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SATURATION IN A MARKOVIAN PARKING PROCESS[1]

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We consider $\mathbb{Z}$ as an infinite lattice street where cars of integer length $m \geq 1$ can park. The parking process is described by a 0–1 interacting particle system such that a site $z \in \mathbb{Z}$ is in state 1 whenever a car has its rear end at z and 0 otherwise. Cars attempt to park after exponential times with parameter λ, leave after exponential times with parameter 1 and are not allowed to touch nor overlap. We define and study a jamming occupation density for this parking process, using the quasi-stationary distribution of a Markov chain related to the reversible measure of the particle system. An extension to a strip in $\mathbb{Z}^2$ is also investigated.

1. Introduction. What is usually known as (irreversible) sequential parking (sometimes called packing), can be informally described as a process in which objects (cars, molecules, adsorbates) of arbitrary sizes and shapes attempt to park sequentially, at random locations within a given region of $\mathbb{R}^d$, $d = 1, 2, 3$. Objects are supposed to be rigid, impenetrable and are only allowed to park in a nonoverlapping fashion. A parking attempt that violates this rule is simply rejected, otherwise it is successful and the object is parked forever at the chosen location. The process continues until some sort of saturation condition is attained, for instance, no more room to park. Random variables depending on the saturated configuration, such as the jamming density, are studied.

Random packing first arose as a model for the adsorption of molecules on crystal surfaces and in the theory of liquids, but applications have been found in such diverse areas as granulometry [14], elections in Japan [16], crystallization of polymer chains [22], condensation and coagulation [9]. More recently, interest in these models has been stimulated by their applications in communication networks; see [5, 8, 15, 28].

A. Rényi [27] was among the first to provide a mathematically rigorous treatment of a parking model on the line. According to Renyi's model, cars (segments) of length 1 make i.i.d. uniformly distributed attempts to park in the interval $[0, x]$, until there is no longer any space to park because all the gaps between the parked cars are of length less than 1. Conditioning on the location of the first parked car, Rényi obtained an integral equation which yields, using a Tauberian argument, the limiting value of the expected jamming density $c \sim 0.7476$, as $x \to \infty$. Asymptotics for higher moments and a central limit

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theorem for the jamming density were obtained in [23] and [13], exploiting the same conditioning argument. At about the same epoch, results for the lattice version of Rényi's model were published in [24], [11] and [20]. Also, a first attempt in [26] to analyze Rényi's model in dimensions 2 or higher yielded a conjecture only supported by simulations; the problem remains open but a number of heuristic arguments and massive simulations have been reported, both in physics and probability journals. The common feature of all these models is that cars, once parked, stay forever. However, it seems natural that cars, after being parked for a while, want to leave, thus vacating some space which can be used by other cars.

In this paper we consider a continuous time process on the lattice $\mathbb{Z}$ where cars leave as well as arrive and study a jamming density for it. Roughly speaking, at each point of the lattice $\mathbb{Z}$, cars of a fixed length try to park after exponential times with parameter λ; if a car parks (because it does not overlap with any other presently parked car), it stays there for an exponential time with parameter 1. Under this evolution scheme, our model behaves as an interacting particle system as described in [12] or [19]. Inspired by the sequential parking model, we say that a fixed segment is saturated when there is no space available for parking, but in our model this condition evolves with time and no segment will remain saturated forever. Nevertheless, at any given time $t > 0$, we will certainly find saturated segments of arbitrary length L, so we define the jamming density, as for the sequential scheme, computing the expected proportion of occupied space in these segments, with $L \to \infty$. In order to get a definition of the jamming density which is independent of the initial conditions, we consider the process in equilibrium. As we shall see, this leads us to the study of the quasi-stationary distribution of a finite Markov chain related to the reversible measure of the parking process.

Our model can be seen as a loss network. Following [18], a one-dimensional loss network can be described as a large number of stations placed along a line; a request for a call from station r to station s is accepted if there are no more than C calls already using any part of the segment $[r, s]$ (if a call is not accepted, it is lost). Times between arrivals of calls and times of completion of calls are exponentially distributed. Loss networks have many applications in computer, communication and information systems (see [2, 3, 18, 21] and references therein). In this setting, our jamming density is a measure of the efficiency of the network under saturation since it accounts for the proportion of used lines in a big collapsed area (i.e., where no more calls are accepted) and could be used to compare the performance of different network designs.

Another reversible parking scheme, with interesting applications in computer science, is first-fit storage allocation [6, 7], where cars park in $[0, \infty)$ occupying the leftmost available position. In addition to its dynamics, this model differs from ours in that the measure of efficiency, relating the total number of parked cars to the position of the rightmost one, is not taken under saturation.

The paper is organized as follows. In Section 2 we introduce the model and state the main results, considering first the process on $\mathbb{Z}$ and then, through

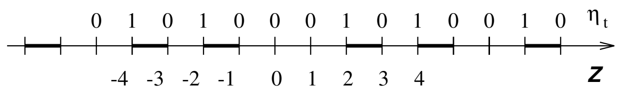

FIG. 1. *Unit cars parked on* $\mathbb{Z}$.

a limit argument, a continuous version of the model; we also study an extension to the two-dimensional strip $\mathbb{Z} \times \{0, 1, 2, 3\}$. In all these cases, we obtain explicit expressions of the jamming density. In Section 3, we present the proofs of the results and some discussion, leaving technical lemmas for the Appendix.

2. Definition of the model and main results. We describe a one-dimensional model on $S = \mathbb{Z}$ and cars of fixed length $m \in \mathbb{Z}_+$. A car is said to be parked at $x \in \mathbb{Z}$ whenever its left end is exactly at x, thus occupying the $m+1$ sites $x, \dots, x+m$. When a car is parked at x we give x the value 1 and $x+1, \dots, x+m$ the value 0; unoccupied sites are also given the value 0. Cars are neither allowed to overlap nor touch. Thus, the set of possible configurations is

$$\mathbb{X}_1 = \{\eta \in \{0, 1\}^{\mathbb{Z}} | \eta(x) + \cdots + \eta(x+m) \le 1, \ \forall \ x \in \mathbb{Z}\}. \tag{2.1}$$

See Figure 1. Parking attempts at each site take place after exponential times with parameter λ. Whenever such an attempt would result in the current state exiting $\mathbb{X}_1$, the car is rejected, otherwise it stays there for an exponential time with parameter 1, and then leaves. Attempts at each site are independent of attempts at other sites and of the past. Also, the exponential times that cars spend in the parking are independent, and independent of every other exponential time.

The above situation corresponds to an interacting particle system η_t, where particles or sites in $\mathbb{Z}$ can be in states 0 or 1, with rates of change $c(x, \eta)$ given by

$$c(x, \eta) = \begin{cases} \lambda, & \text{if } \eta(x-m) = \cdots = \eta(x) = \cdots = \eta(x+m) = 0, \\ 1, & \text{if } \eta(x) = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

This process is a continuous time strong Markov process, with state space $\mathbb{X}_1$ defined in (2.1). For the usual notation and definitions in interacting particle systems, the reader can consult [12] or [19]. Observe that when a car is allowed to park with its left end at x, only that site will change from 0 to 1 while neighboring sites from $x - m$ to $x + m$ will remain in state 0. This coding implies that sites change their states only one at a time.

As we shall see, this process is reversible with unique reversible measure ν shown in (3.5). We take ν, which is invariant for (η_t), as the starting measure. Kelly [17] has shown the relevance of this measure for one-dimensional loss networks since it characterizes the stationary measures when the network has a finite number of stations. Recall that a measure ν is said to be invariant for a process if the process starting with measure ν has distribution ν for all

$t > 0$; it is said to be reversible if the distribution of the process, starting with ν, does not change when time is reversed.

In Rényi's nonreversible parking model [27], the jamming density is defined as the limiting expected proportion of occupied space in a saturated segment of length L, with $L \to \infty$. A straightforward translation of this definition is not possible here, given the dynamic nature of our process. In order to define the jamming density, we consider the number $J_t(x)$ of cars parked at time t, starting from site x to the right, up to the first site $T_t(x)$ available for parking. That is,

$$J_t(x) = \sum_{y=x}^{T_t(x)} \eta_t(y),$$

where

$$T_t(x) = \min\left\{ y \geq x \,\middle|\, \sum_{|z-y| \leq m} \eta_t(z) = 0 \right\}$$

and $[x, T_t(x)]$ is the longest jammed interval beginning at x. See Figure 3. Next, we take the expected proportion of occupied space in $[x, T_t(x)]$, conditioned on the event $T_t(x) - x > L$, that is,

$$E_\nu\left(\frac{mJ_t(x)}{T_t(x) - x} \,\middle|\, L < T_t(x) - x < \infty \right), \tag{2.3}$$

where E_ν denotes the expectation when starting with measure ν, and finally, we study the asymptotic behavior of (2.3) as $L \to \infty$. Since ν is invariant for the process and, as we will see, translation invariant, the expression in (2.3) is independent of x and t; therefore, we fix $x = 0$, $t = 0$ and define

$$D_m(\lambda, L) = E_\nu\left(\frac{mJ_0(0)}{T_0(0)} \,\middle|\, L < T_0(0) < \infty \right). \tag{2.4}$$

THEOREM 2.1. *Let (η_t) be the parking process on $\mathbb{Z}$ with cars of length $m \geq 1$, starting with measure ν. Then*

$$\lim_{L\to\infty} D_m(\lambda, L) = \left(\frac{2m+1}{m} - \frac{(m+1)(1-a)}{m\rho^m(\rho-a)} + \frac{(1-a)(\rho^{m+1} - a^{m+1})}{m(\rho-a)^2\rho^{2m}} \right)^{-1}, \tag{2.5}$$

where a is the unique solution of $\lambda u^{m+1} = 1 - u$, $u \in (0,1)$ and ρ is the Perron eigenvalue (i.e., the eigenvalue of greatest modulus) of the $(m+(m+1)) \times (m + (m+1))$ matrix

$$Q = \begin{pmatrix} 0 & 1 & 0 & & & \cdots & 0 \\ 0 & 0 & 1 & 0 & & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & & \cdots & & 1 & \cdots & 0 \\ 1-a & & \cdots & & & a\ \cdots & 0 \\ \vdots & & & & & \ddots & \vdots \\ 1-a & & \cdots & & & \cdots & a \\ 1-a & & \cdots & & & \cdots & 0 \end{pmatrix}.$$

The asymptotic expected proportion (2.5) is called the jamming density of the parking process and is denoted by $D_m(\lambda)$.

Another useful interpretation of process (η_t) is that of unit cars parking on $\mathbb{Z}/m$. Since λm is the expected number of attempts to park in a unit length interval per unit time, λ should be replaced by $\lambda_m = c/m$ in order to get comparable parking processes for different m's.

In our next result we let $m \to \infty$. The parking process approaches the continuous scheme where cars are allowed to park at any real point.

THEOREM 2.2. *For $c > 0$,*

$$\lim_{m\to\infty} D_m\left(\frac{c}{m}\right) = \begin{cases} \left(2 - \dfrac{1-(1-k)e^k}{k(e^k-1)}\right)^{-1}, & \text{if } c \neq 1, \\ 2/3, & \text{if } c = 1, \end{cases} \tag{2.6}$$

where k is the unique nonzero solution of $ue^{2u} = c(e^u - 1)$.

REMARK. In Figure 2, the jamming density $D_m(c/m)$ is plotted against c for several values of m. Obviously, as m increases the parking tends to be more efficient since cars are allowed to park on $\mathbb{Z}/m$. It is also interesting to observe that the jamming density both in the discrete and in the continuous scheme increases with c. Furthermore, the limiting value in (2.5) for fixed m, as $c \to \infty$, is $m/m+1$ since $a \to 0$ and $\rho \to 1$, while the jamming density in (2.6) converges to 1 as $c \to \infty$ since $k \to \infty$. This shows that the space available for parking is optimized as the intensity increases.

We finally consider a two-dimensional parking model, where unit square cars can park on the infinite band $\mathbb{Z} \times \{0, 1, 2, 3\}$. As before, touching or overlapping is not allowed. Observe that our band has height 3 and that we do not study the simpler case of height 2 because it is clearly equivalent to the one-dimensional model.

In this model, unit square cars try to park setting their southwest vertices at $(x, y) \in \mathbb{Z}\times\{0, 1, 2\}$ after exponential times with parameter λ. If an attempt is successful, the car remains there for an exponential time with parameter 1. As in the one-dimensional case, sites corresponding to the southwest vertices of cars are given the value 1; the remaining sites are set to 0.

The two-dimensional parking process (η_t) can be seen as an interacting particle system on $S = \mathbb{Z} \times \{0, 1, 2\}$ with space state $\mathbb{X}_2 \subseteq \{0, 1\}^S$, given by

$$\mathbb{X}_2 = \left\{\eta \in \{0,1\}^S \,\middle|\, \sum_{\|z'-z\|_\infty \le 1} \eta(z') \le 1,\ \forall\, z \in S \text{ s.t. } \eta(z) = 1\right\}$$

and rates

$$c(z, \eta) = \begin{cases} \lambda, & \text{if } \displaystyle\sum_{\|z'-z\|_\infty \le 1} \eta(z') = 0, \\ 1, & \text{if } \eta(z) = 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\|\cdot\|_\infty$ stands for the supremum norm on $\mathbb{Z}^2$.

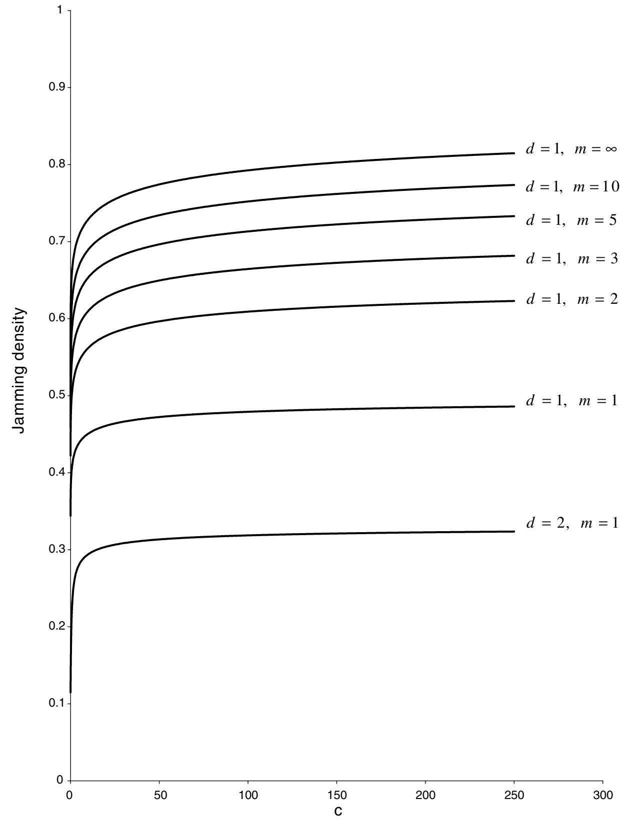

FIG. 2. *Jamming density as a function of the intensity c.*

We will show that this process has a unique reversible measure ν given in (3.13) and use it as the starting measure. The jamming density is defined as before, as the limit of the expected proportion of occupied space in a saturated segment when its length tends to ∞. That is,

$$T_t(x) = \min\left\{x' > x \,\middle|\, \exists\, y' \in \{0, 1, 2\} \text{ s.t. } \sum_{\|z''-z'\|_\infty \le 1} \eta(z'') = 0\right\}$$

and

$$J_t(x) = \sum_{\{z' : x' \in [x, T_t(x)]\}} \eta_t(z')$$

with $z' = (x', y')$.

For this process, we have the following result, similar to Theorem 2.1.

THEOREM 2.3. *Let (η_t) be the parking process on $\mathbb{Z} \times \{0, 1, 2, 3\}$ with unit square cars, starting with measure ν. Then*

$$\lim_{L\to\infty} E_\nu\left(\frac{J_0(0)}{3T_0(0)}\Big| L < T_0(0) < \infty\right) = \frac{v_1 w_1 + 2v_2 w_2}{3},$$

with $v = (v_j)$ and $w = (w_j)$, the respective left and right eigenvectors of

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ a_M & a_{BT} & 0 & a & 0 & 0 & 2a_B \\ a_M & a_{BT} & 0 & 0 & 0 & 0 & 0 \\ a_M & a_{BT} & 0 & 0 & 0 & 0 & a_B \\ 0 & 0 & 0 & 0 & b & 1-b & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-b & 0 \end{pmatrix},$$

corresponding to the Perron eigenvalue ρ, normalized to make $\sum v_i = \sum v_i w_i = 1$ and a_M, a_B, a_{BT}, a, b the unique constants satisfying

$$\begin{aligned} 2a_B + a_M + a_{BT} &= 1 - a, \\ \lambda a^2 &= a_M, \\ \lambda a^2 &= a_B b, \\ \lambda a &= 1 - b, \\ \lambda a_B b &= a_{BT}, \\ 0 < a_M, a_B, a_{BT}, &a, b < 1. \end{aligned} \tag{2.7}$$

REMARK. As observed for the parking process on $\mathbb{Z}$, increasing the intensity of arrivals yields a tighter parking of cars. See Figure 2.

3. Proofs of theorems. To prove Theorem 2.1, it is instructive to begin with $m = 1$. The reasoning in the general case is basically the same but the notation and some equations of the model get more involved.

PROOF OF THEOREM 2.1 FOR $m = 1$. In order to define the starting measure ν of the process, we consider a Markov chain $(X_n)_{n\ge 0}$ with state space $\{0, 1\}$ and transition matrix

$$P = \begin{pmatrix} a & 1-a \\ 1 & 0 \end{pmatrix},$$

where a is the unique solution of $\lambda u^2 = 1 - u$, $u \in (0, 1)$; that is, $a = (\sqrt{1+4\lambda} - 1)/2\lambda$. The stationary probability $\pi = (\pi(0), \pi(1))$ of the chain is readily obtained as $\pi(0) = 1/(2-a)$ and $\pi(1) = (1-a)/(2-a)$.

Let ν be the measure on $\{0, 1\}^{\mathbb{Z}}$ defined as

$$\begin{aligned} \nu\{\eta\colon \eta(x) = i_0,\ \eta(x+1) = i_1, \ldots, \eta(x+n) = i_n\} \\ = \pi(i_0)P(i_0, i_1)\cdots P(i_{n-1}, i_n), \end{aligned}$$

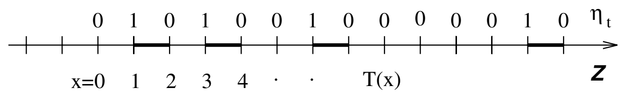

FIG. 3. *Jammed segment starting at* $x = 0$.

for $n \geq 0$ and $i_0, i_1, \ldots, i_n \in \{0, 1\}$. It can be easily shown that ν is a reversible measure for the particle system defined by (2.2) and is therefore invariant. See, for example, Chapter IV of [19]. In [31] Ycart introduces the measure ν and shows ergodicity of the process (η_t) in a problem dealing with philosophers eating Chinese food.

The stationary probability $\pi(1)$ can be seen as the car density in a very long segment of the parking, under measure ν. For instance, when the arrival rate is $\lambda = 2$, we get $a = 1/2$ and $\pi(1) = 1/3$. Observe that, given the restrictions of the parking process, the occupation density is bounded above by 0.5, which is the limiting value of $\pi(1)$ when $\lambda \to \infty$.

We turn our attention to the expectation in (2.3) which is independent of x and t since ν is translation invariant and invariant for the process. Furthermore, $D_m(\lambda, L)$ in (2.4) can be easily expressed in terms of the Markov chain (X_n), as

$$E_\pi\left(\frac{1}{T-1}\sum_{n=1}^{T} X_n \,\middle|\, T-1 > L\right),$$

where E_π denotes the expectation starting with measure π and T is the first time we see a run of three zeros (0,0,0), that is, $T = \min\{n \geq 1\colon X_{n-1} = X_n = X_{n+1} = 0\}$.

In order to study the limiting behavior of the above conditional expectation, we restate the problem in terms of the quasi-stationary distribution of the Markov chain $(Y_n)_{n\geq 1}$ where

$$Y_n = \begin{cases} (X_{n-1}, X_n, X_{n+1}), & \text{if } 1 \leq n \leq T, \\ (0, 0, 0), & \text{if } n > T. \end{cases}$$

The chain (Y_n) has transient states $\{001, 010, 100, 101\}$, absorbing state 000, and transition matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & a & 1-a & 0 \\ 1-a & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where xyz is a simplified notation for (x, y, z).

A simpler equivalent chain (Z_n) with states $\{001, 010, 100, 000\}$ is obtained by collapsing 001 and 101. Its transition matrix is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1-a & 0 & a & 0 \\ 1-a & 0 & 0 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.1}$$

The following result of Darroch and Seneta, on the behavior of a Markov chain conditionally on not being absorbed, will be used throughout the paper. Notation has been slightly modified. See (14) or (15) in [10].

Consider a Markov chain with states $\{1, \dots, s+1\}$ and transition matrix

$$\begin{pmatrix} Q & p_{s+1} \\ \mathbf{0}' & 1 \end{pmatrix},$$

where Q is an irreducible $s \times s$ matrix, and $p_{s+1} \neq \mathbf{0}$ and $\mathbf{0}$ are both $s \times 1$. Let μ be a probability measure on the set $\mathscr{T} = \{1, \dots, s\}$ of transient states and V_j the number of visits to state $j \in \mathscr{T}$. Let T be the absorption time in state $s+1$. Then, as $t \to \infty$,

$$E_\mu\left(\frac{V_j}{T}\middle| T > t\right) = w_j v_j + O(t^{-1}), \tag{3.2}$$

with $v = (v_j)$ and $w = (w_j)$, the left and right eigenvectors of Q corresponding to the Perron eigenvalue ρ, normalized to make $\sum_{i\in\mathscr{T}} v_i = \sum_{i\in\mathscr{T}} v_i w_i = 1$.

We return to the parking model, taking Q as the upper left 3×3 submatrix in (3.1). The characteristic polynomial of Q is given by $\phi(r) = r^3 - r(1-a) - a(1-a)$, and its normalized left and right Perron eigenvectors are, respectively, $v' = c[\rho^2, \rho, a]$ and $w' = d[\rho, \rho^2, 1-a]$, where $c(\rho^2+\rho+a) = cd(2\rho^3+a(1-a)) = 1$.

Finally, we calculate the jamming density using Darroch and Seneta's formula. Observe that the number of visits to state 001 of (Z_n) actually counts the number of parked cars, except possibly one at the beginning, since this state corresponds to $\{001, 101\}$ of (Y_n). Thus, (3.2) with $j = 001$ yields

$$D_1(\lambda) = \frac{\rho^3}{2\rho^3 + a(1-a)} = \frac{\rho + a}{2\rho + 3a}.$$

As ϕ has degree 3, its unique positive root (see Lemma A.3 of the Appendix) can be computed explicitly in terms of λ, yielding,

$$D_1(\lambda) = \frac{z+6}{2z+18}$$

with $z = \lambda^{1/3}((108 + 12\sqrt{81 - 12\lambda})^{1/3} + (108 - 12\sqrt{81 - 12\lambda})^{1/3})$. $\square$

We now consider the case $m > 1$. The idea of the proof is similar to the case $m = 1$, but now the reversible measure and the auxiliary Markov chains have a more complex state space.

PROOF OF THEOREM 2.1 FOR $m > 1$. As for unit cars, the reversible measure ν will be defined by a Markov chain $(X_n)_{n\geq 0}$. However, the elements of the state space are vectors with the values of m consecutive sites, for all configurations $\eta \in \mathbb{X}_1$. These will be denoted $x_0, x_1, \dots, x_m$, where $x_0 = (0, 0, \dots, 0)$ and $x_i = (0, \dots, \overset{i}{1}, \dots, 0)$, for $i = 1, \dots, m$. The increasing complexity of the state space for the reversible measure, as m grows, is quite intuitive because now we have to look at more places to determine whether a site is available for parking or not.

The transition matrix for (X_n) is given by

$$P = (P(x_i, x_j))_{i,j} = \begin{pmatrix} a & 0 & \cdots & 0 & 1-a \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \tag{3.3}$$

where a is the unique solution of

$$\lambda u^{m+1} = 1 - u, \qquad 0 < u < 1. \tag{3.4}$$

The fact that (3.4) has a unique solution follows easily taking $g(u) = \lambda u^{m+1} + u - 1$ and noting that $g(0) = -1$, $g(1) = \lambda > 0$ and $g'(u) = \lambda(m+1)u^m + 1 > 0$ for $u \in (0, 1)$.

As P is irreducible and aperiodic, the stationary distribution of (X_n) is easily calculated as

$$\pi(x_j) = \frac{1-a}{1+m(1-a)}, \qquad j \neq 0; \qquad \pi(x_0) = \frac{1}{1+m(1-a)},$$

and the measure ν on $\{0, 1\}^{\mathbb{Z}}$ is defined by

$$\begin{aligned} &\nu\{\eta\colon \eta(x) = i_0, \eta(x+1) = i_1, \dots, \eta(x+n+m-1) = i_{n+m-1}\} \\ &\qquad = \pi(\tilde{x}_{i_0}) P(\tilde{x}_{i_0}, \tilde{x}_{i_1}) \cdots P(\tilde{x}_{i_{n-1}}, \tilde{x}_{i_n}), \end{aligned} \tag{3.5}$$

for all $x \in \mathbb{Z}$, $n \geq 0$, $i_0, \dots, i_{n+m-1} \in \{0, 1\}$ and $\tilde{x}_{i_j} = (i_j, \dots, i_{j+m-1})$, for $j = 0, \dots, n$.

It can be directly checked that ν satisfies the following detailed balance equations:

$$\nu^n(\eta^n) c(x, \eta^n) = \nu^n(\eta^n_x) c(x, \eta^n_x)$$

for all $x \in \{-(n-1)m, \dots, (n-1)m\}$, $n \geq 1$, $\eta^n \in \{0, 1\}^{\{-nm, \dots, nm\}}$, where ν^n denotes the projection of ν on $\{0, 1\}^{\{-nm, \dots, nm\}}$ and

$$\eta_x(y) = \begin{cases} \eta(y), & \text{if } y \neq x, \\ 1 - \eta(x), & \text{if } y = x. \end{cases}$$

Equation (3.4) is obtained from the detailed balance equation with $\eta^n = 0$ and $x = 0$. We check this in the particular case $m = 3$ and for $n = 1$ (for $n > 1$, the resulting equations are identical). In this case $\eta^1 = (0, 0, 0, 0, 0, 0, 0)$, $\nu^1(\eta^1) = \pi((0, 0, 0))a^4$, $c(0, \eta^1) = \lambda$, $\eta^1_0 = (0, 0, 0, 1, 0, 0, 0)$, $\nu^1(\eta^1_0) = \pi((0, 0, 0))(1 - a)$

and $c(0, \eta_0^1) = 1$, which yields (3.4). Therefore, ν is reversible for the process (see Lemma 11.18 of [4]).

The uniqueness of ν as reversible measure for (η_t) follows from a slightly modified version of Theorems IV.2.13, IV.2.14 and IV.3.9 in [19], replacing the hypothesis of strictly positive rates by a weaker positivity condition, such as requiring that the rates of change to a specific state (in our case 0) are strictly positive, for all possible configurations. In fact, this latter condition implies that P in (3.3) is irreducible and aperiodic, while the strict positivity of all rates implies P strictly positive.

Now we study the jamming density for (η_t) starting with measure ν. As in the case $m = 1$,

$$D_m(\lambda, L) = m E_\pi\left(\frac{1}{T-m} \sum_{n=m}^{T} \mathbb{1}_{\{X_n = x_1\}} \,\middle|\, T - m > L\right),$$

where $T = \min\{n \geq m\colon X_{n-m} = \cdots = X_n = X_{n+1} = x_0\}$ is the first time we see a run of $m+2$ x_0's and the factor m takes into account the length of a car.

We introduce the associated Markov chain $(Y_n)_{n \geq m}$, where Y_n is defined as the $2m+1$ vector whose ith component is 1 if $X_{n-m-1+i} = x_1$ and 0 otherwise, for $i = 1, \ldots, 2m+1$. The states of (Y_n) have at most 2 ones among the $2m+1$ coordinates and the rest are zero. We code the states of Y_n as follows: $y_0 = (0, \ldots, 0)$, $y_i = (0, \ldots, \overset{i}{1}, \ldots, 0)$, for $1 \leq i \leq 2m+1$ and $y_{i,j} = (0, \ldots \overset{i}{1}, \ldots, 0, \overset{j}{1}, \ldots, 0)$, for $1 \leq i, j \leq 2m+1, j - i > m$. The cardinal of the state space of (Y_n) is $m(m+1)/2 + 2m + 2$.

We make the state y_0 absorbing because it corresponds to a situation where a new car can park. The nonzero entries of the transition matrix of chain (Y_n) are

$$\begin{aligned}
P(y_i, y_{i-1}) &= a && \text{if } 1 \leq i \leq m+1,\\
P(y_i, y_{i-1}) &= 1 && \text{if } m+1 < i \leq 2m+1,\\
P(y_i, y_{i-1,2m+1}) &= 1 - a && \text{if } 2 \leq i \leq m+1,\\
P(y_1, y_{2m+1}) &= 1 - a, &&\\
P(y_{i,j}, y_{i-1,j-1}) &= 1 && \text{if } i > 1,\\
P(y_{1,j}, y_{j-1} &= 1, &&\\
P(y_0, y_0) &= 1. &&
\end{aligned}$$

In order to reduce the cardinality of the state space of (Y_n), we collapse some of the states. Consider the application Φ acting as $\Phi(y_0) = 0$, $\Phi(y_i) = 2m + 2 - i$, for $i > 0$, and $\Phi(y_{i,j}) = 2m + 2 - j$. The process (Z_n) with $Z_n = \Phi(Y_n)$ is a Markov chain with state space $\{0, 1, \ldots, 2m+1\}$ and transition matrix with nonzero entries given by

$$\begin{aligned} P(i, i+1) &= 1 && \text{if } 1 \le i \le m, \\ P(i, i+1) &= a && \text{if } m < i < 2m+1, \\ P(i, 1) &= 1-a && \text{if } m < i \le 2m+1, \\ P(2m+1, 0) &= a, \\ P(0,0) &= 1. \end{aligned}$$

The jamming density can be defined using (Z_n) as

$$D_m(\lambda) = m \lim_{L\to\infty} E_\mu\left(\frac{1}{T}\sum_{n=0}^{T} \mathbb{1}_{\{Z_n=1\}} \middle| T > L\right),$$

where T is the absorption time of (Z_n) and μ is any measure on the transient states. From (3.2) we get $D_m(\lambda) = m v_1 w_1$, with $v = (v_j)$ and $w = (w_j)$ the respective left and right eigenvectors of Q [the transition submatrix of transient states of (Z_n)] corresponding to the Perron eigenvalue ρ, normalized to make $\sum v_i = \sum v_i w_i = 1$.

For $m > 1$, there is no hope of solving the eigenvector problem explicitly in order to write the value of $D_m(\lambda)$ in terms of ρ (as we did for $m = 1$). Instead we use the following result on nonnegative irreducible matrices (Corollary 2, page 8 of [29]), which states that

$$wv' = \frac{\text{Adj }(I\rho - Q)}{\phi'(\rho)} \tag{3.6}$$

where v and w are defined above, Adj is the adjoint matrix, I is the identity and ϕ' is the derivative of the characteristic polynomial of Q. Note that we are interested in the element (1,1) of wv'.

The submatrix Q is the $(2m+1) \times (2m+1)$ matrix given by

$$Q = \begin{pmatrix} 0 & 1 & 0 & & & & \cdots & 0 \\ 0 & 0 & 1 & 0 & & & \cdots & 0 \\ \vdots & & & \ddots & & & & \vdots \\ 0 & & \cdots & & 1 & & \cdots & 0 \\ 1-a & & \cdots & & & a & \cdots & 0 \\ \vdots & & & & & & \ddots & \vdots \\ 1-a & & \cdots & & & & \cdots & a \\ 1-a & & \cdots & & & & \cdots & 0 \end{pmatrix}. \tag{3.7}$$

The element (1,1) of $\text{Adj}(I\rho - Q)$ is ρ^{2m}, since the matrix obtained by deleting row 1 and column 1 is triangular. Also,

$$\phi(x) = x^{2m+1} - (1-a)\frac{x^{m+1} - a^{m+1}}{x - a} \tag{3.8}$$

for $x \neq a$ and $\phi(a) = a^{2m+1} - (m+1)(1-a)a^m$ (see Lemma A.1 of the Appendix). Taking the derivative of ϕ, (3.6) yields

$$D_m(\lambda) = mv_1w_1 = \left(\frac{2m+1}{m} - \frac{(m+1)(1-a)}{m\rho^m(\rho - a)} + \frac{(1-a)(\rho^{m+1} - a^{m+1})}{m(\rho - a)^2\rho^{2m}}\right)^{-1}. \tag{3.9}$$

□

PROOF OF THEOREM 2.2. Before taking the limit in (3.9), note that a [the solution of $\lambda u^{m+1} = 1-u$, $u \in (0, 1)$], as well as ρ [the Perron eigenvalue of the $(2m+1) \times (2m+1)$ matrix Q] depend on m. In order to avoid confusion, from now on a and ρ will be subscripted by m. Also, as pointed out in Section 2, we take $\lambda = c/m$, where c is interpreted as the number of attempts to park per unit time in a segment of length 1. Next, we obtain asymptotic expressions for a_m and ρ_m as $m \to \infty$.

Given that a_m is the unique root in (0,1) of $g_m(x) = cx^{m+1} + mx - m$, a reasonable approximate for a_m is $\sqrt[m+1]{\alpha}$ with $\alpha \in (0, 1)$ such that

$$c\alpha + \log \alpha = 0,$$

since

$$g(\sqrt[m+1]{\alpha}) = c\alpha + m\left(e^{1/(m+1)\log\alpha} - 1\right) \sim c\alpha + \log \alpha.$$

In fact, $a_m = \sqrt[m+1]{\alpha} + o(1/m)$, as shown in Lemma A.2.

We turn our attention to ρ_m, the Perron eigenvalue of matrix Q. In Lemma A.3 we show that ϕ, the characteristic polynomial of Q given in (3.8), has a unique positive root, which is necessarily ρ_m. Furthermore, in Lemma 4.4 we prove the following asymptotic formula for ρ_m:

$$\rho_m = a_m + k/m + o(1/m),$$

as $m \to \infty$, where k is the unique nonzero solution of

$$ue^{2u} = c(e^u - 1), \tag{3.10}$$

if $c \neq 1$ and $k = 0$ otherwise.

Note that (3.10) is obtained from

$$\phi(a_m + k/m) = a_m^{2m+1}\left(1 + \frac{k}{ma_m}\right)^{2m+1} - \frac{ca_m^{2m+2}}{m}\left(\frac{\left(1 + \frac{k}{ma_m}\right)^{m+1} - 1}{\frac{k}{m}}\right)$$
$$\sim \alpha^2\left(e^{2k} - \frac{c}{k}(e^k - 1)\right).$$

From the above, $\lim_{m\to\infty} D_m(c/m) = \lim_{m\to\infty} m\rho_m^{2m}/\phi'(\rho_m)$ is easily computed since

$$\rho_m^{2m} = a_m^{2m}\left(1 + \frac{k}{ma_m} + o\left(\frac{1}{m}\right)\right)^{2m} \to \alpha^2 e^{2k},$$

$$\left(\frac{\rho_m}{a_m}\right)^{m+1} = \left(1 + \frac{k}{ma_m} + o\left(\frac{1}{m}\right)\right)^{m+1} \to e^k$$

and

$$\begin{aligned}\frac{\phi'(\rho_m)}{m} &= \frac{(2m+1)\rho_m^{2m}}{m} - \frac{ca_m^{2m+2}}{m}\left(\frac{(m+1)\frac{\rho_m^m}{a_m^{m+1}}\left(\frac{k}{m}+o\left(\frac{1}{m}\right)\right) - \left(\frac{\rho_m^{m+1}}{a_m^{m+1}}-1\right)}{m\left(\frac{k}{m}+o\left(\frac{1}{m}\right)\right)^2}\right)\\ &\to 2\alpha^2 e^{2k} - \frac{c\alpha^2}{k^2}(ke^k - e^k + 1).\end{aligned}$$

Finally,

$$\lim_{m\to\infty} D_m\left(\frac{c}{m}\right) = \left(2 - \frac{1-(1-k)e^k}{k(e^k-1)}\right)^{-1}. \qquad \square$$

PROOF OF THEOREM 2.3. In order to study the parking process (η_t) with $S = \mathbb{Z}\times\{0, 1, 2\}$ we begin by defining an equivalent particle system $(\tilde{\eta}_t)$ with $S = \mathbb{Z}$, where the states of each site are $\{0, B, T, BT, M\}$. The process $(\tilde{\eta}_t)$ is constructed from (η_t) as follows:

$$\begin{aligned}
\tilde{\eta}(x) = 0 &\iff \eta(x,0)=0,\ \eta(x,1)=0,\ \eta(x,2)=0,\\
\tilde{\eta}(x) = B &\iff \eta(x,0)=1,\ \eta(x,1)=0,\ \eta(x,2)=0,\\
\tilde{\eta}(x) = T &\iff \eta(x,0)=0,\ \eta(x,1)=0,\ \eta(x,2)=1,\\
\tilde{\eta}(x) = BT &\iff \eta(x,0)=1,\ \eta(x,1)=0,\ \eta(x,2)=1,\\
\tilde{\eta}(x) = M &\iff \eta(x,0)=0,\ \eta(x,1)=1,\ \eta(x,2)=0.
\end{aligned}$$

In words, the process $(\tilde{\eta}_t)$ carries the number and position of cars parked at $x\times\{0, 1, 2\}$, for each integer x. Symbols B, M and T stand for bottom, middle and top, respectively, while BT corresponds to the situation where two cars are parked at the same x coordinate.

Consider the following nonnegative irreducible matrix (with rows and columns corresponding to 0, B, T, BT and M):

$$F = \begin{pmatrix} 1 & \lambda & \lambda & \lambda^2 & \lambda\\ 1 & 0 & \lambda & 0 & 0\\ 1 & \lambda & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\end{pmatrix}$$

and the stochastic matrix P with elements

$$P(i, j) = F(i, j)\frac{v_j}{\Lambda v_i}, \tag{3.11}$$

where Λ is the Perron eigenvalue of F and v the corresponding right eigenvector.

An inspection of F reveals that $v_B = v_T$ and, therefore, $P(0, B) = P(0, T)$ and $P(B, T) = P(T, B)$. In what follows, we write P as

$$P = \begin{pmatrix} a & a_B & a_B & a_{BT} & a_M \\ b & 0 & 1-b & 0 & 0 \\ b & 1-b & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.12}$$

with $2a_B + a_{BT} + a_M = 1 - a$ and $0 < a, a_B, a_M, a_{BT} < 1$. Matrix P is irreducible and aperiodic and, therefore, the Markov chain $(X_n)_{n\geq 0}$ with P as its transition matrix has a unique stationary measure π.

Define measure ν on $\{0, B, T, BT, M\}^{\mathbb{Z}}$ as

$$\begin{aligned} &\nu\{\eta : \eta(x) = i_0, \eta(x+1) = i_1, \dots \eta(x+n) = i_n\} \\ &\quad = \pi(i_0)P(i_0, i_1)\cdots P(i_{n-1}, i_n) \end{aligned} \tag{3.13}$$

for $n \geq 0$ and $i_0, i_1, \dots, i_n \in \{0, B, T, BT, M\}$.

As in the one-dimensional case, the reversibility of ν follows from Lemma 11.18 in [4], which is stated for spin systems (i.e., when sites have two values and only one particle changes its value at each transition), but can be easily extended to the case where the sites have a finite number of values. That is, ν is reversible for the process if and only if the following detailed balance equations are satisfied:

$$\nu^n(\tilde{\eta}^n)c_{ij}(x, \tilde{\eta}^n) = \nu^n(\tilde{\eta}^n_{xij})c_{ij}(x, \tilde{\eta}^n_{xij}) \tag{3.14}$$

for all $i \neq j$, $x \in \{-n+1, \dots, n-1\}$, $n \geq 1$ and $\tilde{\eta}^n \in \{0, B, T, BT, M\}^{\{-n,\dots,n\}}$. Here $c_{ij}(x, \eta)$ is defined as the rate of change of the process $\tilde{\eta}_t$ at site x from value i to j and vice versa,

$$\tilde{\eta}_{xij}(y) = \begin{cases} \tilde{\eta}(y), & \text{if } y \neq x, \\ j, & \text{if } y = x \text{ and } \tilde{\eta}(x) = i, \\ i, & \text{if } y = x \text{ and } \tilde{\eta}(x) = j, \\ \tilde{\eta}(x), & \text{otherwise} \end{cases}$$

and ν^n is the projection of ν on $\{0, B, T, BT, M\}^{\{-n,\dots,n\}}$.

In order to check that (3.14) holds if and only if a, a_B, a_M, a_{BT} and b satisfy (2.7), we must consider the detailed balance equations for all the configurations. The table below shows the configurations (for $n = 1$ and $x = 0$) and their corresponding detailed balance equations, which are exactly those of (2.7).

$\tilde{\eta}^1$	$\tilde{\eta}^1_{0ij}$	Detailed balance equation
$(0, 0, 0)$	$(0, M, 0)$	$\lambda a^2 = a_M$
$(0, 0, 0)$	$(0, B, 0)$	$\lambda a^2 = a_B b$
$(0, 0, B)$	$(0, T, B)$	$\lambda a = 1 - b$
$(0, B, 0)$	$(0, BT, 0)$	$\lambda a_B b = a_{BT}$

The detailed balance equations for the remaining configurations are a direct consequence of those in the table above.

On the other hand, the uniqueness of the reversible measure ν follows from an extension of the argument given for the one-dimensional case. Here, the potential has five states instead of two and the positivity assumptions on the rates can be reformulated as the existence of a value (in our case 0) such that a site can change from every other value to that specific value in a finite number of transitions (note that our positivity condition implies condition ω_0 of [30] for the reversible measure). The uniqueness of the reversible measure implies the uniqueness of a, a_B, a_M, a_{BT}, b as stated in Theorem 2.3.

Consider now the process $(\tilde{\eta}_t)$ starting from its reversible measure ν and define the Markov chain $(Y_n)_{n\geq 1}$ as

$$Y_n = \begin{cases} (X_{n-1}, X_n, X_{n+1}), & \text{if } n < T, \\ (0,0,0), & \text{if } n \geq T, \end{cases}$$

where $T = \min\{n \geq 1\colon (X_{n-1}, X_n, X_{n+1}) \in A\}$ and $A = \{(0,0,0), (0,0,B), (0,B,0), (B,0,0), (B,0,B), (0,0,T), (0,T,0), (T,0,0), (T,0,T)\}$, the set of states where a new car can park. The chain (Y_n) has 27 states and, as before, we collapse some states to define a new chain (Z_n). Let $Z_n = \Phi(Y_n)$, where Φ is defined by

$$\begin{aligned}
&\Phi(x,0,M) = 1, && x \in \{0, B, T, BT, M\},\\
&\Phi(x,0,BT) = 2, && x \in \{0, B, T, BT, M\},\\
&\Phi(0,x,0) = 3, && x \in \{BT, M\},\\
&\Phi(x,0,0) = 4, && x \in \{BT, M\},\\
&\Phi(x,y,0) = 5, && x, y \in \{B, T\},\ x \neq y,\\
&\Phi(x,y,z) = 6, && x \in \{0, B, T\},\ y,\ z \in \{B, T\},\ x \neq y,\ y \neq z,\\
&\Phi(x,0,y) = 7, && x \in \{B, T, BT, M\},\ y \in \{B, T\},\ x \neq y,\\
&\Phi(x,y,z) = 8, && \text{otherwise.}
\end{aligned}$$

It follows that the transition matrix of (Z_n) is given by

$$\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
a_M & a_{BT} & 0 & a & 0 & 0 & 2a_B & 0\\
a_M & a_{BT} & 0 & 0 & 0 & 0 & 0 & 1 - a_{BT} - a_M\\
a_M & a_{BT} & 0 & 0 & 0 & 0 & a_B & 1 - a_{BT} - a_M - a_B\\
0 & 0 & 0 & 0 & b & 1-b & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1-b & 0 & b\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

The jamming density can be written as

$$\lim_{L\to\infty} E_\mu\biggl(\frac{1}{3T}\sum_{n=0}^{T} V_n \biggm| T > L\biggr),$$

where μ is any probability measure on the transient states and $V_n = \mathbb{1}_{\{Z_n=1\}} + 2\,\mathbb{1}_{\{Z_n=2\}}$. Observe that V_n actually counts the number of parked cars while 3 in the denominator takes into account the height of the band.

Therefore, from (3.2), the jamming density is given by

$$\frac{v_1 w_1 + 2 v_2 w_2}{3},$$

with $v = (v_j)$ and $w = (w_j)$, the respective left and right eigenvectors of Q (the above matrix with the last row and column deleted) corresponding to the Perron eigenvalue ρ, normalized to make $\sum v_i = \sum v_i w_i = 1$. $\square$

REMARK. Unlike the one-dimensional case, the asymptotic behavior of Rényi's model in a band of height 3 with unit square cars is an open problem. The main difficulty is related to the uselessness of the conditioning argument since the first parked car does not break the band into two independent parking processes. See [1].

CONCLUDING REMARKS. We believe that an interesting feature of the model presented in this paper is the use of the theory of quasi-stationary distributions in the study of the jamming density. We can therefore expect to apply results from this theory (see, e.g., [25] and references therein) to obtain, for instance, convergence of higher order moments and central limit theorems. Furthermore, following this approach, other measures of asymptotic efficiency could be defined by considering different functions of the saturated configuration and different absorption conditions.

Our methods could be easily extended to handle more complex situations such as cars with bounded random lengths or one-dimensional loss networks with capacity $C > 1$. Also, the parking process on $\mathbb{Z} \times \{0, 1, \ldots, k\}$, with $k \geq 4$, can be analyzed following the general strategy of Theorem 2.3 although the number of states of the process $(\tilde{\eta}_t)$ grows exponentially with k; notice, however, that the problem remains essentially one-dimensional and its extension to a full two-dimensional lattice would require different methods.

APPENDIX

LEMMA A.1. *The characteristic polynomial of the matrix* Q *defined in* (3.7) *is*

$$\phi(x) = \begin{cases} x^{2m+1} - (1-a)\dfrac{x^{m+1} - a^{m+1}}{x - a}, & \text{if } x \neq a, \\ a^{2m+1} - (m+1)(1-a)a^m, & \text{if } x = a. \end{cases} \tag{A.1}$$

PROOF. We compute

$$|xI - Q| = \begin{vmatrix} x & -1 & 0 & & & & \cdots & 0 \\ 0 & x & -1 & 0 & & & \cdots & 0 \\ \vdots & & \ddots & \ddots & & & & \vdots \\ 0 & & \cdots & x & -1 & & \cdots & 0 \\ -1+a & & & \cdots & x & -a & \cdots & 0 \\ \vdots & & & & & \ddots & \ddots & \vdots \\ -1+a & & \cdots & & & \cdots & x & -a \\ -1+a & & \cdots & & & & \cdots & x \end{vmatrix},$$

by successively expanding the determinant about its first row to get $|xI - Q| = x^{2m+1} - (1-a)\Delta_{m+1}$, where

$$\Delta_{m+1} = \begin{vmatrix} 1 & -a & & \cdots & 0 \\ 1 & x & -a & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 1 & & \cdots & x & -a \\ 1 & & & \cdots & x \end{vmatrix}.$$

Expanding about the first row, we obtain the recursive equation $\Delta_{m+1} = x^m + a\Delta_m$, for $m \geq 1$ and $\Delta_1 = 1$, which yields

$$\phi(x) = x^{2m+1} - (1-a)\left(x^m + ax^{m-1} + a^2x^{m-2} + \cdots + a^m\right). \qquad \square$$

LEMMA A.2. *Let a_m be the unique solution of*

$$cu^{m+1} = m(1-u), \qquad 0 < u < 1,$$

then $a_m = \sqrt[m+1]{\alpha} + o(\frac{1}{m})$ as $m \to \infty$, where α is the unique solution of

$$cu + \log u = 0, \qquad 0 < u < 1.$$

PROOF. Let $g_m(x) = cx^{m+1} + mx - m$. Then, $f_m(x) = g_m(\sqrt[m+1]{x}) \to \varphi(x) = cx + \log x$, as $m \to \infty$, for all $x > 0$. Since f_m and φ are continuous and increasing, they have unique roots in $(0, 1)$, a_m^{m+1} and α, respectively.

Given $\varepsilon > 0$, let $\alpha_1 < \alpha < \alpha_2$ such that $\log(\frac{\alpha_2}{\alpha_1}) < \varepsilon/2$. Since $f_m(\alpha_1) \to \varphi(\alpha_1) < 0$ and $f_m(\alpha_2) \to \varphi(\alpha_2) > 0$ then, for all $m \geq m_0$, $f_m(\alpha_1) < 0$, $f_m(\alpha_2) > 0$ and, therefore, $\alpha_1 < a_m^{m+1} < \alpha_2$. Also, since $\sqrt[m+1]{\alpha_1} < \sqrt[m+1]{\alpha} < \sqrt[m+1]{\alpha_2}$, we get

$$\begin{aligned} m\left|a_m - \sqrt[m+1]{\alpha}\right| &\leq m\left(\sqrt[m+1]{\alpha_2} - \sqrt[m+1]{\alpha_1}\right) \\ &\leq m\left(\sqrt[m+1]{\frac{\alpha_2}{\alpha_1}} - 1\right) \\ &\leq \frac{m}{m+1} 2 \log\left(\frac{\alpha_2}{\alpha_1}\right) < \varepsilon, \end{aligned}$$

for all $m \geq m_0$, where the third inequality follows from $e^x - 1 < 2x$ for small positive x. $\square$

LEMMA A.3. *There is a unique positive root of $\phi(x)$ defined in* (A.1).

PROOF. Since $\phi(x)$ is the characteristic polynomial of a nonnegative matrix, the Perron–Frobenius theorem (see, e.g., Theorem 1.1, page 3 of [29]) guarantees the existence of at least one positive root. For the uniqueness, we rewrite $\phi(x) = x^{2m+1} - (1-a)(x^m + ax^{m-1} + a^2x^{m-2} + \cdots + a^m)$ and note that ϕ and its first m derivatives $\phi^{(k)}$, $k \le m$, are negative at zero. Suppose that ϕ has more than one positive root, then, as ϕ is negative and decreasing at 0, it has at least a local minimum and a local maximum and, therefore, its derivative has at least two positive roots. Iterating this argument, we find that the first m derivatives of ϕ have at least two positive roots, but this is a contradiction since $\phi^{(m)}(x) = (2m+1)\cdots(m+2)x^{m+1} - (1-a)m!$ has only one positive root. □

LEMMA A.4. *Let ρ_m be the unique positive root of*

$$\phi_m(x) = \begin{cases} x^{2m+1} - (1-a_m)\dfrac{x^{m+1} - a_m^{m+1}}{x - a_m}, & \textit{if } x \neq a_m, \\ a_m^{2m+1} - (m+1)(1-a_m)a_m^m, & \textit{if } x = a_m, \end{cases}$$

where a_m is the unique solution of

$$cu^{m+1} = m(1-u), \qquad 0 < u < 1.$$

Then, $\rho_m = a_m + k/m + o(1/m)$ as $m \to \infty$ where k is the unique nonzero solution of

$$ue^{2u} = c(e^u - 1), \tag{A.2}$$

if $c \neq 1$ and $k = 0$, otherwise.

PROOF. We first check that (A.2) has a unique solution. Let $h(x) = xe^{2x} / (e^x - 1)$ for $x \neq 0$ and $h(0) = 1$. It is easy to see that h is continuous and strictly increasing, with $h(-\infty) = 0$ and $h(+\infty) = +\infty$. Hence, $h(x) = c$ has a unique solution k, for $c > 0$.

Define $f_m(x) = \phi_m(a_m + \frac{x}{m})$. Then, Lemma A.2 yields $f_m(x) \to f(x)$ as $m \to \infty$, for all $x \in \mathbb{R}$, where

$$f(x) = \begin{cases} \alpha^2\Big(e^{2x} - \dfrac{c(e^x - 1)}{x}\Big), & \text{if } x \neq 0, \\ \alpha^2(1-c), & \text{if } x = 0. \end{cases}$$

We verify that $f'(k) > 0$ considering two cases depending on the value of c. If $c = 1$ then $k = 0$ and $f'(0) = 3\alpha^2/2 > 0$. Otherwise, if $c \neq 1$ then $k \neq 0$ and $f'(k) = \alpha^2(2e^{2k} - c\frac{ke^k - e^k + 1}{k^2})$. Substituting $ke^{2k}/(e^k - 1)$ for c and simplifying, we get $f'(k) > 0$.

Since $f'(k) > 0$, there exists $\delta > 0$ such that $f(x) < 0$ for $x \in [k - \delta, k)$ and $f(x) > 0$ for $x \in (k, k + \delta]$. Let $\varepsilon > 0$ ($\varepsilon < \delta$) and $k_1 < k < k_2$ with $k_2 - k_1 < \varepsilon$. Given that $f_m(x) \to f(x)$ and $f(k_1) < f(k) < f(k_2)$, we conclude

$f_m(k_1) < 0 < f_m(k_2)$ for $m \geq m_0$. Also, since $f_m(x) = 0$ has the unique solution $m(\rho_m - a_m)$ (recall that ρ_m is the unique root of $\phi_m(x)$), then

$$k_1 < m(\rho_m - a_m) < k_2 \quad \Longrightarrow \quad m\left|\rho_m - \left(a_m + \frac{k}{m}\right)\right| < \varepsilon. \qquad \square$$

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