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**THE NORM ESTIMATE OF THE DIFFERENCE BETWEEN
THE KAC OPERATOR AND SCHRÖDINGER SEMIGROUP II:
THE GENERAL CASE INCLUDING THE RELATIVISTIC CASE**

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Abstract More thorough results than in our previous paper in Nagoya Math. J. are given on the L_p -operator norm estimates for the Kac operator $e^{-tV/2}e^{-tH_0}e^{-tV/2}$ compared with the Schrödinger semigroup $e^{-t(H_0+V)}$. The Schrödinger operators H_0+V to be treated in this paper are more general ones associated with the Lévy process, including the relativistic Schrödinger operator. The method of proof is probabilistic based on the Feynman-Kac formula. It differs from our previous work in the point of using *the Feynman-Kac formula* not directly for these operators, but instead through *subordination* from the Brownian motion, which enables us to deal with all these operators in a unified way. As an application of such estimates the Trotter product formula in the L_p -operator norm, with error bounds, for these Schrödinger semigroups is also derived.

Keywords Schrödinger operator, Schrödinger semigroup, relativistic Schrödinger operator, Trotter product formula, Lie-Trotter-Kato product formula, Feynman-Kac formula, subordination of Brownian motion, Kato's inequality

AMS subject classification 47D07, 35J10, 47F05, 60J65, 60J35

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1. Introduction

By the *Kac operator* we mean an operator of the kind $K(t) = e^{-tV/2}e^{-tH_0}e^{-tV/2}$, where $H = H_0 + V \equiv -\Delta/2 + V(x)$ is the nonrelativistic Schrödinger operator in $L_2(\mathbb{R}^d)$ with mass 1 with scalar potential $V(x)$ bounded from below. This $K(t)$ may correspond to the transfer operator for a lattice model in statistical mechanics studied by M. Kac [Ka]. There it is one of the important problems to know asymptotic spectral properties of $K(t)$ for $t \downarrow 0$. To this end, in [H1, H2] Helffer estimated the L_2 -operator norm of the difference between $K(t)$ and the Schrödinger semigroup e^{-tH} to be of order $O(t^2)$ for small $t > 0$, if $V(x)$ satisfies $|\partial^\alpha V(x)| \leq C_\alpha(1 + |x|^2)^{(2-|\alpha|)_+/2}$ for every multi-index α with a constant C_α . Then such norm estimates may be applied to get spectral properties of $K(t)$ in comparison with those of H .

In [I-Tak1] and [I-Tak2] we have extended his result to the case of more general scalar potentials $V(x)$ even in the L_p -operator norm, $1 \leq p \leq \infty$, making a probabilistic approach based on the Feynman-Kac formula. In [I-Tak2] we have also considered this problem for both the nonrelativistic Schrödinger operator $H = H_0 + V$ and the relativistic Schrödinger operator $H^r = H_0^r + V \equiv \sqrt{-\Delta + 1} - 1 + V(x)$ with light velocity 1. The L_p -operator norm of this difference is estimated to be of order $O(t^a)$ of small $t > 0$ with $a \geq 1$, though the relativistic case shows for small $t > 0$ a slightly different behavior from the nonrelativistic case. As another application of these results the Trotter product formula for the nonrelativistic and relativistic Schrödinger operators in the L_p -operator norm with error bounds is obtained. There are also related L_2 results with operator-theoretic methods, for which we refer to [D-I-Tam].

The aim of this paper is to generalize and refine the result of [I-Tak2] in the relativistic case, admitting of *more general operators* than the free relativistic Schrödinger operator $H_0^r = \sqrt{-\Delta + 1} - 1$ as well as relaxing the conditions for the potentials $V(x)$. We use the probabilistic method with Feynman-Kac formula, though observing everything in a unified way through *subordination* from the Brownian motion. In this respect the present method differs from that in [I-Tak2] used for the relativistic Schrödinger operator H^r , which made the best of the explicit expression of the integral kernel of $e^{-tH_0^r}$.

The *more general operator* we have in mind is the following operator

$$H_0^\psi = \psi\left(\frac{1}{2}(-\Delta + 1)\right) - \psi\left(\frac{1}{2}\right), \quad (1.1)$$

which will play the same role as the relativistic Schrödinger operator

$$H_0^r = \sqrt{-\Delta + 1} - 1 \quad (1.2)$$

in [I-Tak2]. Obviously, H_0^ψ is a selfadjoint operator in $L_2(\mathbb{R}^d)$. Here $\psi(\lambda)$ is a continuous increasing function on $[0, \infty)$ with $\psi(0) = 0$ and $\psi(\infty) = \infty$ expressed as

$$\psi(\lambda) = \int_{(0, \infty)} (1 - e^{-\lambda l}) n(dl), \quad \lambda \geq 0, \quad (1.3)$$

where $n(dl)$ is a Lévy measure on $(0, \infty)$ (i.e. a measure on $(0, \infty)$ such that $\int_{(0, \infty)} l \wedge 1 n(dl) < \infty$) with $n((0, \infty)) = \infty$. It is clear that

$$\psi\left(\lambda + \frac{1}{2}\right) - \psi\left(\frac{1}{2}\right) = \int_{(0, \infty)} (1 - e^{-\lambda l}) e^{-l/2} n(dl). \quad (1.4)$$

As a special case of H_0^ψ we have for $\psi(\lambda) = (2\lambda)^\alpha$, $0 < \alpha < 1$, the operator

$$H_0^{(\alpha)} = (-\Delta + 1)^\alpha - 1, \quad (1.5)$$

which reduces to the relativistic Schrödinger operator when $\alpha = 1/2$: $H_0^{(1/2)} = H_0^r$. In this case the Lévy measure is $n(dl) = \{2^\alpha \alpha / \Gamma(1 - \alpha)\} l^{-1-\alpha} dl$.

To formulate our results we are going to describe what kind of function $V(x)$ is. Let $0 < \gamma, \delta \leq 1$, $0 \leq \kappa \leq 1$, $0 \leq \mu, \nu, \rho < \infty$, $0 \leq C_1, C_2, c_1, c_2 < \infty$ and $0 < c < \infty$. Let $V : \mathbb{R}^d \rightarrow [0, \infty)$ be a continuous function satisfying one of the following five conditions:

$$(A)_0 \quad |V(x) - V(y)| \leq C_1 |x - y|^\gamma;$$

(A)₁ V is a C^1 -function such that

$$(i) \quad |\nabla V(z)| \leq C_1(1 + V(z)^{1-\delta}), \quad (ii) \quad |\nabla V(x) - \nabla V(y)| \leq C_2 |x - y|^\kappa;$$

(A)₂ V is a C^1 -function such that

$$(i) \quad |\nabla V(z)| \leq C_1(1 + V(z)^{1-\delta}), \\ (ii) \quad |\nabla V(x) - \nabla V(y)| \\ \leq C_2 \left\{ V(x)^{(1-2\delta)+} (1 + |x - y|^\mu) + 1 + |x - y|^\nu \right\} |x - y|;$$

(V)₁ V is a C^1 -function such that

$$(i) \quad V(z) \geq c \langle z \rangle^\rho, \quad (ii) \quad |\nabla V(z)| \leq c_1 \langle z \rangle^{(\rho-1)+};$$

(V)₂ V is a C^2 -function such that

$$(i) \quad V(z) \geq c \langle z \rangle^\rho, \quad (ii) \quad |\nabla V(z)| \leq c_1 \langle z \rangle^{(\rho-1)+}, \\ (iii) \quad |\nabla^2 V(z)| \leq c_2 \langle z \rangle^{(\rho-2)+}.$$

Here $\langle z \rangle := \sqrt{1 + |z|^2}$.

Conditions (A)₀, (A)₁ and (A)₂ on $V(x)$ are used in [Tak] and are more general than in [I-Tak1,2], while conditions (V)₁ and (V)₂ are used in [D-I-Tam]. But these conditions may not be best possible. A simple example of a function which has property (A)₀, (A)₁ or (A)₂ is, needless to say, $V(x) = |x|^r$ ($0 < r < \infty$), and a slightly complicated one $V(x) = |x|^r (2 + \sin \log |x|)$, according as $0 < r \leq 1$, $1 < r < 2$ or $r \geq 2$. Also $V(x) = 1 + |x_1 - x_2|^r$ ($x = (x_1, x_2, \dots, x_d)$) satisfies (A)₀, (A)₁ or (A)₂ with the same r as above, but neither (V)₁ nor (V)₂. To the contrary $V(x) = 1 + |x| \int_0^{|x|} (1 + \sin(\theta^2)) d\theta$ satisfies (V)₁, but neither (V)₂, (A)₀, (A)₁ nor (A)₂.

The operator $H_0^\psi + V$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$, and so its unique selfadjoint extension is also denoted by the same $H_0^\psi + V$. The semigroup $e^{-t(H_0^\psi + V)}$ on $L_2(\mathbb{R}^d)$ is extended to a strongly continuous semigroup on $L_p(\mathbb{R}^d)$ ($1 \leq p < \infty$) and $C_\infty(\mathbb{R}^d)$, to be denoted by the same $e^{-t(H_0^\psi + V)}$. Here $C_\infty(\mathbb{R}^d)$ is the Banach space of the continuous functions on \mathbb{R}^d vanishing at infinity. To be complete, these and further facts are proved in Appendix.

As for the Lévy measure $n(dl)$ introduced in (1.3) and (1.4), we make the following assumption:

- (L) For some $\alpha \in [0, 1]$, $n((\cdot, \infty))$ is regularly varying at zero with exponent $-\alpha$, i.e., there exists a slowly varying function $L(\lambda)$ at infinity such that

$$n((t, \infty)) \sim t^{-\alpha} L\left(\frac{1}{t}\right) \quad \text{as } t \downarrow 0. \quad (1.6)$$

Here a positive function $L(\cdot)$ is called *slowly varying at infinity* if for any $c > 0$,

$$\lim_{\lambda \uparrow \infty} \frac{L(c\lambda)}{L(\lambda)} = 1.$$

Let $\phi^{-1}(\cdot)$ be the inverse function of $\phi(\lambda) := \psi(\lambda + 1/2) - \psi(1/2)$. (Note that ϕ is strictly increasing.) Under the above assumption, set

$$L_1(\lambda) := \begin{cases} \Gamma(1 - \alpha)L(\lambda) & \text{if } 0 \leq \alpha < 1 \\ \int_0^{1/\lambda} n((s, \infty))ds & \text{if } \alpha = 1, \end{cases}$$

$$L_2(x) := L_1(\phi^{-1}(x))^{-1/\alpha} \quad \text{if } 0 < \alpha \leq 1.$$

These two functions are slowly varying at infinity, and we have $\phi(\lambda) \sim \lambda^\alpha L_1(\lambda)$ as $\lambda \rightarrow \infty$ and $\phi^{-1}(x) \sim x^{1/\alpha} L_2(x)$ as $x \rightarrow \infty$, as will be seen from Fact in Section 6, so that $\int_0^\cdot (\phi^{-1}(\theta))^{-\alpha} d\theta$ ($0 < \alpha < 1$) is also slowly varying at infinity.

Now we state the main results of this paper, which generalize the results in [I-Tak2]. In the following $\|\cdot\|_{p \rightarrow p}$ stands for the L_p -operator norm for $1 \leq p < \infty$ and the supremum norm on $C_\infty(\mathbb{R}^d)$ for $p = \infty$.

Theorem 1. *Suppose assumption (L) and let $1 \leq p \leq \infty$. Then the following estimates (i), (ii) and (iii) hold for small $t > 0$.*

(i) Under $(A)_0$,

$$\begin{aligned} & \|e^{-tV/2} e^{-tH_0^\psi} e^{-tV/2} - e^{-t(H_0^\psi + V)}\|_{p \rightarrow p}, \\ & \|e^{-tV} e^{-tH_0^\psi} - e^{-t(H_0^\psi + V)}\|_{p \rightarrow p}, \\ & \|e^{-tH_0^\psi/2} e^{-tV} e^{-tH_0^\psi/2} - e^{-t(H_0^\psi + V)}\|_{p \rightarrow p} \\ & = \begin{cases} O(t^2) & \text{if } \alpha < \gamma/2 \\ O(t^2 \int_0^{1/t} (\phi^{-1}(\theta))^{-\alpha} d\theta) & \text{if } \alpha = \gamma/2 \\ O(t^{1+\gamma/2\alpha} L_2(\frac{1}{t})^{-\gamma/2}) & \text{if } \gamma/2 < \alpha. \end{cases} \end{aligned}$$

(ii) Under $(A)_1$,

$$\|e^{-tV/2} e^{-tH_0^\psi} e^{-tV/2} - e^{-t(H_0^\psi + V)}\|_{p \rightarrow p}$$

$$= \begin{cases} O(t^{1+1\wedge 2\delta}) & \text{if } \alpha < (1 + \kappa)/2 \text{ or } \kappa = 1 \\ O(t^{1+2\delta}) + O(t^2 \int_0^{1/t} (\phi^{-1}(\theta))^{-\alpha} d\theta) & \text{if } \alpha = (1 + \kappa)/2 < 1 \\ O(t^{1+2\delta}) + O(t^{1+(1+\kappa)/2\alpha} L_2(\frac{1}{t})^{-(1+\kappa)/2}) & \text{if } (1 + \kappa)/2 < \alpha, \end{cases}$$

$$\|e^{-tV} e^{-tH_0^\psi} - e^{-t(H_0^\psi + V)}\|_{p \rightarrow p},$$

$$\|e^{-tH_0^\psi/2} e^{-tV} e^{-tH_0^\psi/2} - e^{-t(H_0^\psi + V)}\|_{p \rightarrow p}$$

$$= \begin{cases} O(t^{1+\delta}) & \text{if } \alpha < 1/2 \\ O(t^{1+\delta} \int_0^{1/t} (\phi^{-1}(\theta))^{-\alpha} d\theta) & \text{if } \alpha = 1/2 \\ O(t^{\delta+1/2\alpha} L_2(\frac{1}{t})^{-1/2}) & \text{if } 1/2 < \alpha. \end{cases}$$

(iii) Under (A)₂,

$$\|e^{-tV/2} e^{-tH_0^\psi} e^{-tV/2} - e^{-t(H_0^\psi + V)}\|_{p \rightarrow p} = O(t^{1+1\wedge 2\delta}),$$

$$\|e^{-tV} e^{-tH_0^\psi} - e^{-t(H_0^\psi + V)}\|_{p \rightarrow p},$$

$$\|e^{-tH_0^\psi/2} e^{-tV} e^{-tH_0^\psi/2} - e^{-t(H_0^\psi + V)}\|_{p \rightarrow p}$$

$$= \begin{cases} O(t^{1+\delta}) & \text{if } \alpha < 1/2 \\ O(t^{1+\delta} \int_0^{1/t} (\phi^{-1}(\theta))^{-\alpha} d\theta) & \text{if } \alpha = 1/2 \\ O(t^{\delta+1/2\alpha} L_2(\frac{1}{t})^{-1/2}) & \text{if } 1/2 < \alpha. \end{cases}$$

In fact, the first estimate in (iii) holds independent of (L).

A consequence of Theorem 1 is the following Trotter product formula in the L_p -operator norm with error bounds.

Theorem 2. *Suppose assumption (L) and let $1 \leq p \leq \infty$. Then the following estimates (i), (ii), (iii) and (iv) hold uniformly on each finite t -interval on $[0, \infty)$.*

(i) Under (A)₀,

$$\|(e^{-tV/2n} e^{-tH_0^\psi/n} e^{-tV/2n})^n - e^{-t(H_0^\psi + V)}\|_{p \rightarrow p},$$

$$\|(e^{-tV/n} e^{-tH_0^\psi/n})^n - e^{-t(H_0^\psi + V)}\|_{p \rightarrow p},$$

$$\|(e^{-tH_0^\psi/2n} e^{-tV/n} e^{-tH_0^\psi/2n})^n - e^{-t(H_0^\psi + V)}\|_{p \rightarrow p}$$

$$= \begin{cases} O(n^{-1}) & \text{if } \alpha < \gamma/2 \\ O(n^{-1} \int_0^n (\phi^{-1}(\theta))^{-\alpha} d\theta) & \text{if } \alpha = \gamma/2 \\ O(n^{-\gamma/2\alpha} L_2(n)^{-\gamma/2}) & \text{if } \gamma/2 < \alpha. \end{cases}$$

(ii) Under (A)₁,

$$\begin{aligned}
& \|(e^{-tV/2n} e^{-tH_0^\psi/n} e^{-tV/2n})^n - e^{-t(H_0^\psi+V)}\|_{p \rightarrow p}, \\
& \|(e^{-tV/n} e^{-tH_0^\psi/n})^n - e^{-t(H_0^\psi+V)}\|_{p \rightarrow p}, \\
& \|(e^{-tH_0^\psi/2n} e^{-tV/n} e^{-tH_0^\psi/2n})^n - e^{-t(H_0^\psi+V)}\|_{p \rightarrow p} \\
& = \begin{cases} O(n^{-1 \wedge 2\delta}) & \text{if } \alpha < (1 + \kappa)/2 \text{ or } \kappa = 1 \\ O(n^{-2\delta}) + O(n^{-1} \int_0^n (\phi^{-1}(\theta))^{-\alpha} d\theta) & \text{if } \alpha = (1 + \kappa)/2 < 1 \\ O(n^{-2\delta}) + O(n^{-(1+\kappa)/2\alpha} L_2(n)^{-(1+\kappa)/2}) & \text{if } (1 + \kappa)/2 < \alpha. \end{cases}
\end{aligned}$$

(iii) Under (A)₂,

$$\begin{aligned}
& \|(e^{-tV/2n} e^{-tH_0^\psi/n} e^{-tV/2n})^n - e^{-t(H_0^\psi+V)}\|_{p \rightarrow p}, \\
& \|(e^{-tV/n} e^{-tH_0^\psi/n})^n - e^{-t(H_0^\psi+V)}\|_{p \rightarrow p}, \\
& \|(e^{-tH_0^\psi/2n} e^{-tV/n} e^{-tH_0^\psi/2n})^n - e^{-t(H_0^\psi+V)}\|_{p \rightarrow p} \\
& = O(n^{-1 \wedge 2\delta}).
\end{aligned}$$

(iv) Under (V)_i ($i = 1, 2$),

$$\begin{aligned}
& \|(e^{-tV/2n} e^{-tH_0^\psi/n} e^{-tV/2n})^n - e^{-t(H_0^\psi+V)}\|_{p \rightarrow p}, \\
& \|(e^{-tV/n} e^{-tH_0^\psi/n})^n - e^{-t(H_0^\psi+V)}\|_{p \rightarrow p}, \\
& \|(e^{-tH_0^\psi/2n} e^{-tV/n} e^{-tH_0^\psi/2n})^n - e^{-t(H_0^\psi+V)}\|_{p \rightarrow p} \\
& = O(n^{-i/2 \vee \rho}).
\end{aligned}$$

In fact, the asymptotic estimates (iii) and (iv) hold independent of (L).

Notice here that though the estimates with small t , in Theorem 1, for $e^{-tV} e^{-tH_0^\psi}$ and $e^{-tH_0^\psi/2} e^{-tV} e^{-tH_0^\psi/2}$ are of worse order than that for $e^{-tV/2} e^{-tH_0^\psi} e^{-tV/2}$, one has, in Theorem 2, the same error bounds with large n for these three products.

Finally we give a comment on what kind of operators are to be covered by our $H_0^\psi + V$. To this end we briefly illustrate how our result reads on the Trotter product formula in the case $H_0^{(\alpha)} + V$ with $H_0^{(\alpha)} = (-\Delta + 1)^\alpha - 1$, $0 < \alpha < 1$, in (1.5). In this case, we have $n((t, \infty)) = (2^\alpha / \Gamma(1 - \alpha)) t^{-\alpha}$, or $L_2(\cdot) \equiv 2^{-1}$, so that

$$\int_0^x (\phi^{-1}(\theta))^{-\alpha} d\theta \sim 2^\alpha \log x \quad \text{as } x \rightarrow \infty.$$

Therefore Theorem 2 says that for $1 \leq p \leq \infty$ and uniformly on each finite t -interval in $[0, \infty)$,

$$\begin{aligned}
& \|(e^{-tV/2n} e^{-tH_0^{(\alpha)}/n} e^{-tV/2n})^n - e^{-t(H_0^{(\alpha)}+V)}\|_{p \rightarrow p}, \\
& \|(e^{-tV/n} e^{-tH_0^{(\alpha)}/n})^n - e^{-t(H_0^{(\alpha)}+V)}\|_{p \rightarrow p}, \\
& \|(e^{-tH_0^{(\alpha)}/2n} e^{-tV/n} e^{-tH_0^{(\alpha)}/2n})^n - e^{-t(H_0^{(\alpha)}+V)}\|_{p \rightarrow p}
\end{aligned}$$

$$\begin{aligned}
&= \begin{cases} O(n^{-1}) & \text{if } \alpha < \gamma/2 \\ O(n^{-1} \log n) & \text{if } \alpha = \gamma/2 \\ O(n^{-\gamma/2\alpha}) & \text{if } \gamma/2 < \alpha \end{cases} \quad \text{under (A)}_0, \\
&= \begin{cases} O(n^{-1 \wedge 2\delta}) & \text{if } \alpha < (1 + \kappa)/2 \\ O(n^{-1} \log n) & \text{if } \alpha = (1 + \kappa)/2 \text{ and } 1/2 \leq \delta \leq 1 \\ O(n^{-2\delta}) & \text{if } \alpha = (1 + \kappa)/2 \text{ and } 0 < \delta < 1/2 \\ O(n^{-2\delta \wedge (1+\kappa)/2\alpha}) & \text{if } (1 + \kappa)/2 < \alpha \end{cases} \quad \text{under (A)}_1.
\end{aligned}$$

An important remark is the following. In the above example, the case $\alpha = 1$ is missing. This is equivalent to the nonrelativistic case $H_0 + V = -\Delta/2 + V(x)$, treated in [Tak] (cf. [I-Tak1,2]). However we may think that this case is also implicitly contained in our results, Theorems 1 and 2, for $\alpha = 1/2$. Indeed, by using $H_0^r(c) = \sqrt{-c^2\Delta + c^4} - c^2$ with light velocity c restored in place of H_0^r in (1.2), we can obtain the case $\alpha = 1/2$ so as to involve the parameter c (light velocity). Since, in the nonrelativistic limit $c \rightarrow \infty$, the relativistic Schrödinger semigroup $e^{-t(H_0^r(c)+V)}$ is strongly convergent to the nonrelativistic Schrödinger semigroup $e^{-t(H_0+V)}$ uniformly on each finite t -interval in $[0, \infty)$ (e.g. [I2]), we can reproduce the nonrelativistic result in [Tak] (cf. Remark following Theorem 2.3).

In Section 2, we state our results in more general form: we generalize Theorems 1 and 2 to Theorems 2.1 and 2.2 / 2.3 by introducing the subordinator σ_t , namely, a time-homogeneous Lévy process associated with the Lévy measure $e^{-l/2}n(dl)$. Moreover we state Theorem 2.4 on asymptotics of the moments of the process σ_t . Once we know these asymptotics, we can obtain Theorems 1 and 2 from Theorems 2.1 and 2.2 / 2.3. These four theorems are proved in Sections 3–6.

In Appendix, we give a full study of the semigroups $e^{-t(H_0^\psi+V)}$, $t \geq 0$, on $L_p(\mathbb{R}^d)$, $1 \leq p < \infty$ and $C_\infty(\mathbb{R}^d)$ defined through the Feynman-Kac formula. We show they constitute a strongly continuous contraction semigroup there. It is also shown that its infinitesimal generator $\mathfrak{G}_p^{\psi,V}$ has $C_0^\infty(\mathbb{R}^d)$ as a core, by establishing Kato's inequality for the operator H_0^ψ . Some of these results seem to be new.

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2. General results

In this section we shall prove the theorems in a little more general setting based on probability theory. To describe it we introduce some notations and notions. For a continuous function $V : \mathbb{R}^d \rightarrow [0, \infty)$, set

$$\begin{aligned}
K(t) &:= e^{-tV/2} e^{-tH_0^\psi} e^{-tV/2}, \\
G(t) &:= e^{-tV} e^{-tH_0^\psi},
\end{aligned}$$

$$R(t) := e^{-tH_0^\psi/2} e^{-tV} e^{-tH_0^\psi/2}$$

and

$$Q_K(t) := K(t) - e^{-t(H_0^\psi + V)},$$

$$Q_G(t) := G(t) - e^{-t(H_0^\psi + V)},$$

$$Q_R(t) := R(t) - e^{-t(H_0^\psi + V)}.$$

Suppose we are given the independent random objects $N(\cdot)$ and $B(\cdot)$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$:

- (i) $N(dsdl)$ is a Poisson random measure on $[0, \infty) \times (0, \infty)$ such that $\mathbb{E}[N(dsdl)] = dse^{-l/2}n(dl)$;
- (ii) $(B(t))_{t \geq 0}$ is a d -dimensional Brownian motion starting at 0.

Set

$$\sigma_t := \int_0^{t+} \int_{(0, \infty)} l N(dsdl). \quad (2.1)$$

Then $(\sigma_t)_{t \geq 0}$ is a time-homogeneous Lévy process with increasing paths such that

$$\mathbb{E}[e^{-\lambda \sigma_t}] = e^{-t(\psi(\lambda + 1/2) - \psi(1/2))} \quad (2.2)$$

(e.g. Note 1.7.1 in [It-MK]). Note that σ_t has moments of all order (cf. (6.1)), which is to be seen at the beginning of Section 6. We use a subordination of $B(\cdot)$ by a subordinator σ_\cdot , i.e., a process $(B(\sigma_t))_{t \geq 0}$ on \mathbb{R}^d . This is a Lévy process such that

$$\mathbb{E}[e^{\sqrt{-1}\langle p, B(\sigma_t) \rangle}] = e^{-t(\psi((|p|^2 + 1)/2) - \psi(1/2))},$$

which corresponds to the semigroup $\{e^{-tH_0^\psi}\}_{t \geq 0}$ with generator H_0^ψ in (1.1).

We prove the following generalization of Theorems 1 and 2.

Theorem 2.1. *Let $1 \leq p \leq \infty$ and $t \geq 0$.*

(i) *Under $(A)_0$,*

$$\|Q_K(t)\|_{p \rightarrow p}, \|Q_G(t)\|_{p \rightarrow p}, \|Q_R(t)\|_{p \rightarrow p} \leq \text{const}(\gamma, d) C_1 t \mathbb{E}[\sigma_t^{\gamma/2}].$$

(ii) *Under $(A)_1$,*

$$\|Q_K(t)\|_{p \rightarrow p} \leq \text{const}(\delta, \kappa, d) \left[C_1^2 (t^2 + t^{2\delta}) \mathbb{E}[\sigma_t] + \sum_{j=1}^2 (C_2 t)^j \mathbb{E}[\sigma_t^{j(1+\kappa)/2}] \right],$$

$$\|Q_G(t)\|_{p \rightarrow p}, \|Q_R(t)\|_{p \rightarrow p} \leq \text{const}(\delta, \kappa, d) \sum_{j=1}^2 \left\{ C_1^j (t^j + t^{j\delta}) \mathbb{E}[\sigma_t^{j/2}] + (C_2 t)^j \mathbb{E}[\sigma_t^{j(1+\kappa)/2}] \right\}.$$

(iii) Under (A)₂,

$$\begin{aligned} \|Q_K(t)\|_{p \rightarrow p} &\leq \text{const}(\delta, \mu, \nu, d) \left[C_1^2(t^2 + t^{2\delta})\mathbb{E}[\sigma_t] + \sum_{j=1}^2 \left\{ (C_2 t)^j \mathbb{E}[\sigma_t^j] \right. \right. \\ &\quad \left. \left. + (C_2 t)^j \mathbb{E}[\sigma_t^{j(1+\nu/2)}] + (C_2 t^{1\wedge 2\delta})^j \mathbb{E}[\sigma_t^j] + (C_2 t^{1\wedge 2\delta})^j \mathbb{E}[\sigma_t^{j(1+\mu/2)}] \right\} \right], \\ \|Q_G(t)\|_{p \rightarrow p}, \|Q_R(t)\|_{p \rightarrow p} &\leq \text{const}(\delta, \mu, \nu, d) \sum_{j=1}^2 \left\{ C_1^j(t^j + t^{j\delta})\mathbb{E}[\sigma_t^{j/2}] + (C_2 t)^j \mathbb{E}[\sigma_t^j] \right. \\ &\quad \left. + (C_2 t)^j \mathbb{E}[\sigma_t^{j(1+\nu/2)}] + (C_2 t^{1\wedge 2\delta})^j \mathbb{E}[\sigma_t^j] \right. \\ &\quad \left. + (C_2 t^{1\wedge 2\delta})^j \mathbb{E}[\sigma_t^{j(1+\mu/2)}] \right\}. \end{aligned}$$

Theorem 2.2. Let $1 \leq p \leq \infty$, $t \geq 0$ and $n \in \mathbb{N}$.

(i) Under (A)₀,

$$\begin{aligned} &\|(e^{-tV/2n} e^{-tH_0^\psi/n} e^{-tV/2n})^n - e^{-t(H_0^\psi+V)}\|_{p \rightarrow p}, \\ &\|(e^{-tV/n} e^{-tH_0^\psi/n})^n - e^{-t(H_0^\psi+V)}\|_{p \rightarrow p}, \\ &\|(e^{-tH_0^\psi/2n} e^{-tV/n} e^{-tH_0^\psi/2n})^n - e^{-t(H_0^\psi+V)}\|_{p \rightarrow p} \\ &\leq \text{const}(\gamma, d) C_1 t \mathbb{E}[\sigma_{t/n}^{\gamma/2}]. \end{aligned}$$

(ii) Under (A)₁,

$$\begin{aligned} &\|(e^{-tV/2n} e^{-tH_0^\psi/n} e^{-tV/2n})^n - e^{-t(H_0^\psi+V)}\|_{p \rightarrow p} \\ &\leq \text{const}(\delta, \kappa, d) \left[C_1^2 \left(\left(\frac{t}{n}\right)^2 + \left(\frac{t}{n}\right)^{2\delta} \right) n \mathbb{E}[\sigma_{t/n}] + \sum_{j=1}^2 (C_2 \frac{t}{n})^j n \mathbb{E}[\sigma_{t/n}^{j(1+\kappa)/2}] \right], \\ &\|(e^{-tV/n} e^{-tH_0^\psi/n})^n - e^{-t(H_0^\psi+V)}\|_{p \rightarrow p}, \\ &\|(e^{-tH_0^\psi/2n} e^{-tV/n} e^{-tH_0^\psi/2n})^n - e^{-t(H_0^\psi+V)}\|_{p \rightarrow p} \\ &\leq \text{const}(\delta, \kappa, d) \left[\frac{1}{n} \left(C_1(t + t^\delta) \mathbb{E}[\sigma_t^{1/2}] + C_1 t \mathbb{E}[\sigma_t^{(1+\kappa)/2}] \right) \right. \\ &\quad \left. + C_1 \left(\frac{t}{n} + \left(\frac{t}{n}\right)^\delta \right) \mathbb{E}[\sigma_{t/n}^{1/2}] + C_1^2 \left(\left(\frac{t}{n}\right)^2 + \left(\frac{t}{n}\right)^{2\delta} \right) n \mathbb{E}[\sigma_{t/n}] + \sum_{j=1}^2 (C_2 \frac{t}{n})^j n \mathbb{E}[\sigma_{t/n}^{j(1+\kappa)/2}] \right]. \end{aligned}$$

(iii) Under (A)₂,

$$\begin{aligned} &\|(e^{-tV/2n} e^{-tH_0^\psi/n} e^{-tV/2n})^n - e^{-t(H_0^\psi+V)}\|_{p \rightarrow p} \\ &\leq \text{const}(\delta, \mu, \nu, d) \left[C_1^2 \left(\left(\frac{t}{n}\right)^2 + \left(\frac{t}{n}\right)^{2\delta} \right) n \mathbb{E}[\sigma_{t/n}] + \sum_{j=1}^2 \left\{ (C_2 \frac{t}{n})^j n \mathbb{E}[\sigma_{t/n}^j] \right. \right. \\ &\quad \left. \left. + (C_2 \frac{t}{n})^j n \mathbb{E}[\sigma_{t/n}^{j(1+\nu/2)}] + (C_2 (\frac{t}{n})^{1\wedge 2\delta})^j n \mathbb{E}[\sigma_{t/n}^j] + (C_2 (\frac{t}{n})^{1\wedge 2\delta})^j n \mathbb{E}[\sigma_{t/n}^{j(1+\mu/2)}] \right\} \right], \end{aligned}$$

$$\begin{aligned}
& \| (e^{-tV/n} e^{-tH_0^\psi/n})^n - e^{-t(H_0^\psi+V)} \|_{p \rightarrow p}, \\
& \| (e^{-tH_0^\psi/2n} e^{-tV/n} e^{-tH_0^\psi/2n})^n - e^{-t(H_0^\psi+V)} \|_{p \rightarrow p} \\
& \leq \text{const}(\delta, \mu, \nu, d) \left[\frac{1}{n} \left(C_1(t + t^\delta) \mathbb{E}[\sigma_t^{1/2}] + C_2 t^{1 \wedge 2\delta} (\mathbb{E}[\sigma_t] + \mathbb{E}[\sigma_t^{1+\mu/2}]) \right. \right. \\
& \quad \left. \left. + C_2 t (\mathbb{E}[\sigma_t] + \mathbb{E}[\sigma_t^{1+\nu/2}]) \right) + C_1 \left(\frac{t}{n} + \left(\frac{t}{n} \right)^\delta \right) \mathbb{E}[\sigma_{t/n}^{1/2}] + C_1^2 \left(\left(\frac{t}{n} \right)^2 + \left(\frac{t}{n} \right)^{2\delta} \right) n \mathbb{E}[\sigma_{t/n}] \right. \\
& \quad \left. + \sum_{j=1}^2 \left\{ (C_2 \frac{t}{n})^j n \mathbb{E}[\sigma_{t/n}^j] + (C_2 \frac{t}{n})^j n \mathbb{E}[\sigma_{t/n}^{j(1+\nu/2)}] + (C_2 (\frac{t}{n})^{1 \wedge 2\delta})^j n \mathbb{E}[\sigma_{t/n}^j] \right. \right. \\
& \quad \left. \left. + (C_2 (\frac{t}{n})^{1 \wedge 2\delta})^j n \mathbb{E}[\sigma_{t/n}^{j(1+\mu/2)}] \right\} \right].
\end{aligned}$$

Theorem 2.3. *Let $1 \leq p \leq \infty$ and $t \geq 0$.*

(i) *Under (V)₁ for $n \geq 2^{2(2\nu\rho)}$,*

$$\begin{aligned}
& \| (e^{-tV/2n} e^{-tH_0^\psi/n} e^{-tV/2n})^n - e^{-t(H_0^\psi+V)} \|_{p \rightarrow p} \\
& \leq \text{const}(\rho, c, c_1, d) n^{-1/2\nu\rho} \left[t^{2/(\rho \wedge 2)\vee 1 - 1} + (t^2 + t^{2(1 \wedge ((\rho \wedge 2)\vee 1)/2\rho)}) n \mathbb{E}[\sigma_{t/n}] \right. \\
& \quad \left. + \sum_{j=1}^2 \left((t^j + t^{j2/2\nu\rho}) n \mathbb{E}[\sigma_{t/n}^j] + t^j n \mathbb{E}[\sigma_{t/n}^{j(2\nu\rho)/2}] \right) \right],
\end{aligned}$$

$$\begin{aligned}
& \| (e^{-tV/n} e^{-tH_0^\psi/n})^n - e^{-t(H_0^\psi+V)} \|_{p \rightarrow p}, \\
& \| (e^{-tH_0^\psi/2n} e^{-tV/n} e^{-tH_0^\psi/2n})^n - e^{-t(H_0^\psi+V)} \|_{p \rightarrow p} \\
& \leq \text{const}(\rho, c, c_1, d) n^{-1/2\nu\rho} \left[t^{2/(\rho \wedge 2)\vee 1 - 1} + (t + t^{1 \wedge ((\rho \wedge 2)\vee 1)/2\rho}) \mathbb{E}[\sigma_t^{1/2}] \right. \\
& \quad \left. + t^{2/2\nu\rho} \mathbb{E}[\sigma_t] + t (\mathbb{E}[\sigma_t] + \mathbb{E}[\sigma_t^{(2\nu\rho)/2}]) + (t^2 + t^{2(1 \wedge ((\rho \wedge 2)\vee 1)/2\rho)}) n \mathbb{E}[\sigma_{t/n}] \right. \\
& \quad \left. + \sum_{j=1}^2 \left\{ (t^j + t^{j2/2\nu\rho}) n \mathbb{E}[\sigma_{t/n}^j] + t^j n \mathbb{E}[\sigma_{t/n}^{j(2\nu\rho)/2}] \right\} \right].
\end{aligned}$$

(ii) *Under (V)₂ for $n \geq 1$,*

$$\begin{aligned}
& \| e^{-tV/2n} e^{-tH_0^\psi/n} e^{-tV/2n})^n - e^{-t(H_0^\psi+V)} \|_{p \rightarrow p} \\
& \leq \text{const}(\rho, c, c_1, c_2, d) n^{-2/2\nu\rho} \left[(t^2 + t^{2/1\nu\rho}) n \mathbb{E}[\sigma_{t/n}] \right. \\
& \quad \left. + \sum_{j=1}^2 \left((t^j + t^{j2/2\nu\rho}) n \mathbb{E}[\sigma_{t/n}^j] + t^j n \mathbb{E}[\sigma_{t/n}^{j(2\nu\rho)/2}] \right) \right],
\end{aligned}$$

$$\begin{aligned}
& \| (e^{-tV/n} e^{-tH_0^\psi/n})^n - e^{-t(H_0^\psi+V)} \|_{p \rightarrow p}, \\
& \| (e^{-tH_0^\psi/2n} e^{-tV/n} e^{-tH_0^\psi/2n})^n - e^{-t(H_0^\psi+V)} \|_{p \rightarrow p} \\
& \leq \text{const}(\rho, c, c_1, c_2, d) \left[n^{-2/2\nu\rho} \left((t + t^{1/1\nu\rho}) \mathbb{E}[\sigma_t^{1/2}] + (t + t^{2/2\nu\rho}) \mathbb{E}[\sigma_t] + t \mathbb{E}[\sigma_t^{(2\nu\rho)/2}] \right) \right. \\
& \quad \left. + (t^2 + t^{2/1\nu\rho}) n \mathbb{E}[\sigma_{t/n}] + \sum_{j=1}^2 \left\{ (t^j + t^{j2/2\nu\rho}) n \mathbb{E}[\sigma_{t/n}^j] + t^j n \mathbb{E}[\sigma_{t/n}^{j(2\nu\rho)/2}] \right\} \right]
\end{aligned}$$

$$+ n^{-1/1\nu\rho} \mathbb{E}[\sigma_{t/n}^{1/2}](t + t^{1/1\nu\rho}).$$

Remark. As noted at the end of Section 1, the *nonrelativistic* case for $H_0 + V = -\Delta/2 + V$, being equivalent to the case $\alpha = 1$ which Theorems 1 and 2 fail to cover, can be thought to be implicitly contained in the *relativistic* case, of the above three theorems, for the relativistic Schrödinger operator $H_0^r(c) \equiv \sqrt{-c^2\Delta + c^4} - c^2$ with the light velocity $c \geq 1$ restored. We have $H_0^\psi = H_0^r(c)$, where this $\psi(\lambda)$ is a c -dependent function (1.3) given by $\psi(\lambda) := \psi(\lambda; c) = \sqrt{2c^2\lambda + c^4 - c^2} - \sqrt{c^4 - c^2}$ associated with the c -dependent Lévy measure $e^{-l/2}n(dl; c) = (2\pi)^{-1/2}ce^{-c^2l/2}l^{-3/2}dl$. In this case, Theorem 2.1 and Theorems 2.2/2.3 hold with the corresponding c -dependent subordinator $\sigma_t(c)$, just as they stand, namely, only with $\mathbb{E}[\sigma_s^a]$ replaced by $\mathbb{E}[\sigma_s(c)^a]$ for each respective $s > 0$ and $a > 0$. Then the nonrelativistic case in question is obtained as the nonrelativistic limit $c \rightarrow \infty$ of this c -dependent relativistic case, turning out to be just Theorems 2.1 and 2.2/2.3 with $\mathbb{E}[\sigma_s^a]$ replaced by s^a . This is because one can show that, as $c \rightarrow \infty$, the relativistic Schrödinger semigroup $e^{-t(H_0^r(c)+V)}$ on the LHS converges strongly to the nonrelativistic Schrödinger semigroup $e^{-t(H_0+V)}$ uniformly on each finite t -interval in $[0, \infty)$ (cf. [I2]), and $\mathbb{E}[\sigma_t(c)^a]$ on the RHS tends to t^a . Then taking the most dominant contribution on the RHS for small t or large n reproduces the same nonrelativistic result as in [Tak].

Theorems 1 and 2 follow immediately from Theorems 2.1 and 2.2/2.3, if one knows the asymptotics for $t \downarrow 0$ of the moments of σ_t to investigate which of the terms on the RHS makes a dominant contribution for small t or large n . These asymptotics are given by the following theorem.

Theorem 2.4. *Suppose assumption (L). Let $a > 0$.*

(i) *If $\alpha < a$ or $a \geq 1$, then $\int_{(0,\infty)} l^\alpha e^{-l/2}n(dl) < \infty$ and*

$$\mathbb{E}[\sigma_t^a] \sim t \int_{(0,\infty)} l^\alpha e^{-l/2}n(dl) \quad \text{as } t \downarrow 0.$$

In fact, for $a \geq 1$ this always holds independent of (L).

(ii) *If $\alpha = a$ and $a < 1$, then*

$$\mathbb{E}[\sigma_t^a] \sim \frac{1}{\Gamma(1-\alpha)} t \int_0^{1/t} (\phi^{-1}(\theta))^{-\alpha} d\theta \quad \text{as } t \downarrow 0.$$

(iii) *If $0 < a < \alpha$, then*

$$\mathbb{E}[\sigma_t^a] \sim \frac{\Gamma(1-\frac{a}{\alpha})}{\Gamma(1-a)} t^{a/\alpha} L_2(\frac{1}{t})^{-a} \quad \text{as } t \downarrow 0.$$

The proofs of Theorems 2.1, 2.2, 2.3 and 2.4 are given in Sections 3, 4, 5 and 6, respectively. To show Theorem 2.1, in fact, we prove estimates of the *integral kernels* of $Q_K(t)$, $Q_G(t)$ and $Q_R(t)$ by a finite positive linear combination of $t^c \mathbb{E}[|x-y|^\alpha \sigma_t^b p(\sigma_t, x-y)]$, where $p(t, x-y)$ is the heat kernel (see (A.2)). Such estimates of the integral kernels of the three operators of difference in Theorems 2.2/2.3 also can be obtained (cf. [Tak]), but are omitted.

3. Proof of Theorem 2.1

It is easily seen (see (A.6)) that for $f \in C_0^\infty(\mathbb{R}^d)$

$$K(t)f(x) = \mathbb{E}\left[\exp\left(-\frac{t}{2}(V(x) + V(x + X_t))\right)f(x + X_t)\right], \quad (3.1)$$

$$G(t)f(x) = \mathbb{E}\left[\exp\left(-tV(x)\right)f(x + X_t)\right], \quad (3.2)$$

$$R(t)f(x) = \mathbb{E}\left[\exp\left(-tV(x + X_{t/2})\right)f(x + X_t)\right] \quad (3.3)$$

and generally

$$K\left(\frac{t}{n}\right)^n f(x) = \mathbb{E}\left[\exp\left(-\frac{t}{2n} \sum_{k=1}^n (V(x + X_{(k-1)t/n}) + V(x + X_{kt/n}))\right)f(x + X_t)\right], \quad (3.4)$$

$$G\left(\frac{t}{n}\right)^n f(x) = \mathbb{E}\left[\exp\left(-\frac{t}{n} \sum_{k=1}^n V(x + X_{(k-1)t/n})\right)f(x + X_t)\right], \quad (3.5)$$

$$R\left(\frac{t}{n}\right)^n f(x) = \mathbb{E}\left[\exp\left(-\frac{t}{n} \sum_{k=1}^n V(x + X_{(2k-1)t/2n})\right)f(x + X_t)\right]. \quad (3.6)$$

Further, for $f \in C_0^\infty(\mathbb{R}^d)$ we have (see (A.13))

$$Q_K(t)f(x) = \int_{\mathbb{R}^d} dy f(y) \mathbb{E}_\sigma \left[\mathbb{E}_B [v_K(t, x, y; \sigma)] p(\sigma_t, x - y) \right], \quad (3.7)$$

$$Q_G(t)f(x) = \int_{\mathbb{R}^d} dy f(y) \mathbb{E}_\sigma \left[\mathbb{E}_B [v_G(t, x, y; \sigma)] p(\sigma_t, x - y) \right], \quad (3.8)$$

$$Q_R(t)f(x) = \int_{\mathbb{R}^d} f(y) dy \mathbb{E}_\sigma \left[\mathbb{E}_B [v_R(t, x, y; \sigma)] p(\sigma_t, x - y) \right], \quad (3.9)$$

where \mathbb{E}_σ and \mathbb{E}_B are the expectations with respect to σ . and B ., respectively,

$$v_K(t, x, y; \sigma) := \exp\left(-\frac{t}{2}(V(x) + V(y))\right) - \exp\left(-\int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_s)) ds\right), \quad (3.10)$$

$$v_G(t, x, y; \sigma) := \exp\left(-tV(x)\right) - \exp\left(-\int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_s)) ds\right), \quad (3.11)$$

$$v_R(t, x, y; \sigma) := \exp\left(-tV(B_{0,x}^{\sigma_t,y}(\sigma_{t/2}))\right) - \exp\left(-\int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_s)) ds\right), \quad (3.12)$$

and, for $\tau > 0$, $x, y \in \mathbb{R}^d$ and $0 \leq \theta \leq \tau$

$$\begin{aligned} B_{0,x}^{\tau,y}(\theta) &:= x + \frac{\theta}{\tau}(y - x) + B_0^\tau(\theta) \\ B_0^\tau(\theta) &:= B(\theta) - \frac{\theta}{\tau}B(\tau). \end{aligned} \quad (3.13)$$

Since

$$e^a - e^b = (a - b)e^b + (a - b)^2 \int_0^1 (1 - \theta) e^{\theta a} e^{(1-\theta)b} d\theta, \quad a, b \in \mathbb{R},$$

we have

$$\begin{aligned}
v_K(t, x, y; \sigma) &= w_K(t, x, y; \sigma) \exp\left(-\frac{t}{2}(V(x) + V(y))\right) \\
&\quad - w_K(t, x, y; \sigma)^2 \int_0^1 (1 - \theta) \exp\left(-\theta \int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_s)) ds\right) \\
&\quad \quad \times \exp\left(-(1 - \theta)\frac{t}{2}(V(x) + V(y))\right) d\theta \\
&=: v_{K1}(t, x, y; \sigma) + v_{K2}(t, x, y; \sigma),
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
v_G(t, x, y; \sigma) &= w_G(t, x, y; \sigma) \exp\left(-tV(x)\right) \\
&\quad - w_G(t, x, y; \sigma)^2 \int_0^1 (1 - \theta) \exp\left(-\theta \int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_s)) ds\right) \\
&\quad \quad \times \exp\left(-(1 - \theta)tV(x)\right) d\theta \\
&=: v_{G1}(t, x, y; \sigma) + v_{G2}(t, x, y; \sigma),
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
v_R(t, x, y; \sigma) &= w_R(t, x, y; \sigma) \exp\left(-tV(B_{0,x}^{\sigma_t,y}(\sigma_{t/2}))\right) \\
&\quad - w_R(t, x, y; \sigma)^2 \int_0^1 (1 - \theta) \exp\left(-\theta \int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_s)) ds\right) \\
&\quad \quad \times \exp\left(-(1 - \theta)tV(B_{0,x}^{\sigma_t,y}(\sigma_{t/2}))\right) d\theta \\
&=: v_{R1}(t, x, y; \sigma) + v_{R2}(t, x, y; \sigma),
\end{aligned} \tag{3.16}$$

where

$$w_K(t, x, y; \sigma) := -\frac{t}{2}(V(x) + V(y)) + \int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_s)) ds, \tag{3.17}$$

$$w_G(t, x, y; \sigma) := -tV(x) + \int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_s)) ds, \tag{3.18}$$

$$w_R(t, x, y; \sigma) := -tV(B_{0,x}^{\sigma_t,y}(\sigma_{t/2})) + \int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_s)) ds. \tag{3.19}$$

When V is further a C^1 -function, since

$$V(z) - V(w) = \langle \nabla V(w), z - w \rangle + \int_0^1 \langle \nabla V(w + \theta(z - w)) - \nabla V(w), z - w \rangle d\theta,$$

we have

$$\begin{aligned}
w_K(t, x, y; \sigma) &= \frac{1}{2} \langle \nabla V(x) - \nabla V(y), y - x \rangle \int_0^t \frac{\sigma_s}{\sigma_t} ds \\
&\quad + \frac{1}{2} \langle \nabla V(y), y - x \rangle \left(\int_0^t \frac{\sigma_s}{\sigma_t} ds - \int_0^t \frac{\sigma_t - \sigma_s}{\sigma_t} ds \right) \\
&\quad + \frac{1}{2} \left\langle \nabla V(x) + \nabla V(y), \int_0^t B_0^{\sigma_t}(\sigma_s) ds \right\rangle
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \int_0^t ds \int_0^1 \left\langle \nabla V(x + \theta(\frac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s))) - \nabla V(x), \right. \\
& \quad \left. \frac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s) \right\rangle d\theta \\
& + \frac{1}{2} \int_0^t ds \int_0^1 \left\langle \nabla V(y + \theta(\frac{\sigma_t - \sigma_s}{\sigma_t}(x-y) + B_0^{\sigma_t}(\sigma_s))) - \nabla V(y), \right. \\
& \quad \left. \frac{\sigma_t - \sigma_s}{\sigma_t}(x-y) + B_0^{\sigma_t}(\sigma_s) \right\rangle d\theta \\
& =: \sum_{j=1}^5 w_{Kj}(t, x, y; \sigma), \tag{3.20}
\end{aligned}$$

$$\begin{aligned}
w_G(t, x, y; \sigma) & = \left\langle \nabla V(x), \int_0^t (\frac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s)) ds \right\rangle \\
& + \int_0^t ds \int_0^1 \left\langle \nabla V(x + \theta(\frac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s))) - \nabla V(x), \right. \\
& \quad \left. \frac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s) \right\rangle d\theta \\
& =: w_{G1}(t, x, y; \sigma) + w_{G2}(t, x, y; \sigma), \tag{3.21}
\end{aligned}$$

$$\begin{aligned}
w_R(t, x, y; \sigma) & = \left\langle \nabla V(B_{0,x}^{\sigma_t,y}(\sigma_{t/2})), \int_0^t (B_{0,x}^{\sigma_t,y}(\sigma_s) - B_{0,x}^{\sigma_t,y}(\sigma_{t/2})) ds \right\rangle \\
& + \int_0^t ds \int_0^1 \left\langle \nabla V(B_{0,x}^{\sigma_t,y}(\sigma_{t/2}) + \theta(B_{0,x}^{\sigma_t,y}(\sigma_s) - B_{0,x}^{\sigma_t,y}(\sigma_{t/2}))) \right. \\
& \quad \left. - \nabla V(B_{0,x}^{\sigma_t,y}(\sigma_{t/2})), B_{0,x}^{\sigma_t,y}(\sigma_s) - B_{0,x}^{\sigma_t,y}(\sigma_{t/2}) \right\rangle d\theta \\
& =: w_{R1}(t, x, y; \sigma) + w_{R2}(t, x, y; \sigma). \tag{3.22}
\end{aligned}$$

In the following we shall prove Theorem 2.1 only in Cases (A)₂ and (A)₀. The proof of Case (A)₁ is omitted; it is similar to that of (A)₂.

3.1. Case (A)₂

In this subsection, we suppose condition (A)₂ on $V(x)$.

Claim 3.1.

$$\begin{aligned}
& \left| \mathbb{E}_\sigma \left[\mathbb{E}_B [v_{K1}(t, x, y; \sigma)] p(\sigma_t, x - y) \right] \right| \\
& \leq \text{const}(\delta, \mu, \nu, d) C_2 \left[t^{1 \wedge 2\delta} \left(\mathbb{E}_\sigma [|x - y|^2 p(\sigma_t, x - y)] + \mathbb{E}_\sigma [\sigma_t p(\sigma_t, x - y)] \right) \right. \\
& \quad \left. + \mathbb{E}_\sigma [|x - y|^{2+\mu} p(\sigma_t, x - y)] + \mathbb{E}_\sigma [\sigma_t^{1+\mu/2} p(\sigma_t, x - y)] \right) \\
& \quad + t \left(\mathbb{E}_\sigma [|x - y|^2 p(\sigma_t, x - y)] + \mathbb{E}_\sigma [\sigma_t p(\sigma_t, x - y)] + \mathbb{E}_\sigma [|x - y|^{2+\nu} p(\sigma_t, x - y)] \right. \\
& \quad \left. + \mathbb{E}_\sigma [\sigma_t^{1+\nu/2} p(\sigma_t, x - y)] \right).
\end{aligned}$$

Proof. In view of (3.14) and (3.20), we set

$$\begin{aligned} v_{K1}(t, x, y; \sigma) &= \sum_{j=1}^5 w_{Kj}(t, x, y; \sigma) e^{-t(V(x)+V(y))/2} \\ &=: \sum_{j=1}^5 v_{K1j}(t, x, y; \sigma). \end{aligned} \quad (3.23)$$

Clearly

$$\mathbb{E}_B[w_{K3}(t, x, y; \sigma)] = \frac{1}{2} \left\langle \nabla V(x) + \nabla V(y), \int_0^t \mathbb{E}_B[B_0^{\sigma_t}(\sigma_s)] ds \right\rangle = 0,$$

and hence $\mathbb{E}_B[v_{K13}(t, x, y; \sigma)] = 0$. By the fact $(\sigma_t - \sigma_{t-s})_{0 \leq s \leq t} \stackrel{\mathcal{L}}{\sim} (\sigma_s)_{0 \leq s \leq t}$,

$$\begin{aligned} &\mathbb{E}_\sigma \left[w_{K2}(t, x, y; \sigma) p(\sigma_t, x - y) \right] \\ &= \frac{1}{2} \langle \nabla V(y), y - x \rangle \left(\mathbb{E}_\sigma \left[\int_0^t \frac{\sigma_s}{\sigma_t} ds p(\sigma_t, x - y) \right] - \mathbb{E}_\sigma \left[\int_0^t \frac{\sigma_t - \sigma_{t-s}}{\sigma_t - \sigma_{t-t}} ds p(\sigma_t - \sigma_{t-t}, x - y) \right] \right) \\ &= 0, \end{aligned}$$

and hence $\mathbb{E}_\sigma \left[\mathbb{E}_B[v_{K12}(t, x, y; \sigma)] p(\sigma_t, x - y) \right] = \mathbb{E}_\sigma \left[v_{K12}(t, x, y; \sigma) p(\sigma_t, x - y) \right] = 0$. By (A)₂(ii)

$$\begin{aligned} |v_{K11}(t, x, y; \sigma)| &= |w_{K1}(t, x, y; \sigma)| e^{-t(V(x)+V(y))/2} \\ &\leq \frac{1}{2} |\nabla V(x) - \nabla V(y)| |x - y| t e^{-t(V(x)+V(y))/2} \\ &\leq \frac{C_2}{2} \left\{ V(x)^{(1-2\delta)_+} (1 + |x - y|^\mu) + 1 + |x - y|^\nu \right\} |x - y|^2 t e^{-tV(x)/2} \\ &\leq \frac{C_2}{2} \left\{ V(x)^{(1-2\delta)_+} e^{-tV(x)/2} t (|x - y|^2 + |x - y|^{2+\mu}) + t (|x - y|^2 + |x - y|^{2+\nu}) \right\} \\ &\leq \frac{C_2}{2} \left\{ \left(\frac{2(1-2\delta)_+}{e} \right)^{(1-2\delta)_+} t^{1 \wedge 2\delta} (|x - y|^2 + |x - y|^{2+\mu}) + t (|x - y|^2 + |x - y|^{2+\nu}) \right\}. \end{aligned} \quad (3.24)$$

Here (and hereafter) the following inequality has been (will be) used:

$$t^b e^{-t} \leq \left(\frac{b}{e}\right)^b, \quad t \geq 0, b \geq 0, \quad (3.25)$$

where for $b = 0$ we understand $(0/e)^0 := 1$. By (A)₂(ii) and (3.25) again

$$\begin{aligned} |v_{K14}(t, x, y; \sigma)| &= |w_{K4}(t, x, y; \sigma)| e^{-t(V(x)+V(y))/2} \\ &\leq \frac{1}{2} \int_0^t ds \int_0^1 |\nabla V(x + \theta(\frac{\sigma_s}{\sigma_t}(y - x) + B_0^{\sigma_t}(\sigma_s))) - \nabla V(x)| \\ &\quad \times \left| \frac{\sigma_s}{\sigma_t}(y - x) + B_0^{\sigma_t}(\sigma_s) \right| d\theta e^{-tV(x)/2} \\ &\leq \frac{C_2}{2} \int_0^t \left\{ V(x)^{(1-2\delta)_+} e^{-tV(x)/2} \left(\left| \frac{\sigma_s}{\sigma_t}(y - x) + B_0^{\sigma_t}(\sigma_s) \right|^2 + \left| \frac{\sigma_s}{\sigma_t}(y - x) + B_0^{\sigma_t}(\sigma_s) \right|^{2+\mu} \right) \right\} \end{aligned}$$

$$\begin{aligned}
& + \left| \frac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s) \right|^2 + \left| \frac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s) \right|^{2+\nu} \Big\} ds \\
\leq & \frac{C_2}{2} \int_0^t \left\{ \left(\frac{2(1-2\delta)_+}{e} \right)^{(1-2\delta)_+} t^{-(1-2\delta)_+} \right. \\
& \times \left(\left| \frac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s) \right|^2 + \left| \frac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s) \right|^{2+\mu} \right) \\
& \left. + \left| \frac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s) \right|^2 + \left| \frac{\sigma_s}{\sigma_t}(y-x) + B_0^{\sigma_t}(\sigma_s) \right|^{2+\nu} \right\} ds. \tag{3.26}
\end{aligned}$$

Similarly

$$\begin{aligned}
& |v_{K15}(t, x, y; \sigma)| \\
\leq & \frac{C_2}{2} \int_0^t \left\{ \left(\frac{2(1-2\delta)_+}{e} \right)^{(1-2\delta)_+} t^{-(1-2\delta)_+} \right. \\
& \times \left(\left| \frac{\sigma_t - \sigma_s}{\sigma_t}(x-y) + B_0^{\sigma_t}(\sigma_s) \right|^2 + \left| \frac{\sigma_t - \sigma_s}{\sigma_t}(x-y) + B_0^{\sigma_t}(\sigma_s) \right|^{2+\mu} \right) \\
& \left. + \left| \frac{\sigma_t - \sigma_s}{\sigma_t}(x-y) + B_0^{\sigma_t}(\sigma_s) \right|^2 + \left| \frac{\sigma_t - \sigma_s}{\sigma_t}(x-y) + B_0^{\sigma_t}(\sigma_s) \right|^{2+\nu} \right\} ds. \tag{3.27}
\end{aligned}$$

Note that for $a > 0$ and $0 \leq \theta \leq \tau$ ($\tau > 0$)

$$\begin{aligned}
\mathbb{E}_B \left[\left| \frac{\theta}{\tau} z + B_0^\tau(\theta) \right|^a \right] & \leq 3^{(a-1)_+} \left(|z|^a + 2C(a, d)\tau^{a/2} \right), \\
\mathbb{E}_B \left[\left| \frac{\tau - \theta}{\tau} z + B_0^\tau(\theta) \right|^a \right] & \leq 3^{(a-1)_+} \left(|z|^a + 2C(a, d)\tau^{a/2} \right) \tag{3.28}
\end{aligned}$$

where $C(a, d) := \mathbb{E}_B[|B(1)|^a] = \int_{\mathbb{R}^d} |y|^a p(1, y) dy$. Thus, taking expectation \mathbb{E}_B in (3.26) and (3.27), we have

$$\begin{aligned}
& \mathbb{E}_B \left[|v_{K14}(t, x, y; \sigma)| \right] + \mathbb{E}_B \left[|v_{K15}(t, x, y; \sigma)| \right] \\
\leq & C_2 \left\{ \left(\frac{2(1-2\delta)_+}{e} \right)^{(1-2\delta)_+} t^{1 \wedge 2\delta} \right. \\
& \times \left(3|x-y|^2 + 6C(2, d)\sigma_t + 3^{1+\mu}|x-y|^{2+\mu} + 3^{1+\mu}2C(2+\mu, d)\sigma_t^{1+\mu/2} \right) \\
& \left. + t \left(3|x-y|^2 + 6C(2, d)\sigma_t + 3^{1+\nu}|x-y|^{2+\nu} + 3^{1+\nu}2C(2+\nu, d)\sigma_t^{1+\nu/2} \right) \right\}.
\end{aligned}$$

Collecting all the above into (3.23) yields the estimate in Claim 3.1 and the proof is complete. \square

Claim 3.2.

$$\begin{aligned}
& \mathbb{E}_\sigma \left[\mathbb{E}_B[|v_{K2}(t, x, y; \sigma)|] p(\sigma_t, x-y) \right] \\
\leq & \text{const}(\delta, \mu, \nu, d) \left[C_1^2(t^2 + t^{2\delta}) \left(\mathbb{E}_\sigma[|x-y|^2 p(\sigma_t, x-y)] + \mathbb{E}_\sigma[\sigma_t p(\sigma_t, x-y)] \right) \right. \\
& + C_2^2 t^{2(1 \wedge 2\delta)} \left(\mathbb{E}_\sigma[|x-y|^4 p(\sigma_t, x-y)] + \mathbb{E}_\sigma[\sigma_t^2 p(\sigma_t, x-y)] \right) \\
& \left. + \mathbb{E}_\sigma[|x-y|^{4+2\mu} p(\sigma_t, x-y)] + \mathbb{E}_\sigma[\sigma_t^{2+\mu} p(\sigma_t, x-y)] \right]
\end{aligned}$$

$$\begin{aligned}
& + C_2^2 t^2 \left(\mathbb{E}_\sigma [|x - y|^4 p(\sigma_t, x - y)] + \mathbb{E}_\sigma [\sigma_t^2 p(\sigma_t, x - y)] \right. \\
& \left. + \mathbb{E}_\sigma [|x - y|^{4+2\nu} p(\sigma_t, x - y)] + \mathbb{E}_\sigma [\sigma_t^{2+\nu} p(\sigma_t, x - y)] \right).
\end{aligned}$$

Proof. By (A)₂(i)

$$\begin{aligned}
\left| \sum_{j=1}^3 w_{Kj}(t, x, y; \sigma) \right| & = \left| \frac{1}{2} \left\langle \nabla V(x), \int_0^t \left(\frac{\sigma_s}{\sigma_t} (y - x) + B_0^{\sigma_t}(\sigma_s) \right) ds \right\rangle \right. \\
& \quad \left. + \frac{1}{2} \left\langle \nabla V(y), \int_0^t \left(\frac{\sigma_t - \sigma_s}{\sigma_t} (x - y) + B_0^{\sigma_t}(\sigma_s) \right) ds \right\rangle \right| \\
& \leq \frac{C_1}{2} \left\{ (1 + V(x)^{1-\delta}) \int_0^t \left| \frac{\sigma_s}{\sigma_t} (y - x) + B_0^{\sigma_t}(\sigma_s) \right| ds \right. \\
& \quad \left. + (1 + V(y)^{1-\delta}) \int_0^t \left| \frac{\sigma_t - \sigma_s}{\sigma_t} (x - y) + B_0^{\sigma_t}(\sigma_s) \right| ds \right\}. \quad (3.29)
\end{aligned}$$

This estimate together with (3.26) and (3.27) gives us that

$$\begin{aligned}
& |w_K(t, x, y; \sigma)| e^{-\theta t(V(x)+V(y))/4} \\
& \leq \left| \sum_{j=1}^3 w_{Kj}(t, x, y; \sigma) \right| e^{-\theta t(V(x)+V(y))/4} + \sum_{j=4}^5 |w_{Kj}(t, x, y; \sigma)| e^{-\theta t(V(x)+V(y))/4} \\
& \leq \frac{C_1}{2} \left(1 + \left(\frac{4(1-\delta)}{e} \right)^{1-\delta} \theta^{-1+\delta} t^{-1+\delta} \right) \\
& \quad \times \int_0^t \left(\left| \frac{\sigma_s}{\sigma_t} (y - x) + B_0^{\sigma_t}(\sigma_s) \right| + \left| \frac{\sigma_t - \sigma_s}{\sigma_t} (x - y) + B_0^{\sigma_t}(\sigma_s) \right| \right) ds \\
& + \frac{C_2}{2} \int_0^t ds \left\{ \theta^{-(1-2\delta)} + t^{-(1-2\delta)} + \left(\frac{4(1-2\delta)_+}{e} \right)^{(1-2\delta)_+} \right. \\
& \quad \times \left(\left| \frac{\sigma_s}{\sigma_t} (y - x) + B_0^{\sigma_t}(\sigma_s) \right|^2 + \left| \frac{\sigma_t - \sigma_s}{\sigma_t} (x - y) + B_0^{\sigma_t}(\sigma_s) \right|^{2+\mu} \right. \\
& \quad \left. + \left| \frac{\sigma_t - \sigma_s}{\sigma_t} (x - y) + B_0^{\sigma_t}(\sigma_s) \right|^2 + \left| \frac{\sigma_t - \sigma_s}{\sigma_t} (x - y) + B_0^{\sigma_t}(\sigma_s) \right|^{2+\mu} \right) \\
& \quad + \left| \frac{\sigma_s}{\sigma_t} (y - x) + B_0^{\sigma_t}(\sigma_s) \right|^2 + \left| \frac{\sigma_s}{\sigma_t} (y - x) + B_0^{\sigma_t}(\sigma_s) \right|^{2+\nu} \\
& \quad \left. + \left| \frac{\sigma_t - \sigma_s}{\sigma_t} (x - y) + B_0^{\sigma_t}(\sigma_s) \right|^2 + \left| \frac{\sigma_t - \sigma_s}{\sigma_t} (x - y) + B_0^{\sigma_t}(\sigma_s) \right|^{2+\nu} \right\}.
\end{aligned}$$

By the Schwarz inequality, it follows that

$$\begin{aligned}
& \left(|w_K(t, x, y; \sigma)| e^{-\theta t(V(x)+V(y))/4} \right)^2 \\
& \leq 12 \left[\left(\frac{C_1}{2} \right)^2 \left(t + \left(\frac{4(1-\delta)}{e} \right)^{2(1-\delta)} \theta^{-2+2\delta} t^{-1+2\delta} \right) \right. \\
& \quad \times \left(\int_0^t \left| \frac{\sigma_s}{\sigma_t} (y - x) + B_0^{\sigma_t}(\sigma_s) \right|^2 ds + \int_0^t \left| \frac{\sigma_t - \sigma_s}{\sigma_t} (x - y) + B_0^{\sigma_t}(\sigma_s) \right|^2 ds \right) \\
& \quad \left. + \left(\frac{C_2}{2} \right)^2 \left\{ \left(\frac{4(1-2\delta)_+}{e} \right)^{2(1-2\delta)_+} \theta^{-2(1-2\delta)_+} t^{2(1\wedge 2\delta)-1} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \times \left(\int_0^t \left| \frac{\sigma_s}{\sigma_t} (y-x) + B_0^{\sigma_t}(\sigma_s) \right|^4 ds + \int_0^t \left| \frac{\sigma_s}{\sigma_t} (y-x) + B_0^{\sigma_t}(\sigma_s) \right|^{4+2\mu} ds \right. \\
& \quad + \int_0^t \left| \frac{\sigma_t - \sigma_s}{\sigma_t} (x-y) + B_0^{\sigma_t}(\sigma_s) \right|^4 ds + \int_0^t \left| \frac{\sigma_t - \sigma_s}{\sigma_t} (x-y) + B_0^{\sigma_t}(\sigma_s) \right|^{4+2\mu} ds \Big) \\
& + t \left(\int_0^t \left| \frac{\sigma_s}{\sigma_t} (y-x) + B_0^{\sigma_t}(\sigma_s) \right|^4 ds + \int_0^t \left| \frac{\sigma_s}{\sigma_t} (y-x) + B_0^{\sigma_t}(\sigma_s) \right|^{4+2\nu} ds \right. \\
& \quad \left. + \int_0^t \left| \frac{\sigma_t - \sigma_s}{\sigma_t} (x-y) + B_0^{\sigma_t}(\sigma_s) \right|^4 ds + \int_0^t \left| \frac{\sigma_t - \sigma_s}{\sigma_t} (x-y) + B_0^{\sigma_t}(\sigma_s) \right|^{4+2\nu} ds \right) \Big].
\end{aligned}$$

Take expectation \mathbb{E}_B above, and integrate in θ . Then

$$\begin{aligned}
& \mathbb{E}_B [|v_{K2}(t, x, y; \sigma)|] \\
& \leq \mathbb{E}_B \left[w_K(t, x, y; \sigma)^2 \int_0^1 \theta e^{-\theta t(V(x)+V(y))/2} d\theta \right] \\
& = \int_0^1 \theta \mathbb{E}_B \left[\left(|w_K(t, x, y; \sigma)| e^{-\theta t(V(x)+V(y))/4} \right)^2 \right] d\theta \\
& \leq 12 \left[\left(\frac{C_1}{2} \right)^2 3(t^2 + \left(\frac{4(1-\delta)}{e} \right)^{2(1-\delta)} \frac{1}{\delta} t^{2\delta}) (|x-y|^2 + 2C(2, d)\sigma_t) \right. \\
& \quad \left. + \left(\frac{C_2}{2} \right)^2 \left\{ \left(\frac{4(1-2\delta)_+}{e} \right)^{2(1-2\delta)_+} + \frac{1}{1 \wedge 2\delta} t^{2(1 \wedge 2\delta)} \right. \right. \\
& \quad \quad \times [3^3(|x-y|^4 + 2C(4, d)\sigma_t^2) + 3^{3+2\mu}(|x-y|^{4+2\mu} + 2C(4+2\mu, d)\sigma_t^{2+\mu})] \\
& \quad \quad \left. \left. + t^2 [3^3(|x-y|^4 + 2C(4, d)\sigma_t^2) + 3^{3+2\nu}(|x-y|^{4+2\nu} + 2C(4+2\nu, d)\sigma_t^{2+\nu})] \right\} \right],
\end{aligned}$$

whence follows immediately the estimate in Claim 3.2. \square

Claim 3.3.

$$\begin{aligned}
& \mathbb{E}_\sigma \left[\mathbb{E}_B [|v_G(t, x, y; \sigma)|] p(\sigma_t, x-y) \right], \quad \mathbb{E}_\sigma \left[\mathbb{E}_B [|v_R(t, x, y; \sigma)|] p(\sigma_t, x-y) \right] \\
& \leq \text{const}(\delta, \mu, \nu, d) \sum_{j=1}^2 \left[C_1^j (t^j + t^{j\delta}) \left(\mathbb{E}_\sigma [|x-y|^j p(\sigma_t, x-y)] + \mathbb{E}_\sigma [\sigma_t^{j/2} p(\sigma_t, x-y)] \right) \right. \\
& \quad + C_2^j t^{j(1 \wedge 2\delta)} \left(\mathbb{E}_\sigma [|x-y|^{2j} p(\sigma_t, x-y)] + \mathbb{E}_\sigma [\sigma_t^j p(\sigma_t, x-y)] \right) \\
& \quad + \mathbb{E}_\sigma [|x-y|^{j(2+\mu)} p(\sigma_t, x-y)] + \mathbb{E}_\sigma [\sigma_t^{j(1+\mu/2)} p(\sigma_t, x-y)] \Big) \\
& \quad + C_2^j t^j \left(\mathbb{E}_\sigma [|x-y|^{2j} p(\sigma_t, x-y)] + \mathbb{E}_\sigma [\sigma_t^j p(\sigma_t, x-y)] \right) \\
& \quad \left. + \mathbb{E}_\sigma [|x-y|^{j(2+\nu)} p(\sigma_t, x-y)] + \mathbb{E}_\sigma [\sigma_t^{j(1+\nu/2)} p(\sigma_t, x-y)] \right].
\end{aligned}$$

Proof. Similarly to what is done in (3.29), (3.26) and (3.27), we have

$$\begin{aligned}
& |w_{G1}(t, x, y; \sigma)| e^{-rtV(x)} \\
& \leq C_1 \left(1 + \left(\frac{1-\delta}{e} \right)^{1-\delta} (rt)^{-1+\delta} \right) \int_0^t \left| \frac{\sigma_s}{\sigma_t} (y-x) + B_0^{\sigma_t}(\sigma_s) \right| ds, \tag{3.30}
\end{aligned}$$

$$\begin{aligned}
& |w_{G2}(t, x, y; \sigma)| e^{-rtV(x)} \\
& \leq C_2 \left[\left(\frac{(1-2\delta)_+}{e} \right)^{(1-2\delta)_+} (rt)^{-(1-2\delta)_+} \right. \\
& \quad \times \int_0^t \left(\left| \frac{\sigma_s}{\sigma_t} (y-x) + B_0^{\sigma_t}(\sigma_s) \right|^2 + \left| \frac{\sigma_s}{\sigma_t} (y-x) + B_0^{\sigma_t}(\sigma_s) \right|^{2+\mu} \right) ds \\
& \quad \left. + \int_0^t \left(\left| \frac{\sigma_s}{\sigma_t} (y-x) + B_0^{\sigma_t}(\sigma_s) \right|^2 + \left| \frac{\sigma_s}{\sigma_t} (y-x) + B_0^{\sigma_t}(\sigma_s) \right|^{2+\nu} \right) ds \right], \tag{3.31}
\end{aligned}$$

$$\begin{aligned}
& |w_{R1}(t, x, y; \sigma)| e^{-rtV(B_{0,x}^{\sigma_t,y}(\sigma_{t/2}))} \\
& \leq C_1 \left(1 + \left(\frac{1-\delta}{e} \right)^{1-\delta} (rt)^{-1+\delta} \right) \int_0^t |B_{0,x}^{\sigma_t,y}(\sigma_s) - B_{0,x}^{\sigma_t,y}(\sigma_{t/2})| ds, \tag{3.32}
\end{aligned}$$

$$\begin{aligned}
& |w_{R2}(t, x, y; \sigma)| e^{-rtV(B_{0,x}^{\sigma_t,y}(\sigma_{t/2}))} \\
& \leq C_2 \left[\left(\frac{(1-2\delta)_+}{e} \right)^{(1-2\delta)_+} (rt)^{-(1-2\delta)_+} \right. \\
& \quad \times \int_0^t \left(|B_{0,x}^{\sigma_t,y}(\sigma_s) - B_{0,x}^{\sigma_t,y}(\sigma_{t/2})|^2 + |B_{0,x}^{\sigma_t,y}(\sigma_s) - B_{0,x}^{\sigma_t,y}(\sigma_{t/2})|^{2+\mu} \right) ds \\
& \quad \left. + \int_0^t \left(|B_{0,x}^{\sigma_t,y}(\sigma_s) - B_{0,x}^{\sigma_t,y}(\sigma_{t/2})|^2 + |B_{0,x}^{\sigma_t,y}(\sigma_s) - B_{0,x}^{\sigma_t,y}(\sigma_{t/2})|^{2+\nu} \right) ds \right]. \tag{3.33}
\end{aligned}$$

By (3.15), (3.16), (3.21) and (3.22), note that

$$\begin{aligned}
& |v_G(t, x, y; \sigma)| \\
& \leq |w_{G1}(t, x, y; \sigma)| e^{-tV(x)} + |w_{G2}(t, x, y; \sigma)| e^{-tV(x)} \\
& \quad + \int_0^1 \theta \left(|w_{G1}(t, x, y; \sigma)| e^{-\theta tV(x)/2} + |w_{G2}(t, x, y; \sigma)| e^{-\theta tV(x)/2} \right)^2 d\theta, \tag{3.34}
\end{aligned}$$

$$\begin{aligned}
& |v_R(t, x, y; \sigma)| \\
& \leq |w_{R1}(t, x, y; \sigma)| e^{-tV(B_{0,x}^{\sigma_t,y}(\sigma_{t/2}))} + |w_{R2}(t, x, y; \sigma)| e^{-tV(B_{0,x}^{\sigma_t,y}(\sigma_{t/2}))} \\
& \quad + \int_0^1 \theta \left(|w_{R1}(t, x, y; \sigma)| e^{-\theta tV(B_{0,x}^{\sigma_t,y}(\sigma_{t/2}))/2} \right. \\
& \quad \left. + |w_{R2}(t, x, y; \sigma)| e^{-\theta tV(B_{0,x}^{\sigma_t,y}(\sigma_{t/2}))/2} \right)^2 d\theta. \tag{3.35}
\end{aligned}$$

Also note that for $a > 0$ and $0 \leq \theta_1, \theta_2 \leq \tau$ ($\tau > 0$) (cf. (3.28))

$$\mathbb{E}_B \left[|B_{0,x}^{\tau,y}(\theta_1) - B_{0,x}^{\tau,y}(\theta_2)|^a \right] \leq 3^{(a-1)_+} (|x-y|^a + 2C(a, d)\tau^{a/2}). \tag{3.36}$$

Collecting all the above yields the estimate in Claim 3.3 immediately. \square

We are now in a position to prove Theorem 2.1(iii). To do so, we need the following lemma.

Lemma 3.1. *Let $1 \leq p \leq \infty$. Then, for $a, b \geq 0$ with $C(a, d) = \int_{\mathbb{R}^d} |y|^a p(1, y) dy$,*

$$f_{a,b}(t) := \left\| \int_{\mathbb{R}^d} |f(y)| \mathbb{E}_\sigma [|\cdot - y|^a \sigma_t^b p(\sigma_t, \cdot - y)] dy \right\|_p$$

$$\leq C(a, d) \mathbb{E}_\sigma [\sigma_t^{a/2+b}] \|f\|_p, \quad f \in L_p(\mathbb{R}^d).$$

Proof. For $p = \infty$, the described estimate is obvious. So let $1 \leq p < \infty$. First we note the Minkowski inequality for integrals: If $h(x, y)$ is a measurable function on a σ -finite product measure space $(\mathcal{X} \times \mathcal{Y}, \alpha(dx) \times \beta(dy))$, then

$$\left(\int_{\mathcal{Y}} \left(\int_{\mathcal{X}} |h(x, y)| \alpha(dx) \right)^p \beta(dy) \right)^{1/p} \leq \int_{\mathcal{X}} \left(\int_{\mathcal{Y}} |h(x, y)|^p \beta(dy) \right)^{1/p} \alpha(dx).$$

Note also that for $c \geq 0$

$$\left\| \int_{\mathbb{R}^d} |f(y)| |\cdot - y|^c p(\tau, \cdot - y) dy \right\|_p \leq C(c, d) \tau^{c/2} \|f\|_p.$$

By these inequalities, the estimate is obtained as follows:

$$\begin{aligned} \left\| \int_{\mathbb{R}^d} |f(y)| \mathbb{E}_\sigma [|\cdot - y|^a \sigma_t^b p(\sigma_t, \cdot - y)] dy \right\|_p &\leq \mathbb{E}_\sigma \left[\left\| \int_{\mathbb{R}^d} |f(y)| |\cdot - y|^a \sigma_t^b p(\sigma_t, \cdot - y) dy \right\|_p \right] \\ &\leq C(a, d) \mathbb{E}_\sigma [\sigma_t^{a/2+b}] \|f\|_p. \end{aligned} \quad \square$$

Proof of Theorem 2.1(iii). By Claims 3.1, 3.2 with (3.7)

$$\begin{aligned} \|Q_K(t)f\|_p &\leq \left\| \int_{\mathbb{R}^d} |f(y)| |\mathbb{E}_\sigma [\mathbb{E}_B [v_{K1}(t, \cdot, y; \sigma)] p(\sigma_t, \cdot - y)]| dy \right. \\ &\quad \left. + \int_{\mathbb{R}^d} |f(y)| \mathbb{E}_\sigma [\mathbb{E}_B [v_{K2}(t, \cdot, y; \sigma)] p(\sigma_t, \cdot - y)] dy \right\|_p \\ &\leq \text{const}(\delta, \mu, \nu, d) \left[C_1^2(t^2 + t^{2\delta})(f_{2,0}(t) + f_{0,1}(t)) \right. \\ &\quad \left. + \sum_{j=1}^2 \left\{ C_2^j t^{j(1 \wedge 2\delta)} (f_{2j,0}(t) + f_{0,j}(t) + f_{j(2+\mu),0}(t) + f_{0,j(1+\mu/2)}(t)) \right. \right. \\ &\quad \left. \left. + C_2^j t^j (f_{2j,0}(t) + f_{0,j}(t) + f_{j(2+\nu),0}(t) + f_{0,j(1+\nu/2)}(t)) \right\} \right]. \end{aligned}$$

By Claim 3.3 with (3.8), (3.9)

$$\begin{aligned} \|Q_R^G(t)f\|_p &\leq \left\| \int_{\mathbb{R}^d} |f(y)| \mathbb{E}_\sigma [\mathbb{E}_B [v_R^G(t, \cdot, y; \sigma)] p(\sigma_t, \cdot - y)] dy \right\|_p \\ &\leq \text{const}(\delta, \mu, \nu, d) \sum_{j=1}^2 \left[C_1^j (t^j + t^{j\delta})(f_{j,0}(t) + f_{0,j/2}(t)) \right. \\ &\quad \left. + C_2^j t^{j(1 \wedge 2\delta)} (f_{2j,0}(t) + f_{0,j}(t) + f_{j(2+\mu),0}(t) + f_{0,j(1+\mu/2)}(t)) \right. \\ &\quad \left. + C_2^j t^j (f_{2j,0}(t) + f_{0,j}(t) + f_{j(2+\nu),0}(t) + f_{0,j(1+\nu/2)}(t)) \right]. \end{aligned}$$

Combining these with Lemma 3.1 we have the assertion of Theorem 2.1(iii). \square

3.2. Case (A)₀

In this subsection, we suppose condition (A)₀ on $V(x)$. In this case

$$\begin{aligned}
|v_K(t, x, y; \sigma)| &\leq |w_K(t, x, y; \sigma)| \\
&\leq \frac{C_1}{2} \int_0^t \left| \frac{\sigma_s}{\sigma_t} (y - x) + B_0^{\sigma_t}(\sigma_s) \right|^\gamma ds + \frac{C_1}{2} \int_0^t \left| \frac{\sigma_t - \sigma_s}{\sigma_t} (x - y) + B_0^{\sigma_t}(\sigma_s) \right|^\gamma ds, \\
|v_G(t, x, y; \sigma)| &\leq |w_G(t, x, y; \sigma)| \\
&\leq C_1 \int_0^t \left| \frac{\sigma_s}{\sigma_t} (y - x) + B_0^{\sigma_t}(\sigma_s) \right|^\gamma ds, \\
|v_R(t, x, y; \sigma)| &\leq |w_R(t, x, y; \sigma)| \\
&\leq C_1 \int_0^t |B_{0,x}^{\sigma_t, y}(\sigma_s) - B_{0,x}^{\sigma_t, y}(\sigma_{t/2})|^\gamma ds.
\end{aligned}$$

Here taking expectation \mathbb{E}_B , we have by (3.28) or (3.36),

$$\begin{aligned}
&\mathbb{E}_B[|v_K(t, x, y; \sigma)|], \mathbb{E}_B[|v_G(t, x, y; \sigma)|], \mathbb{E}_B[|v_R(t, x, y; \sigma)|] \\
&\leq C_1 t (|x - y|^\gamma + 2C(\gamma, d) \sigma_t^{\gamma/2})
\end{aligned}$$

and hence, by (3.7), (3.8) and (3.9)

$$\begin{aligned}
&|Q_K(t)f(x)|, |Q_G(t)f(x)|, |Q_R(t)f(x)| \\
&\leq C_1 t \left\{ \int_{\mathbb{R}^d} |f(y)| \mathbb{E}_\sigma \left[|x - y|^\gamma p(\sigma_t, x - y) \right] dy \right. \\
&\quad \left. + 2C(\gamma, d) \int_{\mathbb{R}^d} |f(y)| \mathbb{E}_\sigma \left[\sigma_t^{\gamma/2} p(\sigma_t, x - y) \right] dy \right\}.
\end{aligned}$$

From this and Lemma 3.1 the assertion of Theorem 2.1(i) follows immediately.

4. Proof of Theorem 2.2

For notational simplicity we set $H_0 := H_0^\psi$ and $H := H_0 + V$, in the following, so that $K(t) = e^{-tV/2} e^{-tH_0} e^{-tV/2}$, $G(t) = e^{-tV} e^{-tH_0}$ and $R(t) = e^{-tH_0/2} e^{-tV} e^{-tH_0/2}$.

4.1. Proof of Theorem 2.2 for $K(t)$

Since $K(t)$ and e^{-sH} are contractions, we have

$$\begin{aligned}
\|K(\frac{t}{n})^n - e^{-tH}\|_{p \rightarrow p} &= \left\| \sum_{k=0}^{n-1} K(\frac{t}{n})^{n-1-k} (K(\frac{t}{n}) - e^{-tH/n}) e^{-ktH/n} \right\|_{p \rightarrow p} \\
&\leq \sum_{k=0}^{n-1} \|K(\frac{t}{n}) - e^{-tH/n}\|_{p \rightarrow p} \\
&= n \|Q_K(\frac{t}{n})\|_{p \rightarrow p}.
\end{aligned}$$

Combined with the estimates for $Q_K(t)$ in Theorem 2.1, the desired bound for $K(t/n)^n - e^{-tH}$ in Case (A)₀, (A)₁ or (A)₂ is obtained immediately.

4.2. Proof of Theorem 2.2 for $G(t)$ and $R(t)$ in Case (A)₀

In the same way as above

$$\begin{aligned} \|G(\frac{t}{n})^n - e^{-tH}\|_{p \rightarrow p} &\leq n \|Q_G(\frac{t}{n})\|_{p \rightarrow p}, \\ \|R(\frac{t}{n})^n - e^{-tH}\|_{p \rightarrow p} &\leq n \|Q_R(\frac{t}{n})\|_{p \rightarrow p}, \end{aligned}$$

from which together with Theorem 2.1(i), the desired bounds follow immediately.

4.3. Proof of Theorem 2.2 for $G(t)$ and $R(t)$ in Case (A)₁ or (A)₂

In this subsection we suppose that $V(x)$ satisfies (A)₁ or (A)₂.

We first observe that for $t \geq 0$ and $n \in \mathbb{N}$

$$\begin{aligned} G(\frac{t}{n})^n - e^{-tH} &= e^{-tV/2n} (K(\frac{n-1}{n}t\frac{1}{n-1})^{n-1} - e^{-(n-1)tH/n}) e^{-tV/2n} e^{-tH_0/n} \\ &\quad + [e^{-tV/2n}, e^{-(n-1)tH/n}] e^{-tV/2n} e^{-tH_0/n} + e^{-(n-1)tH/n} Q_G(\frac{t}{n}), \\ R(\frac{t}{n})^n - e^{-tH} &= e^{-tH_0/2n} e^{-tV/2n} (K(\frac{n-1}{n}t\frac{1}{n-1})^{n-1} - e^{-(n-1)tH/n}) e^{-tV/2n} e^{-tH_0/2n} \\ &\quad + e^{-tH_0/2n} [e^{-tV/2n}, e^{-(n-1)tH/n}] e^{-tV/2n} e^{-tH_0/2n} \\ &\quad + [e^{-tH_0/2n}, e^{-(n-1)tH/n}] e^{-tV/2n} e^{-tH_0/2n} + e^{-(n-1)tH/n} Q_R(\frac{t}{n}), \end{aligned}$$

where $[A, B] = AB - BA$. Hence

$$\begin{aligned} \|G(\frac{t}{n})^n - e^{-tH}\|_{p \rightarrow p} &\leq \|K(\frac{n-1}{n}t\frac{1}{n-1})^{n-1} - e^{-(n-1)tH/n}\|_{p \rightarrow p} \\ &\quad + \|[e^{-tV/2n}, e^{-(n-1)tH/n}]\|_{p \rightarrow p} + \|Q_G(\frac{t}{n})\|_{p \rightarrow p}, \end{aligned} \tag{4.1}$$

$$\begin{aligned} \|R(\frac{t}{n})^n - e^{-tH}\|_{p \rightarrow p} &\leq \|K(\frac{n-1}{n}t\frac{1}{n-1})^{n-1} - e^{-(n-1)tH/n}\|_{p \rightarrow p} \\ &\quad + \|[e^{-tV/2n}, e^{-(n-1)tH/n}]\|_{p \rightarrow p} + \|[e^{-tH_0/2n}, e^{-(n-1)tH/n}]\|_{p \rightarrow p} \\ &\quad + \|Q_R(\frac{t}{n})\|_{p \rightarrow p}. \end{aligned} \tag{4.2}$$

As for the first term on the RHS of (4.1) and (4.2), we see by Theorem 2.2 which was proved in Section 4.1

$$\begin{aligned} &\|K(\frac{n-1}{n}t\frac{1}{n-1})^{n-1} - e^{-(n-1)tH/n}\|_{p \rightarrow p} \\ &\leq \begin{cases} \text{const}(\delta, \kappa, d) \left[C_1^2 \left((\frac{t}{n})^2 + (\frac{t}{n})^{2\delta} \right) (n-1) \mathbb{E}[\sigma_{t/n}] + \sum_{j=1}^2 (C_2 \frac{t}{n})^j (n-1) \mathbb{E}[\sigma_{t/n}^{j(1+\kappa)/2}] \right], \\ \hspace{15em} \text{in Case (A)}_1, \\ \\ \text{const}(\delta, \mu, \nu, d) \left[C_1^2 \left((\frac{t}{n})^2 + (\frac{t}{n})^{2\delta} \right) (n-1) \mathbb{E}[\sigma_{t/n}] + \sum_{j=1}^2 \left\{ (C_2 \frac{t}{n})^j \left((n-1) \mathbb{E}[\sigma_{t/n}^j] \right. \right. \right. \\ \left. \left. \left. + (n-1) \mathbb{E}[\sigma_{t/n}^{j(1+\nu/2)}] \right) + (C_2 (\frac{t}{n})^{1 \wedge 2\delta})^j \left((n-1) \mathbb{E}[\sigma_{t/n}^j] + (n-1) \mathbb{E}[\sigma_{t/n}^{j(1+\mu/2)}] \right) \right\} \right], \\ \hspace{15em} \text{in Case (A)}_2. \end{cases} \end{aligned}$$

As for the third term on the RHS of (4.1) and the fourth term of (4.2), we see by Theorem 2.1

$$\begin{aligned} & \|Q_G(\frac{t}{n})\|_{p \rightarrow p}, \|Q_R(\frac{t}{n})\|_{p \rightarrow p} \\ & \leq \begin{cases} \text{const}(\delta, \kappa, d) \sum_{j=1}^2 \left\{ C_1^j \left((\frac{t}{n})^j + (\frac{t}{n})^{j\delta} \right) \mathbb{E}[\sigma_{t/n}^{j/2}] + (C_2 \frac{t}{n})^j \mathbb{E}[\sigma_{t/n}^{j(1+\kappa)/2}] \right\}, & \text{in Case (A)}_1, \\ \text{const}(\delta, \mu, \nu, d) \sum_{j=1}^2 \left\{ C_1^j \left((\frac{t}{n})^j + (\frac{t}{n})^{j\delta} \right) \mathbb{E}[\sigma_{t/n}^{j/2}] \right. \\ \quad \left. + (C_2 (\frac{t}{n})^{1 \wedge 2\delta})^j (\mathbb{E}[\sigma_{t/n}^j] + \mathbb{E}[\sigma_{t/n}^{j(1+\mu/2)}]) \right. \\ \quad \left. + (C_2 \frac{t}{n})^j (\mathbb{E}[\sigma_{t/n}^j] + \mathbb{E}[\sigma_{t/n}^{j(1+\nu/2)}]) \right\}, & \text{in Case (A)}_2. \end{cases} \end{aligned}$$

Therefore we need to estimate the middle terms of (4.1) and (4.2).

Claim 4.1. *Let $s \geq 0$ and $t > 0$. Then*

$$\begin{aligned} & \|[e^{-sV}, e^{-tH}]\|_{p \rightarrow p}, \|[e^{-sH_0}, e^{-tH}]\|_{p \rightarrow p} \\ & \leq \begin{cases} \text{const}(\delta, \kappa, d) s \left[C_1 (1 + t^{-1+\delta}) \mathbb{E}[\sigma_t^{1/2}] + C_2 \mathbb{E}[\sigma_t^{(1+\kappa)/2}] \right], & \text{in Case (A)}_1, \\ \text{const}(\delta, \mu, \nu, d) s \left[C_1 (1 + t^{-1+\delta}) \mathbb{E}[\sigma_t^{1/2}] + C_2 t^{-(1-2\delta)+} (\mathbb{E}[\sigma_t] + \mathbb{E}[\sigma_t^{1+\mu/2}]) \right. \\ \quad \left. + C_2 (\mathbb{E}[\sigma_t] + \mathbb{E}[\sigma_t^{1+\nu/2}]) \right], & \text{in Case (A)}_2. \end{cases} \end{aligned}$$

Proof. First we estimate the L_p -operator norm of $[e^{-sV}, e^{-tH}]$. We have (by (A.13)) that for $f \in C_0(\mathbb{R}^d)$

$$\begin{aligned} & [e^{-sV}, e^{-tH}]f(x) \\ & = \int_{\mathbb{R}^d} f(y) (e^{-sV(x)} - e^{-sV(y)}) \mathbb{E} \left[\exp \left(- \int_0^t V(B_{0,x}^{\sigma_t, y}(\sigma_r)) dr \right) p(\sigma_t, x - y) \right] dy. \end{aligned}$$

Hence we have

$$\begin{aligned} & |[e^{-sV}, e^{-tH}]f(x)| \\ & \leq s \int_{\mathbb{R}^d} |f(y)| \mathbb{E} \left[|V(y) - V(x)| \exp \left(- \int_0^t V(B_{0,x}^{\sigma_t, y}(\sigma_r)) dr \right) p(\sigma_t, x - y) \right] dy. \end{aligned} \quad (4.3)$$

To estimate the integrand in (4.3), note by Taylor's theorem that

$$\begin{aligned} V(y) - V(x) & = \int_0^t \langle \nabla V(B_{0,x}^{\sigma_t, y}(\sigma_r)), y - x \rangle \frac{dr}{t} \\ & \quad + \int_0^1 d\theta \int_0^t \langle \nabla V(x + \theta(y - x)) - \nabla V(B_{0,x}^{\sigma_t, y}(\sigma_r)), y - x \rangle \frac{dr}{t}. \end{aligned}$$

In Case (A)₁, it follows that

$$\begin{aligned}
|V(y) - V(x)| &\leq \int_0^t C_1(1 + V(B_{0,x}^{\sigma_t,y}(\sigma_r)))^{1-\delta} \frac{dr}{t} |x - y| \\
&\quad + \int_0^1 d\theta \int_0^t C_2 |(\frac{\sigma_r}{\sigma_t} - \theta)(y - x) + B_0^{\sigma_t}(\sigma_r)|^\kappa dr \frac{|x-y|}{t} \\
&\leq C_1 \left(1 + t^{-1+\delta} \left(\int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_r)) dr\right)^{1-\delta}\right) |x - y| \\
&\quad + C_2 \frac{1}{t} \int_0^1 d\theta \int_0^t |(\frac{\sigma_r}{\sigma_t} - \theta)(y - x) + B_0^{\sigma_t}(\sigma_r)|^\kappa dr |x - y| \tag{4.4}
\end{aligned}$$

where the last inequality is due to Jensen's inequality. In Case (A)₂

$$\begin{aligned}
&|V(y) - V(x)| \\
&\leq \int_0^t C_1(1 + V(B_{0,x}^{\sigma_t,y}(\sigma_r)))^{1-\delta} \frac{dr}{t} |x - y| \\
&\quad + \int_0^1 d\theta \int_0^t C_2 \left\{ V(B_{0,x}^{\sigma_t,y}(\sigma_r))^{(1-2\delta)+} (1 + |(\frac{\sigma_r}{\sigma_t} - \theta)(y - x) + B_0^{\sigma_t}(\sigma_r)|^\mu) \right. \\
&\quad \quad \left. + 1 + |(\frac{\sigma_r}{\sigma_t} - \theta)(y - x) + B_0^{\sigma_t}(\sigma_r)|^\nu \right\} |(\frac{\sigma_r}{\sigma_t} - \theta)(y - x) + B_0^{\sigma_t}(\sigma_r)| \frac{dr}{t} |x - y| \\
&\leq C_1 \left(1 + t^{-1+\delta} \left(\int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_r)) dr\right)^{1-\delta}\right) |x - y| \\
&\quad + C_2 t^{-(1-2\delta)+} \left(\int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_r)) dr\right)^{(1-2\delta)+} \int_0^1 \left(\max_{0 \leq \sigma \leq \sigma_t} |(\frac{\sigma}{\sigma_t} - \theta)(y - x) + B_0^{\sigma_t}(\sigma)| \right. \\
&\quad \quad \left. + \max_{0 \leq \sigma \leq \sigma_t} |(\frac{\sigma}{\sigma_t} - \theta)(y - x) + B_0^{\sigma_t}(\sigma)|^{1+\mu}\right) d\theta |x - y| \\
&\quad + C_2 \frac{1}{t} \int_0^1 d\theta \int_0^t \left(|(\frac{\sigma_r}{\sigma_t} - \theta)(y - x) + B_0^{\sigma_t}(\sigma_r)| \right. \\
&\quad \quad \left. + |(\frac{\sigma_r}{\sigma_t} - \theta)(y - x) + B_0^{\sigma_t}(\sigma_r)|^{1+\nu} \right) dr |x - y|. \tag{4.5}
\end{aligned}$$

By (3.25), (4.4) and (4.5) imply the desired estimate:

$$|V(y) - V(x)| \exp\left(-\int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_r)) dr\right)$$

$$\leq \begin{cases} C_1(1 + (\frac{1-\delta}{e})^{1-\delta}t^{-1+\delta})|x-y| \\ + C_2\frac{1}{t}\int_0^1 d\theta \int_0^t |(\frac{\sigma_r}{\sigma_t} - \theta)(y-x) + B_0^{\sigma_t}(\sigma_r)|^\kappa dr |x-y|, & \text{in Case (A)}_1, \\ C_1(1 + (\frac{1-\delta}{e})^{1-\delta}t^{-1+\delta})|x-y| \\ + C_2(\frac{(1-2\delta)_+}{e})^{(1-2\delta)_+}t^{-(1-2\delta)_+} \\ \times \int_0^1 \left(\max_{0 \leq \sigma \leq \sigma_t} |(\frac{\sigma}{\sigma_t} - \theta)(y-x) + B_0^{\sigma_t}(\sigma)| \right. \\ \left. + \max_{0 \leq \sigma \leq \sigma_t} |(\frac{\sigma}{\sigma_t} - \theta)(y-x) + B_0^{\sigma_t}(\sigma)|^{1+\mu} \right) d\theta |x-y| \\ + C_2\frac{1}{t}\int_0^1 d\theta \int_0^t \left(|(\frac{\sigma_r}{\sigma_t} - \theta)(y-x) + B_0^{\sigma_t}(\sigma_r)| \right. \\ \left. + |(\frac{\sigma_r}{\sigma_t} - \theta)(y-x) + B_0^{\sigma_t}(\sigma_r)|^{1+\nu} \right) dr |x-y|, & \text{in Case (A)}_2. \end{cases}$$

We take expectation \mathbb{E}_B in the above. This time we use the following moment estimate: For $a > 0$, $\tau > 0$, $0 \leq \theta \leq 1$ and $z \in \mathbb{R}^d$

$$\begin{aligned} \mathbb{E}_B \left[\left| (\frac{t}{\tau} - \theta)z + B_0^\tau(t) \right|^a \right] &\leq 3^{(a-1)_+} (|z|^a + 2C(a, d)\tau^{a/2}), \\ \mathbb{E}_B \left[\max_{0 \leq t \leq \tau} \left| (\frac{t}{\tau} - \theta)z + B_0^\tau(t) \right|^a \right] &\leq 3^{(a-1)_+} (|z|^a + 2\tilde{C}(a, d)\tau^{a/2}) \end{aligned} \quad (4.6)$$

where $C(a, d) = \mathbb{E}_B [|B(1)|^a]$ and $\tilde{C}(a, d) = \mathbb{E}_B [\max_{0 \leq t \leq 1} |B(t)|^a]$, and thereby we have

$$\begin{aligned} &\mathbb{E}_B \left[|V(y) - V(x)| \exp \left(- \int_0^t V(B_{0,x}^{\sigma_t, y}(\sigma_r)) dr \right) \right] \\ &\leq \begin{cases} C_1(1 + (\frac{1-\delta}{e})^{1-\delta}t^{-1+\delta})|x-y| + C_2(|x-y|^{1+\kappa} + 2C(\kappa, d)\sigma_t^{\kappa/2}|x-y|), \\ & \text{in Case (A)}_1, \\ C_1(1 + (\frac{1-\delta}{e})^{1-\delta}t^{-1+\delta})|x-y| \\ + C_2(\frac{(1-2\delta)_+}{e})^{(1-2\delta)_+}t^{-(1-2\delta)_+} \\ \times \left(|x-y| + 2\tilde{C}(1, d)\sigma_t^{1/2} + 3^\mu (|x-y|^{1+\mu} + 2\tilde{C}(1+\mu, d)\sigma_t^{(1+\mu)/2}) \right) |x-y| \\ + C_2 \left(|x-y| + 2C(1, d)\sigma_t^{1/2} + 3^\nu (|x-y|^{1+\nu} + 2C(1+\nu, d)\sigma_t^{(1+\nu)/2}) \right) |x-y|, \\ & \text{in Case (A)}_2. \end{cases} \end{aligned} \quad (4.7)$$

Hence follows the desired bound for $[e^{-sV}, e^{-tH}]$ by Lemma 3.1 with (4.3).

Next we estimate the L_p -operator norm of $[e^{-sH_0}, e^{-tH}]$.

First we suppose that $V : \mathbb{R}^d \rightarrow [0, \infty)$ is in C^∞ and all its derivatives have polynomial growth. Then it is easily verified that (cf. Claim A.2 and its Remark)

- (i) $e^{-tH}(\mathcal{S}(\mathbb{R}^d)) \subset \mathcal{S}(\mathbb{R}^d)$, in particular, $e^{-tH_0}(\mathcal{S}(\mathbb{R}^d)) \subset \mathcal{S}(\mathbb{R}^d)$, and
- (ii) $\mathcal{S}(\mathbb{R}^d) \subset \bigcap_{1 \leq p \leq \infty} \mathfrak{D}(\mathfrak{G}_p^{\psi, V}) \cap \bigcap_{1 \leq p \leq \infty} \mathfrak{D}(\mathfrak{G}_p^{\psi, 0})$ and $\mathfrak{G}_p^{\psi, V} = \mathfrak{G}_p^{\psi, 0} - V$ on $\mathcal{S}(\mathbb{R}^d)$.

Here $\mathfrak{G}_p^{\psi, V}$ ($1 \leq p < \infty$) is the infinitesimal generator of $\{e^{-t(H_0+V)}\}$ on $L_p(\mathbb{R}^d)$ and $\mathfrak{G}_\infty^{\psi, V}$ the one on $C_\infty(\mathbb{R}^d)$. By these facts the following formula holds in $L_p(\mathbb{R}^d)$ ($1 \leq p < \infty$) and $C_\infty(\mathbb{R}^d)$:

For each $f \in \mathcal{S}(\mathbb{R}^d)$

$$[e^{-sH_0}, e^{-tH}]f = \int_0^s e^{-uH_0} [V, e^{-tH}] e^{-(s-u)H_0} f du.$$

Hence, taking L_p -norm in the above yields that for each $f \in \mathcal{S}(\mathbb{R}^d)$

$$\|[e^{-sH_0}, e^{-tH}]f\|_p \leq \int_0^s \|[V, e^{-tH}]e^{-(s-u)H_0} f\|_p du. \quad (4.8)$$

Now let V satisfy (A)₁ or (A)₂. In this case V is not necessarily smooth. So, take a nonnegative $h \in C_0^\infty$ with support in $\{x \in \mathbb{R}^d; |x| \leq 1\}$ and $\int_{\mathbb{R}^d} h(x) dx = 1$. Set $V^\varepsilon = V * h_\varepsilon$ with $h_\varepsilon(x) = (1/\varepsilon)^d h(x/\varepsilon)$. Then V^ε is in $C^\infty(\mathbb{R}^d \rightarrow [0, \infty))$, and satisfies condition (A)₁ or (A)₂ with the same const's as V does. Further, by (A)₁(i) or (A)₂(ii) all the derivatives of V^ε have polynomial growth. Hence, by (4.7) and Lemma 3.1 it holds that for $g \in \mathcal{S}(\mathbb{R}^d)$

$$\begin{aligned} & \|[V^\varepsilon, e^{-t(H_0+V^\varepsilon)}]g\|_p \\ & \leq \begin{cases} \text{const}(\delta, \kappa, d) \left[C_1(1+t^{-1+\delta})\mathbb{E}[\sigma_t^{1/2}] + C_2\mathbb{E}[\sigma_t^{(1+\kappa)/2}] \right] \|g\|_p, & \text{in Case (A)}_1, \\ \text{const}(\delta, \mu, \nu, d) \left[C_1(1+t^{-1+\delta})\mathbb{E}[\sigma_t^{1/2}] + C_2t^{-(1-2\delta)+}(\mathbb{E}[\sigma_t] + \mathbb{E}[\sigma_t^{1+\mu/2}]) \right. \\ \quad \left. + C_2(\mathbb{E}[\sigma_t] + \mathbb{E}[\sigma_t^{1+\nu/2}]) \right] \|g\|_p, & \text{in Case (A)}_2. \end{cases} \end{aligned}$$

Since (4.8) holds with $V = V^\varepsilon$, by combining this with the above we have

$$\begin{aligned} & \|[e^{-sH_0}, e^{-t(H_0+V^\varepsilon)}]f\|_p \\ & \leq \begin{cases} \text{const}(\delta, \kappa, d)s \left[C_1(1+t^{-1+\delta})\mathbb{E}[\sigma_t^{1/2}] + C_2\mathbb{E}[\sigma_t^{(1+\kappa)/2}] \right] \|f\|_p, & \text{in Case (A)}_1, \\ \text{const}(\delta, \mu, \nu, d)s \left[C_1(1+t^{-1+\delta})\mathbb{E}[\sigma_t^{1/2}] + C_2t^{-(1-2\delta)+}(\mathbb{E}[\sigma_t] + \mathbb{E}[\sigma_t^{1+\mu/2}]) \right. \\ \quad \left. + C_2(\mathbb{E}[\sigma_t] + \mathbb{E}[\sigma_t^{1+\nu/2}]) \right] \|f\|_p, & \text{in Case (A)}_2. \end{cases} \end{aligned}$$

Finally let $\varepsilon \downarrow 0$. Since $V^\varepsilon \rightarrow V$ compact uniformly, we see by the Feynman-Kac formula (A.6) that $e^{-t(H_0+V^\varepsilon)}f \rightarrow e^{-t(H_0+V)}f$ boundedly pointwise, so that $[e^{-sH_0}, e^{-t(H_0+V^\varepsilon)}]f \rightarrow [e^{-sH_0}, e^{-t(H_0+V)}]f$ pointwise. Hence the desired bound for $[e^{-sH_0}, e^{-t(H_0+V)}]$ follows immediately by the Fatou inequality. \square

We return to estimate $G(t/n)^n - e^{-tH}$ and $R(t/n)^n - e^{-tH}$. By Claim 4.1

$$\begin{aligned} & \|[e^{-tV/2n}, e^{-(n-1)tH/n}]\|_{p \rightarrow p}, \|[e^{-tH_0/2n}, e^{-(n-1)tH/n}]\|_{p \rightarrow p} \\ & \leq \begin{cases} \text{const}(\delta, \kappa, d)\frac{1}{n} \left[C_1(t+t^\delta)\mathbb{E}[\sigma_t^{1/2}] + C_2t\mathbb{E}[\sigma_t^{(1+\kappa)/2}] \right], & \text{in Case (A)}_1, \\ \text{const}(\delta, \mu, \nu, d)\frac{1}{n} \left[C_1(t+t^\delta)\mathbb{E}[\sigma_t^{1/2}] + C_2t^{1\wedge 2\delta}(\mathbb{E}[\sigma_t] + \mathbb{E}[\sigma_t^{1+\mu/2}]) \right. \\ \quad \left. + C_2t(\mathbb{E}[\sigma_t] + \mathbb{E}[\sigma_t^{1+\nu/2}]) \right], & \text{in Case (A)}_2. \end{cases} \end{aligned}$$

Therefore, collecting all the estimates above yields the desired bounds for $G(t/n)^n - e^{-tH}$ and $R(t/n)^n - e^{-tH}$.

5. Proof of Theorem 2.3

As in the previous section, we are setting $H_0 = H_0^\psi$ and $H = H_0 + V$.

5.1. Case (V)₂

Condition (V)₂ implies (A)₂ with $\delta = 1 \wedge 1/\rho$, $C_1 = c_1 c^{-(1-1 \wedge 1/\rho)}$, $C_2 = c_2 2^{(\rho-3)_+} ((1/2)c^{-(1-2(1 \wedge 1/\rho))_+} \vee 1)$, $\mu = 0$ and $\nu = (\rho - 2)_+$. So this case follows immediately from Theorem 2.2(iii).

5.2. Case (V)₁

In this subsection we suppose condition (V)₁ on $V(x)$.

Let us adopt an idea in [D-I-Tam]. Take again a nonnegative $h \in C_0^\infty$ with support in $\{x \in \mathbb{R}^d; |x| \leq 1\}$ and $\int_{\mathbb{R}^d} h(x) dx = 1$. For $0 < \varepsilon \leq 1/4$, set

$$V_\varepsilon(x) := \left(\frac{1}{\varepsilon \langle x \rangle^\eta} \right)^d \int_{\mathbb{R}^d} h\left(\frac{x-y}{\varepsilon \langle x \rangle^\eta}\right) V(y) dy,$$

where $\eta := ((\rho - 1) \vee 0) \wedge 1$. Then V_ε is a smooth function and it satisfies the following:

Lemma 5.1. (i) $V_\varepsilon(x) \geq c' \langle x \rangle^\rho$ where $c' = c/4^\rho$.

(ii) $|V_\varepsilon(x) - V(x)| \leq C' \varepsilon \langle x \rangle^{(\rho-1)_+ + \eta}$ where $C' = c_1(5/4)^{(\rho-1)_+}$.

(iii) $|\nabla V_\varepsilon(x)| \leq c'_1 \langle x \rangle^{(\rho-1)_+}$ where $c'_1 = c_1(5/4)^{\rho \vee 1}$.

(iv) $|\nabla V_\varepsilon(x) - \nabla V_\varepsilon(y)| \leq (1/\varepsilon) c'_2 \{ \langle x \rangle^{(\rho-2\lambda)_+} + |x-y|^{(\rho-2\lambda)_+} \} |x-y|$ where $\lambda := (1+\eta)/2$ and $c'_2 = c_1(5/4)^{(\rho-1)_+} 2^{(\rho-3)_+} (5d/16 + 2)$.

The proof is not difficult, so is omitted (cf. [Tak]).

As a consequence of Lemma 5.1, it is easily seen that V_ε satisfies condition (A)₂, i.e.

$$\begin{aligned} \text{(A)}_{2,\varepsilon} \quad |\nabla V_\varepsilon(x)| &\leq C'_1 V_\varepsilon(x)^{1-1 \wedge \lambda/\rho}, \\ |\nabla V_\varepsilon(x) - \nabla V_\varepsilon(y)| &\leq \frac{1}{\varepsilon} C'_2 \left\{ V_\varepsilon(x)^{(1-2(1 \wedge \lambda/\rho))_+} + |x-y|^{(\rho-2)_+} \right\} |x-y| \end{aligned}$$

where $C'_1 = c'_1 c'^{-(1-1 \wedge \lambda/\rho)}$ and $C'_2 = c'_2 (c'^{-(1-2(1 \wedge \lambda/\rho))_+} \vee 1)$. Indeed, by the definition of λ , we have $\rho - \rho \wedge \lambda \geq (\rho - 1)_+$, $(\rho - 2(\rho \wedge \lambda))_+ = (\rho - 2\lambda)_+ = (\rho - 2)_+$. Hence (A)_{2, ε} follows, because, by (i) with $\langle x \rangle \geq 1$,

$$\begin{aligned} V_\varepsilon(x)^{1-1 \wedge \lambda/\rho} &\geq (c')^{1-1 \wedge \lambda/\rho} \langle x \rangle^{\rho - \rho \wedge \lambda} \geq (c')^{1-1 \wedge \lambda/\rho} \langle x \rangle^{(\rho-1)_+}, \\ V_\varepsilon(x)^{(1-2(1 \wedge \lambda/\rho))_+} &\geq (c')^{(1-2(1 \wedge \lambda/\rho))_+} \langle x \rangle^{(\rho-2(\rho \wedge \lambda))_+} = (c')^{(1-2(1 \wedge \lambda/\rho))_+} \langle x \rangle^{(\rho-2\lambda)_+}. \end{aligned}$$

In what follows we write c, C, c_1, c_2, C_1 and C_2 simply for c', C', c'_1, c'_2, C'_1 and C'_2 .

Now let $K_\varepsilon(t) := e^{-tV_\varepsilon/2} e^{-tH_0} e^{-tV_\varepsilon/2}$, $G_\varepsilon(t) := e^{-tV_\varepsilon} e^{-tH_0}$ and $R_\varepsilon(t) := e^{-tH_0/2} e^{-tV_\varepsilon} e^{-tH_0/2}$.

Claim 5.1. *Let $t \geq 0$ and $n \in \mathbb{N}$. Then with $H^\varepsilon = H_0 + V_\varepsilon$*

$$\begin{aligned}
& \|K_\varepsilon(\frac{t}{n})^n - e^{-tH^\varepsilon}\|_{p \rightarrow p} \\
& \leq \text{const}(\rho, d) \left[C_1^2 \left((\frac{t}{n})^2 + (\frac{t}{n})^{2(1 \wedge \lambda/\rho)} \right) n \mathbb{E}[\sigma_{t/n}] + \sum_{j=1}^2 \left\{ (C_2 \frac{1}{\varepsilon} \frac{1}{n} t)^j n \mathbb{E}[\sigma_{t/n}^j] \right. \right. \\
& \quad \left. \left. + (C_2 \frac{1}{\varepsilon} \frac{1}{n} t)^j n \mathbb{E}[\sigma_{t/n}^{j(2 \vee \rho)/2}] + (C_2 \frac{1}{\varepsilon} (\frac{1}{n})^{1 \wedge 2\lambda/\rho} t^{1 \wedge 2\lambda/\rho})^j n \mathbb{E}[\sigma_{t/n}^j] \right\} \right], \\
& \|G_\varepsilon(\frac{t}{n})^n - e^{-tH^\varepsilon}\|_{p \rightarrow p}, \|R_\varepsilon(\frac{t}{n})^n - e^{-tH^\varepsilon}\|_{p \rightarrow p} \\
& \leq \text{const}(\rho, d) \left[\frac{1}{n} C_1 (t + t^{1 \wedge \lambda/\rho}) \mathbb{E}[\sigma_t^{1/2}] + C_2 \frac{1}{\varepsilon} \frac{1}{n} t^{1 \wedge 2\lambda/\rho} \mathbb{E}[\sigma_t] \right. \\
& \quad \left. + C_2 \frac{1}{\varepsilon} \frac{1}{n} t (\mathbb{E}[\sigma_t] + \mathbb{E}[\sigma_t^{(2 \vee \rho)/2}]) + C_1 (\frac{t}{n} + (\frac{t}{n})^{1 \wedge \lambda/\rho}) \mathbb{E}[\sigma_{t/n}^{1/2}] \right. \\
& \quad \left. + C_1^2 \left((\frac{t}{n})^2 + (\frac{t}{n})^{2(1 \wedge \lambda/\rho)} \right) n \mathbb{E}[\sigma_{t/n}] + \sum_{j=1}^2 \left\{ (C_2 \frac{1}{\varepsilon} \frac{1}{n} t)^j n \mathbb{E}[\sigma_{t/n}^j] + (C_2 \frac{1}{\varepsilon} \frac{1}{n} t)^j n \mathbb{E}[\sigma_{t/n}^{j(2 \vee \rho)/2}] \right. \right. \\
& \quad \left. \left. + (C_2 \frac{1}{\varepsilon} (\frac{1}{n})^{1 \wedge 2\lambda/\rho} t^{1 \wedge 2\lambda/\rho})^j n \mathbb{E}[\sigma_{t/n}^j] \right\} \right].
\end{aligned}$$

This is obvious from (A)_{2,ε} and Theorem 2.2(iii).

Claim 5.2. *Let $t \geq 0$ and $n \in \mathbb{N}$. Then*

$$\begin{aligned}
& \|e^{-tH} - e^{-tH^\varepsilon}\|_{p \rightarrow p}, \\
& \|K(\frac{t}{n})^n - K_\varepsilon(\frac{t}{n})^n\|_{p \rightarrow p}, \|G(\frac{t}{n})^n - G_\varepsilon(\frac{t}{n})^n\|_{p \rightarrow p}, \|R(\frac{t}{n})^n - R_\varepsilon(\frac{t}{n})^n\|_{p \rightarrow p} \\
& \leq \text{const}(C, c, \rho) \varepsilon t^{2/((\rho \wedge 2) \vee 1) - 1}.
\end{aligned}$$

Proof. Let $f \in C_0^\infty(\mathbb{R}^d)$. By (3.4), (3.5) and (3.6) with (A.6),

$$\begin{aligned}
& |(e^{-tH} - e^{-tH^\varepsilon})f(x)| \\
& \leq \mathbb{E} \left[\left| \exp\left(-\int_0^t V(x + X_s) ds\right) - \exp\left(-\int_0^t V_\varepsilon(x + X_s) ds\right) \right| |f(x + X_t)| \right], \tag{5.1}
\end{aligned}$$

$$\begin{aligned}
& |(K(\frac{t}{n})^n - K_\varepsilon(\frac{t}{n})^n)f(x)| \\
& \leq \mathbb{E} \left[\left| \exp\left(-\frac{t}{2n} \sum_{k=1}^n (V(x + X_{(k-1)t/n}) + V(x + X_{kt/n}))\right) \right. \right. \\
& \quad \left. \left. - \exp\left(-\frac{t}{2n} \sum_{k=1}^n (V_\varepsilon(x + X_{(k-1)t/n}) + V_\varepsilon(x + X_{kt/n}))\right) \right| |f(x + X_t)| \right], \tag{5.2}
\end{aligned}$$

$$\begin{aligned}
& |(G(\frac{t}{n})^n - G_\varepsilon(\frac{t}{n})^n)f(x)| \\
& \leq \mathbb{E} \left[\left| \exp\left(-\frac{t}{n} \sum_{k=1}^n V(x + X_{(k-1)t/n})\right) \right. \right. \\
& \quad \left. \left. - \exp\left(-\frac{t}{n} \sum_{k=1}^n V_\varepsilon(x + X_{(k-1)t/n})\right) \right| |f(x + X_t)| \right], \tag{5.3}
\end{aligned}$$

$$\begin{aligned}
& |(R(\frac{t}{n})^n - R_\varepsilon(\frac{t}{n})^n)f(x)| \\
& \leq \mathbb{E} \left[\left| \exp\left(-\frac{t}{n} \sum_{k=1}^n V(x + X_{(2k-1)t/2n})\right) \right. \right. \\
& \quad \left. \left. - \exp\left(-\frac{t}{n} \sum_{k=1}^n V_\varepsilon(x + X_{(2k-1)t/2n})\right) \right| |f(x + X_t)| \right]. \tag{5.4}
\end{aligned}$$

By a formula

$$e^{-a} - e^{-b} = \int_0^1 (b-a)e^{-\theta a} e^{-(1-\theta)b} d\theta, \quad a, b \in \mathbb{R}$$

and Lemma 5.1, we have

$$\begin{aligned}
& \left| \exp\left(-\int_0^t V(x + X_s) ds\right) - \exp\left(-\int_0^t V_\varepsilon(x + X_s) ds\right) \right| \\
& \leq \int_0^1 d\theta \int_0^t |V_\varepsilon(x + X_s) - V(x + X_s)| ds \\
& \quad \times \exp\left(-\theta \int_0^t V(x + X_s) ds\right) \exp\left(-(1-\theta) \int_0^t V_\varepsilon(x + X_s) ds\right) \\
& \leq C\varepsilon \int_0^t \langle x + X_s \rangle^{(\rho-1)++\eta} ds \exp\left(-c \int_0^t \langle x + X_s \rangle^\rho ds\right), \\
& \left| \exp\left(-\frac{t}{2n} \sum_{k=1}^n (V(x + X_{(k-1)t/n}) + V(x + X_{kt/n}))\right) \right. \\
& \quad \left. - \exp\left(-\frac{t}{2n} \sum_{k=1}^n (V_\varepsilon(x + X_{(k-1)t/n}) + V_\varepsilon(x + X_{kt/n}))\right) \right| \\
& \leq \int_0^1 d\theta \frac{t}{2n} \sum_{k=1}^n \left(|V_\varepsilon(x + X_{(k-1)t/n}) - V(x + X_{(k-1)t/n})| \right. \\
& \quad \left. + |V_\varepsilon(x + X_{kt/n}) - V(x + X_{kt/n})| \right) \\
& \quad \times \exp\left(-\theta \frac{t}{2n} \sum_{k=1}^n (V(x + X_{(k-1)t/n}) + V(x + X_{kt/n}))\right) \\
& \quad \times \exp\left(-(1-\theta) \frac{t}{2n} \sum_{k=1}^n (V_\varepsilon(x + X_{(k-1)t/n}) + V_\varepsilon(x + X_{kt/n}))\right) \\
& \leq C\varepsilon \left(\frac{t}{2n} \sum_{k=1}^n \langle x + X_{(k-1)t/n} \rangle^{(\rho-1)++\eta} + \frac{t}{2n} \sum_{k=1}^n \langle x + X_{kt/n} \rangle^{(\rho-1)++\eta} \right) \\
& \quad \times \exp\left(-c \frac{t}{2n} \sum_{k=1}^n \langle x + X_{(k-1)t/n} \rangle^\rho\right) \exp\left(-c \frac{t}{2n} \sum_{k=1}^n \langle x + X_{kt/n} \rangle^\rho\right).
\end{aligned}$$

Similarly

$$\begin{aligned}
& \left| \exp\left(-\frac{t}{n} \sum_{k=1}^n V(x + X_{(k-1)t/n})\right) - \exp\left(-\frac{t}{n} \sum_{k=1}^n V_\varepsilon(x + X_{(k-1)t/n})\right) \right| \\
& \leq C\varepsilon \frac{t}{n} \sum_{k=1}^n \langle x + X_{(k-1)t/n} \rangle^{(\rho-1)_+ + \eta} \exp\left(-c\frac{t}{n} \sum_{k=1}^n \langle x + X_{(k-1)t/n} \rangle^\rho\right), \\
& \left| \exp\left(-\frac{t}{n} \sum_{k=1}^n V(x + X_{(2k-1)t/2n})\right) - \exp\left(-\frac{t}{n} \sum_{k=1}^n V_\varepsilon(x + X_{(2k-1)t/2n})\right) \right| \\
& \leq C\varepsilon \frac{t}{n} \sum_{k=1}^n \langle x + X_{(2k-1)t/2n} \rangle^{(\rho-1)_+ + \eta} \exp\left(-c\frac{t}{n} \sum_{k=1}^n \langle x + X_{(2k-1)t/2n} \rangle^\rho\right).
\end{aligned}$$

By Jensen's inequality and (3.25),

$$\begin{aligned}
& \left| \exp\left(-\int_0^t V(x + X_s) ds\right) - \exp\left(-\int_0^t V_\varepsilon(x + X_s) ds\right) \right|, \\
& \left| \exp\left(-\frac{t}{n} \sum_{k=1}^n V(x + X_{(k-1)t/n})\right) - \exp\left(-\frac{t}{n} \sum_{k=1}^n V_\varepsilon(x + X_{(k-1)t/n})\right) \right|, \\
& \left| \exp\left(-\frac{t}{n} \sum_{k=1}^n V(x + X_{(2k-1)t/2n})\right) - \exp\left(-\frac{t}{n} \sum_{k=1}^n V_\varepsilon(x + X_{(2k-1)t/2n})\right) \right| \\
& \leq C\varepsilon t^{1 - ((\rho-1)_+ + \eta)/\rho} \left(\frac{(\rho-1)_+ + \eta}{\rho} \frac{1}{c\varepsilon}\right)^{((\rho-1)_+ + \eta)/\rho}, \\
& \left| \exp\left(-\frac{t}{2n} \sum_{k=1}^n (V(x + X_{(k-1)t/n}) + V(x + X_{kt/n}))\right) \right. \\
& \quad \left. - \exp\left(-\frac{t}{2n} \sum_{k=1}^n (V_\varepsilon(x + X_{(k-1)t/n}) + V_\varepsilon(x + X_{kt/n}))\right) \right| \\
& \leq C\varepsilon \left(\frac{t}{2}\right)^{1 - ((\rho-1)_+ + \eta)/\rho} 2 \left(\frac{(\rho-1)_+ + \eta}{\rho} \frac{1}{c\varepsilon}\right)^{((\rho-1)_+ + \eta)/\rho},
\end{aligned}$$

where for $\rho = 0$ we understand $((\rho - 1)_+ + \eta)/\rho = 0$. Substituting these into (5.1), (5.2), (5.3) and (5.4), respectively, we have

$$\begin{aligned}
& |(e^{-tH} - e^{-tH^\varepsilon})f(x)|, \\
& |(K(\frac{t}{n})^n - K_\varepsilon(\frac{t}{n})^n)f(x)|, |(G(\frac{t}{n})^n - G_\varepsilon(\frac{t}{n})^n)f(x)|, |(R(\frac{t}{n})^n - R_\varepsilon(\frac{t}{n})^n)f(x)| \\
& \leq \text{const}(C, c, \rho) \varepsilon t^{2/((\rho \wedge 2) \vee 1) - 1} \mathbb{E}[|f(x + X_t)|],
\end{aligned}$$

which imply the estimates in Claim 5.2 and the proof is complete. \square

Proof of Theorem 2.3(i). By Claims 5.1 and 5.2

$$\|K(\frac{t}{n})^n - e^{-tH}\|_{p \rightarrow p}$$

$$\begin{aligned}
&\leq \|K(\frac{t}{n})^n - K_\varepsilon(\frac{t}{n})^n\|_{p \rightarrow p} + \|K_\varepsilon(\frac{t}{n})^n - e^{-tH^\varepsilon}\|_{p \rightarrow p} + \|e^{-tH^\varepsilon} - e^{-tH}\|_{p \rightarrow p} \\
&\leq \text{const}(\rho, C, c, d) \left[\varepsilon t^{2/((\rho \wedge 2) \vee 1) - 1} + C_1^2 \left((\frac{t}{n})^2 + (\frac{t}{n})^{2(1 \wedge \lambda/\rho)} \right) n \mathbb{E}[\sigma_{t/n}] \right. \\
&\quad + \sum_{j=1}^2 \left\{ (C_2 \frac{1}{\varepsilon} \frac{1}{n} t)^j n \mathbb{E}[\sigma_{t/n}^j] + (C_2 \frac{1}{\varepsilon} \frac{1}{n} t)^j n \mathbb{E}[\sigma_{t/n}^{j(2 \vee \rho)/2}] \right. \\
&\quad \left. \left. + (C_2 \frac{1}{\varepsilon} (\frac{1}{n})^{1 \wedge 2\lambda/\rho} t^{1 \wedge 2\lambda/\rho})^j n \mathbb{E}[\sigma_{t/n}^j] \right\} \right], \\
&\|G(\frac{t}{n})^n - e^{-tH}\|_{p \rightarrow p}, \|R(\frac{t}{n})^n - e^{-tH}\|_{p \rightarrow p} \\
&\leq \text{const}(\rho, C, c, d) \left[\varepsilon t^{2/((\rho \wedge 2) \vee 1) - 1} + \frac{1}{n} C_1 (t + t^{1 \wedge \lambda/\rho}) \mathbb{E}[\sigma_t^{1/2}] \right. \\
&\quad + C_2 \frac{1}{\varepsilon} \frac{1}{n} t^{1 \wedge 2\lambda/\rho} \mathbb{E}[\sigma_t] + C_2 \frac{1}{\varepsilon} \frac{1}{n} t (\mathbb{E}[\sigma_t] + \mathbb{E}[\sigma_t^{(2 \vee \rho)/2}]) \\
&\quad + C_1 \left(\frac{t}{n} + (\frac{t}{n})^{1 \wedge \lambda/\rho} \right) \mathbb{E}[\sigma_{t/n}^{1/2}] + C_1^2 \left((\frac{t}{n})^2 + (\frac{t}{n})^{2(1 \wedge \lambda/\rho)} \right) n \mathbb{E}[\sigma_{t/n}] \\
&\quad + \sum_{j=1}^2 \left\{ (C_2 \frac{1}{\varepsilon} \frac{1}{n} t)^j n \mathbb{E}[\sigma_{t/n}^j] + (C_2 \frac{1}{\varepsilon} \frac{1}{n} t)^j n \mathbb{E}[\sigma_{t/n}^{j(2 \vee \rho)/2}] \right. \\
&\quad \left. \left. + (C_2 \frac{1}{\varepsilon} (\frac{1}{n})^{1 \wedge 2\lambda/\rho} t^{1 \wedge 2\lambda/\rho})^j n \mathbb{E}[\sigma_{t/n}^j] \right\} \right].
\end{aligned}$$

Now let $n \geq 2^{2(2 \vee \rho)}$ and $\varepsilon := n^{-(1/2) \wedge (\lambda/\rho)} = n^{-1/2 \vee \rho}$. Then $\varepsilon \leq 1/4$, $\varepsilon^{-1} n^{-1 \wedge 2\lambda/\rho} = n^{-1/2 \vee \rho}$ and $\varepsilon^{-1} n^{-1} \leq n^{-1/2 \vee \rho}$. Therefore we have

$$\begin{aligned}
&\|K(\frac{t}{n})^n - e^{-tH}\|_{p \rightarrow p} \\
&\leq \text{const}(\rho, C, c, C_1, C_2, d) \left(\frac{1}{n} \right)^{1/2 \vee \rho} \left[t^{2/((\rho \wedge 2) \vee 1) - 1} + (t^2 + t^{2(1 \wedge ((\rho \wedge 2) \vee 1)/2\rho)}) n \mathbb{E}[\sigma_{t/n}] \right. \\
&\quad \left. + \sum_{j=1}^2 \left\{ (t^j + t^{j2/2 \vee \rho}) n \mathbb{E}[\sigma_{t/n}^j] + t^j n \mathbb{E}[\sigma_{t/n}^{j(2 \vee \rho)/2}] \right\} \right], \\
&\|G(\frac{t}{n})^n - e^{-tH}\|_{p \rightarrow p}, \|R(\frac{t}{n})^n - e^{-tH}\|_{p \rightarrow p} \\
&\leq \text{const}(\rho, C, c, C_1, C_2, d) \left(\frac{1}{n} \right)^{1/2 \vee \rho} \left[t^{2/((\rho \wedge 2) \vee 1) - 1} + (t + t^{1 \wedge ((\rho \wedge 2) \vee 1)/2\rho}) \mathbb{E}[\sigma_t^{1/2}] \right. \\
&\quad + t^{2/2 \vee \rho} \mathbb{E}[\sigma_t] + t (\mathbb{E}[\sigma_t] + \mathbb{E}[\sigma_t^{(2 \vee \rho)/2}]) + (t^2 + t^{2(1 \wedge ((\rho \wedge 2) \vee 1)/2\rho)}) n \mathbb{E}[\sigma_{t/n}] \\
&\quad \left. + \sum_{j=1}^2 \left\{ (t^j + t^{j2/2 \vee \rho}) n \mathbb{E}[\sigma_{t/n}^j] + t^j n \mathbb{E}[\sigma_{t/n}^{j(2 \vee \rho)/2}] \right\} \right],
\end{aligned}$$

and the proof is complete. \square

6. Proof of Theorem 2.4

For $a > 0$, the proof will be given, divided into the three cases $a = 1$, $a > 1$ and $0 < a < 1$.

First we note that for every $a > 0$

$$\mathbb{E}[\sigma_t^a] < \infty. \tag{6.1}$$

In fact, it is enough to show when $a = \nu$ is a positive integer. To do so, let φ_t be the characteristic function of σ_t , i.e., $\varphi_t(\xi) = \mathbb{E}[e^{\sqrt{-1}\xi\sigma_t}]$. We have $\varphi_t(\xi) = e^{-tf(\xi)}$, where

$$f(\xi) = \int_{(0,\infty)} (1 - e^{\sqrt{-1}\xi l})e^{-l/2}n(dl).$$

Since smoothness of $\varphi_t(\xi)$ near $\xi = 0$ implies existence of moments of σ_t (cf. Exercise 2.6(viii) in [It]), we have only to show that φ_t or f is in C^∞ near $\xi = 0$. But this is obvious, because, by a property of the Lévy measure n , the integral $\int_{(0,\infty)} l^\nu e^{-l/2}n(dl)$ is convergent, so that by the Lebesgue convergence theorem

$$\left(\frac{d}{d\xi}\right)^\nu f(\xi) = - \int_{(0,\infty)} (\sqrt{-1}l)^\nu e^{\sqrt{-1}\xi l} e^{-l/2}n(dl).$$

By Itô's formula (e.g. [Ik-Wa]),

$$\begin{aligned} \sigma_t^a &= \int_0^{t+} \int_{(0,\infty)} \left\{ (\sigma_{s-} + l)^a - \sigma_{s-}^a \right\} N(dsdl) \\ &= \int_0^{t+} \int_{(0,\infty)} a \int_0^1 (\sigma_{s-} + \theta l)^{a-1} d\theta l N(dsdl), \end{aligned}$$

and hence, by taking expectation \mathbb{E}

$$\mathbb{E}[\sigma_t^a] = \int_0^t ds \int_{(0,\infty)} l e^{-l/2}n(dl) a \int_0^1 \mathbb{E}[(\sigma_s + \theta l)^{a-1}] d\theta. \quad (6.2)$$

This is further, by the change of variable $r = \frac{s}{t}$, rewritten as

$$\frac{1}{t} \mathbb{E}[\sigma_t^a] = \int_0^1 dr \int_{(0,\infty)} l e^{-l/2}n(dl) a \int_0^1 \mathbb{E}[(\sigma_{tr} + \theta l)^{a-1}] d\theta. \quad (6.3)$$

6.1. The case $a = 1$

By (6.3), it is clear that

$$\frac{1}{t} \mathbb{E}[\sigma_t] = \int_{(0,\infty)} l e^{-l/2}n(dl) \in (0, \infty). \quad (6.4)$$

6.2. The case $a > 1$

By (6.1) and (6.3), $\mathbb{E}[(\sigma_r + \theta l)^{a-1}]$ is of course integrable on $(0, \infty) \times [0, 1] \times [0, 1]$ w.r.t. $l e^{-l/2}n(dl) dr d\theta$. Since σ_t is increasing in t with $\sigma_{0+} = \sigma_0 = 0$ and $a - 1 > 0$, we have $(\sigma_{tr} + \theta l)^{a-1} \downarrow \theta^{a-1} l^{a-1}$ as $t \downarrow 0$. It follows by the Lebesgue convergence theorem that

$$\frac{1}{t} \mathbb{E}[\sigma_t^a] \downarrow \int_{(0,\infty)} l^a e^{-l/2}n(dl) \in (0, \infty). \quad (6.5)$$

6.3. The case $0 < a < 1$

By the same reason as above (but in this case, $a - 1 < 0$), we have $(\sigma_{tr} + \theta l)^{a-1} \uparrow \theta^{a-1} l^{a-1}$ as $t \downarrow 0$, and hence, by the monotone convergence theorem

$$\frac{1}{t} \mathbb{E}[\sigma_t^a] \uparrow \int_{(0, \infty)} l^a e^{-l/2} n(dl) \in (0, \infty]. \quad (6.6)$$

This time the integral on the RHS is not always convergent. To find the exact asymptotics we suppose assumption (L).

We start with a remark on (L) and $\psi(\lambda)$ defined by (1.3):

Fact. (i) *If $0 \leq \alpha < 1$, then*

$$\psi(\lambda) \sim \Gamma(1 - \alpha) \lambda^\alpha L(\lambda) \quad \text{as } \lambda \uparrow \infty.$$

(ii) *If $\alpha = 1$, then $\int_0^\cdot n((s, \infty)) ds$ is slowly varying at zero, $L(1/t) = o(\int_0^t n((s, \infty)) ds)$ as $t \downarrow 0$ and*

$$\psi(\lambda) \sim \lambda \int_0^{1/\lambda} n((s, \infty)) ds \quad \text{as } \lambda \uparrow \infty.$$

Proof. First of all note that

$$\infty > \int_{(0, \infty)} l \wedge 1 n(dl) = \int_0^1 n((t, \infty)) dt, \quad (6.7)$$

$$\psi(\lambda) = \lambda \int_0^\infty e^{-\lambda t} d\left(\int_0^t n((s, \infty)) ds\right). \quad (6.8)$$

By (1.6), $n((1/y, \infty)) \sim y^\alpha L(y)$ as $y \uparrow \infty$, and by (6.7),

$$\int_x^\infty \frac{1}{y^2} n\left(\frac{1}{y}, \infty\right) dy = \int_0^{1/x} n((s, \infty)) ds < \infty \quad \text{for any } x > 0.$$

Let us apply Lemma and Theorem 1 of §VIII.9 in [Fe]. These say that $\int_x^\infty 1/y^2 n((1/y, \infty)) dy$ is regularly varying with exponent $-1 + \alpha$ and

$$\frac{(1/x)n((1/x, \infty))}{\int_x^\infty 1/y^2 n((1/y, \infty)) dy} \longrightarrow 1 - \alpha \quad \text{as } x \uparrow \infty.$$

Combining these with (1.6), we see that when $0 \leq \alpha < 1$

$$\int_0^t n((s, \infty)) ds \sim \frac{1}{1-\alpha} t n((t, \infty)) \sim \frac{1}{1-\alpha} t^{1-\alpha} L\left(\frac{1}{t}\right) \quad \text{as } t \downarrow 0,$$

and that when $\alpha = 1$, $\int_0^\cdot n((s, \infty)) ds$ is slowly varying at zero and $L(1/t) = o(\int_0^t n((s, \infty)) ds)$ as $t \downarrow 0$.

By virtue of (6.8), if we apply the Abelian theorem (cf. Theorem 2 of §XIII.5 in [Fe]), the asymptotics of ψ follow from those of $\int_0^\cdot n((s, \infty)) ds$. \square

Remark. Conversely, when $0 \leq \alpha < 1$, we have (1.6) by Fact (i) by the Tauberian theorem.

Recall functions ϕ , L_1 and L_2 around assumption (L) in Section 1. By Fact, L_1 is slowly varying at infinity and

$$\phi(\lambda) \sim \lambda^\alpha L_1(\lambda) \quad \text{as } \lambda \uparrow \infty. \quad (6.9)$$

As ψ is strictly increasing with $\psi(0) = 0$ and $\psi(\infty) = \infty$, so is ϕ , so that the inverse ϕ^{-1} exists. By (6.9), if $0 < \alpha \leq 1$,

$$\phi^{-1}(x) \sim x^{1/\alpha} L_2(x) \quad \text{as } x \uparrow \infty. \quad (6.10)$$

Since, by (6.9) again, ϕ is regularly varying with exponent α , so is ϕ^{-1} with exponent $1/\alpha$, and hence L_2 and $\int_0^\cdot (\phi^{-1}(\theta))^{-\alpha} d\theta$ ($0 < \alpha < 1$) are also slowly varying at infinity.

Now we are in a position to show the asymptotics of $\mathbb{E}[\sigma_t^a]$ for $0 < a < 1$.

Claim 6.1. (i) If $0 < a < \alpha$,

$$\mathbb{E}[\sigma_t^a] \sim \frac{\Gamma(1 - a/\alpha)}{\Gamma(1 - a)} t^{a/\alpha} L_2(\frac{1}{t})^{-a} \sim \frac{\Gamma(1 - a/\alpha)}{\Gamma(1 - a)} \phi^{-1}(\frac{1}{t})^{-a} \quad \text{as } t \downarrow 0.$$

(ii) If $a = \alpha$,

$$\mathbb{E}[\sigma_t^\alpha] \sim \frac{1}{\Gamma(1 - \alpha)} t \int_0^{1/t} (\phi^{-1}(\theta))^{-\alpha} d\theta \quad \text{as } t \downarrow 0.$$

(iii) If $\alpha < a < 1$, then $\int_0^\infty \lambda^{-1-a} \phi(\lambda) d\lambda \in (0, \infty)$ and

$$\mathbb{E}[\sigma_t^a] \sim t \frac{a}{\Gamma(1 - a)} \int_0^\infty \lambda^{-1-a} \phi(\lambda) d\lambda \quad \text{as } t \downarrow 0.$$

Proof. To rewrite (6.2), we see first with (2.2)

$$\begin{aligned} \mathbb{E}[(\sigma_s + \theta l)^{a-1}] &= \frac{1}{\Gamma(1 - a)} \int_0^\infty \lambda^{-a} e^{-\lambda \theta l} \mathbb{E}[e^{-\lambda \sigma_s}] d\lambda \\ &= \frac{1}{\Gamma(1 - a)} \int_0^\infty \lambda^{-a} e^{-\lambda \theta l} e^{-s\phi(\lambda)} d\lambda, \end{aligned}$$

and then we have

$$\begin{aligned} \mathbb{E}[\sigma_t^a] &= \frac{a}{\Gamma(1 - a)} \int_0^t ds \int_0^\infty \lambda^{-1-a} e^{-s\phi(\lambda)} d\lambda \int_{(0, \infty)} (1 - e^{-\lambda l}) e^{-l/2} n(dl) \\ &= \frac{a}{\Gamma(1 - a)} \int_0^t ds \int_0^\infty \lambda^{-1-a} \phi(\lambda) e^{-s\phi(\lambda)} d\lambda. \end{aligned}$$

The λ -integral in the last line is further computed by the change of variable $\lambda = \phi^{-1}(x)$ as follows:

$$\begin{aligned}
& \int_0^\infty \lambda^{-1-a} \phi(\lambda) e^{-s\phi(\lambda)} d\lambda \\
&= \int_0^\infty (\phi^{-1}(x))^{-1-a} x e^{-sx} (\phi^{-1})'(x) dx \\
&= \left[-\frac{1}{a} (\phi^{-1}(x))^{-a} x e^{-sx} \right]_0^\infty + \frac{1}{a} \int_0^\infty (\phi^{-1}(x))^{-a} (e^{-sx} - s x e^{-sx}) dx \\
&= \frac{1}{a} \left\{ \int_0^\infty (\phi^{-1}(x))^{-a} e^{-sx} dx - s \int_0^\infty (\phi^{-1}(x))^{-a} x e^{-sx} dx \right\} \\
&= \frac{1}{a} \left\{ \int_0^\infty e^{-sx} d\left(\int_0^x (\phi^{-1}(\theta))^{-a} d\theta \right) - s \int_0^\infty e^{-sx} d\left(\int_0^x (\phi^{-1}(\theta))^{-a} \theta d\theta \right) \right\} \\
&= \frac{1}{a} \left\{ L\left(s, \int_0^\cdot (\phi^{-1}(\theta))^{-a} d\theta\right) - s L\left(s, \int_0^\cdot (\phi^{-1}(\theta))^{-a} \theta d\theta\right) \right\}.
\end{aligned}$$

Here $L(\cdot, G)$ denotes the Laplace transform of a right-continuous increasing function $G : [0, \infty) \rightarrow [0, \infty)$: $L(s, G) := \int_0^\infty e^{-sx} dG(x)$. The last fourth and third equalities are respectively because $0 \leq (\phi^{-1}(x))^{-a} x e^{-sx} \leq (\psi'(1/2))^a x^{1-a} e^{-sx} \rightarrow 0$ as $x \downarrow 0$, and because for $b > a - 1$, $\int_0^\infty (\phi^{-1}(x))^{-a} x^b e^{-sx} dx \leq (\psi'(1/2))^a \int_0^\infty x^{b-a} e^{-sx} dx = (\psi'(1/2))^a s^{a-b-1} \Gamma(b-a+1) < \infty$. Hence (6.2) is rewritten as follows:

$$\mathbb{E}[\sigma_t^a] = \frac{1}{\Gamma(1-a)} \int_0^t \left(L\left(s, \int_0^\cdot (\phi^{-1}(\theta))^{-a} d\theta\right) - s L\left(s, \int_0^\cdot (\phi^{-1}(\theta))^{-a} \theta d\theta\right) \right) ds. \quad (6.11)$$

1° The case $0 < a < \alpha$. Then $0 < \alpha \leq 1$. By (6.10), $(\phi^{-1}(\cdot))^{-a}$ is regularly varying with exponent $-a/\alpha \in (-1, 0)$. By Theorem 1 of §VIII.9 in [Fe],

$$\frac{x^2 (\phi^{-1}(x))^{-a}}{\int_0^x (\phi^{-1}(\theta))^{-a} \theta d\theta} \rightarrow 2 - \frac{a}{\alpha} > 0, \quad \frac{x (\phi^{-1}(x))^{-a}}{\int_0^x (\phi^{-1}(\theta))^{-a} d\theta} \rightarrow \frac{\alpha-a}{\alpha} > 0$$

as $x \uparrow \infty$. Hence, by combining this with (6.10),

$$\begin{aligned}
\int_0^x (\phi^{-1}(\theta))^{-a} \theta d\theta &\sim \frac{1}{2 - a/\alpha} x^2 (\phi^{-1}(x))^{-a} \sim \frac{1}{2 - a/\alpha} x^{2-a/\alpha} L_2(x)^{-a}, \\
\int_0^x (\phi^{-1}(\theta))^{-a} d\theta &\sim \frac{\alpha}{\alpha-a} x (\phi^{-1}(x))^{-a} \sim \frac{\alpha}{\alpha-a} x^{1-a/\alpha} L_2(x)^{-a}
\end{aligned}$$

as $x \uparrow \infty$. By applying the Abelian theorem (cf. Theorem 2 of §XIII.5 in [Fe]), this implies that

$$\begin{aligned}
L\left(s, \int_0^\cdot (\phi^{-1}(\theta))^{-a} \theta d\theta\right) &\sim \frac{\Gamma(2 - a/\alpha + 1)}{2 - a/\alpha} s^{-2+a/\alpha} L_2\left(\frac{1}{s}\right)^{-a} = \Gamma\left(2 - \frac{a}{\alpha}\right) s^{-2+a/\alpha} L_2\left(\frac{1}{s}\right)^{-a}, \\
L\left(s, \int_0^\cdot (\phi^{-1}(\theta))^{-a} d\theta\right) &\sim \frac{\alpha}{\alpha-a} \Gamma\left(2 - \frac{a}{\alpha}\right) s^{-1+a/\alpha} L_2\left(\frac{1}{s}\right)^{-a}
\end{aligned}$$

as $s \downarrow 0$, and hence

$$\begin{aligned}
L\left(s, \int_0^\cdot (\phi^{-1}(\theta))^{-a} d\theta\right) - s L\left(s, \int_0^\cdot (\phi^{-1}(\theta))^{-a} \theta d\theta\right) &\sim \left(\frac{\alpha}{\alpha-a} - 1\right) \Gamma\left(2 - \frac{a}{\alpha}\right) s^{-1+a/\alpha} L_2\left(\frac{1}{s}\right)^{-a} \\
&= \frac{a}{\alpha} \Gamma\left(\frac{\alpha-a}{\alpha}\right) s^{-1+a/\alpha} L_2\left(\frac{1}{s}\right)^{-a} \quad \text{as } s \downarrow 0.
\end{aligned}$$

Now if, for simplicity, we set

$$Z(x) := L\left(\frac{1}{x}, \int_0^{\cdot} (\phi^{-1}(\theta))^{-a} d\theta\right) - \frac{1}{x} L\left(\frac{1}{x}, \int_0^{\cdot} (\phi^{-1}(\theta))^{-a} \theta d\theta\right),$$

then, by (6.11)

$$\mathbb{E}[\sigma_t^a] = \frac{1}{\Gamma(1-a)} \int_0^t Z\left(\frac{1}{s}\right) ds = \frac{1}{\Gamma(1-a)} \int_{1/t}^{\infty} x^{-2} Z(x) dx$$

and also,

$$Z(x) \sim \frac{a}{\alpha} \Gamma\left(\frac{\alpha-a}{\alpha}\right) x^{1-a/\alpha} L_2(x)^{-a} \quad \text{as } x \uparrow \infty.$$

Therefore, applying Theorem 1 of §VIII.9 in [Fe] again, we have

$$\frac{(1/t)^{-2+1} Z(1/t)}{\mathbb{E}[\sigma_t^a]} \longrightarrow \Gamma(1-a) \frac{a}{\alpha} \quad \text{as } t \downarrow 0,$$

and consequently

$$\mathbb{E}[\sigma_t^a] \sim \frac{\Gamma((\alpha-a)/\alpha)}{\Gamma(1-a)} t^{a/\alpha} L_2\left(\frac{1}{t}\right)^{-a},$$

which is just the assertion (i).

2° The case $a = \alpha$. Then $0 < \alpha < 1$ and hence, by (6.10), $(\phi^{-1}(\cdot))^{-\alpha}$ is regularly varying with exponent -1 . Once again, by Theorem 1 of §VIII.9 in [Fe],

$$\frac{x^2 (\phi^{-1}(x))^{-\alpha}}{\int_0^x (\phi^{-1}(\theta))^{-\alpha} \theta d\theta} \longrightarrow 1, \quad \frac{x (\phi^{-1}(x))^{-\alpha}}{\int_0^x (\phi^{-1}(\theta))^{-\alpha} d\theta} \longrightarrow 0$$

as $x \uparrow \infty$, and $\int_0^{\cdot} (\phi^{-1}(\theta))^{-\alpha} d\theta$ is slowly varying at infinity. By combining this with (6.10)

$$\int_0^x (\phi^{-1}(\theta))^{-\alpha} \theta d\theta \sim x^2 (\phi^{-1}(x))^{-\alpha} \sim x L_2(x)^{-\alpha},$$

$$L_2(x)^{-\alpha} \sim x (\phi^{-1}(x))^{-\alpha} = o\left(\int_0^x (\phi^{-1}(\theta))^{-\alpha} d\theta\right)$$

as $x \uparrow \infty$, and hence, by the Abelian theorem

$$L\left(s, \int_0^{\cdot} (\phi^{-1}(\theta))^{-\alpha} \theta d\theta\right) \sim s^{-1} L_2\left(\frac{1}{s}\right)^{-\alpha},$$

$$L\left(s, \int_0^{\cdot} (\phi^{-1}(\theta))^{-\alpha} d\theta\right) \sim \int_0^{1/s} (\phi^{-1}(\theta))^{-\alpha} d\theta$$

as $s \downarrow 0$. Therefore

$$\begin{aligned} Z\left(\frac{1}{s}\right) &= L\left(s, \int_0^{\cdot} (\phi^{-1}(\theta))^{-\alpha} d\theta\right) - s L\left(s, \int_0^{\cdot} (\phi^{-1}(\theta))^{-\alpha} \theta d\theta\right) \\ &\sim \int_0^{1/s} (\phi^{-1}(\theta))^{-\alpha} d\theta \quad \text{as } s \downarrow 0. \end{aligned}$$

In exactly the same way as in 1° we eventually have

$$\frac{(1/t)^{-2+1}Z(1/t)}{\mathbb{E}[\sigma_t^\alpha]} \longrightarrow \Gamma(1-\alpha) \quad \text{as } t \downarrow 0,$$

from which the assertion (ii) is easily seen.

3° The case $\alpha < a < 1$. Then $0 \leq \alpha < 1$. By (6.6), it is enough to show that

$$\int_{(0,\infty)} l^a e^{-l/2} n(dl) = \frac{a}{\Gamma(1-a)} \int_0^\infty \lambda^{-1-a} \phi(\lambda) d\lambda < \infty.$$

First this identity is seen from the following computation:

$$\begin{aligned} \int_0^\infty \lambda^{-1-a} \phi(\lambda) d\lambda &= \int_0^\infty \lambda^{-1-a} \left(\psi(\lambda + \tfrac{1}{2}) - \psi(\tfrac{1}{2}) \right) d\lambda \\ &= \int_{(0,\infty)} e^{-l/2} n(dl) \int_0^\infty \lambda^{-1-a} (1 - e^{-\lambda l}) d\lambda \\ &= \frac{\Gamma(1-a)}{a} \int_{(0,\infty)} l^a e^{-l/2} n(dl). \end{aligned}$$

Next this integral is convergent. Indeed, since $\phi(\lambda) \leq \psi'(1/2)\lambda$ ($\lambda \geq 0$),

$$\int_0^R \lambda^{-1-a} \phi(\lambda) d\lambda \leq \psi'(\tfrac{1}{2}) \int_0^R \lambda^{-a} d\lambda = \psi'(\tfrac{1}{2}) \frac{R^{1-a}}{1-a} < \infty$$

for any $R > 0$. On the other hand, since $\phi(\lambda) \sim \lambda^\alpha L_1(\lambda)$ as $\lambda \uparrow \infty$, and $L_1(\cdot)$ is slowly varying at infinity, there exists an $R_\varepsilon > 0$ for $0 < \varepsilon < a - \alpha$ (cf. Lemma 2 of §VIII.8 in [Fe]) such that $\phi(\lambda) \leq 2\lambda^\alpha L_1(\lambda)$ and $L_1(\lambda) < \lambda^\varepsilon$ for any $\lambda \geq R_\varepsilon$. Hence

$$\int_{R_\varepsilon}^\infty \lambda^{-1-a} \phi(\lambda) d\lambda \leq \int_{R_\varepsilon}^\infty \lambda^{-1-a} 2\lambda^{\alpha+\varepsilon} d\lambda = \frac{2}{a-\alpha-\varepsilon} \left(\frac{1}{R_\varepsilon} \right)^{a-\alpha-\varepsilon} < \infty. \quad \square$$

Appendix: Semigroups $e^{-t(H_0^\psi + V)}$ and their generators in $L_p(\mathbb{R}^d)$ and $C_\infty(\mathbb{R}^d)$

In this appendix we suppose only that $V : \mathbb{R}^d \rightarrow [0, \infty)$ is a continuous function. The main result is Theorem A.1, which follows from Lemma A.2 (Kato's inequality).

Let $M(dsdx)$ be a Poisson random measure on $[0, \infty) \times (\mathbb{R}^d \setminus \{0\})$ with intensity measure $dsJ(dx)$, where

$$J(dx) := \int_{(0,\infty)} e^{-l/2} p(l, x) n(dl) dx, \quad (\text{A.1})$$

$$p(l, x) := \left(\frac{1}{2\pi l} \right)^{d/2} \exp\left(-\frac{|x|^2}{2l}\right). \quad (\text{A.2})$$

This $M(\cdot)$ may be defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as in Section 2. Note that for $p \in [1, \infty)$ the $2p$ -th order absolute moment of J is finite, i.e.,

$$\int_{\mathbb{R}^d \setminus \{0\}} |x|^{2p} J(dx) < \infty. \quad (\text{A.3})$$

Following the notation in [Ik-Wa], we set

$$\widehat{M}(dsdx) := dsJ(dx), \quad \widetilde{M}(dsdx) := M(dsdx) - \widehat{M}(dsdx)$$

and define an \mathbb{R}^d -valued right-continuous process $(X_t)_{t \geq 0}$ by

$$X_t := \int_0^{t+} \int_{\mathbb{R}^d \setminus \{0\}} x 1_{|x| \geq 1} M(dsdx) + \int_0^{t+} \int_{\mathbb{R}^d \setminus \{0\}} x 1_{|x| < 1} \widetilde{M}(dsdx), \quad (\text{A.4})$$

where the second term on the RHS is a stochastic integral w.r.t. \widetilde{M} . This is a d -dimensional time-homogeneous Lévy process starting at the origin such that

$$\mathbb{E}[e^{\sqrt{-1}\langle p, X_t \rangle}] = e^{-t(\psi(|p|^2+1)/2) - \psi(1/2)},$$

which is easily seen by Itô's formula (cf. [Ik-Wa]), so that

$$(X_t)_{t \geq 0} \stackrel{\mathcal{L}}{\sim} (B(\sigma_t))_{t \geq 0}. \quad (\text{A.5})$$

We now define a system of operators $P_t^{\psi, V}, t \geq 0$, by the Feynman-Kac formula:

$$P_t^{\psi, V} f(x) := \mathbb{E} \left[\exp \left(- \int_0^t V(x + X_s) ds \right) f(x + X_t) \right]. \quad (\text{A.6})$$

From this definition the following is easily seen:

(i) If f is a nonnegative Borel measurable function, so is $P_t^{\psi, V} f$, and it satisfies

$$P_t^{\psi, V} (P_s^{\psi, V} f) = P_{t+s}^{\psi, V} f, \quad (\text{A.7})$$

$$\int_{\mathbb{R}^d} |P_t^{\psi, V} f(x)|^p dx \leq \int_{\mathbb{R}^d} |f(x)|^p dx, \quad 1 \leq p < \infty. \quad (\text{A.8})$$

(ii) If $f \in C_\infty(\mathbb{R}^d)$, then $P_t^{\psi, V} f \in C_\infty(\mathbb{R}^d)$ and

$$\|P_t^{\psi, V} f\|_\infty \leq \|f\|_\infty, \quad (\text{A.9})$$

$$\|P_t^{\psi, V} f - f\|_\infty \rightarrow 0 \quad \text{as } t \downarrow 0. \quad (\text{A.10})$$

(iii) For two nonnegative Borel measurable functions f, g

$$\int_{\mathbb{R}^d} P_t^{\psi, V} f(x) g(x) dx = \int_{\mathbb{R}^d} f(x) P_t^{\psi, V} g(x) dx. \quad (\text{A.11})$$

By (i) and (ii), $\{P_t^{\psi, V}\}_{t \geq 0}$ is a strongly continuous contraction semigroup on $C_\infty(\mathbb{R}^d)$. By the Riesz-Banach theorem there exists a finite measure $P^{\psi, V}(t, x, dy)$ on \mathbb{R}^d such that

$$P_t^{\psi, V} f(x) = \int_{\mathbb{R}^d} f(y) P^{\psi, V}(t, x, dy), \quad f \in C_\infty(\mathbb{R}^d). \quad (\text{A.12})$$

Indeed, by noting (A.5), $P^{\psi,V}(t, x, dy)$ is absolutely continuous w.r.t. the Lebesgue measure dy on \mathbb{R}^d and expressed as

$$P^{\psi,V}(t, x, dy) = \mathbb{E}\left[\exp\left(-\int_0^t V(B_{0,x}^{\sigma_t,y}(\sigma_s))ds\right)p(\sigma_t, x-y)\right]dy, \quad (\text{A.13})$$

where $B_{0,x}^{\tau,y}(\theta)$ is defined in (3.13).

By (i) and (ii) again $P_t^{\psi,V}$ is uniquely extended to a bounded operator on $L_p(\mathbb{R}^d)$, which is denoted by the same $P_t^{\psi,V}$, and thus $\{P_t^{\psi,V}\}_{t \geq 0}$ is a strongly continuous contraction semigroup on $L_p(\mathbb{R}^d)$. Clearly, for $f \in L_p(\mathbb{R}^d)$

$$P_t^{\psi,V}f(x) = \mathbb{E}\left[\exp\left(-\int_0^t V(x+X_s)ds\right)f(x+X_t)\right] \quad \text{a.e. } x$$

and, when $p = 2$, $P_t^{\psi,V}$ is symmetric.

Let $\mathfrak{G}_p^{\psi,V}$ be the infinitesimal generator of $\{P_t^{\psi,V}\}_{t \geq 0}$ on $L_p(\mathbb{R}^d)$ for $1 \leq p < \infty$, and on $C_\infty(\mathbb{R}^d)$ for $p = \infty$. Their domains are denoted by $\mathfrak{D}(\mathfrak{G}_p^{\psi,V})$.

Put

$$H_0^\psi f(x) := -\int_{\mathbb{R}^d \setminus \{0\}} \{f(x+y) - f(x) - \langle y, \nabla f(x) \rangle 1_{|y| < 1}\} J(dy), \quad (\text{A.14})$$

$$H^\psi f(x) := H_0^\psi f(x) + V(x)f(x). \quad (\text{A.15})$$

Claim A.1. (i) For $f \in \mathcal{S}(\mathbb{R}^d)$, $H_0^\psi f$ is in $\mathcal{S}(\mathbb{R}^d)$, and hence, for $f \in C_0^\infty(\mathbb{R}^d)$, $H^\psi f \in C_\infty(\mathbb{R}^d) \cap \bigcap_{1 \leq p < \infty} L_p(\mathbb{R}^d)$.

(ii) For $f \in C^\infty(\mathbb{R}^d) \cap L_p(\mathbb{R}^d)$ (where $1 \leq p < \infty$), $H_0^\psi f$ is well-defined, i.e., the integral in (A.14) is convergent for a.e. x , and $H_0^\psi f \in L_p^{loc}(\mathbb{R}^d)$. Also, for $f \in C^\infty(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$, the integral in (A.14) is convergent for every x and $H_0^\psi f \in C(\mathbb{R}^d)$.

For the proof, cf. [I1].

Claim A.2. $C_0^\infty(\mathbb{R}^d) \subset \bigcap_{1 \leq p \leq \infty} \mathfrak{D}(\mathfrak{G}_p^{\psi,V})$, and for $f \in C_0^\infty(\mathbb{R}^d)$, $\mathfrak{G}_p^{\psi,V}f = -H^\psi f$.

Proof. Let $f \in C_0^\infty(\mathbb{R}^d)$.

We start with the proof that

$$\frac{1}{t}(P_t^{\psi,V}f - f) \xrightarrow[t \downarrow 0]{} -H^\psi f \quad \text{in } C_\infty(\mathbb{R}^d). \quad (\text{A.16})$$

Since $H^\psi f \in C_\infty(\mathbb{R}^d)$ by Claim A.1, it is enough to check pointwise convergence (cf. Lemma 31.7 in [Sa]). To do so we apply Itô's formula for (A.4) to obtain

$$\exp\left(-\int_0^t V(x+X_s)ds\right)f(x+X_t)$$

$$\begin{aligned}
&= f(x) - \int_0^t \exp\left(-\int_0^s V(x+X_r)dr\right) V(x+X_s) f(x+X_s) ds \\
&+ \int_0^{t+} \int_{\mathbb{R}^d \setminus \{0\}} \exp\left(-\int_0^s V(x+X_r)dr\right) \left(f(x+X_{s-}+y) - f(x+X_{s-})\right) 1_{|y| \geq 1} M(dsdy) \\
&+ \int_0^{t+} \int_{\mathbb{R}^d \setminus \{0\}} \exp\left(-\int_0^s V(x+X_r)dr\right) \left(f(x+X_{s-}+y) - f(x+X_{s-})\right) 1_{|y| < 1} \widetilde{M}(dsdy) \\
&+ \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} \exp\left(-\int_0^s V(x+X_r)dr\right) \left(f(x+X_s+y) - f(x+X_s)\right. \\
&\quad \left. - \langle y, \nabla f(x+X_s) \rangle\right) 1_{|y| < 1} \widehat{M}(dsdy).
\end{aligned}$$

Note that the third term on the RHS is a martingale, so that the expectation is zero. Taking expectation and changing the variable $s = t\sigma$ we have

$$\begin{aligned}
&\frac{1}{t}(P_t^{\psi, V} f(x) - f(x)) + \int_0^1 \mathbb{E}\left[\exp\left(-\int_0^{t\sigma} V(x+X_r)dr\right) (Vf)(x+X_{t\sigma})\right] d\sigma \\
&= \int_0^1 d\sigma \int_{|y| \geq 1} \mathbb{E}\left[\exp\left(-\int_0^{t\sigma} V(x+X_r)dr\right) \left(f(x+X_{t\sigma}+y) - f(x+X_{t\sigma})\right)\right] J(dy) \\
&+ \int_0^1 d\sigma \int_{0 < |y| < 1} \mathbb{E}\left[\exp\left(-\int_0^{t\sigma} V(x+X_r)dr\right) \left(f(x+X_{t\sigma}+y) - f(x+X_{t\sigma})\right.\right. \\
&\quad \left.\left. - \langle y, \nabla f(x+X_{t\sigma}) \rangle\right)\right] J(dy) \\
&= \int_0^1 d\sigma \int_{\mathbb{R}^d \setminus \{0\}} \mathbb{E}\left[\exp\left(-\int_0^{t\sigma} V(x+X_r)dr\right)\right. \\
&\quad \left.\times \int_0^1 (1-\theta) \langle y, \nabla^2 f(x+X_{t\sigma}+\theta y) y \rangle d\theta\right] J(dy), \tag{A.17}
\end{aligned}$$

where the second equality is due to Taylor's theorem with the aid of symmetry of $J(dy)$. On letting $t \downarrow 0$ in the first equality of (A.17) we have (A.16) pointwise.

Next we prove for $1 \leq p < \infty$ that

$$\frac{1}{t}(P_t^{\psi, V} f - f) \xrightarrow[t \downarrow 0]{} -H^\psi f \quad \text{in } L_p(\mathbb{R}^d). \tag{A.18}$$

Since $H^\psi f \in L_p(\mathbb{R}^d)$ by Claim A.1, it is enough to check weak convergence (cf. Lemma 32.3 in [Sa]).

First of all, we note by (A.17) that

$$\sup_{t > 0} \|\frac{1}{t}(P_t^{\psi, V} f - f)\|_p < \infty \quad \text{for } 1 \leq p < \infty \tag{A.19}$$

and that

$$\lim_{R \rightarrow \infty} \limsup_{t \downarrow 0} \int_{|x| > R} |\frac{1}{t}(P_t^{\psi, V} f(x) - f(x))| dx = 0. \tag{A.20}$$

Indeed, by the second equality of (A.17)

$$\begin{aligned} |\frac{1}{t}(P_t^{\psi,V}f(x) - f(x))| &\leq \int_0^1 \mathbb{E}[|(Vf)(x + X_{t\sigma})|]d\sigma \\ &\quad + \int_0^1 d\sigma \int_{\mathbb{R}^d \setminus \{0\}} |y|^2 J(dy) \int_0^1 (1 - \theta) \mathbb{E}[|\nabla^2 f(x + X_{t\sigma} + \theta y)|]d\theta. \end{aligned} \tag{A.21}$$

Hence, by Minkowski's inequality, Jensen's inequality and Fubini's theorem

$$\left(\int_{\mathbb{R}^d} |\frac{1}{t}(P_t^{\psi,V}f(x) - f(x))|^p dx \right)^{1/p} \leq \|Vf\|_p + \frac{1}{2} \int_{\mathbb{R}^d \setminus \{0\}} |y|^2 J(dy) \|\nabla^2 f\|_p,$$

which shows (A.19). To show (A.20), take $R_0 > 0$ such that $\text{supp } f \subset \{x \in \mathbb{R}^d; |x| < R_0\}$, and let $R > R_0$. Note that $1_{|x|>R}h(x+y) = 1_{|x|>R}h(x+y)1_{|y|\geq R-R_0}$ for $h = f, \nabla f$ or $\nabla^2 f$. Hence, by (A.21),

$$\begin{aligned} &\int_{|x|>R} |\frac{1}{t}(P_t^{\psi,V}f(x) - f(x))|dx \\ &\leq \int_0^1 \mathbb{E} \left[\int_{|x|>R} |(Vf)(x + X_{t\sigma})|dx; |X_{t\sigma}| \geq R - R_0 \right] d\sigma \\ &\quad + \int_0^1 d\sigma \int_{\mathbb{R}^d \setminus \{0\}} |y|^2 J(dy) \int_0^1 (1 - \theta) d\theta \\ &\quad \quad \times \mathbb{E} \left[\int_{|x|>R} |\nabla^2 f(x + X_{t\sigma} + \theta y)|dx; |X_{t\sigma} + \theta y| \geq R - R_0 \right] \\ &\leq \|Vf\|_1 \int_0^1 \mathbb{P}(|X_{t\sigma}| \geq R - R_0) d\sigma \\ &\quad + \frac{1}{2} \|\nabla^2 f\|_1 \int_0^1 d\sigma \int_{\mathbb{R}^d \setminus \{0\}} |y|^2 J(dy) \mathbb{P}(|X_{t\sigma}| + |y| \geq R - R_0). \end{aligned}$$

Since $\lim_{t \downarrow 0} X_{t\sigma} = 0$ a.s., by the Lebesgue-Fatou inequality

$$\begin{aligned} &\limsup_{t \downarrow 0} \int_{|x|>R} |\frac{1}{t}(P_t^{\psi,V}f(x) - f(x))|dx \\ &\leq \frac{1}{2} \|\nabla^2 f\|_1 \int_0^1 d\sigma \int_{\mathbb{R}^d \setminus \{0\}} |y|^2 J(dy) \limsup_{t \downarrow 0} \mathbb{P}(|X_{t\sigma}| + |y| \geq R - R_0) \\ &\leq \frac{1}{2} \|\nabla^2 f\|_1 \int_{|y|\geq R-R_0} |y|^2 J(dy), \end{aligned}$$

and thus (A.20) follows.

Now we show weak convergence in $L_p(\mathbb{R}^d)$ of (A.18). When $1 < p < \infty$, let q be the conjugate exponent of p . For each $g \in L_q(\mathbb{R}^d)$ and $R > 0$

$$|\langle \frac{1}{t}(P_t^{\psi,V}f - f) + H^\psi f, g \rangle| \leq \|\frac{1}{t}(P_t^{\psi,V}f - f) + H^\psi f\|_\infty \int_{|x|\leq R} |g(x)|dx$$

$$+ (\sup_{t>0} \|\frac{1}{t}(P_t^{\psi,V}f - f)\|_p + \|H^\psi f\|_p) \left(\int_{|x|>R} |g(x)|^q dx \right)^{1/q}.$$

By (A.16), the first term tends to zero as $t \downarrow 0$ for fixed $R > 0$, and the second term tends to zero as $R \uparrow \infty$. This shows weak convergence in $L_p(\mathbb{R}^d)$. Next, when $p = 1$, for each $g \in L_\infty(\mathbb{R}^d)$ and $R > 0$,

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} \left(\frac{1}{t}(P_t^{\psi,V}f(x) - f(x)) + H^\psi f(x) \right) g(x) dx \right| \\ & \leq \|\frac{1}{t}(P_t^{\psi,V}f - f) + H^\psi f\|_\infty \int_{|x|\leq R} |g(x)| dx \\ & \quad + \left(\int_{|x|>R} |\frac{1}{t}(P_t^{\psi,V}f(x) - f(x))| dx + \int_{|x|>R} |H^\psi f(x)| dx \right) \|g\|_\infty. \end{aligned}$$

Therefore, by (A.16) and (A.20), similarly we can show weak convergence in $L_1(\mathbb{R}^d)$. The proof of Claim A.2 is complete. \square

Remark. When V is further a C^∞ -function and all its derivatives have polynomial growth, it can be shown in exactly the same way as above that $\mathcal{S}(\mathbb{R}^d) \subset \bigcap_{1 \leq p \leq \infty} \mathfrak{D}(\mathfrak{G}_p^{\psi,V})$ and $\mathfrak{G}_p^{\psi,V} = -H^\psi$ on $\mathcal{S}(\mathbb{R}^d)$.

By Claim A.2, H^ψ on $C_0^\infty(\mathbb{R}^d)$ is closable as an operator in $L_p(\mathbb{R}^d)$, $1 \leq p < \infty$, or $C_\infty(\mathbb{R}^d)$. It is natural to ask whether or not its smallest closed extension agrees with $-\mathfrak{G}_p^{\psi,V}$. The following theorem is an affirmative answer.

Theorem A.1. *The smallest closed extension of $H^\psi = -\mathfrak{G}_p^{\psi,V}|_{C_0^\infty(\mathbb{R}^d)}$ in $L_p(\mathbb{R}^d)$ ($1 \leq p < \infty$) (resp. $H^\psi = -\mathfrak{G}_p^{\psi,V}|_{C_0^\infty(\mathbb{R}^d)}$ in $C_\infty(\mathbb{R}^d)$) agrees with $-\mathfrak{G}_p^{\psi,V}$ (resp. $-\mathfrak{G}_\infty^{\psi,V}$). In other words, $C_0^\infty(\mathbb{R}^d)$ is a core of $\mathfrak{G}_p^{\psi,V}$ ($1 \leq p \leq \infty$).*

Needless to say, this theorem for $p = 2$, the L_2 -case, says nothing but that $H_0^\psi + V$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$.

In the same way as in [I1] and [I-Tsu] we prove this theorem. Take $\rho \in C_0^\infty(\mathbb{R}^d)$ such that $\rho \geq 0$, $\text{supp } \rho \subset \{x \in \mathbb{R}^d; |x| \leq 1\}$ and $\int_{\mathbb{R}^d} \rho(x) dx = 1$. For $0 < \delta \leq 1$ set $\rho_\delta(x) := (1/\delta)^d \rho(x/\delta)$. For $u \in L_1^{loc}(\mathbb{R}^d)$, we denote the convolution $u * \rho_\delta$ by u^δ . Clearly $u^\delta \in C^\infty(\mathbb{R}^d)$ and $u^\delta \rightarrow u$ in $L_1^{loc}(\mathbb{R}^d)$ as $\delta \downarrow 0$.

Lemma A.1. *Let $1 \leq q \leq \infty$. Suppose $u \in L_q(\mathbb{R}^d)$ is such that $H_0^\psi u \in L_1^{loc}(\mathbb{R}^d)$, i.e., for some $f \in L_1^{loc}(\mathbb{R}^d)$ it holds that for any $\varphi \in C_0^\infty(\mathbb{R}^d)$*

$$\int_{\mathbb{R}^d} f(x) \varphi(x) dx = \int_{\mathbb{R}^d} u(x) H_0^\psi \varphi(x) dx. \quad (\text{A.22})$$

Then $H_0^\psi u^\delta \rightarrow H_0^\psi u$ in $L_1^{loc}(\mathbb{R}^d)$ as $\delta \downarrow 0$.

Proof. Since $u \in L_q(\mathbb{R}^d)$, $u^\delta \in C^\infty(\mathbb{R}^d) \cap L_q(\mathbb{R}^d)$. By Claim A.1, $H_0^\psi u^\delta \in L_q^{loc}(\mathbb{R}^d)$ or $\in C(\mathbb{R}^d)$ according as $1 \leq q < \infty$ or $q = \infty$, and hence $H_0^\psi u^\delta \in L_1^{loc}(\mathbb{R}^d)$. For the proof, it is enough to check that $H_0^\psi u^\delta = (H_0^\psi u)^\delta$.

By (A.22)

$$\begin{aligned}
(H_0^\psi u)^\delta(x) &= \int_{\mathbb{R}^d} (H_0^\psi u)(y) \rho_\delta(x-y) dy \\
&= \int_{\mathbb{R}^d} u(y) H_0^\psi \rho_\delta(x-\cdot)(y) dy \\
&= \int_{\mathbb{R}^d} u(y) dy \left(- \int_{\mathbb{R}^d \setminus \{0\}} \left\{ \rho_\delta(x-y-z) - \rho_\delta(x-y) \right. \right. \\
&\quad \left. \left. - \langle z, \nabla \rho_\delta(x-\cdot)(y) \rangle 1_{|z|<1} \right\} J(dz) \right). \tag{A.23}
\end{aligned}$$

The integral on the RHS is convergent, because with $\rho_\delta(x-\cdot) =: g \in C_0^\infty(\mathbb{R}^d)$, it is bounded by $\int_{|z| \geq 1} J(dz) \|u\|_q 2 \|g\|_{q/(q-1)} + (1/2) \int_{0 < |z| < 1} |z|^2 J(dz) \|u\|_q \|\nabla^2 g\|_{q/(q-1)}$. Here when $q = 1$ or ∞ we understand $\|\cdot\|_{q/(q-1)} = \|\cdot\|_\infty$ or $\|\cdot\|_1$. Hence by noting that $\nabla \rho_\delta(x-\cdot)(y) = -(\nabla \rho_\delta)(x-y)$, Fubini's theorem gives us that

$$\begin{aligned}
(H_0^\psi u)^\delta(x) &= - \int_{\mathbb{R}^d \setminus \{0\}} J(dz) \left(\int_{\mathbb{R}^d} u(y) \rho_\delta(x-z-y) dy - \int_{\mathbb{R}^d} u(y) \rho_\delta(x-y) dy \right. \\
&\quad \left. - 1_{|z|<1} \left\langle -z, \int_{\mathbb{R}^d} u(y) (\nabla \rho_\delta)(x-y) dy \right\rangle \right) \\
&= - \int_{\mathbb{R}^d \setminus \{0\}} (u^\delta(x+z) - u^\delta(x) - 1_{|z|<1} \langle z, \nabla u^\delta(x) \rangle) J(dz) \\
&= H_0^\psi u^\delta(x),
\end{aligned}$$

where the symmetry of $J(dz)$ has been used. The proof is complete. \square

Lemma A.2. (Kato's inequality). *Let $1 \leq q \leq \infty$. Suppose $u \in L_q(\mathbb{R}^d)$ is such that $H_0^\psi u \in L_1^{loc}(\mathbb{R}^d)$. Then the following distributional inequality holds:*

$$\operatorname{sgn} u H_0^\psi u \geq H_0^\psi |u|,$$

i.e. for any $\varphi \in C_0^\infty(\mathbb{R}^d)$ with $\varphi \geq 0$,

$$\int_{\mathbb{R}^d} (\operatorname{sgn} u)(x) H_0^\psi u(x) \varphi(x) dx \geq \int_{\mathbb{R}^d} |u(x)| H_0^\psi \varphi(x) dx.$$

Here $\operatorname{sgn} u$ is a bounded function on \mathbb{R}^d defined by

$$(\operatorname{sgn} u)(x) := \begin{cases} \frac{u(x)}{|u(x)|} & \text{if } u(x) \neq 0 \\ 0 & \text{if } u(x) = 0. \end{cases}$$

Proof. First let $u \in C^\infty(\mathbb{R}^d) \cap L_q(\mathbb{R}^d)$. By Claim A.1 $H_0^\psi u \in L_q^{loc}(\mathbb{R}^d)$ or $\in C(\mathbb{R}^d)$ according as $1 \leq q < \infty$ or $q = \infty$. For $\varepsilon > 0$, set $u_\varepsilon(x) := \sqrt{|u(x)|^2 + \varepsilon^2}$. Clearly $u_\varepsilon \in C^\infty(\mathbb{R}^d)$ and $u_\varepsilon \geq \varepsilon$. Since $|u(x)||u(x+y)| \leq u_\varepsilon(x)u_\varepsilon(x+y) - \varepsilon^2$, we have

$$-|u(x)||u(x+y)| + u(x)^2 \geq -u_\varepsilon(x)u_\varepsilon(x+y) + u_\varepsilon(x)^2.$$

By noting that $2u(x)\nabla u(x) = \nabla|u(x)|^2 = \nabla u_\varepsilon(x)^2 = 2u_\varepsilon(x)\nabla u_\varepsilon(x)$, this inequality gives us that

$$\begin{aligned} & -u(x)\{u(x+y) - u(x) - \langle y, \nabla u(x) \rangle 1_{|y|<1}\} \\ & = -u(x)u(x+y) + u(x)^2 + \langle y, u(x)\nabla u(x) \rangle 1_{|y|<1} \\ & \geq -u_\varepsilon(x)u_\varepsilon(x+y) + u_\varepsilon(x)^2 + \langle y, u_\varepsilon(x)\nabla u_\varepsilon(x) \rangle 1_{|y|<1} \\ & = -u_\varepsilon(x)\{u_\varepsilon(x+y) - u_\varepsilon(x) - \langle y, \nabla u_\varepsilon(x) \rangle 1_{|y|<1}\}. \end{aligned}$$

Integrating both sides by $J(dy)$, we have $u(x)H_0^\psi u(x) \geq u_\varepsilon(x)H_0^\psi u_\varepsilon(x)$, or

$$\frac{u(x)}{u_\varepsilon(x)} H_0^\psi u(x) \geq H_0^\psi u_\varepsilon(x). \quad (\text{A.24})$$

Second let $u \in L_q(\mathbb{R}^d)$ be such that $H_0^\psi u \in L_1^{loc}(\mathbb{R}^d)$. Since $u^\delta = u * \rho_\delta \in C^\infty(\mathbb{R}^d) \cap L_q(\mathbb{R}^d)$, it holds by (A.24) that

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{u^\delta(x)}{(u^\delta)_\varepsilon(x)} H_0^\psi u^\delta(x) \varphi(x) dx & \geq \int_{\mathbb{R}^d} H_0^\psi (u^\delta)_\varepsilon(x) \varphi(x) dx \\ & = \int_{\mathbb{R}^d} (u^\delta)_\varepsilon(x) H_0^\psi \varphi(x) dx \end{aligned} \quad (\text{A.25})$$

for any nonnegative $\varphi \in C_0^\infty(\mathbb{R}^d)$. In (A.25) let $\delta \downarrow 0$ first and $\varepsilon \downarrow 0$ second. As $\delta \downarrow 0$, $H_0^\psi u^\delta \rightarrow H_0^\psi u$ in $L_1^{loc}(\mathbb{R}^d)$ by Lemma A.1, and $u^\delta \rightarrow u$ in $L_1^{loc}(\mathbb{R}^d)$. By taking a subsequence if necessary we may suppose that $u^\delta \rightarrow u$ a.e. Since $|(u^\delta)_\varepsilon - u_\varepsilon| \leq |u^\delta - u|$ and $|u^\delta / (u^\delta)_\varepsilon| \leq 1$, $u^\delta / (u^\delta)_\varepsilon \rightarrow u / u_\varepsilon$ boundedly a.e. Hence, letting $\delta \downarrow 0$ in (A.25), we have

$$\int_{\mathbb{R}^d} \frac{u(x)}{u_\varepsilon(x)} H_0^\psi u(x) \varphi(x) dx \geq \int_{\mathbb{R}^d} u_\varepsilon(x) H_0^\psi \varphi(x) dx. \quad (\text{A.26})$$

Finally, by $|u_\varepsilon - |u|| \leq \varepsilon$ and $|u/u_\varepsilon| \leq 1$, we obtain that $u/u_\varepsilon \rightarrow \text{sgn } u$ boundedly as $\varepsilon \downarrow 0$. Consequently, letting $\varepsilon \downarrow 0$ in (A.26) yields that

$$\int_{\mathbb{R}^d} (\text{sgn } u)(x) H_0^\psi u(x) \varphi(x) dx \geq \int_{\mathbb{R}^d} |u(x)| H_0^\psi \varphi(x) dx$$

and the proof is complete. \square

Proof of Theorem A.1. First consider the L_p -case, $1 \leq p < \infty$. It suffices to show that $\text{Im}(H_0^\psi + V + 1) = (H_0^\psi + V + 1)(C_0^\infty(\mathbb{R}^d))$ is dense in $L_p(\mathbb{R}^d)$. By the Hahn-Banach theorem, this is further reduced to show the following: Let q be the conjugate exponent of p . If $v \in L_q(\mathbb{R}^d)$ satisfies that $\langle v, (H_0^\psi + V + 1)\varphi \rangle = 0$ for any $\varphi \in C_0^\infty(\mathbb{R}^d)$, then $v = 0$ in $L_q(\mathbb{R}^d)$.

Let $v \in L_q(\mathbb{R}^d)$ be as above. Then $H_0^\psi v = -(V+1)v$ and hence $H_0^\psi v \in L_1^{loc}(\mathbb{R}^d)$. By Lemma A.2, it is seen that for any nonnegative $\varphi \in C_0^\infty(\mathbb{R}^d)$

$$\begin{aligned} \int_{\mathbb{R}^d} |v(x)| H_0^\psi \varphi(x) dx &\leq \int_{\mathbb{R}^d} (\text{sgn } v)(x) H_0^\psi v(x) \varphi(x) dx \\ &= - \int_{\mathbb{R}^d} (V(x) + 1) |v(x)| \varphi(x) dx, \end{aligned}$$

and hence

$$\int_{\mathbb{R}^d} |v(x)| (H_0^\psi + 1) \varphi(x) dx \leq 0. \quad (\text{A.27})$$

Each $\varphi \in \mathcal{S}(\mathbb{R}^d)$ can be approximated by a sequence $\{\varphi_n\}_{n=1}^\infty$ of $C_0^\infty(\mathbb{R}^d)$ in the sense that $\varphi_n \rightarrow \varphi$ and $(H_0^\psi + 1)\varphi_n \rightarrow (H_0^\psi + 1)\varphi$ in $L_p(\mathbb{R}^d)$. If φ is moreover nonnegative, so are $\{\varphi_n\}_{n=1}^\infty$. Therefore (A.27) is valid for nonnegative $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Now note that the resolvent $(1 - \mathfrak{G}_p^{\psi,0})^{-1}$ is expressed as

$$(1 - \mathfrak{G}_p^{\psi,0})^{-1} f(x) = \int_0^\infty e^{-t} \mathbb{E}[f(x + X_t)] dt.$$

Then it is not difficult to check that if $f \in \mathcal{S}(\mathbb{R}^d)$, then $(1 - \mathfrak{G}_p^{\psi,0})^{-1} f \in \mathcal{S}(\mathbb{R}^d)$ and further, if f is nonnegative, so is $(1 - \mathfrak{G}_p^{\psi,0})^{-1} f$. Also, by Remark following Claim A.2 (with $V(x) \equiv 0$) $f = (1 - \mathfrak{G}_p^{\psi,0})(1 - \mathfrak{G}_p^{\psi,0})^{-1} f = (H_0^\psi + 1)(1 - \mathfrak{G}_p^{\psi,0})^{-1} f$. Hence, by (A.27)

$$\int_{\mathbb{R}^d} |v(x)| f(x) dx \leq 0,$$

whence it immediately follows that $v = 0$ and the proof in the L_p -case is complete.

Next let us consider the C_∞ -case. In the same reason as above we have only to show that $(H_0^\psi + V + 1)(C_0^\infty(\mathbb{R}^d))$ is dense in $C_\infty(\mathbb{R}^d)$. For this let $\nu \in C_\infty(\mathbb{R}^d)^*$, the dual of $C_\infty(\mathbb{R}^d)$, be such that $\langle \nu, (H_0^\psi + V + 1)\varphi \rangle = 0$ for any $\varphi \in C_0^\infty(\mathbb{R}^d)$. By the Riesz-Banach theorem, ν is regarded as a finite signed Borel measure on \mathbb{R}^d , and thus

$$\int_{\mathbb{R}^d} (H_0^\psi + V + 1)\varphi(x) \nu(dx) = 0 \quad \text{for any } \varphi \in C_0^\infty(\mathbb{R}^d). \quad (\text{A.28})$$

Let $\nu^\delta = \nu * \rho_\delta$, i.e., $\nu^\delta(x) := \int_{\mathbb{R}^d} \rho_\delta(x-y) \nu(dy)$, $x \in \mathbb{R}^d$. Then $\nu^\delta \in C_b^\infty(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$. It follows by Claim A.1 and by (A.28) that $H_0^\psi \nu^\delta \in C(\mathbb{R}^d)$ and

$$H_0^\psi \nu^\delta = -\nu^\delta(1+V) - \int_{\mathbb{R}^d} (V(y) - V(\cdot)) \rho_\delta(\cdot - y) \nu(dy).$$

By Lemma A.2, this implies that for nonnegative $\varphi \in C_0^\infty(\mathbb{R}^d)$

$$\begin{aligned} &\int_{\mathbb{R}^d} |\nu^\delta(x)| H_0^\psi \varphi(x) dx \\ &\leq \int_{\mathbb{R}^d} (\text{sgn } \nu^\delta)(x) H_0^\psi \nu^\delta(x) \varphi(x) dx \\ &\leq - \int_{\mathbb{R}^d} |\nu^\delta(x)| \varphi(x) dx + \int_{\mathbb{R}^d} \varphi(x) dx \int_{\mathbb{R}^d} |V(y) - V(x)| \rho_\delta(x-y) |\nu|(dy), \end{aligned}$$

where $|\nu|$ is the total variation of ν , and hence we have that for nonnegative $\varphi \in C_0^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} |\nu^\delta(x)|(H_0^\psi + 1)\varphi(x)dx \leq \int_{\mathbb{R}^d} \varphi(x)dx \int_{\mathbb{R}^d} |V(y) - V(x)|\rho_\delta(x-y)|\nu|(dy). \quad (\text{A.29})$$

Let $f \in \mathcal{S}(\mathbb{R}^d)$ be nonnegative. Then, $\varphi := (1 - \mathfrak{G}_\infty^{\psi,0})^{-1}f$ is in $\mathcal{S}(\mathbb{R}^d)$ and nonnegative, and $(H_0^\psi + V)\varphi = f$. Set $\varphi_n(x) := \varphi(x)\chi(|x|^2/n^2)$ with a $\chi \in C^\infty([0, \infty) \rightarrow \mathbb{R})$ such that $0 \leq \chi \leq 1$, $\chi(t) = 1$ ($0 \leq t \leq 1$) and $\chi(t) = 0$ ($t \geq 2$). Clearly $\varphi_n \in C_0^\infty(\mathbb{R}^d)$, $0 \leq \varphi_n \leq \varphi$ and $\text{supp } \varphi_n \subset \{x; |x| \leq \sqrt{2}n\}$. Moreover $\|\varphi_n - \varphi\|_\infty$ and $\|H_0^\psi \varphi_n - H_0^\psi \varphi\|_\infty \rightarrow 0$ as $n \rightarrow \infty$. From (A.29) and this observation it follows that

$$\begin{aligned} \int_{\mathbb{R}^d} |\nu^\delta(x)|f(x)dx &= \int_{\mathbb{R}^d} |\nu^\delta(x)|(H_0^\psi + 1)\varphi(x)dx \\ &= \int_{\mathbb{R}^d} |\nu^\delta(x)|(H_0^\psi + 1)\varphi_n(x)dx + \int_{\mathbb{R}^d} |\nu^\delta(x)|(H_0^\psi + 1)(\varphi - \varphi_n)(x)dx \\ &\leq \int_{\mathbb{R}^d} \varphi_n(x)dx \int_{\mathbb{R}^d} |V(y) - V(x)|\rho_\delta(x-y)|\nu|(dy) \\ &\quad + \|(H_0^\psi + 1)(\varphi - \varphi_n)\|_\infty |\nu|(\mathbb{R}^d) \\ &\leq \|\varphi\|_\infty \int_{|x| \leq \sqrt{2}n} dx \int_{\mathbb{R}^d} |V(y) - V(x)|\rho_\delta(x-y)|\nu|(dy) \\ &\quad + \|(H_0^\psi + 1)(\varphi - \varphi_n)\|_\infty |\nu|(\mathbb{R}^d). \end{aligned}$$

Here, recalling that $\rho_\delta(z)$ has support in $\{z; |z| \leq \rho\}$, we see that for each $n \in \mathbb{N}$

$$\begin{aligned} &\int_{|x| \leq \sqrt{2}n} dx \int_{\mathbb{R}^d} |V(y) - V(x)|\rho_\delta(x-y)|\nu|(dy) \\ &\leq \int_{|y| \leq \sqrt{2}n+\delta} |\nu|(dy) \int_{\mathbb{R}^d} |V(y) - V(x)|\rho_\delta(x-y)dx \\ &= \int_{|y| \leq \sqrt{2}n+\delta} |\nu|(dy) \int_{|x| \leq 1} |V(y) - V(y + \delta x)|\rho(x)dx \\ &\rightarrow 0 \quad \text{as } \delta \downarrow 0. \end{aligned}$$

On the other hand, noting that $\nu^\delta(x)dx \rightarrow \nu(dx)$ weakly, we see that

$$\int_{\mathbb{R}^d} |\nu^\delta(x)|f(x)dx \geq \left| \int_{\mathbb{R}^d} f(x)\nu^\delta(x)dx \right| \rightarrow \left| \int_{\mathbb{R}^d} f(x)\nu(dx) \right| \quad \text{as } \delta \downarrow 0.$$

Therefore it follows that $\int_{\mathbb{R}^d} f(x)\nu(dx) = 0$ for $f \in \mathcal{S}(\mathbb{R}^d)$, $f \geq 0$, which implies that $\nu = 0$, and the proof in the C_∞ -case is complete. \square

In this paper we have denoted the semigroups $P_t^{\psi,0}$ and $P_t^{\psi,V}$ by $e^{-tH_0^\psi}$ and $e^{-t(H_0^\psi + V)}$, respectively, taking Theorem A.1 into account. With the general theory ([Trot], [Ch]) we have taken for granted that the Trotter product formula holds in the strong topology of $L_p(\mathbb{R}^d)$ or $C_\infty(\mathbb{R}^d)$:

$$\begin{aligned} s\text{-}\lim_{n \rightarrow \infty} (e^{-tH_0^\psi/n} e^{-tV/n})^n &= s\text{-}\lim_{n \rightarrow \infty} (e^{-tV/2n} e^{-tH_0^\psi/n} e^{-tV/2n})^n \\ &= s\text{-}\lim_{n \rightarrow \infty} (e^{-tH_0^\psi/2n} e^{-tV/n} e^{-tH_0^\psi/2n})^n = e^{-t(H_0^\psi + V)}. \end{aligned}$$

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