$

*where* $G(a) = \{\mathbf{m} \in D\colon\ f(\mathbf{m}) \leq a\}$, *and*

$$\sup_{\mathbf{m} \in D \setminus G(\bar{a})} d_{\mathbf{x}} f(\mathbf{m}) < 0 \quad \textit{for all } \mathbf{x} \in X'. \tag{2.17}$$

*Suppose also that* $\mathbf{x}(t) \in X'$ *for all* $t$ *in some interval* $T$. *Then, in (2.6) and (2.7), we have* $\pi_t(F) = 1$ *for almost all* $t \in T$.

PROOF. As in Hunt and Kurtz (1994), we take the convergent subsequence of $\{(\mathbf{x}^N(\cdot), \nu^N)\}$ and its limit $(\mathbf{x}(\cdot), \nu)$ to be defined on a common probability space $(\Omega, \mathcal{F}, P)$ in such a way that the convergence is almost sure. We show first that, given any closed interval $[t_1, t_2] \subseteq T$ and $0 < \lambda < 1$, for all sufficiently large $a$,

$$\lim_{N \to \infty} P\left[\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} I_{\{\mathbf{m}^N(t) \in G(a)\}}\, dt > \lambda\right] = 1, \tag{2.18}$$

where, here and elsewhere, $N$ indexes the convergent subsequence. It follows from (2.12), (2.13) and the continuity of $\mathbf{x}(\cdot)$ that the function

$$\sup_{\mathbf{m} \in D \setminus G(\bar{a})} d_{\mathbf{x}(\cdot)} f(\mathbf{m})$$

is continuous. It further follows from (2.17), (2.1), the definition of the function $d_{\mathbf{x}}^N f$ and the weak convergence of $\mathbf{x}^N(\cdot)$ to $\mathbf{x}(\cdot)$ that, given $\delta > 0$, there exist $b > 0$ and $\theta > 0$ such that, for all sufficiently large $N$, on a set $\overline{\Omega}^N \subset \Omega$ with $P(\overline{\Omega}^N) > 1 - \delta$,

$$\sup_{\mathbf{m} \in D \setminus G(\bar{a})} d_{\mathbf{x}^N(t)}^N f(\mathbf{m}) \leq -b \quad \text{for all } t \in [t_1, t_2], \tag{2.19}$$

and also [by considering the Taylor expansion of $e^{\theta f}$ and using (2.13) and (2.19)]

$$\sup_{\mathbf{m}\in D\setminus G(\bar{a})} (d^N_{\mathbf{x}^N(t)} e^{\theta f})(\mathbf{m}) \le 0 \quad \text{for all } t \in [t_1, t_2]. \tag{2.20}$$

Note that the relationships (2.19) and (2.20) are supermartingale properties for the processes $f(\mathbf{m}^N(\cdot))$ and $e^{f(\mathbf{m}^N(\cdot))}$, respectively, while the process $\mathbf{m}^N(\cdot)$ remains within the set $D \setminus G(\bar{a})$. It follows from (2.3) that, for all $N$ and for all $j \in J$,

$$\begin{aligned}|m_j^N(t_1)| \le N\left|\left(C_j - \sum_r A_{jr}x_r(t_1)\right)\right| + |C_j(N) - NC_j|\\ + N\sum_r A_{jr}|x_r^N(t_1) - x_r(t_1)|.\end{aligned}$$

Now apply this result to $j \in K$: it follows from (2.2), (2.13), (2.14) and (2.15) that there exists $c > 0$ such that, for any $N$, if $|\mathbf{x}^N(t_1) - \mathbf{x}(t_1)| < \varepsilon$, then $|f(\mathbf{m}^N(t_1)) - \bar{a}| < cN\varepsilon$. The weak convergence of $\mathbf{x}^N(t_1)$ to $\mathbf{x}(t_1)$, together with (2.19), now ensures that the time from $t_1$ until the process $f(\mathbf{m}^N(\cdot))$ first enters the set $G(\bar{a})$ converges in probability to 0 as $N \to \infty$ [see, e.g., Fayolle, Malyshev and Menshikov (1995), Theorem 2.1.1]. For any $a > \bar{a}$, consider the durations of the subsequent successive crossings of the interval $[\bar{a}, a]$ by the process $f(\mathbf{m}^N(\cdot))$, where upcrossings and downcrossings are achieved at the times of alternate entries into the sets $D \setminus G(a)$ and $G(\bar{a})$, respectively. Standard martingale arguments, together with (2.13), show that, while (2.20) obtains, the expected time of each upcrossing is bounded below by $k_1 e^{\theta a}/N$ for some $k_1 > 0$ independent of $N$ and $a$. [Consider, e.g., any fixed $\hat{a} > \bar{a}$. The upcrossing time of the process $f(\mathbf{m}^N(\cdot))$ associated with the interval $[\bar{a}, \hat{a}]$ is bounded below by $c_1/N$ for some $c_1 > 0$ independent of $N$. The optional stopping theorem, (2.13) and (2.20) show that, conditional upon the completion of such an upcrossing and for all sufficiently large $a > \hat{a}$, the probability that the process then exits the interval $[\bar{a}, a]$ above $a$ is bounded above by $c_2 e^{-\theta a}$, for some $c_2 > 0$ independent of $N$ and $a$.] Similarly, while (2.19) obtains, the expected time of each downcrossing of $[\bar{a}, a]$ is bounded above by $k_2(a - \bar{a})/N$ for some $k_2 > 0$ independent of $N$ and $a$ [again see Fayolle, Malyshev and Menshikov (1995), Theorem 2.1.1]. Since $\delta$ may be taken arbitrarily small, routine probability estimates now show that, given $0 < \lambda < 1$, there exists $a(\lambda)$ such that (2.18) holds for all $a \ge a(\lambda)$.

Now, for each $t$, the convergence (for each sample path in $\Omega$) of the measure $\nu^N([0, t] \times \cdot)$ on $E$ to the measure $\nu([0, t] \times \cdot)$ on $E$ is then that of weak convergence of finite measures. It follows from (2.16), (2.18) and (2.4) that, again given $0 < \lambda < 1$, for all sufficiently large $a$,

$$\lim_{N\to\infty} P\left[\frac{1}{t_2 - t_1}\nu^N\big([t_1, t_2] \times F(a)\big) > \lambda\right] = 1,$$

and so, since $F(a)$ is compact,

$$\frac{1}{t_2 - t_1} \nu\big([t_1, t_2] \times F(a)\big) \geq \lambda \quad \text{a.s.}$$

Hence

$$\frac{1}{t_2 - t_1} \nu\big([t_1, t_2] \times F\big) = 1 \quad \text{a.s.}$$

Since $[t_1, t_2]$ is an arbitrary interval within $T$, the result now follows from (2.7).

□

REMARK 2.2. The function $f$ of Theorem 2.1 is a Liapounov function which ensures a form of tightness for the sequence of random measures $\{\nu^N\}$. In applications it is often convenient to define $f$ on the set $F$ rather than simply on $D \subseteq F$. The compact set $F(a)$ may then usually be taken to be $F(a) = \{\mathbf{m} \in F\colon\ f(\mathbf{m}) \leq a\}$. This is the case both in the application below and in that of Section 3.

REMARK 2.3. In some applications—in particular, that of Section 3—it may be that there exists some proper subset $D' \subset D$ such that $\mathbf{m}^N(\cdot) \in D'$ for all sufficiently large $N$. In this case, we clearly only require (2.17) to hold with $D$ replaced by $D'$.

REMARK 2.4. Other (mild) generalizations of the theorem are possible—though not required in the rest of this paper. Note that, in particular, (2.14) and (2.15) are required simply to ensure that, in the above proof, $f(\mathbf{m}^N(t_1))$ is sufficiently close to $\bar{a}$.

Theorem 2.1 goes some considerable way to closing the gap in the Hunt–Kurtz theory with respect to the unique identification of the limiting dynamics $\mathbf{x}(\cdot)$. In applications it is usually possible, for each $\mathbf{x} \in X$, to guess that part of the state space $E$ on which is concentrated the relevant stationary distribution of the process $\mathbf{m}_\mathbf{x}(\cdot)$, and then to verify this result using Theorem 2.1 and an appropriate choice of one or more Liapounov functions. (There is, of course, great flexibility here.) We give a detailed example of this strategy in the study of the virtual partitioning problem in Section 3. We complete the present section by giving a fairly simple example which nevertheless allows us to make some important points.

EXAMPLE 2.1. Suppose that $J = J_1$, so that $D = \mathbb{Z}_+^J$ and the state space $E = (\mathbb{Z}_+ \cup \{\infty\})^J$ of each of the processes $\mathbf{m}_\mathbf{x}(\cdot)$ fragments into $2^J$ closed components, on each of which there may be concentrated a stationary distribution of $\mathbf{m}_\mathbf{x}(\cdot)$. Let $R^* = \{r \in R\colon\ (\infty, \dots, \infty) \in \mathcal{A}_r\}$. For each $\mathbf{x} \in X$ and $j \in J$, let

$\alpha_j(\mathbf{x}) = \sum_{r\in R} A_{jr}(\kappa_r I_{\{r\in R^*\}} - \mu_r x_r)$ and let $\alpha(\mathbf{x}) = \min_{j\in J} \alpha_j(\mathbf{x})$. Consider first the problem of identifying $\pi_t$ (insofar as it is unique) for $t$ such that $\mathbf{x}(t) \in Y = \{\mathbf{x} \in X\colon \sum_{r\in R} A_{jr}x_r = C_j$ for all $j \in J$ and $\alpha(\mathbf{x}) > 0\}$. (Note, in particular, that, for $t$ such that $\sum_{r\in R} A_{jr}x_r < C_j$ for some $j$, then, by Remark 2.1, the problem effectively reduces to one of smaller dimension.) Apply Theorem 2.1 with $X' = Y$, $F = E \setminus \{(\infty, \dots, \infty)\}$, $f$ defined on all of $F$ (see Remark 2.2) by $f(\mathbf{m}) = \min_{j\in J} m_j$ and $K = J$. Conditions (2.13)–(2.16) are trivially satisfied with, in (2.16), $F(a) = G(a)$. Further, the earlier condition $\mathcal{A}_r \in \mathcal{C}$, $r \in R$, implies that there exists $M \in \mathbb{Z}_+$ such that, for all $\mathbf{m}$ with $f(\mathbf{m}) \geq M$, we have $\mathbf{m} \in \mathcal{A}_r$ for $r \in R^*$ and $\mathbf{m} \notin \mathcal{A}_r$ for $r \notin R^*$. Now, for any such $\mathbf{m}$, let $j$ be such that $f(\mathbf{m}) = m_j$; then, from (2.12) and the definitions of $f$ and $\alpha(\mathbf{x})$, it follows that $d_\mathbf{x} f(\mathbf{m}) \leq d_\mathbf{x} m_j \leq -\alpha(\mathbf{x})$. Thus, from the definition of $Y$, (2.17) is satisfied by taking $\bar{a} = M$. We thus conclude that

$$\pi_t\big((\infty, \dots, \infty)\big) = 0 \quad \text{for all } t \text{ such that } \mathbf{x}(t) \in Y. \tag{2.21}$$

In the single-resource case $J = \{1\}$, (2.21) shows that, for $t$ such that $\mathbf{x}(t) \in Y$, we have $\pi_t = \pi'_{\mathbf{x}(t)}$, where $\pi'_\mathbf{x}$ is the stationary distribution for $\mathbf{m}_\mathbf{x}(\cdot)$ on $\mathbb{Z}_+$. [For $t$ such that $\mathbf{x}(t) \notin Y$, we have $\pi_t(\mathcal{A}_r) = I_{\{r\in R^*\}}$ for all $r$—see Bean, Gibbens and Zachary (1997)—thus completing the unique identification of limiting dynamics in the single-resource case.] For $J = \{1\}$, (2.21) may also be deduced from the observation that the limiting dynamics $\mathbf{x}(\cdot)$ must remain within the set $X$—see Section 3 of Hunt and Kurtz (1994). However, in the present example, this latter observation is, in general, insufficient to deduce (2.21) even in the two-resource case $J = \{1, 2\}$. Hunt (1995) gives an example of pathological behavior for the two-resource case, where the limiting dynamics are not uniquely defined (so that we may, for example, have different limits in different subsequences). The result (2.21) is nevertheless required to identify limiting dynamics insofar as this is possible and, for his example, is proved by Hunt using a quite complex coupling argument. This again does not extend to the more general situation considered here, where some result such as Theorem 2.1 is essential.

Given the result (2.21), a general treatment of the two-resource case $J = \{1, 2\}$ is completed, using more elementary arguments, as in Bean, Gibbens and Zachary (1997). In higher dimensions, any treatment is necessarily more complex. We see an example in the analysis of the virtual partitioning problem of Section 3, where we effectively have $|J_1| = 1$, $|J_2| = 2$, and the choice of Liapounov function for Theorem 2.1 is nontrivial.

**3. Loss systems under virtual partitioning.** We study the virtual partitioning problem described in the Introduction. We consider a sequence of networks as described in Section 2 and show the existence of a velocity field for the limiting dynamics. We then show that all trajectories of these dynamics converge to a single fixed point, thereby also establishing equilibrium behavior.

The call acceptance rule for the sequence is as follows. Associate with each call type $r$ a positive-integer capacity requirement $e_r$. Associate with the $N$th member of the sequence a total capacity $C(N)$ and, for each $r$, a *threshold* $C_r(N)$ and a positive (integer) *reservation parameter* $t_r(N)$. In the $N$th member of the sequence, a call of type $r$ is accepted when the process is in state $\mathbf{n}$ if and only if *either*

$$\sum_{s\in R} e_s n_s + e_r \le C(N) \quad \text{and} \quad e_r(n_r+1) \le C_r(N) \tag{3.1}$$

or

$$\sum_{s\in R} e_s n_s + e_r \le C(N) - t_r(N). \tag{3.2}$$

As in (2.2) we have

$$\lim_{N\to\infty} \frac{C(N)}{N} = C, \qquad \lim_{N\to\infty} \frac{C_r(N)}{N} = C_r \quad \text{for all } r \in R. \tag{3.3}$$

We also assume

$$\lim_{N\to\infty} t_r(N) = \infty, \qquad \lim_{N\to\infty} \frac{t_r(N)}{N} = 0 \quad \text{for all } r \in R. \tag{3.4}$$

Condition (3.4) includes the optimal growth rate $\log N$ of the reservation parameter $t(N)$ [see Key (1990)].

To model the problem as in Section 2, for each $N$ define the process $\mathbf{m}^N(\cdot) = (\hat{m}^N(\cdot),\ m_r^N(\cdot),\ \bar{m}_r^N(\cdot),\ r \in R)$ by

$$\hat{m}^N(\cdot) = C(N) - \sum_{r\in R} e_r n_r^N(\cdot), \tag{3.5}$$

$$m_r^N(\cdot) = C_r(N) - e_r n_r^N(\cdot), \tag{3.6}$$

$$\bar{m}_r^N(\cdot) = C(N) - t_r(N) - \sum_{r\in R} e_r n_r^N(\cdot) \quad \big(= \hat{m}^N(\cdot) - t_r(N)\big). \tag{3.7}$$

The process $\mathbf{m}^N(\cdot)$ takes values in the space $D = \{\mathbf{m} = (\hat{m}, m_r, \bar{m}_r)\colon\ \hat{m} \in \mathbb{Z}_+, m_r \in \mathbb{Z}, \bar{m}_r \in \mathbb{Z}, r \in R\}$. The space $D$ is compactified to $E$ as described in Section 2. The acceptance sets $\mathcal{A}_r \in E$, $r \in R$, are given by

$$\mathcal{A}_r = \{\mathbf{m}\colon\ m_r \ge e_r, \hat{m} \ge e_r\} \cup \{\mathbf{m}\colon\ m_r < e_r, \bar{m}_r \ge e_r\}. \tag{3.8}$$

We now consider the limiting dynamics $\mathbf{x}(\cdot)$ defined in Section 2. Formally, $\mathbf{x}(\cdot)$ is the limit associated with any convergent subsequence of the sequence $\{(\mathbf{x}^N(\cdot), \nu^N)\}$ defined there; however, it turns out that, for the results of this section, the limit $\mathbf{x}(\cdot)$ is unique. Note that $\mathbf{x}(\cdot)$ takes values in the space $X = \{\mathbf{x} \in$

$\mathbb{R}_+^R\colon \sum_r e_r x_r \le C\}$. Define the function $\alpha$ on $X$ and, for each $r \in R$, the functions $\alpha_r$, $\gamma_r$ on $X$ by

$$\alpha(\mathbf{x}) = \sum_{s\in R} e_s(\kappa_s - \mu_s x_s), \tag{3.9}$$

$$\alpha_r(\mathbf{x}) = \alpha(\mathbf{x}) - e_r\kappa_r, \tag{3.10}$$

$$\gamma_r(\mathbf{x}) = \sum_{s\neq r} e_s(\kappa_s - \mu_s x_s) \quad \bigl(= \alpha(\mathbf{x}) - e_r(\kappa_r - \mu_r x_r)\bigr). \tag{3.11}$$

We now define various subsets of $X$ on which it will be necessary to consider separately the behavior of the process $\mathbf{x}(\cdot)$. Define the sets

$$B = \Bigl\{\mathbf{x} \in X\colon \sum_r e_r x_r = C\Bigr\},$$
$$Y = \bigl\{\mathbf{x} \in B\colon \alpha(\mathbf{x}) > 0\bigr\}.$$

Note that it is clear from (2.6) that, for almost all $t$ such that $\mathbf{x}(t) \in Y$, we have $\pi_t(\mathcal{A}_r) < 1$ for at least one $r \in R$—see also Theorem 3.1. Define also the subsets of $Y$:

$$\begin{aligned}
\widehat{U}_r &= \bigl\{\mathbf{x} \in Y\colon e_r x_r = C_r, \gamma_r(\mathbf{x}) \le 0\bigr\}, && r \in R,\\
\widehat{V}_r &= \bigl\{\mathbf{x} \in Y\colon e_r x_r = C_r, \gamma_r(\mathbf{x}) > 0\bigr\}, && r \in R,\\
U_r &= \bigl\{\mathbf{x} \in Y\colon e_r x_r > C_r\bigr\} \cup \widehat{U}_r, && r \in R,\\
V &= Y \setminus \bigcup_r U_r.
\end{aligned}$$

For the present we assume that

$$C_r + C_s > C \quad \text{for all } r, s \in R \text{ with } r \neq s, \tag{3.12}$$

so that no two of the (limiting) *reservation regions* $\{\mathbf{x} \in B\colon e_r x_r \ge C_r\}$ overlap. (It is natural for this to always be the case for $|R| = 2$.) We analyze the limiting dynamics $\mathbf{x}(\cdot)$ as in Section 2. We show in Theorem 3.1 that there exists a unique velocity field for these dynamics. Then, in Theorem 3.2, we show that there exists a unique fixed point to which all trajectories of these dynamics converge (as $t \to \infty$). We thus deduce also the limiting stationary behavior, including call acceptance probabilities. Finally, we informally generalize these results to the case where reservation regions may be permitted to overlap.

In order to state and prove Theorem 3.1, we first define some simpler related control strategies. Consider first the modified control in which a call of any type $r$ is accepted if and only if condition (3.1) holds. This corresponds to the replacement of each acceptance region $\mathcal{A}_r$ by $\mathcal{A}_r^* = \{\mathbf{m}\colon m_r \ge e_r, \hat{m} \ge e_r\}$. For each $N$, denote the corresponding normalized dynamics by $(\mathbf{x}^*)^N(\cdot)$. Here, for each $N$ and for each $r$, the threshold $C_r(N)$ is treated as a "hard-constraint" on the number of type-$r$ calls in progress. This is an instance of the classical control strategy studied by Kelly (1986) where, for each $N$, calls are accepted subject only to a set of linear constraints on the resulting process $(\mathbf{x}^*)^N(\cdot)$. The results of Bean, Gibbens and Zachary (1997) show that, for the corresponding limiting dynamics $\mathbf{x}^*(\cdot)$, there exists a unique velocity field $\mathbf{v}^*$ on $\{\mathbf{x} \in X\colon\ e_r x_r \le C_r \text{ for all } r \in R\}$. This velocity field may be determined as in that paper, but what will be important here is that there is a unique point $\mathbf{x}^*$, defined following (3.23), to which all trajectories of $\mathbf{x}^*(\cdot)$ converge—see the proof of Theorem 3.2.

For each fixed $r \in R$, consider also the simple "trunk reservation" control strategy in which a call of type $r' \ne r$ is accepted if and only if

$$\sum_{s \in R} e_s n_s + e_{r'} \le C(N)$$

(i.e., if and only if there is sufficient capacity to do so), while a call of type $r$ is accepted if and only if (3.2) holds. In a straightforward generalization of Example 2 of Section 3 of Hunt and Kurtz (1994) [and by recalling (3.4)], it follows that there exists a unique velocity field $\mathbf{v}^r$ on $X$ for the associated limiting dynamics $\mathbf{x}^r(\cdot)$. In particular, for $\mathbf{x} \in Y$, $\mathbf{v}^r(\mathbf{x})$ satisfies the conditions

$$v_s^r(\mathbf{x}) = \kappa_s - \mu_s x_s, \qquad s \ne r, \quad \alpha_r(\mathbf{x}) \le 0, \tag{3.13}$$

$$v_r^r(\mathbf{x}) = -\mu_r x_r, \qquad \alpha_r(\mathbf{x}) > 0, \tag{3.14}$$

$$\sum_{s \in R} e_s v_s^r(\mathbf{x}) = 0. \tag{3.15}$$

[See, e.g., Hunt and Laws (1997).] Note also that, for $\mathbf{x} \in Y$ and $\alpha_r(\mathbf{x}) \le 0$, conditions (3.13) and (3.15) determine $\mathbf{v}^r$ uniquely.

We now return to consideration of the limiting dynamics $\mathbf{x}(\cdot)$ corresponding to the virtual partitioning controls described at the beginning of this section.

THEOREM 3.1. *There exists a unique velocity field* $\mathbf{v}$ *on* $X$ *for the limiting dynamics* $\mathbf{x}(\cdot)$, *given by*

$$v_r(\mathbf{x}) = \kappa_r - \mu_r x_r, \qquad r \in R, \quad \mathbf{x} \in X \setminus Y, \tag{3.16}$$

$$\mathbf{v}(\mathbf{x}) = \mathbf{v}^r(\mathbf{x}), \qquad \mathbf{x} \in U_r, \quad r \in R, \tag{3.17}$$

$$\mathbf{v}(\mathbf{x}) = \mathbf{v}^*(\mathbf{x}), \qquad \mathbf{x} \in V. \tag{3.18}$$

PROOF. The result (3.16) is clear for $\mathbf{x} \in X \setminus B$, since here, for the limit process $\mathbf{x}(\cdot)$, neither the capacity constraint nor the controls associated with the virtual partitioning thresholds are operative—see Remark 2.1. [Formally, the result follows from (2.8), applied to the coordinate of $D$ corresponding to $\hat{m}$.] To prove

(3.16) for $\mathbf{x} \in B \setminus Y$, let $T$ be any measurable set, contained within a finite interval, such that, for all $t \in T$, $\mathbf{x}(t) \in B \setminus Y$. Then, from (2.6) and with $\pi_t$ as defined there,

$$\int_T \sum_r e_r \big(\kappa_r \pi_t(\mathcal{A}_r) - \mu_r x_r(t)\big)\, dt = 0.$$

Since $\alpha(\mathbf{x}(t)) \le 0$ for all $t \in T$ (from the definition of $Y$), it follows from (3.9) that $\pi_t(\mathcal{A}_r) = 1$ for all $r$ [and $\alpha(\mathbf{x}(t)) = 0$] for almost all such $t$. Thus we may define a velocity field $\mathbf{v}$ on $B \setminus Y$ by $v_r(\mathbf{x}) = \kappa_r - \mu_r x_r$ for all $r$, and the result (3.16) now also follows for $\mathbf{x} \in B \setminus Y$.

To prove (3.17), note that, for any $r$, the result is clear for $\mathbf{x} \in U_r \setminus \widehat{U}_r$, since here the limiting dynamics are as for the simple trunk reservation control strategy described above. To show (3.17) for $\mathbf{x} \in \widehat{U}_r$, we argue similarly to the proof of (3.16). Let $T$ now be any measurable set, contained within a finite interval, such that, for all $t \in T$, $\mathbf{x}(t) \in \widehat{U}_r$. Then, from (2.6) and the definition of $\widehat{U}_r$,

$$\int_T \sum_{s \ne r} e_s \big(\kappa_s \pi_t(\mathcal{A}_s) - \mu_s x_s(t)\big)\, dt = 0,$$

and, since $\gamma_r(\mathbf{x}) \le 0$ for all $\mathbf{x} \in \widehat{U}_r$, it follows that $\pi_t(\mathcal{A}_s) = 1$ for all $s \ne r$ [and $\gamma_r(\mathbf{x}) = 0$] for almost all $t \in T$. Thus we may define a velocity field $\mathbf{v}$ on $\widehat{U}_r$ by $v_s(\mathbf{x}) = \kappa_s - \mu_s x_s$ for $s \ne r$ and $\sum_{s \in R} e_s v_s(\mathbf{x}) = 0$. From a comparison of (3.10) and (3.11), we also have $\alpha_r(\mathbf{x}) \le 0$ for all $\mathbf{x} \in \widehat{U}_r$. Since also $\widehat{U}_r \subseteq Y$, the result (3.17) now follows by a comparison with (3.13) and (3.15) above.

To prove (3.18), note that the result is once again straightforward for $\mathbf{x} \in V \setminus \bigcup_{r \in R} \widehat{V}_r$, since here it is clear that the limiting dynamics are as for the classical hard-constraint control strategy described above. It thus remains to consider, for any fixed $r$, the behavior of $\mathbf{x}(\cdot)$ on the set $\widehat{V}_r$. Here, to apply the theory of Section 2, we need only consider the components $\hat{m}_{\mathbf{x}}(\cdot)$, $m_{\mathbf{x},r}(\cdot)$ and $\bar{m}_{\mathbf{x},r}(\cdot)$ of the process $\mathbf{m}_{\mathbf{x}}(\cdot)$. Since, from (3.12), the reservation regions do not overlap, these components correspond to the only controls relevant to the dynamics of the process $\mathbf{x}(\cdot)$ on the set $\widehat{V}_r$—again Remark 2.1. That the component $\bar{m}_{\mathbf{x},r}(\cdot)$ is relevant here follows since, from (3.3) and (3.4), $\lim_{N\to\infty}(C(N) - t_r(N))/N = C$ while $\sum_{s \in R} e_s x_s = C$ for all $\mathbf{x} \in \widehat{V}_r$. From (2.5), these components have joint transition rates given by

$$(3.19)\quad (\hat{m}, m_r, \bar{m}_r) \to \begin{cases} (\hat{m} - e_r, m_r - e_r, \bar{m}_r - e_r), & \text{at rate } \kappa_r I_{\{\mathbf{m} \in \mathcal{A}_r\}}, \\ (\hat{m} - e_s, m_r, \bar{m}_r - e_s), & \text{at rate } \kappa_s I_{\{\hat{m} \ge e_s\}}, \\ & \quad s \ne r, \\ (\hat{m} + e_r, m_r + e_r, \bar{m}_r + e_r), & \text{at rate } \mu_r x_r, \\ (\hat{m} + e_s, m_r, \bar{m}_r + e_s), & \text{at rate } \mu_s x_s, \quad s \ne r \end{cases}$$

[where $\mathcal{A}_r$ is as given by (3.8)]. Figure 1 illustrates these transition rates where we take just one $s \ne r$; note, from (3.5) and (3.7), the informal coupling of the $\hat{m}$

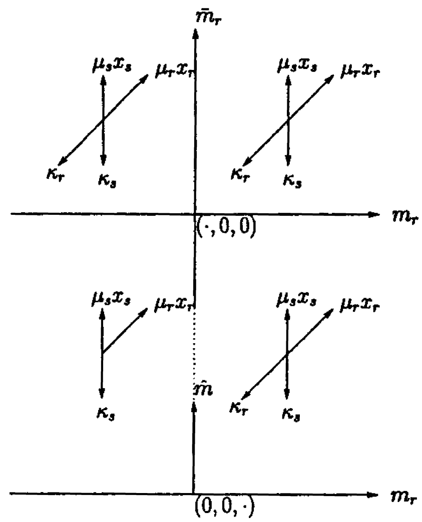


FIG. 1. *Transition rates for the control process* $(\hat{m}_{\mathbf{x}}(\cdot), m_{\mathbf{x},r}(\cdot), \bar{m}_{\mathbf{x},r}(\cdot))$.

and $\bar{m}_r$ axes, which are shown coincident, although with different origins. We shall show that

$$\pi_t\{\mathbf{m}\colon \hat{m} < \infty, m_r > -\infty\} = 1 \quad \text{for almost all } t \text{ such that } \mathbf{x}(t) \in \widehat{V}_r \tag{3.20}$$

[where again $\pi_t$ is as defined by (2.6)]. We also have

$$\pi_t\{\mathbf{m}\colon \hat{m} < \infty, \bar{m}_r > -\infty\} = 0 \quad \text{for almost all } t. \tag{3.21}$$

This latter result follows from (3.5) and (3.7), together with the assumption of (3.4) that $\lim_{N\to\infty} t_r(N) = \infty$; it is implicit in, for example, the work of Hunt and Laws (1997) on trunk reservation and is formally proved as in Lemma 2.1(a) of Alanyali (1999). It then follows from (3.20), (3.21) and the definition of $\mathcal{A}_r$ that $\pi_t\{\mathbf{m}\colon \hat{m} < \infty, \bar{m}_r = -\infty, m_r \geq 0\} = 1$ for almost all $t$ such that $\mathbf{x}(t) \in \widehat{V}_r$.

It now follows that the limiting dynamics $\mathbf{x}(\cdot)$ on the set $\widehat{V}_r$ are the same as would be obtained if the set $\mathcal{A}_r$ were replaced by the set $\mathcal{A}_r^*$ defined above and corresponding to the hard-constraint control strategy. That a unique velocity field for these dynamics now exists on $\widehat{V}_r$, and is given by $\mathbf{v}(\mathbf{x}) = \mathbf{v}^*(\mathbf{x})$, follows from the results of Bean, Gibbens and Zachary (1997). (Indeed the results there show that, on $\widehat{V}_r$, the velocity field is identified by the further conditions $\pi_t\{\mathbf{m}\colon m_r < \infty\} = 1$ if $\mu_r x_r/\kappa_r < \sum_{s\in R} \mu_s x_s/\kappa_s$ and $\pi_t\{\mathbf{m}\colon m_r = \infty\} = 1$ otherwise.)

It thus remains to establish the result (3.20). We shall do this by applying Theorem 2.1, for each $\varepsilon > 0$, with the sets $X'$ and $F$ of that theorem given by $X' = \widehat{V}_{r,\varepsilon} = \{\mathbf{x} \in \widehat{V}_r\colon \alpha(\mathbf{x}) \wedge \gamma_r(\mathbf{x}) > \varepsilon\}$ and $F = \{\mathbf{m}\colon \hat{m} < \infty, m_r > -\infty\}$. Since $\widehat{V}_r = \bigcup_{\varepsilon>0} \widehat{V}_{r,\varepsilon}$, the result (3.20) will then follow.

Thus fix $\varepsilon > 0$. To define the Liapounov function $f$, define first the functions $\hat{h}$ and $\hat{f}$ on $\mathbb{Z}_+$ and the functions $h_r$ and $f_r$ on $\mathbb{Z}$ by

$$\hat{h}(\hat{m}) = 0 \vee \frac{\hat{m} - \hat{e}}{k} \wedge 1,$$

$$\hat{f}(\hat{m}) = \sum_{i=1}^{\hat{m}} \hat{h}(i), \qquad \hat{m} \in \mathbb{Z}_+,$$

$$h_r(m_r) = -1 \vee \left(\frac{m_r - e_r}{k} - 1\right) \wedge 0,$$

$$f_r(0) = 0, \qquad f_r(m_r) - f_r(m_r - 1) = h_r(m_r), \qquad m_r \in \mathbb{Z},$$

where $\hat{e} = \max_{s \in R} e_s$ and where the positive constant $k$ satisfies $k > \hat{e}$. The Liapounov function $f$ will be defined on all of $F$ (see Remark 2.2) by $f(\mathbf{m}) = \hat{f}(\hat{m}) + f_r(m_r)$, where $k$ is taken sufficiently large. Note that the function $f$ does not depend on $\bar{m}_r$. We thus take the set $K$ of Theorem 2.1 to index the components $(\hat{m}, m_r)$ of $\mathbf{m}$. Conditions (2.13)–(2.16) of Theorem 2.1 are trivially satisfied, with, in (2.16), $F(a) = \{\mathbf{m} \in F\colon f(\mathbf{m}) \le a\}$ for all $a$. We therefore require only to verify condition (2.17) for a suitable constant $\bar{a}$, and with $D$ replaced by a set $D'$ such that $\mathbf{m}^N(\cdot) \in D'$ for all sufficiently large $N$—see Remark 2.3.

It is now easy to check that there exists a constant $c$, independent of $k$, such that

$$|d_{\mathbf{x}} f(\mathbf{m}) - \phi_{\mathbf{x}}(\mathbf{m})| \le \frac{c}{k} \quad \text{for all } \mathbf{m} \in D, \mathbf{x} \in \widehat{V}_{r,\varepsilon}, \tag{3.22}$$

where, for each $\mathbf{x} \in \widehat{V}_{r,\varepsilon}$, the function $\phi_{\mathbf{x}}$ on $D$ is given by

$$\begin{aligned}\phi_{\mathbf{x}}(\mathbf{m}) = &-\left[\sum_{s \in R} e_s(\kappa_s I_{\{\mathbf{m} \in \mathcal{A}_s\}} - \mu_s x_s)\right]\hat{h}(\hat{m}) \\ &- e_r(\kappa_r I_{\{\mathbf{m} \in \mathcal{A}_r\}} - \mu_r x_r) h_r(m_r).\end{aligned}$$

Now define $D_1 = \{\mathbf{m} \in D \setminus \mathcal{A}_r\colon m_r < e_r\}$, $D_2 = \{\mathbf{m} \in \mathcal{A}_r\colon \hat{m} \ge \hat{e} + k\}$ and $D_3 = \{\mathbf{m} \in D\colon m_r \ge e_r, \hat{m} < \hat{e} + k\}$. Note that $D_2 \subseteq \mathcal{A}_s$ for all $s \ne r$. For $\mathbf{m} \in D_1$, we have $h_r(m_r) = -1$ and so, for all such $\mathbf{m}$ and for all $\mathbf{x} \in \widehat{V}_{r,\varepsilon}$,

$$\phi_{\mathbf{x}}(\mathbf{m}) = -\left(\gamma_r(\mathbf{x}) - \sum_{s \ne r} e_s \kappa_s I_{\{\hat{m} < e_s\}}\right)\hat{h}(\hat{m}) - e_r \mu_r x_r\big(1 - \hat{h}(\hat{m})\big).$$

Similarly, for all $\mathbf{m} \in D_2$, we have $\hat{h}(\hat{m}) = 1$ and so, for all such $m$ and for all $\mathbf{x} \in \widehat{V}_{r,\varepsilon}$,

$$\phi_{\mathbf{x}}(\mathbf{m}) = -\alpha(\mathbf{x})\big(1 + h_r(m_r)\big) + \gamma_r(\mathbf{x}) h_r(m_r).$$

Since $\alpha(\mathbf{x}) \wedge \gamma_r(\mathbf{x}) > \varepsilon$ for $\mathbf{x} \in \widehat{V}_{r,\varepsilon}$, $-1 \le h_r(m_r) \le 0$ for all $m_r$, $0 \le \hat{h}(\hat{m}) \le 1$ for all $\hat{m}$ and $\hat{h}(\hat{m}) = 0$ whenever $\hat{m} < \hat{e}$, it follows from the above expressions for $\phi_{\mathbf{x}}(\mathbf{m})$ and from (3.22) that we may choose a sufficiently small $\varepsilon' > 0$ and $k$ sufficiently large so that $d_{\mathbf{x}} f(\mathbf{m}) \le -\varepsilon'$ for all $\mathbf{m} \in D_1 \cup D_2$, $\mathbf{x} \in \widehat{V}_{r,\varepsilon}$. Now define $D' = \{\mathbf{m} \in D: \bar{m}_r \le \hat{m} - k\}$. Observe that $\mathbf{m}^N(\cdot) \in D'$ for all sufficiently large $N$, that $D' \subset D_1 \cup D_2 \cup D_3$ and that the function $f$ is bounded on the set $D_3$. Condition (2.17) of Theorem 2.1 is now satisfied with $D$ replaced by $D'$ and $\bar{a}$ any constant satisfying $\bar{a} > \sup_{\mathbf{m} \in D_3} f(\mathbf{m})$. The conclusion (3.20) follows as previously described. □

REMARK 3.1. It is readily verified from (3.10), (3.11) and (3.13)–(3.15) that, for $\mathbf{x} \in U_r \setminus \widehat{U}_r$, $v_r(\mathbf{x}) = v_r^r(\mathbf{x})$ has opposite sign to $\gamma_r(\mathbf{x})$. It is therefore not entirely surprising that, for $\mathbf{x}$ belonging to the boundary set $\widehat{U}_r \cup \widehat{V}_r$, the threshold control associated with calls of type $r$ behaves as a "hard-constraint" if and only if $\gamma_r(\mathbf{x}) > 0$, that is, $\mathbf{x} \in \widehat{V}_r$. What is now interesting is that, from Theorem 3.1 and for $\mathbf{x} \in \widehat{V}_r$, the above threshold control behaves *exactly* as the classical hard-constraint control.

We are now in a position to characterize the behavior of the limiting dynamics. Define $\bar{\mathbf{x}} = (\bar{x}_r,\ r \in R) \in \mathbb{R}_+^R$ by $\bar{x}_r = \kappa_r/\mu_r$, $r \in R$. For each $r \in R$, define also $\mathbf{x}^r \in \mathbb{R}^R$ by $x_s^r = \bar{x}_s$ for $s \ne r$ and $\sum_{s \in R} e_s x_s^r = C$. Define also the function $g$ on $X$ by $g(\mathbf{x}) = \sum_{r \in R} g_r(x_r)$, where

$$g_r(x_r) = x_r \log \kappa_r - x_r \log \mu_r x_r + x_r, \qquad r \in R. \tag{3.23}$$

Note that each of the functions $g_r$, and so also the function $g$, is strictly concave on $X$. Let $\mathbf{x}^*$ maximize $g$ subject to the constraints

$$\sum_{r \in R} e_r x_r \le C, \qquad e_r x_r \le C_r, \qquad r \in R.$$

For the classical hard-constraint control strategy, in which the acceptance sets are the regions $\mathcal{A}_r^*$ considered earlier in this section, Kelly (1986) shows $\mathbf{x}^*$ to be the point on which is concentrated the limiting equilibrium distribution of the corresponding normalized dynamics $(\mathbf{x}^*)^N(\cdot)$. Zachary (2000), Theorem 2.2, further shows that all trajectories of these dynamics converge to $\mathbf{x}^*$.

Now define the *heavy-traffic condition* [see Bean, Gibbens and Zachary (1995)]

$$\sum_{r \in R} e_r \bar{x}_r > C, \tag{3.24}$$

or, equivalently, $\bar{\mathbf{x}} \notin X$.

THEOREM 3.2. *We have* $\lim_{t \to \infty} \mathbf{x}(t) = \hat{\mathbf{x}}$ *a.s., where* $\hat{\mathbf{x}}$ *is given as follows.*

(i) *If the heavy-traffic condition* (3.24) *does* not *hold, then* $\hat{\mathbf{x}} = \bar{\mathbf{x}}$.

(ii) *If condition* (3.24) *holds and additionally, for some* (*necessarily unique*) $r$,

$$\sum_{s \ne r} e_s \bar{x}_s + C_r \le C, \tag{3.25}$$

*then* $\hat{\mathbf{x}} = \mathbf{x}^r \in U_r$.

(iii) *If condition* (3.24) *holds and additionally*

$$\sum_{s\neq r} e_s\bar{x}_s + C_r > C \quad \textit{for all } r \in R, \tag{3.26}$$

*then* $\hat{\mathbf{x}} = \mathbf{x}^* \in V$.

PROOF. Let $\mathbf{v}$ again denote the velocity field for the limiting dynamics $\mathbf{x}(\cdot)$, as given by Theorem 3.1. Define $\overline{X} = \{\mathbf{x} \in X\colon\ x_r \le \bar{x}_r \text{ for all } r\}$. We assume until further notice that $\mathbf{x}(0) \in \overline{X}$ and so also, from (2.10), $\mathbf{x}(t) \in \overline{X}$ for all $t \ge 0$. Since, when the heavy-traffic condition (3.24) does *not* hold, $\overline{X} = \{\mathbf{x} \in \mathbb{R}_+^R\colon\ x_r \le \bar{x}_r \text{ for all } r\} \subset X \setminus Y$, the result (i) is here immediate from (3.16).

Now consider any $r \in R$ and $\mathbf{x} \in U_r \cap \overline{X}$; from (3.17), $\mathbf{v}(\mathbf{x}) = \mathbf{v}^r(\mathbf{x})$. Thus, if $\alpha_r(\mathbf{x}) \le 0$, then, from (3.13) and (3.15),

$$\begin{aligned} v_r(\mathbf{x}) &= -\frac{1}{e_r}\sum_{s\neq r} e_s(\kappa_s - \mu_s x_s) \\ &\le -\frac{1}{e_r}\mu_{\min}\sum_{s\neq r} e_s(\bar{x}_s - x_s) \\ &= \mu_{\min}(x_r^r - x_r), \end{aligned}$$

where $\mu_{\min} = \min_{s\in R}\mu_s > 0$ and where the last equation above follows since, from the definition of $\mathbf{x}^r$ and the assumption $\mathbf{x} \in U_r \subseteq B$, we have $\sum_{s\in R} e_s(x_s^r - x_s) = 0$ and $x_s^r = \bar{x}_s$ for $s \neq r$. If, alternatively, $\alpha_r(\mathbf{x}) > 0$, then, from (3.14), $v_r(\mathbf{x}) = -\mu_r x_r \le -\mu_r C_r/e_r$. We thus obtain

$$v_r(\mathbf{x}) \le \left(-\frac{\mu_r C_r}{e_r}\right) \vee \mu_{\min}(x_r^r - x_r), \qquad \mathbf{x} \in U_r \cap \overline{X},\ r \in R. \tag{3.27}$$

Now suppose that (3.24) does hold and continue to assume that $\mathbf{x}(0) \in \overline{X}$. Note that then, from (3.9), $B \cap \overline{X} \subseteq Y$. Hence, easily from (3.16), there exists $t_0$ such that

$$\mathbf{x}(t) \in B \cap \overline{X} \subseteq Y = V \cup \bigcup_{r\in R} U_r \quad \text{for all } t \ge t_0 \text{ a.s.} \tag{3.28}$$

Suppose first that the additional condition (3.25) also holds for some $r \in R$, which, straightforwardly from (3.12) and (3.24), is unique. Then, from the definition of $\mathbf{x}^r$ and (3.25), $e_r x_r^r \ge C_r$. It thus follows from the definition of $U_r$ and (3.28) that, if $\mathbf{x} \in B \cap \overline{X}$ and $x_r > x_r^r$, then $\mathbf{x} \in U_r$. Thus, from (3.27) and (3.28), $\limsup_{t\to\infty} x_r(t) \le x_r^r$. The result (ii) now follows by again using (3.28) and the definition of $\mathbf{x}^r$. Now suppose, alternatively, that condition (3.26) holds [in addition to (3.24)]. Then, for each $r \in R$, $e_r x_r^r < C_r$ and so, from (3.27) and the definition of $U_r$, $v_r(\mathbf{x})$ is negative and bounded away from 0 on the set $U_r$. It now follows from (3.28) that the process $\mathbf{x}(\cdot)$ eventually exits each set $U_r$ forever. Thus the result (3.28) may be strengthened to conclude that there exists $t_1$ such that, almost surely, $\mathbf{x}(t) \in V \cap \overline{X}$ for all $t \geq t_1$. The result (iii) now follows from (3.18), together with the result of Zachary (2000) referred to above that all trajectories determined by the velocity field $\mathbf{v}^*$ converge to $\mathbf{x}^*$.

Finally, when we do not necessarily assume $\mathbf{x}(0) \in \overline{X}$, observe that, since $v_r(\mathbf{x}) \leq \kappa_r - \mu_r x_r$ for all $\mathbf{x} \in X$, $r \in R$, we nevertheless have $\limsup_{t\to\infty} x_r(t) \leq \bar{x}_r$ for all $r \in R$. Thus entirely routine modifications are required to extend the above proof to this more general case. □

REMARK 3.2. Note also that, given $\mathbf{x}(0)$, the limiting dynamics $(\mathbf{x}(t),\ t \geq 0)$ are uniquely determined (almost surely). This follows straightforwardly from the uniqueness of the velocity field $\mathbf{v}$ and its continuity on each of the sets $X \setminus Y$, $U_r$, $\widehat{V}_r$ and $V \setminus \bigcup_{r\in R} \widehat{V}_r$—see also Zachary (1996).

We now describe limiting equilibrium behavior.

THEOREM 3.3. *The stationary distribution of* $\mathbf{x}^N(\cdot)$ *converges, as* $N \to \infty$, *to that concentrated on the single point* $\hat{\mathbf{x}}$, *while the stationary acceptance probability associated with each call type-$r$ converges to* $\mu_r \hat{x}_r/\kappa_r$.

PROOF. The first assertion of the theorem is immediate from Theorem 2.2 of Bean, Gibbens and Zachary (1997). That theorem also shows that the stationary free-capacity distribution is given by $\pi'_{\hat{\mathbf{x}}}$ (where this is as defined in Section 2). The second assertion is now immediate from (2.6) on taking $\mathbf{x}(0) = \hat{\mathbf{x}}$. □

We now use an extremal principle to consider, briefly and informally, the relaxation of condition (3.12) that no two of the reservation regions overlap. For each $r \in R$, define the concave function $\hat{g}_r$ on $\mathbb{R}_+$ by

$$\hat{g}_r(x_r) = \begin{cases} g_r(x_r), & \text{if } x_r \leq C_r/e_r \wedge \bar{x}_r, \\ g_r(C_r/e_r), & \text{if } C_r/e_r < x_r \leq \bar{x}_r, \\ g_r(x_r) - g_r(\bar{x}_r) + \hat{g}_r(\bar{x}_r), & \text{if } x_r > \bar{x}_r \end{cases}$$

[where $g_r$ is as given by (3.23)]. Define also the function $\hat{g}$ on $X$ by $\hat{g}(\mathbf{x}) = \sum_{r\in R} \hat{g}_r(x_r)$. Let $\hat{\mathbf{x}}$ be any fixed point of the velocity field $\mathbf{v}$ (in any convergent subsequence such that this exists). Suppose also that the heavy-traffic condition (3.24) holds—otherwise, it is again trivial that $\hat{\mathbf{x}} = \bar{\mathbf{x}}$. Clearly, $\hat{x}_r \leq \bar{x}_r$ for all $r \in R$. Then, as previously, it follows easily that $\hat{\mathbf{x}} \in Y$. Under the additional condition

$$\hat{x}_r \geq C_r/e_r \wedge \bar{x}_r \quad \text{for all } r \in R, \tag{3.29}$$

it is trivial that, for each $r$, $\hat{x}_r$ maximizes $\hat{g}_r$ over $\mathbb{R}_+$, and so $\hat{\mathbf{x}}$ maximizes $\hat{g}$ over $B$. Now assume, instead of (3.29), the alternative additional condition

$$\hat{x}_r < C_r/e_r \wedge \bar{x}_r \quad \text{for some } r \in R \tag{3.30}$$

[which, of course, implies (3.24)]. If also condition (3.12) *does* hold, then $\hat{\mathbf{x}} \in V$. [This follows since if, instead, $\hat{\mathbf{x}} \in U_r$ for some $r$, then, by Theorem 3.2, $\hat{\mathbf{x}} = \mathbf{x}^r$, implying (3.29).] Thus here, by Theorem 3.2, $\hat{\mathbf{x}} = \mathbf{x}^*$, where $\mathbf{x}^*$ is as defined earlier following (3.23). More generally, under (3.30) but when condition (3.12) does not necessarily hold, we again expect that $\hat{\mathbf{x}} = \mathbf{x}^*$. The informal explanation for this is that condition (3.30) implies that, in the network sequence and at points which converge to $\hat{\mathbf{x}}$ as $N \to \infty$, the proportion of calls of type $r$, and so of every call type, which are rejected due to the total capacity constraint [given by the first equation in (3.1)] tends to a nonzero limit. Thus, from consideration of the limiting regime and, in particular, the condition $\lim_{N\to\infty} t_r(N) = \infty$, we expect the threshold constraints to behave at $\hat{\mathbf{x}}$ as classical "hard constraints;" however, as previously mentioned, $\mathbf{x}^*$ is the unique fixed point of the velocity field defined by these hard constraints, implying $\hat{\mathbf{x}} = \mathbf{x}^*$ as required. Since also, under (3.30), $\hat{\mathbf{x}} \in Y \subseteq B$, it then follows that $\hat{\mathbf{x}}$ then maximizes the function $g$ over $\{\mathbf{x} \in B\colon\ e_r x_r \le C_r \text{ for all } r\}$ and so maximizes $\hat{g}$ over $B$. A rigorous proof of this result would require an appropriate generalization of Theorem 3.1.

We thus expect that, under the heavy-traffic condition (3.24), any fixed point of the limiting dynamics maximizes $\hat{g}$ over $B$. However, in the absence of condition (3.12), the function $\hat{g}$ is no longer necessarily strictly concave within the intersection of two or more of the reservation regions, and so the fixed point $\hat{\mathbf{x}}$ may not be uniquely determined. This is entirely to be expected: to obtain unique behavior here, we would have to be considerably more specific than (3.4) about the relative limiting behavior of the reservation parameters $t_r(N)$. See Hunt and Laws (1997) for a further discussion of these issues.

**4. Examples.** We give two examples for the virtual partitioning scheme described in the previous section. In each example we take $e_r = 1$ for all $r \in R$; the choice of the remaining parameters $C$, $C_r$, $\kappa_r$ and $\mu_r$, $r \in R$, is such that both the condition (3.12) and the heavy-traffic condition (3.24) are satisfied. It thus follows from Theorem 3.2 that (almost surely) the trajectories of the limiting dynamics $\mathbf{x}(\cdot)$ converge to the unique fixed point $\hat{\mathbf{x}} \in Y$ identified in the previous section. For each example we give a figure. The upper panel shows these trajectories for a set of starting points $\mathbf{x}(0)$ and identifies the fixed point $\hat{\mathbf{x}}$. The lower panel shows simulated trajectories for the corresponding (finite) network with the same call characteristics, capacity $C$, thresholds $C_r$ and arrival rates $\kappa_r$, $r \in R$. These simulations use the same set of starting points. The values of $C, C_r$ and $\kappa_r$ are chosen sufficiently large that, provided the reservation parameters $t_r$, $r \in R$ (the values of which are only relevant to the simulated trajectories for the finite network), are chosen not too close to 0, we expect reasonable agreement between the limiting trajectories and the simulated trajectories.

Example 1 has two types of calls with $C = 1500$, $\kappa_1 = 1500$, $\kappa_2 = 800$, $\mu_1 = 1$, $\mu_2 = 2$ and $C_r = 1000$, $t_r = 5$ for $r = 1, 2$. The fixed point $\hat{\mathbf{x}}$ is here given by $\hat{\mathbf{x}} = (1100, 400) \in U_1$. Figure 2 shows the corresponding limiting and

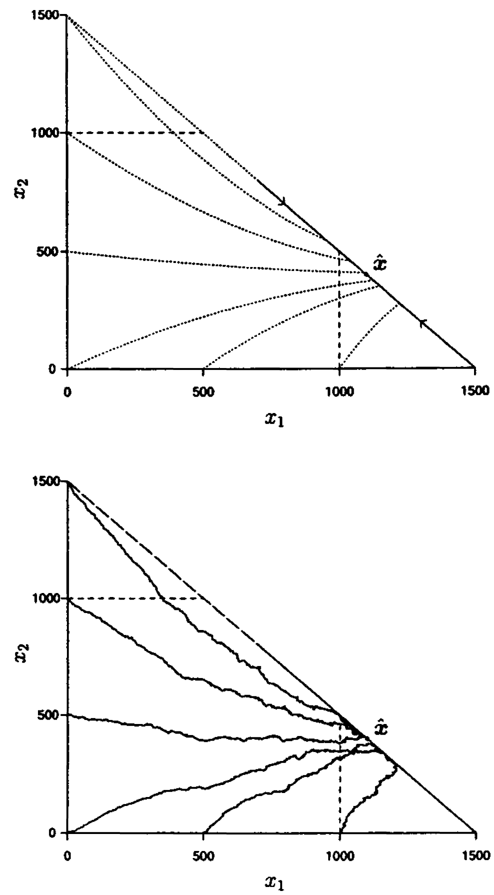


FIG. 2. *Limiting and simulated trajectories for Example* 1.

simulated trajectories in the space $X$—but not their time dependence. However, all trajectories in general move fairly rapidly to the fixed point (within about 2 time units in every case). In this example the set $B \setminus Y = \{\mathbf{x} \in B : x_2 \geq 800\}$ is nonempty and is indicated on the plot as the dotted region on the boundary $B$.

Example 2 has three types of calls with $C = 1500$, $\kappa_1 = 1500$, $\kappa_2 = 960$, $\kappa_3 = 120$, $\mu_1 = \mu_3 = 1$, $\mu_2 = 2$ and $C_r = 1000$, $t_r = 5$ for $r = 1, 2, 3$. The fixed point $\hat{\mathbf{x}}$ for this system is given by $\hat{\mathbf{x}} = (1000, 400, 100) \in \widehat{V}_1$. The limiting and simulated trajectories in Figure 3 are projected onto the boundary plane $B$, by scaling each point $\mathbf{x}$ by $1500/\sum_i x_i$. The figure also shows the boundaries of the reservation regions, corresponding to the thresholds $C_r$. Again, as with the previous example, the set $B \setminus Y = \{\mathbf{x} \in B : x_2 \geq 1080\}$ is nonempty, and limiting trajectories with $\mathbf{x}(0) \in B \setminus Y$ will initially leave the boundary set $B$ before eventually returning to it. On the other hand, it is easy to check that limiting trajectories with initial point $\mathbf{x}(0) \in Y$ never leave the set $Y$. The trajectory shown

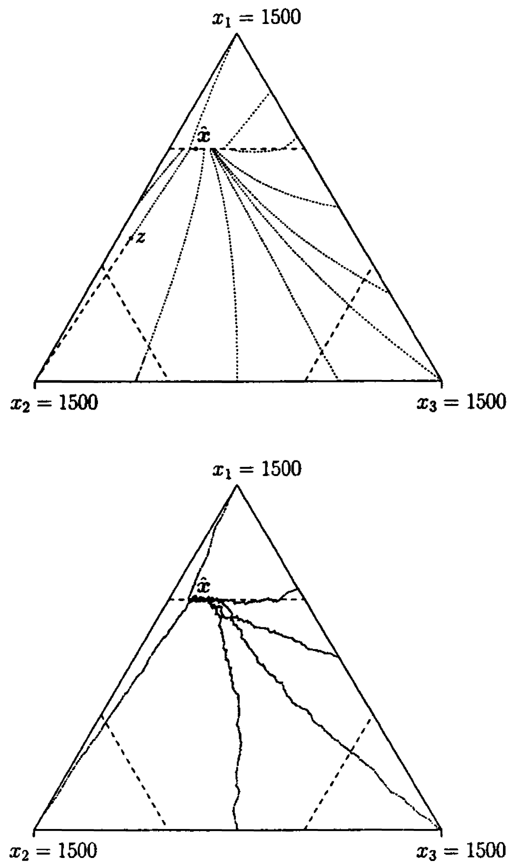


FIG. 3. *Limiting and simulated trajectories for Example* 2.

in the upper panel with the initial point $\mathbf{x}(0) = (0, 1500, 0) \in B \setminus Y$ returns to the set $B$ at the point $\mathbf{z} = (617.65, 832.94, 49.41)$ marked on the plot. Finally, note also that various other kinds of interesting behaviors are possible in this example. Thus the trajectory with initial point $\mathbf{x}(0) = (1050, 0, 450) \in U_1$ shown in the figure passes through the set $\widehat{V}_1$ at the point (1000.00, 78.00, 422.00) into $V \setminus \widehat{V}_1$ before converging to $\hat{\mathbf{x}} \in \widehat{V}_1$.

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## REFERENCES

ALANYALI, M. (1999). Three resource management policies for a resource with partition constraints. *Adv. Appl. Probab.* **31** 795–818.

BEAN, N. G., GIBBENS, R. J. and ZACHARY, S. (1995). Asymptotic analysis of large single resource loss systems under heavy traffic, with applications to integrated networks. *Adv. Appl. Probab.* **27** 273–292.

BEAN, N. G., GIBBENS, R. J. and ZACHARY, S. (1997). Dynamic and equilibrium behavior of controlled loss networks. *Ann. Appl. Probab.* **7** 873–885.

BORST, S. and MITRA, D. (1998). Virtual partitioning for robust resource sharing: computational techniques for heterogeneous traffic. *IEEE Journal on Selected Areas in Communications* **16** 668–678.

FAYOLLE, G., MALYSHEV, V. A. and MENSHIKOV, M. V. (1995). *Topics in the Constructive Theory of Countable Markov Chains.* Cambridge Univ. Press.

HUNT, P. J. (1995). Pathological behaviour in loss networks. *J. Appl. Probab.* **32** 519–533.

HUNT, P. J. and KURTZ, T. G. (1994). Large loss networks. *Stochastic Process. Appl.* **53** 363–378.

HUNT, P. J. and LAWS, C. N. (1997). Optimization via trunk reservation in single resource loss systems under heavy traffic. *Ann. Appl. Probab.* **7** 1058–1079.

KELLY, F. P. (1986). Blocking probabilities in large circuit-switched networks. *Adv. Appl. Probab.* **18** 473–505.

KELLY, F. P. (1991). Loss networks. *Ann. Appl. Probab.* **1** 319–378.

KEY, P. B. (1990). Optimal control and trunk reservation in loss networks. *Probab. Engrg. Inform. Sci.* **4** 203–242.

KUMARAN, K. and MITRA, D. (1998). Performance and fluid simulations of a novel shared buffer management system. In *Proceedings of IEEE INFOCOM 98* 1449–1461. IEEE, New York.

MITRA, D. and ZIEDINS, I. (1996). Virtual partitioning by dynamic priorities: a technique for fairly and efficiently sharing a resource between several services. In *Broadband Communications, Proc. 1996 International Zurich Seminar* (B. Plattner, ed.) 173–185. Springer, New York.

MITRA, D., REIMAN, M. I. and WANG, J. (1997). Robust admission control for heterogeneous ATM systems with both cell and call QoS requirements. In *Teletraffic Contributions for the Information Age* (V. Ramaswami and P. E. Wirth, eds.) 1421–1432. North-Holland, Amsterdam.

ROSS, K. W. (1995). *Multiservice Loss Models for Broadband Telecommunication Networks.* Springer, New York.

ZACHARY, S. (1996). The asymptotic behaviour of large loss networks. In *Stochastic Networks: Theory and Applications* (F. P. Kelly, S. Zachary and I. Ziedins, eds.) 193–203. Oxford University Press.

ZACHARY, S. (2000). Dynamics of large uncontrolled loss networks. *J. Appl. Probab.* **37** 685–695.

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