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ON OCCUPATION TIME FUNCTIONALS FOR DIFFUSION PROCESSES AND BIRTH-AND-DEATH PROCESSES ON GRAPHS

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Occupation time functionals for a diffusion process or a birth-and-death process on the edges of a graph Γ depending only on the values of the process on a part $\Gamma' \subset \Gamma$ of Γ are closely related to so-called eigenvalue depending boundary conditions for the resolvent of the process. Under the assumption that the connected components of $\Gamma \setminus \Gamma'$ are trees, we use the special structure of these boundary conditions to give a procedure that replaces each of the trees by only one edge and that associates this edge with a speed measure such that the respective functional for the appearing process on the simplified graph coincides with the original one.

1. Introduction. Diffusion processes on graphs have been considered first by G. Lumer (see [23] and the references therein). Such processes appear as models, for example, of electrical networks, vibrating elastic nets, nerve impulse propagation or movement of nutrients in the root system of a plant (see [6, 11]). Results on diffusion processes on graphs can also be found in [2, 9, 25, 28]. Investigations of the asymptotic behavior of randomly perturbed dynamical systems lead to such processes as well [7, 8, 9, 10].

A diffusion process on a graph is determined by the following:

1. A (maybe generalized) second-order differential operator for each edge of the graph governing the diffusion process inside the edge until it reaches a vertex of the graph;
2. A gluing condition at each interior vertex of the graph (we will assume that any interior vertex is accessible) and
3. A boundary condition at each regular exterior vertex.

The gluing and boundary conditions determine the domain of the infinitesimal generator A of the process. The infinitesimal generator is locally given by the differential operators. These characteristics describe the diffusion process uniquely.

The processes are generalized diffusion processes which include diffusions, gap diffusions as well as birth-and-death processes. The operators are given by scale functions s and speed measures m, and the state space of the corresponding process consists, roughly speaking, of the points of the graph corresponding to the points of the support of the respective speed measure.

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The spectral theory of generalized second-order derivatives was developed in the early 1950s by M. G. Krein (as the spectral theory of vibrating strings) in order to generalize the theory of the classical Stieltjes moment problem and the Sturm–Liouville boundary problem [13, 14, 19, 20]. In these papers a correspondence is established between the measure m, that is, the mass distribution of the string, and the Titchmarsh–Weyl coefficient of the problem. The Titchmarsh–Weyl coefficient has a representation,

$$Q(z) = a + \int_0^\infty \frac{1}{\lambda - z} d\sigma(\lambda), \tag{1.1}$$

where $a \in [0, \infty]$ and σ is the spectral function of the string.

Starting with W. Feller's work [4, 5], an extensive literature was developed on one-dimensional Markov processes governed by differential operators $D_m D_s^+$, (see, e.g., [3, 12, 21, 22, 24]).

It became clear that there are also useful applications of Krein's theory to problems for one-dimensional Markov processes, and there has been a lot of progress in this direction [16, 17, 18, 22]. For instance, the transition density of the diffusion process governed by $D_m D_s^+$ can be written as

$$p(t, x, y) = \int_0^\infty e^{-tz} \phi(x, z)\, \phi(y, z)\, d\sigma(z), \tag{1.2}$$

with the solution ϕ of $zfdm + d(d/ds)f = 0$ satisfying the initial conditions $\phi(0, z) = 1$, $(d/ds)\phi(0-, z) = 0$ [see (3.1)].

In this paper we make use of Krein's theory to get results for diffusion processes on graphs. Let X_t, $t \geq 0$, be a diffusion process on a graph Γ, and let

$$h(y) = E_y \int_0^\infty e^{-\lambda t} f(X_t)\, dt, \qquad y \in \Gamma. \tag{1.3}$$

Here E_y denotes the expectation with respect to the initial point $y \in \Gamma$. Let us consider functions f which are equal to zero outside a subgraph $\Gamma' \subset \Gamma$. If, for example, f is the indicator function of Γ' then $h(y)$ is the Laplace transform of the probability that $X_t \in \Gamma'$ (depending on the initial point y). Such questions are closely related to projections of the operator A to certain subspaces of functions on Γ, and it is known that they correspond to problems with eigenvalue depending boundary conditions ([15]). Actually, if $I_k \subset \Gamma \backslash \Gamma'$ is an edge that divides Γ into two subgraphs, one containing Γ' and another that we denote by $\widetilde{\Gamma}$, then h satisfies a (boundary) condition in any point $y_0 \in I_k$ that can be (formally) written as

$$\widetilde{Q}(y_0; -\lambda)(D_s^+ h)(y_0) + h(y_0) = 0, \tag{1.4}$$

with a certain function $\widetilde{Q}$ that is independent of f.

We will show that if $\widetilde{\Gamma}$ forms a tree, $\widetilde{Q}(y_0; \cdot)$ is a Titchmarsh–Weyl coefficient. Moreover, $\widetilde{Q}$ can be calculated explicitly from the Titchmarsh–Weyl coefficients corresponding to the edges of $\widetilde{\Gamma}$ and from the Titchmarsh–Weyl coefficient corresponding to the part of the edge I_k between y_0 and $\widetilde{\Gamma}$.

Now let O_j be the vertex that connects I_k and $\widetilde{\Gamma}$, and let $y_0 = O_j$.

If $\widetilde{\Gamma}$ consists of only one edge, then $\widetilde{Q}$ is the Titchmarsh–Weyl coefficient corresponding to the diffusion on that edge (up to a constant factor depending on the gluing condition at O_j).

If $\widetilde{\Gamma}$ consists of more than one edge, then $\widetilde{Q}$ is again a Titchmarsh–Weyl coefficient, and thus, $\widetilde{\Gamma}$ can be replaced by only one edge connected to O_j and equipped with a diffusion according to $\widetilde{Q}$ and a modified gluing condition at O_j, without any change of the corresponding function $h(y)$ for $y \in \Gamma'$. This means, if we modify the graph Γ in this way and define $\tilde{h}$ by (1.3) with the process X_t replaced by the process on the modified graph and f replaced by the function $\tilde{f}$ on the modified graph, where $\tilde{f}$ is equal to f on Γ' and equal to zero outside Γ', then

$$h(y) = \tilde{h}(y) \quad \text{for all } y \in \Gamma'.$$

Thus, if the process is of interest only during the time it spends on a certain part of the graph (for example, if one considers sojourn times), then the graph can be replaced by a simpler one. This can be used to derive formulas for sojourn-time distributions. If Γ' consists of only one edge, then the results for one-dimensional diffusions can be used.

If X_t is a birth-and-death process with a finite number of states, then the speed measures corresponding to edges of treelike parts of $\Gamma \setminus \Gamma'$ are concentrated in a finite number of points. The corresponding Titchmarsh–Weyl coefficients as well as the respective $\widetilde{Q}$ are rational functions. Thus, $\widetilde{Q}$ corresponds to a birth-and-death process on the simplified part of the graph. All parameters of this process, as mean waiting time in any state, and one-step transition probabilities of the embedded Markov chain can be easily obtained from a continued fraction representation of $\widetilde{Q}$ (see [27]). In Section 6 we give the respective results and present some explicitly calculated examples.

If X_t is a diffusion process, then the speed measures corresponding to the edges of the graph can be approximated by measures concentrated in a finite number of points. This is equivalent to an approximation of the diffusion process by a birth-and-death process (see [16]). To these birth-and-death processes the results of Section 6 can be applied to get an approximation of the diffusion process on the simplified graph. This can be useful to simplify simulations as well as to get approximate solutions of sojourn-time problems. In Section 7 we present some explicitly calculated examples illustrating that.

In Sections 2, 3 and 4 we give the necessary definitions and the results from the theory of strings. In Section 5 we present the procedure that replaces treelike components of $\Gamma \setminus \Gamma'$ by only one edge as described above.

2. Preliminaries. Let M^+ denote the set of triples $\mathbf{m} = (m, l, \bar{l})$ where m is a Borel measure on $\mathbb{R}$ such that

$$m((-\infty, 0)) = 0,$$

$$0 \le l = \sup \operatorname{supp} m \le \infty,$$

$$m([0, x)) < \infty \quad \text{for all } x < l,$$

$$l \le \bar{l} \le \infty, \qquad \bar{l} = l \text{ if } m([0, l]) = \infty, \quad m(\{l\}) = 0 \text{ if } l = \bar{l}.$$

Here supp m denotes the support of the measure m.

By M_r^+ we denote the subset of all "regular" elements of M^+, that is,

$$(m, l, \bar{l}) \in M_r^+ \text{ if and only if } l < \infty \quad \text{and} \quad m([0, l]) < \infty.$$

For $0 \le \bar{l} \le \infty$ let $R_{\bar{l}} = (-\infty, \bar{l})$, and let $S(\bar{l})$ be the set of all continuous, strictly increasing functions s on $R_{\bar{l}}$ with $s(0) = 0$ (the scale functions). For $s \in S(\bar{l})$ the derivatives D_s^+ and D_s^- are defined by

$$(D_s^{\pm} f)(x) = \lim_{\varepsilon \to \pm 0} \frac{f(x+\varepsilon) - f(x)}{s(x+\varepsilon) - s(x)}, \quad x \in R_{\bar{l}}, \ f \in C(-\infty, \bar{l}),$$

if the limit exists. If $\mathbf{m} = (m, l, \bar{l}) \in M^+$ and $s \in S(\bar{l})$ let $s(\bar{l}) = \lim_{x \to \bar{l}} s(x)$, $s(\bar{l}) \in [0, \infty]$. According to Feller's classification (see [4, 24]) the (boundary) point l may be regular, exit, entrance or natural. The point $a_0 = \inf \operatorname{supp} m$ is assumed to be always regular.

Let $\mathbf{m} = (m, l, \bar{l}) \in M^+$ and $s \in S(\bar{l})$. By $C_{m,s}$ we denote the complete subspace (with sup-norm) of the Banach space $C(R_{\bar{l}})$ with the property that the elements $f \in C_{m,s}$ are linear in s on the components of $(0, \bar{l}) \setminus \operatorname{supp} m$ [i.e., $f \in C_{m,s}$ has a representation $f(x) = a + bs(x)$, $a, b \in \mathbb{R}$, on each such component], and that $f(x) \to 0$ as $x \to \bar{l}$ if l is not entrance or regular with $s(\bar{l}) = \infty$.

We use the conventions $\infty \cdot 0 = 0$, and $1/\infty = 0$. On $C_{m,s}$ we define operators A^{π_0}, $\pi_0 \in [0, \infty]$, as follows: a function $f \in C_{m,s}$ belongs to the set $\vartheta_{m,s}$ if it has a representation

$$f(x) = \alpha + \beta s(x) + \int_{0-}^{x} (s(x) - s(y)) g(y) m(dy), \tag{2.1}$$

with $\alpha, \beta \in \mathbb{R}$, $x \in R_{\bar{l}}$, and a function $g \in C_{m,s}$. If relation (2.1) holds for two functions $f, g \in C_{m,s}$ then we write $g = D_m D_s^+ f$. We call the operator $D_m D_s^+$ a generalized second-order derivative (see [3, 24]). A function $f \in \vartheta_{m,s}$ belongs to the domain of the operator A^{π_0} if $\Phi_0(f) = 0$, where the (boundary) functional Φ_0 is defined as follows:

$$\Phi_0(f) = \begin{cases} \pi_0 (D_s^- f)(0) - f(0), & \text{if } \pi_0 < \infty, \\ (D_s^- f)(0), & \text{if } \pi_0 = \infty. \end{cases} \tag{2.2}$$

Then with (2.1) $A^{\pi_0} f = g$. If $l < \infty$ is regular then $\Phi_l(f) = 0$ with

$$\Phi_l(f) = \begin{cases} \pi_l (D_s^+ f)(l) + f(l), & \text{if } \bar{l} < \infty, \\ (D_s^+ f)(l), & \text{if } \bar{l} = \infty, \end{cases} \tag{2.3}$$

where $\pi_l = s(\bar{l}) - s(l)$. If $l = \bar{l}$ we denote $D_s^+ f(l) = D_s^- f(l)$.

It is well known that for $\pi_0 \in [0, \infty]$ the operator A^{π_0} is the infinitesimal generator of a strongly continuous contraction semigroup associated with a

Hunt process with right continuous trajectories (see [3, 24]). This process is a generalized diffusion process with state space

$$(\operatorname{supp} m \cap [0, l)) \cup \{l\colon l \text{ is entrance or regular with } \bar{l} > l\}.$$

The process has the speed measure m, the scale function s (and local boundary conditions in regular boundaries). If m is discrete, the process is a birth-and-death process, if m is absolutely continuous with positive density on $(0, l)$, and s is, for example, two times differentiable, the process is an ordinary diffusion on $(0, l)$. For a detailed explanation of behavior at the boundary see [24]. If the operator L is given in the form

$$L = \frac{1}{2}\, a\, \frac{d^2}{dx^2} + b\, \frac{d}{dx},$$

with appropriate functions a and b on $[0, l]$ then $L = D_m D_s^+$ with suitable s and m; see [24].

3. Preliminaries from the theory of strings. We give some results from the theory of strings developed by M. G. Krein (see, e.g., [13, 14]).

In the notion of vibrating strings each triple $(m, l, \bar{l}) \in M^+$ [with scale $s(x) = x$] corresponds to a string of length l with mass distribution m. If $m \in M_r^+$, then the right end of the string is fixed to $\bar{l}$ if $\bar{l} < \infty$, or it is free if $\bar{l} = \infty$. The left end of the string is always supposed to be free. If s is not the identity, then the same holds up to rescaling. If the measure m is discrete there exists also an interpretation in terms of connected masses an springs (see [27]).

Let $(m, l, \bar{l}) \in M^+$ and $s \in S(\bar{l})$. If $l = \infty$ the notations $(x, l]$ and $m(\{l\})$ should be understood as (x, ∞) and 0, respectively. For $\mathbf{m} \in M^+$, $x \in \mathbb{R}$, and $z \in \mathbb{C}$ let the functions ϕ and ψ be defined by (see [14])

$$\phi(x, z) = 1 - z \int_{0-}^{x} (s(x) - s(y))\, \phi(y, z) m(dy), \tag{3.1}$$

$$\psi(x, z) = s(x) - z \int_{0-}^{x} (s(x) - s(y))\, \psi(y, z) m(dy). \tag{3.2}$$

These functions are the unique solutions of the equation

$$D_m D_s^+ u + zu = 0, \tag{3.3}$$

satisfying the initial conditions

$$\phi(0, z) = (D_s^- \psi)(0, z) = 1, \quad (D_s^- \phi)(0, z) = \psi(0, z) = 0.$$

Define the Titchmarsh-Weyl coefficient corresponding to $\mathbf{m} \in M^+$ by

$$Q(z) = \lim_{x \to \bar{l}} \frac{\psi(x, z)}{\phi(x, z)}, \quad z \in \mathbb{C} \setminus [0, \infty). \tag{3.4}$$

The function Q admits a representation

$$Q(z) = a_0 + \int_0^{\infty} \frac{1}{\lambda - z}\, d\sigma(\lambda) \tag{3.5}$$

with some $a_0 \in [0, \infty]$ and a nonnegative Borel measure σ on $[0, \infty)$ [the spectral measure of problem (3.3)] such that $\int_0^\infty (1+\lambda)^{-1}\, d\sigma(\lambda) < \infty$. Note that $a_0 = \infty$, that is, $Q \equiv \infty$, corresponds to $\mathbf{m} = (0, 0, \infty)$.

If $(m, l, \bar{l}) \in M_r^+$ then the measure σ is discrete, and the representation (3.5) becomes

$$Q(z) = a_0 + \sum_{i=1}^{\infty} \frac{\sigma_i}{\lambda_i - z}. \tag{3.6}$$

The square roots of the eigenvalues λ_i give the natural frequencies of the string.

A function Q with representation (3.5) is called an S-function and is characterized by the following properties:

$$\begin{gathered} Q(z) \text{ is holomorphic on } \mathbb{C} \setminus [0, \infty), \\ \operatorname{Im}(z)\operatorname{Im}(Q(z)) \geq 0 \text{ for every } z \in \mathbb{C} \setminus \mathbb{R}, \\ Q(z) \geq 0 \text{ for every } z \in (-\infty, 0) \text{ or } Q \equiv \infty. \end{gathered} \tag{3.7}$$

It is a basic result of M.G. Krein [14] that there is a bijective (up to the choice of the scale s) correspondence between M^+ and the class of S-functions. We denote this correspondence by

$$\mathbf{m} \longleftrightarrow_s Q.$$

If $s(x) = x$, $0 \leq x \leq \bar{l}$, we write $\mathbf{m} \longleftrightarrow_x Q$.

Let $\mathbf{m} = (m, l, \bar{l}) \in M^+$, $s \in S(\bar{l})$ and $\mathbf{m} \longleftrightarrow_s Q$.

We introduce a further solution φ of equation (3.3),

$$\varphi(x, z) = \phi(x, z) - \frac{1}{Q(z)} \psi(x, z), \tag{3.8}$$

$x \in \mathbb{R}$, $z \in \mathbb{C} \setminus (0, \infty)$. This function has the following properties:

$$d(D_s^+ \varphi) + z\varphi\, dm = 0, \tag{3.9}$$

$$\varphi(0, z) = 1, \tag{3.10}$$

$$(s(\bar{l}) - s(l))(D_s^+ \varphi)(l, z) + \varphi(l, z) = 0 \quad \text{if } \mathbf{m} \in M_r^+ \text{ and } \bar{l} < \infty, \tag{3.11}$$

$$(D_s^+ \varphi)(l, z) = 0 \quad \text{if } \mathbf{m} \in M_r^+ \text{ and } \bar{l} = \infty, \tag{3.12}$$

$$(D_s^+ \varphi)(0, z) = -z\, m(\{0\}) - \frac{1}{Q(z)}, \tag{3.13}$$

$$(D_s^- \varphi)(0, z) = -\frac{1}{Q(z)}, \tag{3.14}$$

$$\varphi(\cdot, z) \in \vartheta_{m,s}. \tag{3.15}$$

Let $\mathbf{m} = (m, l, \bar{l}) \in M^+$ and $s \in S(\bar{l})$. We define the measure $\dot{m} \in M^+$ by

$$\dot{m} = m - m(\{0\})\delta_0.$$

We denote by $\dot{A}^0$ and $\dot{A}^\infty$ the infinitesimal generators corresponding to $(\dot{m}, l, \bar{l})$ and s. We note that $\dot{A}^0 = A^0$. In terms of diffusion or birth-and-death processes a superscript 0 or ∞ implies that the process will be killed or reflected, respectively, at the left end of its state space. The resolvents of these generators are

$$\dot{R}_z^i = (z - \dot{A}^i)^{-1}, \quad i = 0, \infty, \; z \in \mathbb{C} \setminus (-\infty, 0]. \tag{3.16}$$

As the resolvents have integral representations with continuous bounded kernels (see, e.g., [17, 22]), these operators can be extended to the space B_m of all bounded m-integrable functions. We note that

$$(R_z^0 f)(0) = 0, \quad (D_s^+ \dot{R}_z^\infty f)(0) = 0, \quad f \in B_m. \tag{3.17}$$

For $f \in B_m$, $f \neq 0$, we consider the function $\dot{\varphi} \in \vartheta_{\dot{m}, s}$

$$\dot{\varphi}(\cdot, -z) = \frac{\dot{R}_z^\infty f - R_z^0 f}{(\dot{R}_z^\infty f)(0)}, \quad z \in \mathbb{C} \setminus (-\infty, 0]. \tag{3.18}$$

This function satisfies the (homogeneous) equation

$$d(D_s^+ \dot{\varphi})(\cdot, z) + z\, \dot{\varphi}(\cdot, z)\; d\dot{m} = 0,$$

and $\dot{\varphi}(0, z) = 1$. These properties determine the function $\dot{\varphi}$ uniquely, and it follows according to (3.13),

$$-\frac{1}{(D_s^+ \dot{\varphi})(0, z)} = \dot{Q}(z), \tag{3.19}$$

where $\dot{Q} \longleftrightarrow_s, (\dot{m}, l, \bar{l})$. Using (3.17), (3.18) and (3.19) we get

$$\begin{aligned} Q(z) &= \frac{\dot{Q}(z)}{\dot{Q}(z)(-m(\{0\}))z + 1} = \frac{1}{-m(\{0\})\; z + 1/\dot{Q}(z)} \\ &= \left(- m(\{0\})\; z + \frac{(D_s^+ R_{-z}^0 f)(0)}{(\dot{R}_{-z}^\infty f)(0)} \right)^{-1}. \end{aligned} \tag{3.20}$$

We will need the following lemma.

Lemma 3.1. *Let* $\mathbf{m}_i = (m_i, l_i, \bar{l}_i) \in M^+$, $s_i \in S(\bar{l}_i)$, $\mathbf{m}_i \longleftrightarrow_{s_i} Q_i$, $\alpha_i > 0$, $i = 1, \ldots, n$, $n \geq 2$ *and* Q *defined by*

$$\frac{1}{Q(z)} = \sum_{i=1}^{n} \frac{\alpha_i}{Q_i(z)}, \quad z \in \mathbb{C} \setminus [0, \infty). \tag{3.21}$$

Then Q *is an* S*-function, and there exists* $\mathbf{m} \in M^+$ *such that* $\mathbf{m} \longleftrightarrow_x Q$.

For the proof, it is easy to check that Q defined by (3.21) satisfies conditions (3.7).

4. Generalized diffusion processes on graphs. Consider a (connected) graph Γ with a finite number of edges I_i, $i = 1, \dots, n$, of finite or infinite length and vertices $O_i, i = 1, \dots, n_v$, $O = \{O_i, i = 1, \dots, n_v\}$, $n, n_v \in N$. Let be given a set of intervals $I_i^* = [0, l_i]$, $i = 1, \dots, n$, such that each I_i is parameterized by its arc length with parameter set I_i^*. In other words, there exist maps P_i: $I_i^* \to \Gamma$ such that

$$P_i((0, l_i)) = I_i, \quad P_i(0) \in O \text{ and } P_i(l_i) \in O \qquad \text{if } l_i < \infty$$

and the length of the arc $P_i(0) \to P_i(x)$ is equal to x, $x \in (0, l_i)$, $i = 1, \dots, n$.

By O_e we denote the set of (finite) end points of the graph. We define $I_\infty = \{j | j \in \{0, \dots, n\}, l_j = \infty\}$ (the indices of edges of infinite length),

$$J_i^0 = \{j|\, j \in \{1, \dots, n\}, P_j(0) = O_i\},$$

$$J_i^l = \{j|\, j \in \{1, \dots, n\},\ P_j(l_j) = O_i\}, i = 1, \dots, n_v.$$

If $j \in J_i^0 \cup J_i^l$, we write $I_j \sim O_i$. For $i = 1, \dots, n$ let $\mathbf{m}_i = (m_i, l_i, \bar{l}_i) \in M^+$ and $s_i \in S(\bar{l}_i)$. We assume that $I_j \sim O_i \in O \setminus O_e$ and $j \in J_i^l$ imply that the point l_j, $j \in \{1, \dots, n\}$, is (m_j, s_j)-regular and $l_j < \bar{l}_j$. Consequently, inner vertices O_i of Γ correspond to regular points with $l_j < \bar{l}_j$ of the edges which meet at O_i. Further we assume that $O_i \in O_e$ implies $O_i = P_j(l_j)$ for a $j \in \{1, \dots, n\}$.

By C_m we denote the set

$$C_m = \{f = (f_1, f_2, \dots, f_n) \text{ such that } f_i \in C_{m_i, s_i},\ D_{s_i}^- f_i(0) = 0$$
$$\text{and } P_i(x) = P_j(y) \text{ implies } f_i(x) = f_j(y),\ x \in I_i^*,\ y \in I_j^*\}.$$

By C_Γ we denote the subset of $C(\Gamma)$, consisting of those elements $f_\Gamma \in C(\Gamma)$ that satisfy

$$f_i(x) = f_\Gamma(P_i(x)) \quad \text{for all } x \in I_i^*, i = 1, \dots, n, \tag{4.1}$$

for some $f = (f_1, \dots, f_n) \in C_m$. The relation (4.1) gives a one-to-one correspondence P_Γ between the functions $f_\Gamma \in C_\Gamma$ and the elements $f \in C_m$. It is easy to check that C_Γ, associated with the sup-norm, is a Banach space. Functions from $C(\Gamma)$ we will indicate by Γ. To each $f_\Gamma \in C_\Gamma$ an $f \in C_m$ is associated through $P_\Gamma : f = (f_1, \dots, f_n) = P_\Gamma f_\Gamma$.

For $O_i \in O \setminus O_e$ we define a gluing functional $G_i : C_\Gamma \to \mathbb{R}$:

$$G_i f_\Gamma = \sum_{j \in J_i^0} \alpha_{i,j} (D_{s_j}^+ f_j)(0) - \sum_{j \in J_i^l} \alpha_{i,j} (D_{s_j}^- f_j)(l_j) \tag{4.2}$$

with $\alpha_{i,j} \geq 0$, $\sum_j \alpha_{i,j} > 0$, $(f_1, \dots, f_n) = P_\Gamma f_\Gamma$. Define

$$\sigma_i = m_j(\{l_j\}) \quad \text{if } O_i \in O_e, I_j \sim O_i, \tag{4.3}$$

$$\sigma_i = \sum_{j \in J_i^0} \alpha_{i,j}\, m_j(\{0\}) + \sum_{j \in J_i^l} \alpha_{i,j}\, m_j(\{l_j\}), \quad \text{if } O_i \in O \setminus O_e. \tag{4.4}$$

Note that σ_i is the (weighted by $\alpha_{i,j}$) sum of the point masses associated with the vertex O_i by the construction of the graph.

Now we define the linear operator A: $C_\Gamma \to C_\Gamma$ with domain

$$\begin{aligned} D(A) = \{f_\Gamma \in C_\Gamma:\ &\text{there exists a } g_\Gamma \in C_\Gamma \text{ such that for } f = P_\Gamma f_\Gamma,\\ &g = P_\Gamma g_\Gamma, f_i \in \vartheta_{m_i, s_i}, D_{m_i} D^+_{s_i} f_i = g_i,\ i = 1, \dots, n,\\ &\text{and } O_i \in O \setminus O_e \text{ implies } G_i(f_\Gamma) = \sigma_i\ g_\Gamma(O_i),\ i = 1, \dots, n_v\}. \end{aligned} \tag{4.5}$$

Then $Af_\Gamma = g_\Gamma$. As in [9] it follows by the Hille–Yosida theory (see [29]) that A is the infinitesimal generator of a strongly continuous contraction semigroup $T_t, t \geq 0$, on C_Γ, corresponding to a generalized diffusion process (X_t, P_x) on Γ. Before the process leaves an edge I_i, it behaves according to the local operator $D_{m_i} D^+_{s_i}$. If $f_\Gamma \in D(A)$, $(f_1, \dots, f_n) = P_\Gamma f_\Gamma$, $O_i \in O_e$, $j \in J^l_i$ and l_j regular, then [see (2.3)]

$$\Phi_{l_j}(f_j) = 0.$$

If $l_j < \bar{l}_j$, this can be written by (2.4) as

$$(D^-_{s_j} f_j)(l_j) + \sigma_j\ (D_{m_j} D^+_{s_j} f_j)(l_j) + \kappa_{l_j} f(l_j) = 0, \tag{4.6}$$

with $\kappa_{l_j} = (\pi_{l_j})^{-1}$.

5. The process on a subgraph. Let Γ be a (connected) graph with vertices O_k, $k = 1, \dots, n_v$, and edges I_i, $i = 1, \dots, n$, and let X_t be a generalized diffusion process on Γ given by an $\mathbf{m}_i = (m_i, l_i, \bar{l}_i) \in M^+$ and a scale function $s_i \in S(\bar{l}_i)$ to each edge I_i (of length l_i), and a positive number $\alpha_{k,i}$ to each pair of an interior vertex O_k and an edge $I_i \sim O_k$. Let A be the infinitesimal generator of this diffusion X_t and

$$R_\lambda = (\lambda - A)^{-1}, \quad \lambda > 0.$$

We mention that there exists an integral representation for R_λ with a continuous bounded kernel such that the operator can be extended to $B_{\mathbf{m}}(\Gamma)$, the space of functions f_Γ on Γ for which there exists $f = (f_1, \dots, f_n)$ with m_i integrable functions f_i, and

$$f_i(x) = f_\Gamma(P_i(x)), \qquad x \in I^*_i, i = 1, \dots, n \tag{5.1}$$

(see [28]). The relation (5.1) gives a correspondence between the functions on Γ and the n-tuples $(f_1, \dots, f_n)$ of functions on I^*_i, $i = 1, \dots, n$, denoted again by P_Γ,

$$f = (f_1, \dots, f_n) = P_\Gamma f_\Gamma.$$

Let $\Gamma' \subset \Gamma$ be a connected subgraph of Γ. We assume that there exists an edge $I_i \subset \Gamma'$ such that I_i divides Γ in two connected subgraphs, one of them is a tree $\widetilde{\Gamma} \subset \Gamma \setminus \Gamma'$.

To prepare the simplification of the graph Γ by replacing $\widetilde{\Gamma}$ by only one edge as indicated in the Introduction, we give now a procedure that defines Titchmarsh–Weyl coefficients $Q_{k,i'}$ for k such that $O_k \sim I_i$, $O_k \in \widetilde{\Gamma}$, and any i' such that $O_k \sim I_{i'}$, $i' \neq i$ (see also Example 5.3 below).

DEFINITION 5.1. The function $Q_{k,i'}$ is defined by the following procedure.

Step 1. For all j such that $P_j(l_j) \in (O_e \cap \widetilde{\Gamma}) \setminus \{O_k\}$ or $l_j = \infty$, $I_j \in \widetilde{\Gamma}$ and k' such that $O_{k'} = P_j(0)$ define

$$Q_{k',j} = \dot{Q}_j. \tag{5.2}$$

Step 2. Check whether $Q_{k,i'}$ is already defined. Otherwise go to Step 3.

Step 3. For any vertex $O_{k''}$ such that $Q_{k'',j'}$ is already defined for all j', $I_{j'} \sim O_{k''}$ with the exception of exactly one j with $I_j \sim O_{k''}$, and $O_{k'} \sim I_j$, and $k' \neq k''$, we define $Q_{k',j}$ by

$$Q_{k',j}(z) = \frac{\alpha_{k'',j} Q^{k'',j}(z)(D^+_{s_j}\psi^{k'}_j)(l_j, z) + \psi^{k'}_j(l_j, z)}{\alpha_{k'',j} Q^{k'',j}(z)(D^+_{s_j}\phi^{k'}_j)(l_j, z) + \phi^{k'}_j(l_j, z)}, \quad z \in \mathbb{C} \setminus [0, \infty), \tag{5.3}$$

where

$$Q^{k'',j}(z) = \left(-z\sigma_{k''} + \sum_{O_{k''} \sim I_l,\, l \neq j} \frac{\alpha_{k'',l}}{\dot{Q}_{k'',l}(z)} \right)^{-1}, \tag{5.4}$$

and $\psi^{k'}_j$, $\phi^{k'}_j$ are the solutions of (3.1) and (3.2), respectively, corresponding to $\mathbf{m}_j$ and s_j if $P_j(0) = O_{k'}$, and they are the respective solutions corresponding to

$$\begin{aligned} \bar{\mathbf{m}}_j = (\bar{m}_j, l_j, \bar{l}_j), \quad d\bar{m}_j(x) = dm_j(l_j - x), \\ \bar{s}_j(x) = s_j(l_j) - s_j(l_j - x), \end{aligned} \tag{5.5}$$

if $P_j(l_j) = O_{k'}$. (Thus, we have changed the direction of the edge I_j in the last case. This is always possible because $l_j < \infty$ for these "inner" edges.)

Go to Step 2.

REMARK 5.2. To calculate $Q_{k,i'}$, only the data of the tree $\widetilde{\Gamma}$ are needed. Obviously, in each loop of the defining procedure at least one new coefficient $Q_{k',j}$ will be defined. Thus, the procedure is finite.

EXAMPLE 5.3. Figure 1 shows a tree, and we calculate $Q_{1,1}$.

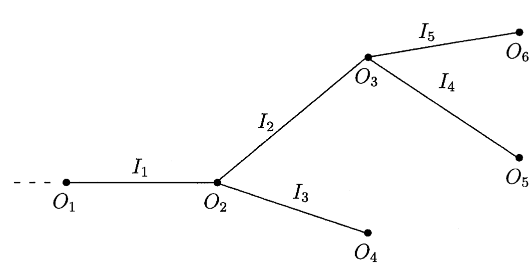

FIG. 1.

Step 1. We define $Q_{3,j} = \dot{Q}_j$, $j = 4, 5$, $Q_{2,3} = \dot{Q}_3$.
Step 2. We are not ready yet.
Step 3. We define

$$Q_{2,2}(z) = \frac{\alpha_{3,2} Q^{3,2}(z)(D^+_{s_2}\psi_2)(l_2, z) + \psi_2(l_2, z)}{\alpha_{3,2} Q^{3,2}(z)(D^+_{s_2}\phi_2)(l_2, z) + \phi_2(l_2, z)}, \qquad z \in \mathbb{C} \setminus [0, \infty),$$

$$Q^{3,2}(z) = \left(-z\sigma_3 + \frac{\alpha_{3,4}}{\dot{Q}_{3,4}} + \frac{\alpha_{3,5}}{\dot{Q}_{3,5}} \right)^{-1} = \left(-z\sigma_3 + \frac{\alpha_{3,4}}{\dot{Q}_4} + \frac{\alpha_{3,5}}{\dot{Q}_5} \right)^{-1}.$$

We go to Step 2 and find that we have to proceed. In a next "Step 3" we finally define

$$Q_{1,1}(z) = \frac{\alpha_{2,1} Q^{2,1}(z)(D^+_{s_1}\psi_1)(l_1, z) + \psi_1(l_1, z)}{\alpha_{2,1} Q^{2,1}(z)(D^+_{s_1}\phi_1)(l_1, z) + \phi_1(l_1, z)}, \qquad z \in \mathbb{C} \setminus [0, \infty),$$

where

$$Q^{2,1}(z) = \left(-z\sigma_2 + \frac{\alpha_{2,2}}{\dot{Q}_{2,2}} + \frac{\alpha_{2,3}}{\dot{Q}_3} \right)^{-1}.$$

LEMMA 5.4. *Let the graph* Γ *be as in Figure* 2, *and let*

$$\mathbf{m}_1 = (m_1, l_1, \bar{l}_1) \in M^+_r, \quad P_1(0) = O_1, \quad \mathbf{m}_2 \in M^+, \quad P_2(0) = O_2, \quad l_2 \le \infty.$$

Further, let α_1, α_2 *be the parameters in the gluing conditions at* O_2, *and define* $\mathbf{m} = \mathbf{m}_{1,2}$ *and* $s = s_{1,2}$ *as follows:*

$$m_{1,2}(dx) = \chi_{(-\infty, l_1]}(x) m_1(dx) + m_2(dx - l_1). \tag{5.6}$$

Here χ_B, $B \subset \mathbb{R}$, *denotes the indicator function of the set* B, *and* $x \in (B - l_1)$ *if and only if* $x + l_1 \in B$. *Further,*

$$s_{1,2}(x) = \begin{cases} s_1(x), & x < l_1, \\ s_1(l_1) + s_2(x - l_1), & x \ge l_1. \end{cases} \tag{5.7}$$

Then $\mathbf{m} \longleftrightarrow_s Q$ *with*

$$Q(z) = \frac{(\alpha_1/\alpha_2) Q_2(z)(D^+_{s_1}\psi_1)(l_1, z) + \psi_1(l_1, z)}{(\alpha_1/\alpha_2) Q_2(z)(D^+_{s_1}\phi_1)(l_1, z) + \phi_1(l_1, z)}, \tag{5.8}$$

where ψ_1, ϕ_1 *are the functions according to* (3.1), (3.2) *with* $\mathbf{m}_1$ *and* s_1.

If $\alpha_1 = \alpha_2 = 1$ the statement of the lemma can be found in [13, 14]. Using this, the statement of the lemma is easy to check.

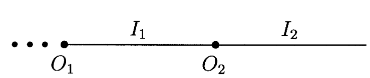

FIG. 2.

THEOREM 5.5. *Let O_k be an interior vertex of the graph Γ and $I_i \sim O_k$, such that I_i divides Γ in two subgraphs, and the one containing O_k forms a tree $\widetilde{\Gamma}$. Let $g_\Gamma \in C_m(\Gamma)$ be such that $g_\Gamma(y) = 0$ for $y \in \widetilde{\Gamma}$. Then, for any $z > 0$, the function $f_i = (P_\Gamma R_z g_\Gamma)_i$ satisfies the (eigenvalue-depending boundary) condition*

$$\pm\, \alpha_{k,i}\; Q^{k,i}(-z)(D^{\pm}_{s_i} f_i)(P_i^{-1}(O_k)) = f_i(P_i^{-1}(O_k)), \tag{5.9}$$

where the + - sign should be taken if $P_i(0) = O_k$, and the − - sign should be taken if $P_i(l_i) = O_k$, and

$$\begin{aligned} Q^{k,i}(z) &= \Bigg(- z\sigma_k + \sum_{I_j \sim O_k,\, j\neq i} \frac{\alpha_{k,j}}{Q_{k,j}(z)} \Bigg)^{-1}, \\ \sigma_k &= \sum_{I_j \sim O_k} \alpha_{k,j} m_j(\{P_j^{-1}(O_k)\}). \end{aligned} \tag{5.10}$$

PROOF. (i) We show the statement of the theorem for a graph Γ as (partially) shown in Figure 3, and $i = k = 1$.

Assume that $O_1 = P_1(0)$ and let $f = P_\Gamma R_z g_\Gamma$, with

$$P_\Gamma g_\Gamma = (g_1, 0, \dots, 0, g_{n_1+1}, \dots, g_n).$$

As the functions f_j, $j = 2, \dots, n_1$, solve the homogeneous equations

$$D_{m_j} D^+_{s_j} f_j - z f_j = 0, \tag{5.11}$$

(and belong to ϑ_{m_j, s_j}) it follows,

$$f_j = C_j \dot{\varphi}_j(\cdot, -z), \qquad C_j \in \mathbb{R},\ \ j = 2, \dots, n_1. \tag{5.12}$$

[Recall that by our assumptions $O_1 = P_j(0)$, $j = 2, \dots, n_1$.] As f_Γ is continuous on Γ, we get

$$C_j = f_1(0), \qquad j = 2, \dots, n_1. \tag{5.13}$$

The gluing condition at O_1 gives with (5.11) and $f \in D(A)$,

$$\sum_{j=1}^{n_1} \alpha_{1,j} (D^+_{s_j} f_j)(0) = \sigma_1 (D_{m_2} D^+_{s_2} f_2)(0) = z\sigma_1 f_2(0) = z\sigma_1 f_1(0). \tag{5.14}$$

By relations (5.12), (5.13) and (3.19) this implies

$$\begin{aligned} \alpha_{1,1} D^+_{s_1} f_1(0) &= z\sigma_1 f_1(0) - \sum_{j=2}^{n_1} \alpha_{1,j} (D^+_{s_j} f_j)(0) \\ &= z\sigma_1 f_1(0) - \sum_{j=2}^{n_1} \alpha_{1,j} f_1(0)(D^+_{s_j} \dot{\varphi}_j)(0, -z) \\ &= f_1(0)\Bigg(z\sigma_1 + \sum_{j=2}^{n_1} \frac{\alpha_{1,j}}{\dot{Q}_j(-z)} \Bigg) = f_1(0) \frac{1}{Q^{1,1}(-z)}. \end{aligned} \tag{5.15}$$

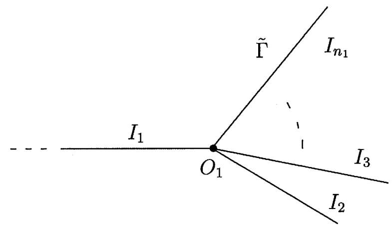

FIG. 3.

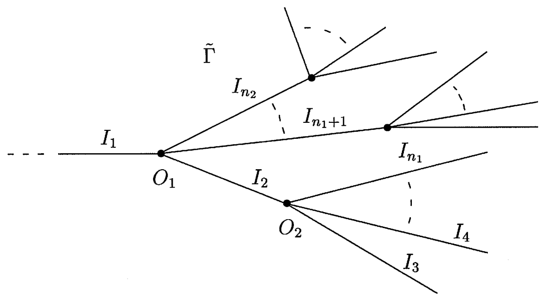

FIG. 4.

If $O_1 = P_1(l_1)$ the proof is analogous. Condition (5.14) becomes

$$-\alpha_{1,1} D^-_{s_1} f_1(l_1) + \sum_{j=2}^{n_1} \alpha_{1,j} D^+_{s_j} f_j(0) = z\sigma_1 f_1(0)$$

in this case. This implies the result by the same arguments.

(ii) Now we consider a graph as (partially) shown in Figure 4, and again $i = k = 1$. Let $f = (f_1, \dots, f_n) = P_\Gamma R_z g_\Gamma$, $P_\Gamma g_\Gamma = (g_1, 0, \dots, 0, g_{n_2+1}, \dots, g_n)$.

It follows from the first part of the proof applied to $i = k = 2$, that the function f_2 satisfies at $P^{-1}(O_2)$ the condition (5.9) with the coefficient

$$Q^{2,2}(z) = -z\sigma_2 + \sum_{j=3}^{n_1} \frac{\alpha_{2,j}}{\dot{Q}_j(z)}.$$

By Lemma 3.1 the function $Q^{2,2}$ is an S-function, and there exists $\mathbf{m}^{2,2} = (m^{2,2}, l^{2,2}, \bar{l}^{2,2}) \in M^+$, $\mathbf{m}^{2,2} \longleftrightarrow_x Q^{2,2}$. Obviously, f_2 satisfies the same boundary condition at $P^{-1}(O_2)$ if we modify the graph Γ by replacing the edges $I_3, \dots, I_{n_1}$ by only one edge $I^{2,2}$ of length $l^{2,2}$ as drawn in Figure 5, and associate to $I^{2,2}$ the scale x and $\mathbf{m}^{2,2}$ (and the coefficient 1 for the gluing condition).

More exactly, denote the modified graph by $\Gamma^{2,2}$ and let $g_{\Gamma^{2,2}} \in C(\Gamma^{2,2})$ be such that $g_{\Gamma^{2,2}}(y) = g_\Gamma(y)$ for $y \in I_j$, $j = 1, 2, n_1 + 1, \dots, n$, and $g_{\Gamma^{2,2}} = 0$ on $I^{2,2}$. Let $R^{2,2}_z$ be the resolvent operator corresponding to the diffusion process

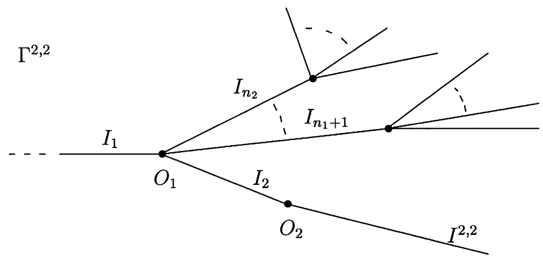

FIG. 5.

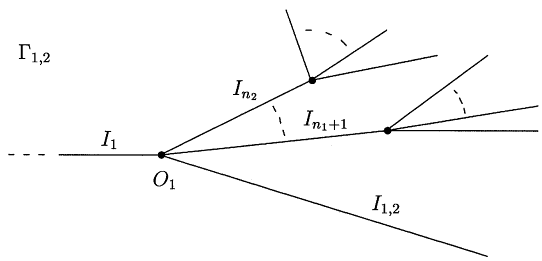

FIG. 6.

on $\Gamma^{2,2}$. Then $(R_z^{2,2} g_{\Gamma^{2,2}})(y) = (R_z g_\Gamma)(y)$ for $y \in I_j$, $j = 1, 2, n_1 + 1, \dots, n$, $z > 0$.

The same remains true if we replace the edges I_2 and $I^{2,2}$ by only one edge $I_{1,2}$ of length $l_2 + l^{2,2}$ as shown in Figure 6, equipped with a scale function and an element of M^+ built from s_2, $\mathbf{m}_2$ and x, $\mathbf{m}_{2,2}$ if $O_2 = P_2(l_2)$ and built from $\bar{s}_2$, $\bar{\mathbf{m}}_2$ and x, $\mathbf{m}_{2,2}$ if $O_2 = P_2(0)$ according to Lemma 5.4. Here $\bar{s}_2$ and $\bar{m}_2$ are defined as $\bar{s}_j$ and $\bar{m}_j$ in (5.5). Thus, if $O_2 = P_2(0)$, we have to change the direction of the edge I_2 as described in Definition 5.1. Then by Lemma 5.4, the diffusion on the edge $I_{1,2}$ corresponds to the Titchmarsh–Weyl coefficient $Q_{1,2}$.

So, we replaced the edge I_2 together with the part of the tree connected with O_1 via I_2 by only one edge $I_{1,2}$ and a generalized diffusion corresponding to $Q_{1,2}$ on that edge without any change of the resolvent at the edge I_1 (and the rest of the tree). The same can be done with the other edges $I_j \sim O_1$, $j \neq 1, 2$, and the parts of the tree which are connected with O_1 via I_j. Each of these parts can be replaced by only one edge $I_{1,j}$, and a generalized diffusion there corresponding to $Q_{1,j}$ without any change of the resolvent at the edge I_1 (and the rest of the tree). This leads to the simplified tree Γ^1 as shown in Figure 7. To the tree Γ^1 we can apply the result of the first part of the proof to get the result of the theorem for the situation as in Figure 3.

(iii) Obviously it follows by induction that also larger parts of a tree can be simplified successively by the same procedure. This procedure matches exactly the procedure for defining Titchmarsh–Weyl coefficients in Definition 5.1. □

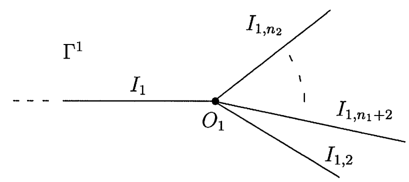

FIG. 7.

COROLLARY 5.6. *As $Q^{k,i}$ is a Titchmarsh–Weyl coefficient, the whole tree $\widetilde{\Gamma}$ can be replaced by only one edge $I^{k,i}$ associated with $\mathbf{m}^{k,i} \longleftrightarrow_x Q^{k,i}$, natural scale, and factor $\alpha = 1$ in the gluing condition such that on the new graph a diffusion process is defined that (if it is started in Γ') cannot be distinguished from the original one by observers that see the process only at times it spends in Γ'.*

A more precise formulation of this corollary is the following.

THEOREM 5.7. *Under the conditions of Theorem 5.5 the following holds: let $f_\Gamma \in B_{\mathbf{m}}(\Gamma)$ and $h_\Gamma \in C(\Gamma)$ be such that*

$$f_\Gamma(y) = 0 \quad \text{if } y \in \widetilde{\Gamma},$$
$$h_\Gamma(y) = E_y \int_0^\infty e^{-\lambda t} f_\Gamma(X_t)\, dt.$$

Let $\Gamma^{k,i}$ be the graph that appears if we replace $\widetilde{\Gamma}$ by only one edge $I^{k,i} \sim O_k$, and let $\widetilde{X}_t$ be the process on $\Gamma^{k,i}$, defined on $\Gamma' = \Gamma \setminus \widetilde{\Gamma}$ by the same scale functions, speed measures, gluing and boundary conditions that define X_t there; on $I^{k,i}$ it is defined by $\mathbf{m}^{k,i} \longleftrightarrow_x Q^{k,i}$, and the behavior at O_k is given by a gluing condition with the factor $\alpha^{k,i} = 1$ for the edge $I^{k,i}$. Define $\tilde{f}_{\Gamma^{k,i}} \in B_{\mathbf{m}}(\Gamma^{k,i})$ and $\tilde{h}_{\Gamma^{k,i}} \in C(\Gamma^{k,i})$ by

$$\tilde{f}_{\Gamma^{k,i}}(y) = \begin{cases} f_\Gamma(y), & \text{if } y \in \Gamma', \\ 0, & \text{otherwise}, \end{cases}$$
$$\tilde{h}_{\Gamma^{k,i}}(y) = E_y \int_0^\infty e^{-\lambda t} \tilde{f}_{\Gamma^{k,i}}(\widetilde{X}_t)\, dt.$$

Then, for all $y \in \Gamma'$,

$$h_\Gamma(y) = \tilde{h}_{\Gamma^{k,i}}(y).$$

The statement of this theorem follows immediately from the following consequence of Theorem 5.5.

THEOREM 5.8. *Let Γ' be a subgraph of a graph Γ satisfying the conditions from Theorem 5.5. Let $f = (f_1, \dots, f_n)$ such that f_i is m_i-integrable, $i = 1, \dots, n$, and*

$$f_i \equiv 0 \quad \text{if } I_i \not\subset \Gamma'.$$

Let $f_\Gamma = P_\Gamma^{-1} f$. *Then for* $\lambda > 0$ *the functions*

$$(5.16)\qquad \begin{aligned} h_i(x) &= (P_\Gamma(R_\lambda f_\Gamma))_i\ (x) \\ &= E_{P_i(x)} \int_0^\infty e^{-\lambda t} f_\Gamma(X_t)\, dt, \qquad i = 1, \dots, n,\ x \in I_i^*, \end{aligned}$$

are on Γ', *that is, for* $x \in I_i^*$ *with* $I_i \subset \Gamma'$, *uniquely determined by the following conditions:*

$$(5.17)\qquad \lambda h_i(x, \lambda) - (D_{m_i} D_{s_i}^+ h_i)(x, \lambda) = f_i(x), I_i \subset \Gamma',\ x \in I_i^*,$$

$$(5.18)\qquad \begin{aligned} & h_i(P_i^{-1}(O_k), \lambda) = h_j(P_j^{-1}(O_k), \lambda), \qquad O_k \in \Gamma',\ O_k \notin O_e', \\ & \quad I_i \sim O_k,\ I_j \sim O_k, \end{aligned}$$

$$(5.19)\qquad \begin{aligned} & \sum_{j \in J_k^0} \alpha_{k,j} (D_{s_j}^+ h_j)(P_j^{-1}(O_k), \lambda) - \sum_{j \in J_k^l} \alpha_{k,j} (D_{s_j}^- h_j)(P_j^{-1}(O_k), \lambda) \\ & \quad = \sigma_k(\lambda h_i(P_i^{-1}(O_k), \lambda) - 1), \qquad O_k \in \Gamma', O_k \notin O_e', \end{aligned}$$

$$(5.20)\qquad h_i(\cdot, \lambda) \in \vartheta_{m_i, s_i}, I_i \subset \Gamma' \quad \textit{and} \quad P_i(l_i) \in O_e \textit{ or } i \in I_\infty,$$

$$(5.21)\qquad \begin{aligned} & \pm \alpha_{k,i} Q^{k,i}(-\lambda)(D_{s_i}^\pm h_i)(P_i^{-1}(O_k), \lambda) \\ & \quad = h_i(P_i^{-1}(O_k), \lambda), \qquad O_k \in O_e', I_i \sim O_k, \end{aligned}$$

with the choice of the sign in (5.21) *according to Theorem* 5.5.

Recall that the resolvent operators can be extended to functions f_Γ on Γ with m_i-measurable $(P_\Gamma f_\Gamma)_i$. Then the statement follows from Theorem 5.5.

COROLLARY 5.9. *Let the conditions be as in Theorem* 5.8. *Then for any* $\Gamma'' \subset \Gamma'$ *the Laplace transform* $h_i(x, \lambda)$, $I_i \subset \Gamma'$, $x \in \operatorname{supp} m_i$, $\lambda > 0$, *of the probability* $P_y(X_t \in \Gamma'')$, $y = P_i(x)$, *is the unique solution of the system* (5.17)–(5.21) *with* $f_i(x) = 1$ *if* $I_i \subset \Gamma'$, $P_i(x) \in \Gamma''$ *and* $f_i(x) = 0$ *otherwise.*

PROOF. The statement follows from Theorem 5.8 with

$$h_i(x, \lambda) P_y(X_t \in \Gamma'')\, dt, \qquad y = P_i(x),$$

as $P_y(X_t \in \Gamma'') = E_y\ \chi_{\Gamma''}(X_t)$. □

6. The case of birth-and-death processes. If the process inside the edges of the graph is a birth-and-death process with a finite number of states, then the corresponding Titchmarsh–Weyl coefficients are rational functions, and the calculations introduced in the foregoing section are very simple and can be done explicitly. We give the formulas relating the Titchmarsh–Weyl coefficient to the distributions of a birth-and-death process and present some examples.

We start with a birth-and-death process with a finite number of states on the real half-line with reflection at the first point a_0 of the state space. In this case $\mathbf{m} = (m, l, \bar{l}) \in M_r^+$, m is a discrete measure concentrated in a finite number of points, and the Titchmarsh-Weyl coefficient has a representation as a continued fraction (see, e.g., [1, 26]):

$$\begin{aligned} Q(z) &= a_0 + \sum_1^n \frac{\sigma_i}{z_i - z} \\ &= a_0 + (1/{-b_1 z}) + (1/a_1) + (1/{-b_2 z}) + \cdots + (1/{-b_n z}) + (1/a_n), \end{aligned} \tag{6.1}$$

where the b_i are the point masses of m, and the a_i are the distances between them. More exactly, let $a_0 \geq 0$, $a_i > 0$, $b_i > 0$, $i = 1, \ldots, n$, $a_n \in (0, \infty]$. Define the measure m as follows:

$$m = \sum_{i=1}^n b_i \delta_{a_0 + a_1 + \cdots + a_{i-1}}. \tag{6.2}$$

Here δ_x denotes the Dirac measure at x. With $l = \sum_{i=0}^{n-1} a_i$, $\bar{l} = l + a_n$, we have $(m, l, \bar{l}) = \mathbf{m} \longleftrightarrow_x Q$, with Q given by (6.1). The corresponding birth-and-death process has the state space

$$\operatorname{supp} m = \left\{ x_i \colon x_i = \sum_{j=0}^{i-1} a_j,\ i = 1, \ldots, n \right\}.$$

$1/b_i$ is the parameter of the exponentially distributed waiting time of the process at x_i, and the probability $p_{i,j}$ of one-step transition from x_i to x_j of the embedded Markov chain is given by

$$p_{1,2} = 1, \tag{6.3}$$

$$p_{i,i+1} = \frac{a_{i-1}}{a_{i-1} + a_i}, \qquad i = 2, \ldots, n-1, \tag{6.4}$$

$$p_{i,i-1} = \frac{a_i}{a_{i-1} + a_i}, \qquad i = 2, \ldots, n. \tag{6.5}$$

Here $p_{n,n-1} = 1$, if $a_n = \infty$. If $a_n < \infty$, the process is not conservative. Note that we can also consider the birth-and-death process on (equidistant) natural numbers. In this case we would have to deal with an appropriate scale function s that is, in general, different from $s(x) = x$.

First, let Γ be as in Figure 3 and $\mathbf{m}_i \leftrightarrow Q_i$, $\alpha_{1,i} = 1$, $i = 2, \dots, n_1$,

$$Q_i(z) = a_0^i + (1/{-b_1^i z}) + (1/a_1^i) + (1/{-b_2^i z}) + \cdots + (1/{-b_{\tilde{n}_i}^i z}) + (1/a_{\tilde{n}_i}^i), \tag{6.6}$$

$a_0^i > 0$. If we consider occupation time problems on $\Gamma \setminus \bigcup_{i=2}^{n_1} I_i$, then the edges $I_2, \dots, I_{n_1}$ can be replaced by an edge $\widetilde{I}$ equipped with

$$\tilde{\mathbf{m}} \leftrightarrow \widetilde{Q}(z) = \left(\sum_{i=2}^{n_1} (Q_i(z))^{-1} \right)^{-1}.$$

EXAMPLE 6.1. Let Γ be as in Figure 3, and let $n_1 = 4$, $\tilde{n}_2 = 2$, $\tilde{n}_3 = 5$, $\tilde{n}_4 = 11$, $a_i^j = b_i^j = 1$, $a_0^j = 1$, $a_{\tilde{n}_j}^j = \infty$, $j = 2, \dots, 4$, $i = 1, \dots, \tilde{n}_j - 1$. That is, $\widetilde{\Gamma}$ consists of three edges, linked at O_1 of the length 2, 5, 11, respectively, and on each edge we have a symmetric birth-and-death process on the 2, 5 or 11 points, respectively, with reflection at the end and mean waiting time equal to 1 in any point. One gets by the results of the foregoing section (see Corollary 5.6) and continued fraction of the respective $Q^{1,1}$ that $\widetilde{\Gamma}$ can be replaced by only one edge with a birth-and-death process with 18 points on it according to Table 1. The values $\tilde{a}_i$ and $\tilde{b}_i$ give the Titchmarsh–Weyl coefficient $Q^{1,1}$ according to (6.1), and the transition probabilities are calculated according to (6.4), (6.5). Note that $\tilde{b}_i$ is the mean waiting time of the process in the point $\tilde{x}_i$. A simple Maple-routine for transformation of a rational function to a continued fraction can be found in [27]. The transition probabilities $p_{1,0}$ describe the transitions from $\tilde{x}_1$ to O_1.

As the Titchmarsh–Weyl coefficient has a finite continued fraction representation, the "linking" of birth-and-death processes as described in Lemma 5.4 obviously is very simple.

LEMMA 6.2. *Let the conditions be as in Lemma* 5.4, *and assume additionally that* Q_1 *has a representation* (6.6). *Then*

$$\begin{aligned} Q(z) = a_0^1 + (1/{-b_1^1 z}) + (1/a_1^1) + (1/{-b_2^1 z}) \\ + \cdots + (1/{-b_n^1 z}) + (1/(\alpha_1/\alpha_2)\ Q_2(z)). \end{aligned} \tag{6.7}$$

REMARK 6.3. We stress that Q_2 in Lemma 6.2 may be a general Titchmarsh–Weyl coefficient.

Using Lemma 6.2 we are able to consider more difficult trees.

EXAMPLE 6.4. The next example is related to Example 5.3 (compare Figure 1). Let $\widetilde{\Gamma}$ be as in Figure 8 and consider a process whose state space is

TABLE 1
Results of Example 6.1

i	0	1	2	3	4	5	6	7	8	9	10
$\tilde{b}_i$		3	3	$\frac{3}{2}$	$\frac{3}{2}$	2	$\frac{9}{8}$	$\frac{625}{536}$	$\frac{101761}{169242}$	$\frac{31205}{66939}$	$\frac{12005}{22101}$
$\tilde{a}_i$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{64}{75}$	$\frac{8978}{7975}$	$\frac{531723}{252010}$	$\frac{8427}{3871}$	$\frac{57963}{42140}$
$p_{i,i+1}$		$\frac{1}{2}$	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{25}{57}$	$\frac{10208}{23675}$	$\frac{1418524}{4077139}$	$\frac{8684809}{17645519}$	$\frac{2415740}{3942099}$
$p_{i,i-1}$		$\frac{1}{2}$	$\frac{3}{5}$	$\frac{4}{7}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{32}{57}$	$\frac{13467}{23675}$	$\frac{2658615}{4077139}$	$\frac{8960710}{17645519}$	$\frac{1526359}{3942099}$

i	11	12	13	14	15	16	17	18
$\tilde{b}_i$	$\frac{14792}{12371}$	$\frac{12482}{25365}$	$\frac{28322}{71535}$	$\frac{123008}{533877}$	$\frac{67081}{214827}$	$\frac{2809}{7878}$	$\frac{25}{442}$	$\frac{1}{17}$
$\tilde{a}_i$	$\frac{23763}{13588}$	$\frac{27075}{9401}$	$\frac{189003}{59024}$	$\frac{502681}{128464}$	$\frac{91809}{27454}$	$\frac{2704}{265}$	$\frac{289}{10}$	∞
$p_{i,i+1}$	$\frac{1526359}{3467004}$	$\frac{942599}{249489}$	$\frac{4476400}{9453479}$	$\frac{6993111}{15538688}$	$\frac{26642093}{49410725}$	$\frac{459045}{1859717}$	$\frac{5408}{20725}$	
$p_{i,i-1}$	$\frac{1940645}{3467004}$	$\frac{1552300}{2494899}$	$\frac{4977079}{9453479}$	$\frac{8545577}{15538688}$	$\frac{22768632}{49410725}$	$\frac{1400672}{1859717}$	$\frac{15317}{20725}$	1

marked by the dots in Figure 8 with transition probabilities equal to 1/2 for the transition to one of the two next neighbors inside the edge, 1 for the transition from O_4, O_5 or O_6 to the next neighbor and 1/3 for the transition from O_2 or O_3 to one of the three next neighbors. Then, using the procedure from Example 5.3, $\widetilde{\Gamma}$ can be replaced in the sense of Corollary 5.6 by one edge equipped with a birth-and-death process as described in Table 2 (compare Example 6.1 for an explanation of the values in this table). The function $m^{2,1}([0,x])$ is drawn in Figure 9 ($m^{2,1}$ is the speed measure on the new edge). Actually, Figure 9 does not contain all points of the support of $m^{2,1}$, the missing points are $m^{2,1}(\{87\}) = \tilde{b}_{13} \approx 0.0081$ and $m^{2,1}(\{1047\}) = \tilde{b}_{14} \approx 0.0062$. Note that we always have reflection at the end of the new edge, that is, $\bar{l}^{\,2,1} = \infty$ in the example under consideration. Figure 10 contains the respective result for the same problem with five times as much (equidistant) points in any edge of the graph in Figure 8.

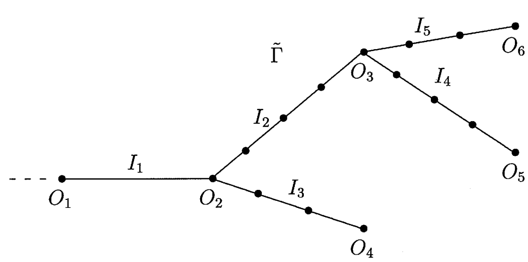

FIG. 8.

TABLE 2
Results of Example 6.4

i	0	1	2	3	4	5	6	7	8
$\tilde{b}_i$		2	2	2	$\frac{2}{3}$	$\frac{50}{51}$	$\frac{1682}{1683}$	$\frac{28561}{23661}$	$\frac{20601}{129777}$
$\tilde{a}_i$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{9}{10}$	$\frac{289}{290}$	$\frac{9801}{9802}$	$\frac{57121}{75881}$	$\frac{294849}{304871}$
$p_{i,\,i+1}$		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{10}{19}$	$\frac{261}{550}$	$\frac{48841}{97846}$	$\frac{4400649}{7713667}$	$\frac{38785159}{88614640}$
$p_{i,\,i-1}$		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{9}{19}$	$\frac{289}{550}$	$\frac{49005}{97846}$	$\frac{3313018}{7713667}$	$\frac{49829481}{88614640}$

i	9	10	11	12	13	14
$\tilde{b}_i$	$\frac{461041}{476211}$	$\frac{1234321}{1165533}$	$\frac{114921}{223715}$	$\frac{32041}{846885}$	$\frac{729}{89999}$	$\frac{1}{161}$
$\tilde{a}_i$	$\frac{769129}{754369}$	$\frac{196249}{125543}$	$\frac{255025}{20227}$	$\frac{312481}{4833}$	$\frac{25921}{27}$	∞
$p_{i,i+1}$	$\frac{327577239}{672916160}$	$\frac{86911577}{220164648}$	$\frac{35128571}{318461346}$	$\frac{6885675}{42196028}$	$\frac{312481}{4952340}$	
$p_{i,i-1}$	$\frac{345338921}{672916160}$	$\frac{133253071}{220164648}$	$\frac{283332775}{318461346}$	$\frac{35310353}{42196028}$	$\frac{4639859}{4952340}$	1

7. Approximation of diffusion processes by birth-and-death processes.

As described in the introduction, a diffusion process on a graph can be approximated by birth-and-death processes via discrete approximations of the speed measure. Example 6.4 can be understood in this way as giving two levels of approximation for a Wiener process on $\widetilde{\Gamma}$ with reflection at the exterior vertices. Figures 9 and 10 then give approximations of the respective speed measure $m^{1,1}$.

EXAMPLE 7.1. Let Γ be as in Figure 3 with $n_1 = 3$, and consider a diffusion process on Γ that is the Wiener process inside the edges, and that has all coefficients in the gluing conditions equal to 1. More exactly, assume that

$$\mathbf{m}_2 = (m_2, 4, \infty), \qquad \mathbf{m}_3 = (m_3, 6, \infty),$$

where m_2 and m_3 denote the Lebesgue measure on $[0, 4]$ and $[0, 6]$, respectively. Figures 11, 12 and 13 contain approximations of the speed measure $m^{1,1}$ (i.e., the function $m^{1,1}([0, x])$) on the edge $I^{1,1}$ that replaces $\widetilde{\Gamma}$. These

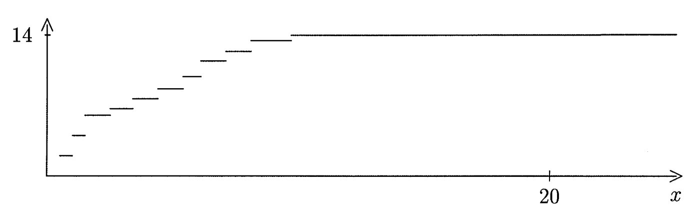

FIG. 9.

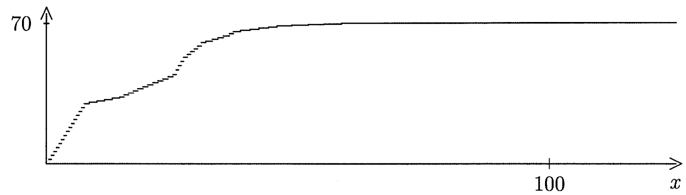

FIG. 10.

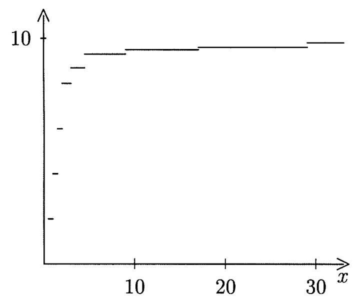

FIG. 11.

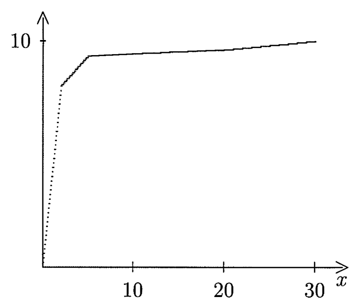

FIG. 12.

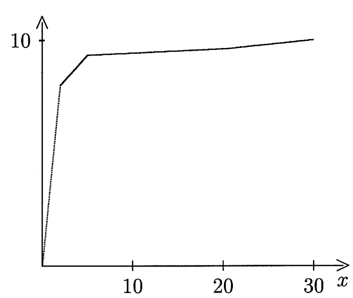

FIG. 13.

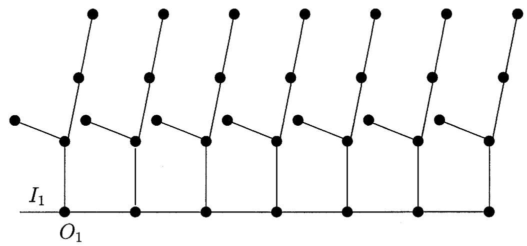

FIG. 14.

approximations are obtained as in the foregoing section using approximations of the speed measures m_2 and m_3 by

$\tilde{n}_2 = 4,$	$\tilde{n}_3 = 6$	Figure 11,
$\tilde{n}_2 = 40,$	$\tilde{n}_3 = 60$	Figure 12,
$\tilde{n}_2 = 100,$	$\tilde{n}_3 = 150$	Figure 13

equidistant point masses of the same size.

EXAMPLE 7.2. Let Γ be as in Figure 14. The vertices are marked by the dots in Figure 14. Assume that any edge has length 15. Consider a diffusion process on Γ that is the Wiener process inside the edges, and that has all coefficients in the gluing conditions equal to 1. Figure 15 contains an approximation of the speed measure $m^{1,1}$ on the edge $I^{1,1}$ that replaces everything drawn in Figure 14 (with exception of I_1). This approximation is obtained using an approximation of the speed measure to any edge, that is, of the Lebesgue measure on $[0, 15]$, by 15 equidistant point masses of size 1.

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FIG. 15.

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