Critical branching Brownian motion with killing

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Abstract

We obtain sharp asymptotic estimates for hitting probabilities of a critical branching Brownian motion in one dimension with killing at 0. We also obtain sharp asymptotic formulas for the tail probabilities of the number of particles killed at 0. In the special case of double-or-nothing branching, we give exact formulas for both the hitting probabilities, in terms of elliptic functions, and the distribution of the number of killed particles.

1 Introduction

Branching Brownian motion is a stochastic particle system in which each individual particle moves along a Brownian trajectory, and at a random, exponentially distributed time independent of its motion is replaced by a random collection of identical offspring particles. The motions, gestation times, and offspring numbers of different particles are conditionally independent, given the times and locations of their births. Thus, conditional on the event that at time t there are Z_t particles at locations $x_1, x_2, \ldots, x_{Z_t}$, the law of the post-$t$ evolution is identical to that of Z_t mutually independent branching Brownian motions started by individual particles at the locations x_i. A formal construction of the process is outlined in section 2 below.

The process Z_t that records the total number of particles at time t is a *continuous-time Galton-Watson process*: see [AN72], ch. 2 for the basic theory of these. Branching Brownian motion is said to be *supercritical*, *critical* or *subcritical* according as the mean of the offspring distribution is greater, equal or less than 1. In the critical and subcritical cases, the particle population eventually dies out, with probability one, provided the population starts with only finitely many particles; in the supercritical case, however, there is positive probability that the population blows up, that is, $Z_t \to \infty$ as $t \to \infty$.

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Thus, the questions that are germane to the supercritical case are different from those of interest in the critical case.

It has been known since the work of McKean [M75] that supercritical branching Brownian motion is intimately related to the behavior of solutions to the *Fisher-KPP equation*. In particular, this equation governs the cumulative distribution function $u(t,x) = \mathbb{P}^0(M_t \leq x)$ of the position M_t of rightmost particle at time t. Using this fact, McKean gave a probabilistic proof of the Kolmogorov, Petrovsky, and Piscounov [KPP37] theorem, which asserts that the solution of the KPP equation with Heaviside initial data stabilizes as a traveling wave of velocity $\sqrt{2}$. Subsequently, Bramson [B78] used the connection with supercritical branching Brownian motion to obtain sharp estimates for the center of the wave, and Lalley and Sellke [LS87] showed that the limiting traveling wave $w(x)$ can be represented as a as a mixture of extreme-value distributions.

When the branching mechanism is *critical* or *sub-critical*, a more natural object of study is the random variable

$$M = \sup_{t>0} M_t,$$

the rightmost location ever reached by a particle of the branching Brownian motion. Critical branching Brownian motion has been proposed as a model for the spatial displacement of alleles without selective advantage or disadvantage, and in this context the distribution of M plays an important role (see, for example, [CG76], [S76], [SF79] and references therein). Sawyer and Fleischman [SF79] proved that if the offspring distribution has mean 1, positive variance σ^2 and finite third moment, then the tail of the distribution of M satisfies the power law

$$\mathbb{P}(M \geq x) \sim \frac{6}{\sigma^2 x^2} \quad \text{as } x \to \infty. \tag{1.1}$$

Modifications of branching Brownian motion and branching random walk in which the laws of reproduction and/or particle motion depend on particle location arise in various contexts. See, for instance, Lalley and Sellke [LS88], [LS89], in which particle reproduction is allowed only in certain favored regions of space; Kesten [K78], Aldous [A], Addario-Berry and Broutin [AB11], Aïdékon, Hu, and Zindy [AHZ13], Berestycki *et al.* [BBS13] and Maillard [M13] where particles are killed upon entering the half-line $(-\infty, 0]$; and Lalley and Sellke [LS92] and Korostelev and Korosteleva [KK03], [K04], [KK04], where particles move according to spatially-inhomogeneous diffusion laws. In the articles [AB11], [AHZ13], [M13], and [BBHM15], the branching law is supercritical, but particle production is balanced by the killing in $(-\infty, 0]$ so that $M_n/n \to 0$.

This paper will focus on the modification of critical branching Brownian motion (that is, where the mean number of offspring at reproduction events is 1) in which particles are killed upon reaching the interval $(-\infty, 0]$. Clearly, the number Z_t of particles alive at time t in this process is dominated by the corresponding random variable for the critical branching Brownian motion with no killing, and so $Z_t = 0$ eventually, with probability 1. Furthermore, the distribution of the maximal particle location M is dominated by that of the maximal particle location in critical branching Brownian motion with no killing, and so the results of Sawyer and Fleischman [SF79] imply that for any $\varepsilon > 0$ and any initial particle location $y > 0$,

$$\mathbb{P}^y\{M \geq x\} \leq \frac{(6+\varepsilon)}{\sigma^2 x^2}$$

for all sufficiently large x.

It is by no means evident, however, that the tail behavior should be the same as for branching Brownian motion with no killing. In fact we will prove that when the branching process is initiated by a single particle at a location $y > 0$ near zero, the

tail follows a power law with exponent 3 rather than 2. In particular, we will prove in Theorem 6.1 that for each fixed $y > 0$,

$$\mathbb{P}^y(M > x) \sim \frac{C_3 y}{x^3} \quad \text{as} \ \ x \to \infty$$

where $C_3 > 0$ is a constant depending on the offspring distribution but not on x or y. On the other hand, we will show that for initial particle locations $y = sx$ whose distances from the killing zone are proportional to the target x, the exponent of the power law reverts to 2; in particular, there exists a continuous function $C_4(s)$ of $s \in (0, 1)$ such that

$$\mathbb{P}^{sx}(M > x) \sim \frac{C_4(s)}{x^2} \quad \text{as} \ \ x \to \infty.$$

Furthermore, we will show that in the *Moranian* case, where the offspring law is double-or-nothing, the tail probability $\mathbb{P}^y(M > x)$ can be explicitly written as a Weierstrass $\mathcal{P}-$function. All of these results will be deduced from an analysis of a boundary value problem in the variable y satisfied by the hitting probability $\mathbb{P}^y(M > x)$.

Also of interest is the total number N of particles killed at 0. For supercritical branching Brownian motion with particle drift and killing at 0, Maillard [M13] and Berestycki *et al.* [BBHM15] have, under various hypotheses concerning the drift and the reproduction mechanism, obtained sharp estimates for the tail of the distribution of N. For critical branching Brownian motion with killing, T. Y. Lee [L90-1] proved a conditional limit theorem for the distribution of N given that $N \geq 1$: in particular, he showed that as the position y of the initial particle $\to \infty$, so that $\mathbb{P}^y(N \geq 1) \to 0$, the $\mathbb{P}^y-$conditional distribution of $N/P^y(N \geq 1)$ converges to a non-degenerate limit distribution. (See also [L90-2] for a time-dependent analogue.)

We will study the distribution of N for critical branching Brownian motion with killing at 0 under a *fixed* $\mathbb{P}^y$. In section 7 we will show that, for offspring distributions with mean 1, positive finite variance σ^2, and finite third moment,

$$\sum_{j=1}^{k} j \mathbb{P}^y(N \geq j) \sim Cy\sqrt{k} \quad \text{as } k \to \infty.$$

Under certain additional hypotheses on the offspring distribution, we will show that the distribution of N obeys a power law with exponent $3/2$, and in addition, we will show that the N obeys an asymptotic *local* limit theorem. In particular, we will prove that

$$\mathbb{P}^y(N \geq k) \sim \frac{C_7 y}{k^{3/2}},$$

and

$$\mathbb{P}^y(N = k) \sim \frac{C_8 y}{k^{5/2}},$$

where $C_7, C_8 > 0$ are constants depending on the offspring distribution. Finally, in the Moranian case, we will give in Theorem 7.9 an explicit formula for the tail distribution of N.

2 Construction and Monotonicity Properties

Branching Brownian motions with initial particle locations at points $y \in \mathbb{R}_+$ can be constructed on any probability space that supports countably many (i) independent standard Wiener processes W^i; (ii) independent, identically distributed unit exponential random variables T_i; and (iii) independent, identically distributed random variables L_i all distributed according to the prescribed offspring distribution. We dub this construction

the *discrete Brownian snake*, as it is the natural discrete analogue of Le Gall's Brownian snake: see [L99] for details.

The construction proceeds by using the random variables $\{L_i\}_{i \geq 0}$ to construct a *Galton-Watson tree*. This construction is standard: see [AN72]. If the offspring distribution has mean 1, as we shall assume throughout, then the resulting Galton-Watson tree is almost surely a finite, rooted tree with vertices arranged in *generations*, beginning with the root at generation 0. To each vertex v is attached one of the random variables L_i, with L_0 attached to the root; for each vertex v the random variable L_i determines the number of offspring vertices. The random variables L_i can be attached to vertices in any number of different ways, the most common being the *breadth-first* rule, in which the values L_i are read successively from the stack generation-by-generation, left-to-right.

Given the realization of the Galton-Watson tree, we attach unit exponential random variables T_i and standard Wiener processes W^i to the *edges* of the tree in such a way that the index $i = i(v)$ matches the index of the random variable L_i attached to the *lower* vertex v of the edge (the incident vertex with higher generation number). The random variable T_i attached to an edge determines the real time elapsed between reproduction events, and the Wiener process W^i determines the displacement of the particle in real time from its position at the last reproduction event. Thus, the particles alive at (real) time t are in one-to-one correspondence with the vertices v of the tree such that

$$\sum_{w < v} T_{i(w)} < t \leq \sum_{w \leq v} T_{i(w)};$$

here the symbols $<$ and $\leq$ indicate the ordering of vertices w along the geodesic path in the tree from the root to v. The spatial position of the particle represented by vertex v at time t is

$$y + \sum_{w < v} W^{i(w)}(T_{i(w)}) + W^v\left(t - \sum_{w < v} T_{i(w)}\right).$$

Observe that these rules yield a *simultaneous* construction of branching Brownian motions from all initial positions y. It is evident from this construction that the distribution of the maximum position M attained by a point of the branching Brownian motion is stochastically monotone in the initial position y.

Branching Brownian motion with killing at 0, or more generally with killing at any point $z \leq 0$, can be constructed using the same marked tree as for branching Brownian motion with no killing. The rule is simple: once a trajectory along an edge enters $(-\infty, z]$, the tree is pruned at that point. This leaves a subtree of the original Galton-Watson tree in which certain edges (those corresponding to particles that are killed at 0) are cut. The vertices of this subtree represent particles of the branching Brownian motion with killing at z. Thus, the set of particles alive in the branching Brownian motion with killing is a subset of the set of particles in the branching Brownian motion with no killing, which we will henceforth refer to as the *enveloping* branching Brownian motion.

This construction makes it obvious that the distribution of M is dominated by that for branching Brownian motion with no killing at z, and that if $z_2 < z_1$ then the distribution of M for branching Brownian motion with killing at z_1 is stochastically dominated by that for branching Brownian motion with killing at z_2. Furthermore, the implied inequalities among the cumulative distribution functions are *strict*: for instance, if $w_2(x)$ and $w_1(x)$ are the tail distributions of M for branching Brownian motions with killing at $z_2 < z_1 \leq 0$, respectively, when both are initiated by a single particle at 0 (that is, $w_i(x)$ is the probability that $M \geq x$) then

$$w_1(x) < w_2(x). \tag{2.1}$$

To see this, observe that there is positive probability that a branch will be pruned when there is killing at z_1 but not when the killing is at z_2, and that this branch will extend

in such a way that it gives rise to a particle that reaches location x. Finally, branching Brownian motions with killing at z converge as $z \to -\infty$ to branching Brownian motion with no killing. Thus, for any $x > 0$,

$$\lim_{z \to \infty} w_z(x) = w_\infty(x), \tag{2.2}$$

where $w_z(x)$ is the probability that $M \geq x$ for branching Brownian motion with killing at $-z$ and $w_\infty(x)$ is the corresponding probability for branching Brownian motion with no killing (both with initial particles located at 0).

It should be obvious that minor variations of the construction just outlined can be used to build a variety of related processes. One that will prove useful in certain of the arguments to follow is *branching Brownian motion with freezing*, in which particles that reach a target point 0 (or, more generally, a closed set B) are frozen in place, ceasing all motion and reproduction thereafter, but not dying. In a critical branching Brownian motion with freezing of particles at location 0, eventually all existing particles will be frozen at 0; moreover, the number N_t of particles frozen at time t is the same as the number of particles killed at 0 up to time t in the corresponding branching Brownian motion with killing at 0.

Henceforth, we shall assume that all branching Brownian motions are *critical* and that the offspring distribution has positive, finite variance σ^2, and we shall denote by

$$\Psi(z) = \sum_{k=0}^{\infty} \mathbb{P}(L = k) z^k \tag{2.3}$$

the probability generating function of the offspring distribution.

3 Product Martingales and Differential Equations

The key to our analysis will be the fact that hitting probabilities and related expectations for critical branching Brownian motion, viewed as functions of the initial point y, are governed by a nonlinear second-order differential equation. This differential equation is well known, but since we will have occasion to consider expectations of complex-valued random variables, we shall spell out the boundary value problems in detail.

Say that a sequence $f : \mathbb{Z}_+ \to \mathbb{C}$ is *multiplicative* if it is a geometric sequence of the form $f(n) = z^n$ for some $z \in \mathbb{C}$. For any $A \in (0, \infty]$, let $\mathbb{P}^y$ be the law of a branching Brownian motion with initial point $y \in [0, A]$ in which particles are frozen upon reaching either 0 or A. For $i = 0$ and $i = A$ define

$$N_i = \text{number of particles frozen at } i. \tag{3.1}$$

Both N_0 and N_A are almost surely finite, since only finitely many particles are born in the course of a critical branching Brownian motion. Clearly, $N_A = 0$ when $A = \infty$.

Proposition 3.1. *If $f, g : \mathbb{Z}_+ \to \mathbb{C}$ are bounded, multiplicative sequences, then the function $\varphi(y) = \mathbb{E}^y \left[f(N_0) g(N_A) \right]$ satisfies the second order differential equation*

$$\frac{1}{2} \varphi''(y) = \varphi(y) - \Psi(\varphi(y)) \quad \text{for all } y \in (0, A). \tag{3.2}$$

In the special case where $A = \infty$ and $f(n) = \delta_0(n)$ this was stated and proved by Sawyer and Fleischman [SF79], and this proof was subsequently cited by Lee [L90-1]. But the proof in [SF79] seems to have a gap: the derivation of the differential equation relies on the smoothness of the function $\varphi(y)$, but to prove this the authors quote the

version of Weyl's Lemma given in [M69] to conclude that a weak solution must be C^∞. We do not understand this argument, as Weyl's Lemma, in the form stated in [M69], applies only to *linear parabolic* differential operators, while the differential operators in [SF79], section 2, and in our Proposition 3.1 are *nonlinear*. Therefore, we will sketch another approach to the proof of Proposition 3.1 that uses an interesting class of *product martingales*. (Similar martingales for supercritical branching Brownian motion were used in [LS88] and [N88]). Let $h : [0, A] \to \mathbb{C}$ be a function bounded in absolute value by 1, and denote by $X_1(t), \ldots, X_{Z(t)}(t)$ the locations of the particles alive at time t (including those frozen at one of the endpoints $0, A$) in a branching Brownian motion with freezing at 0 and A; define

$$Y(t) = Y_h(t) = \prod_{i=1}^{Z(t)} h(X_i(t)). \tag{3.3}$$

Proposition 3.2. *If $h(y)$ satisfies the differential equation $\frac{1}{2}h'' = h - \Psi(h)$ in the interval $(0, A)$ then $Y(t)$ is a bounded martingale, relative to the standard filtration for the branching Brownian motion, under $\mathbb{P}^y$, for any $y \in [0, A]$.*

Proof of Proposition 3.1 (Sketch). Given Proposition 3.2, we proceed as follows. Fix f, g, and let $h : [0, A] \to \mathbb{C}$ be the unique solution to the boundary value problem

$$\frac{1}{2}h'' = h - \Psi(h);$$
$$h(0) = f(1),$$
$$h(A) = g(1).$$

(When $A = \infty$, the boundary condition should be replaced by $h(A) = g(0) = 1$.) The existence and uniqueness of solutions follows by standard arguments in the theory of ordinary differential equations; we omit the details.[1] By Proposition 3.2, the process $Y(t)$ defined by (3.3) is a bounded martingale, and so for any $t < \infty$,

$$\mathbb{E}^y Y(t) = Y(0) = h(y).$$

But for all sufficiently large t, all particles will be frozen at either 0 or A, so eventually $Y(t)$ coincides with $f(N_0)g(N_A)$. (For this the product structure of the martingale is essential.) Therefore, by the bounded convergence theorem,

$$h(y) = \mathbb{E}^y f(N_0)g(N_A).$$

$\square$

Proof of Proposition 3.2 (Sketch). By the Markov property, it suffices to show that for any initial configuration of particles $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ the expectation $\mathbb{E}^{\mathbf{y}} Y(t)$ is constant in time. Since each of the m particles engenders its own independent branching Brownian motion, the expectation $\mathbb{E}^{\mathbf{y}} Y(t)$ factors as

$$\mathbb{E}^{\mathbf{y}} Y(t) = \prod_{i=1}^{m} \mathbb{E}^{y_i} Y(t);$$

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[1] At any rate the argument is routine in the case where $f(1)$ and $g(1)$ take values in the unit interval $[0, 1]$; in this case existence follows by a routine phase-portrait analysis for the associated first-order system, using the nonnegativity of the forcing term $\Psi(h)$. When $f(1)$ and $g(1)$ are complex-valued, however, other methods must be used. See the proof of Lemma 7.14 in section 7 below for a proof in the case needed for the theorems on the distribution of the number of killed particles.

consequently, it suffices to prove that for any $y \in [0, A]$ the expectation $\mathbb{E}^y Y(t)$ is constant in time, and for this it is enough to show that

$$\frac{d}{dt} \mathbb{E}^y Y(t) = 0.$$

But for this another conditioning shows that it is enough to prove that the derivative is zero at $t = 0$. This can be accomplished by a routine argument, by partitioning the expectation into the expectations on the events that the initial particle reproduces or not by time t and using the fact that h is bounded and C^2 and satisfies the differential equation $h''/2 = h - \Psi(h)$. □

4 Weierstrass' $\mathcal{P}-$Functions

In the special case of double-or-nothing branching (the Moranian case), the probability generating function of the offspring distribution is the quadratic function $\Psi(s) = \frac{1}{2}(1 + s^2)$. In this case the differential equation (3.2) reduces, as we will show, to the differential equation of the *Weierstrass $\mathcal{P}-$function*. For a given *period lattice*

$$\mathcal{L} = \{m\omega + n\widetilde{\omega}, m, n \in \mathbb{N}\},$$

where ω and $\tilde{\omega}$ are nonzero complex numbers whose ratio is not real, Weierstrass' $\mathcal{P}-$function with period lattice $\mathcal{L}$ is the meromorphic function on $\mathbb{C}$ defined by

$$\mathcal{P}_{\mathcal{L}}(z) = \frac{1}{z^2} + \sum_{l \in \mathcal{L}, l \neq 0} \left(\frac{1}{(z-l)^2} - \frac{1}{l^2} \right). \tag{4.1}$$

See [K84] or [MM99] for expositions of the basic theory. Clearly, (4.1) defines a doubly-periodic function of z whose periods are the elements of the lattice $\mathcal{L}$. It is also evident from (4.1) that $\mathcal{P}-$functions with proportional period lattices are related by a scaling law: in particular, for any $\beta \neq 0$ and any lattice $\mathcal{L}$,

$$\mathcal{P}_{\beta\mathcal{L}}(\beta z) = \frac{1}{\beta^2} \mathcal{P}_{\mathcal{L}}(z) \quad \text{for all } z \in \mathbb{C}. \tag{4.2}$$

It is known (cf. [K84] or [MM99]) that the restrictions of $\mathcal{P}_{\mathcal{L}}$ and its derivative $\mathcal{P}'_{\mathcal{L}}$ to a fundamental parallelogram are branched covers of the Riemann sphere $\hat{\mathbb{C}}$ of degrees 2 and 3, respectively, and so for all but three exceptional values $w \in \mathbb{C}$ the equation $\mathcal{P}_{\mathcal{L}}(z) = w$ has two solutions z_1, z_2 in each fundamental parallelogram, and $\mathcal{P}'_{\mathcal{L}}(z_1) = -\mathcal{P}'_{\mathcal{L}}(z_2)$. Furthermore, the function $\mathcal{P}_{\mathcal{L}}(z)$ satisfies the differential equation

$$\mathcal{P}'_{\mathcal{L}}(z)^2 = 4\mathcal{P}_{\mathcal{L}}(z)^3 - g_2(\mathcal{L})\mathcal{P}_{\mathcal{L}}(z) - g_3(\mathcal{L}), \tag{4.3}$$

where the constants $g_2(\mathcal{L})$ and $g_3(\mathcal{L})$ are given by the Eisenstein series

$$g_2(\mathcal{L}) = 60 \sum_{l \in \mathcal{L}, l \neq 0} \frac{1}{l^4}$$

$$g_3(\mathcal{L}) = 140 \sum_{l \in \mathcal{L}, l \neq 0} \frac{1}{l^6}.$$

For any two complex numbers A, B such that $A^3 - 27B^2 \neq 0$, there exists (cf. Proposition III.13 in [K84]) a lattice $\mathcal{L}$ such that

$$g_2(\mathcal{L}) = A \quad \text{and} \tag{4.4}$$
$$g_3(\mathcal{L}) = B.$$

Proposition 4.1. *Let A and B be two constants such that $A^3 - 27B^2 \neq 0$, and Let $u(z)$ be a C^1 function on an open interval $J \subset \mathbb{R}$ with derivative $u'(x) \neq 0$ for all $x \in J$ that satisfies the differential equation*

$$u'(z)^2 = 4u(z)^3 - Au(z) - B. \tag{4.5}$$

Then for some lattice $\mathcal{L}$ and some $\alpha \in \mathbb{C}$,

$$u(x) = \mathcal{P}_{\mathcal{L}}(x + \alpha) \quad \text{for all } x \in J. \tag{4.6}$$

Proof. Without loss of generality, assume that $0 \in J$ and that $u'(0) \neq 0$. The differential equation (4.5) implies that in some neighborhood of $x = 0$, for one of the two branches of the square root function,

$$u'(x) = \sqrt{4u(x)^3 - Au(x) - B}. \tag{4.7}$$

Since $u'(0) \neq 0$, the right side of this equation is a Lipshitz continuous function of $u(x)$ for x near 0, and so the Picard-Lindelöf theorem guarantees that the equation (4.7) has a unique solution with initial value $u(0)$.

Let $\mathcal{L}$ be a lattice such that equations (4.4) hold. Because the Weierstrass $\mathcal{P}-$function is a double covering of $\mathbb{C}$, there exist two arguments $\alpha, \alpha' \in \mathbb{C}$ such that $\mathcal{P}_{\mathcal{L}}(\alpha) = \mathcal{P}_{\mathcal{L}}(\alpha') = u(0)$, and for one of these (say α) it must be the case that $\mathcal{P}'_{\mathcal{L}}(\alpha) = u'(0)$. Since the functions $\mathcal{P}_{\mathcal{L}}(x + \alpha)$ and $\mathcal{P}_{\mathcal{L}}(x + \alpha')$ both satisfy the differential equation (4.5), one of them (say $\mathcal{P}_{\mathcal{L}}(x + \alpha)$) must also satisfy (4.7). By the Picard-Lindelöf theorem, the equation (4.6) must hold in J.

$\square$

The connection between the differential equation (3.2) and the Weierstrass $\mathcal{P}-$function is easily explained. If $h(z) = z^2$, then the forcing term in (3.2) is quadratic, and so after a rescaling (3.2) can be written in the form

$$u''(y) = 6u(y)^2. \tag{4.8}$$

Multiplying both sides by $u'(y)$ and integrating yields

$$(u'(y))^2 = 4u(y)^3 + C, \tag{4.9}$$

where C is a constant of integration. This is the characteristic equation for a $\mathcal{P}-$function whose period lattice satisfies $g_2(\mathcal{L}) = 0$.

Proposition 4.2. *The Weierstrass function $u = \mathcal{P}_{\mathcal{L}}$ satisfies the differential equation (4.9) for some $C \in \mathbb{C} \setminus \{0\}$ if and only if the period lattice is of the form*

$$\mathcal{L} = \{m\omega + n\omega e^{\pi i/3}\} \tag{4.10}$$

for some $\omega \neq 0$; furthermore, $C > 0$ in (4.9) if and only if the lattice has the form (4.10) with

$$\omega = |\omega| e^{\pi i/6}. \tag{4.11}$$

In this case, u has real poles at integer multiples of $\sqrt{3}|\omega|$, and takes only real values on $\mathbb{R}$; furthermore, its only zeros in the fundamental parallelogram are at $\sqrt{3}|\omega|/3$ and $2\sqrt{3}|\omega|/3$, and u is strictly increasing on $(2\sqrt{3}|\omega|/3, \sqrt{3}|\omega|)$.

Proof. The Eisenstein series for the lattice $\mathcal{L} = \{m\omega + n\tilde{\omega}\}$ can be written as

$$g_2(\mathcal{L}) = 60 \sum_{m,n}^{*} (m\omega + n\tilde{\omega})^{-4} = 60\omega^{-4} G_4(\xi) \quad \text{and}$$

$$g_3(\mathcal{L}) = 140 \sum_{m,n}^{*} (m\omega + n\tilde{\omega})^{-6} = 140\omega^{-6} G_6(\xi) \tag{4.12}$$

where $\xi = \tilde{\omega}/\omega$ is the ratio of two fundamental periods and the sum is over all pairs of integers except $(0,0)$. By convention, the periods are ordered so that $\Im\xi > 0$; with this convention, G_4 and G_6 are modular forms of weights 4 and 6 (cf. [K84], section III.2). By the residue theorem for modular forms (cf. [K84], Proposition III.2.8), any nonzero modular form of weight 4 has precisely two zeros in the closure of the standard fundamental polygon of the modular group, at the points $\xi_- = e^{\pi i/3}$ and $\xi_+ = e^{2\pi i/3}$. Therefore, any Weierstrass function $u = \mathcal{P}_{\mathcal{L}}$ that satisfies the differential equation (4.9) must have period lattice of the form (4.10) (as the choices ξ_- and ξ_+ lead to the same lattice).

The lattice (4.10) is invariant under rotation by $\pi/3$ (that is, $\mathcal{L} = e^{\pi i/3}\mathcal{L}$), and so by averaging over the six rotations $e^{k\pi i/3}$ one finds that

$$G_6(e^{\pi i/3}) = G_6(e^{2\pi i/3}) = \sum_{m,n}^{*} \frac{1}{m^6 + n^6} > 0. \tag{4.13}$$

Consequently, if $g_3(\mathcal{L}) = -C < 0$ then $\mathcal{L}$ must be of the form (4.10) for some ω such that $\omega^6 < 0$, that is, ω is a positive multiple of a primitive 12th root of unity. Thus, in the case $g_3(\mathcal{L}) = -C < 0$ the lattice $\mathcal{L}$ must have the form (4.10) with $\omega = |\omega|e^{\pi i/6}$.

Assume now that $\mathcal{L}$ is of the form (4.10) for some ω satisfying (4.11). Then by the *addition law* for the elliptic curve $y^2 = 4x^3 - g_3$ (cf. [K84], section I.7; see especially Problem 8),

$$\mathcal{P}_{\mathcal{L}}(\sqrt{3}|\omega|/3) = \mathcal{P}_{\mathcal{L}}(2\sqrt{3}|\omega|/3) = 0.$$

Since $\mathcal{P}_{\mathcal{L}}$ has degree 2, it has only two zeros in a fundamental parallelogram, and by equation (4.9) the derivatives $\mathcal{P}'_{\mathcal{L}}(\sqrt{3}|\omega|/3) = -\mathcal{P}'_{\mathcal{L}}(2\sqrt{3}|\omega|/3)$ must be the two square roots of C. It is easily seen that the unique solution of (4.9) with initial conditions $u(y_0) = 0$ and $u'(y_0) > 0$ must be strictly increasing, with increasing derivative, on any interval (y_0, y_1) on which the solution u is well-defined and finite. This implies that $\mathcal{P}'_{\mathcal{L}}(\sqrt{3}|\omega|/3)$ is *negative*, and hence $\mathcal{P}'_{\mathcal{L}}(2\sqrt{3}|\omega|/3)$ is *positive*. It then follows that u is strictly increasing in $(2\sqrt{3}|\omega|/3, \sqrt{3}|\omega|)$.

$\square$

Remark 4.3. The case where $C = -g_3(\mathcal{L}) = -1$ in equation (4.9) is known as the *equianharmonic* case; cf. [AS72] for further information. In the equianharmonic case the period lattice is of the form (4.10), but with $\omega > 0$, i.e., the lattice is of the same form as in the case where $C = 1$ but rotated by $-\pi/6$. Call the case where $C = -g_3(\mathcal{L}) = +1$ the *anti-anharmonic* case; then by the scaling law, the $\mathcal{P}$−functions for the equianharmonic and the anti-anharmonic cases are related by

$$\mathcal{P}_{AAH}(e^{\pi i/6}z) = e^{-\pi i/3}\mathcal{P}_{EAH}(z) \quad \text{for all} \ \ z \in \mathbb{C}.$$

Thus, mapping properties and special values of the $\mathcal{P}$-function in the anti-anharmonic case can be read off from those for the equianharmonic case, which have been extensively tabulated.

As far as we know, the occurrence of the $\mathcal{P}$−function in critical branching processes was first observed by the first author in [L09], sec. 1.8. However, [L09] mistakenly asserts that the differential equation (4.9) with $C > 0$ falls into the equianharmonic case, and consequently the formulas in [L09], sec. 1.8 are off by factors of $e^{\pi i/6}$ and $e^{\pi i/3}$.

5 Distribution of M: Moranian Case

In this section we consider the Moranian case [SF79], where the number of offspring is either zero or two, each with probability $\frac{1}{2}$. In this case the probability generating

function is $\Psi(t) = \frac{1}{2} + \frac{1}{2}t^2$. For $0 \le y < x$ define

$$u_x(y) = \mathbb{P}^y\{M \ge x\} \tag{5.1}$$

to be the probability that the maximum position M attained by a particle of the branching Brownian motion initiated by a particle at y, with freezing of particles at 0, will exceed x. The function $\varphi_x(y) = 1 - u_x(y)$ is of the form covered by Proposition 3.1, so it satisfies the differential equation (3.2) with $\Psi(z) = (1 + z^2)/2$, and consequently u_x satisfies

$$u_x''(y) = 2\left[\frac{1}{2} + \frac{1}{2}(1 - u_x(y))^2 - (1 - u_x(y))\right] = u_x(y)^2. \tag{5.2}$$

Theorem 5.1. *For branching Brownian motion with Moranian offspring distribution and killing at 0, the tail distribution function $u_x(y) = \mathbb{P}^y(M \ge x)$ is given by*

$$u_x(y) = 6\mathcal{P}_{\mathcal{L}_x}(y + 2\omega_x/3), \tag{5.3}$$

where $\mathcal{P}_{\mathcal{L}_x}(z)$ is the Weierstrass $\mathcal{P}$ function with period lattice

$$\mathcal{L}_x = \left\{ m\frac{\omega_x}{\sqrt{3}}e^{\pi i/6} + n\frac{\omega_x}{\sqrt{3}}e^{\pi i/2} : m, n \in \mathbb{Z} \right\} \tag{5.4}$$

for some $\omega_x > 0$. The positive period ω_x is uniquely determined by the boundary condition

$$6\mathcal{P}_{\mathcal{L}_x}(x + 2\omega_x/3) = 1. \tag{5.5}$$

Remark 5.2. The value of ω_x can be computed numerically, by exploiting the fact that the inverse of the Weierstrass $\mathcal{P}-$function $w = \mathcal{P}_{\mathcal{L}}(z)$ is given by the elliptic integral

$$z = \int_w^\infty \frac{dt}{\sqrt{4t^3 - g_2(\mathcal{L})t - g_3(\mathcal{L})}}. \tag{5.6}$$

For lattices of the form (5.4), we have $g_2(\mathcal{L}) = 0$. Consequently, the boundary condition (5.5) implies that

$$x = \int_0^{\frac{1}{6}} \frac{dt}{\sqrt{4t^3 - g_3(\mathcal{L})}}. \tag{5.7}$$

This equation (5.7) determines $g_3(\mathcal{L})$, and hence, using the identities (4.12), the value of ω_x. For $x = 1$, the values are

$$g_3(\mathcal{L}_1) = -0.023786\cdots \quad \text{and} \quad \omega_1 = 9.88285\cdots.$$

The large x dependence of ω_x on x will be further clarified below, in Corollary 5.4.

The proof of Theorem 5.1 will rely on the uniqueness theorem for solutions of the differential equation for the $\mathcal{P}-$function (Proposition 4.1). For this, it will be necessary to know that $u_x'(y) \ne 0$ for any $y \in [0, x]$.

Lemma 5.3. *The function $y \mapsto u_x(y)$ has strictly positive derivative $u_x'(y)$ on the interval $[0, x]$.*

Proof. Since $u_x(y)$ is a cumulative distribution function, it is non-decreasing, and so its derivative must be nonnegative. Moreover, the differential equation $u_x''(y) = u_x(y)^2$ implies that the derivative is increasing at every y where $u_x(y) > 0$. It is easily seen that a branching Brownian motion started at any $y > 0$ has positive probability of putting a particle at x, so $u_x(y) > 0$ for all $y \in (0, x]$.

It remains to show that $u'_x(0) > 0$. The differential equation $u'' = u^2$ can be rewritten as the autonomous system

$$u' = v,$$
$$v' = u^2.$$

The vector field in this system is clearly Lipshitz continuous, so solutions to the initial value problem are unique. Since $u \equiv v \equiv 0$ is the unique solution with initial conditions $u(0) = v(0) = 0$, it follows that any non-constant solution of $u'' = u^2$ satisfying $u(0) = 0$ cannot have derivative $u'(0) = 0$. $\qquad\square$

Proof of Theorem 5.1. Set $\widetilde{u}_x(y) = \frac{1}{6}u_x(y)$; then (5.2) becomes

$$\widetilde{u}_x(y)'' = 6\widetilde{u}_x(y)^2,$$

the differential equation encountered earlier in (4.8). The integrated form is (4.9). By Lemma 5.3, the derivative $u'_x(y)$ is strictly positive on $y \in [0, x]$, and so the same is obviously true of $\widetilde{u}_x(y)$. Hence, by Proposition 4.1, $\widetilde{u}_x$ must coincide with a translate of a $\mathcal{P}-$function, and so for each $x > 0$ there exists a unique period lattice and a unique $\alpha_x \in \mathbb{C}$ such that

$$\widetilde{u}_x(y) = \mathcal{P}_{\mathcal{L}}(y + \alpha_x) \quad \text{for all} \ \ y \in [0, x].$$

Since there is no linear term in the equation (4.8), the period lattice $\mathcal{L}$ must be of the form (5.4).

Finally, the boundary condition $u_x(0) = 0$ and (5.3) imply that α_x is a zero for $\mathcal{P}_{\mathcal{L}}(z)$. Since $u_x(y)$ is increasing in y and $\mathcal{P}_{\mathcal{L}}(z)$ is doubly periodic, the constant α_x must be the larger zero of $\mathcal{P}_{\mathcal{L}}(z)$ in $(0, \omega_x)$, and in particular, by Proposition 4.2,

$$\alpha_x = 2\omega_x/3. \tag{5.8}$$

$$\square$$

All of the $\mathcal{P}-$functions that occur in Theorem 5.6 are scaled versions of $\mathcal{P}_{\mathcal{L}_1}$. By (4.2), if the positive periods ω_1 and ω_x of the lattices $\mathcal{L}_1$ and $\mathcal{L}_x$, respectively, are related by

$$\lambda_x := \frac{\omega_1}{\omega_x}, \tag{5.9}$$

then

$$\mathcal{P}_{\mathcal{L}_x}(z) = \lambda_x^2 \mathcal{P}_{\mathcal{L}_1}(\lambda_x z). \tag{5.10}$$

It is obvious that $\lim_{x \to \infty} \omega_x = \infty$, because the function $u_x(y)$ is an increasing function on $(0, x)$, and hence cannot have a positive period smaller than x. Although the dependence of ω_x on x is not linear, it is asymptotically linear, as the next corollary shows.

Corollary 5.4.

$$\lim_{x \to \infty} \frac{\omega_x}{x} = \lim_{x \to \infty} \frac{\omega_1}{\lambda_x x} = 3. \tag{5.11}$$

Proof. The scaling law (4.2) and the boundary conditions for the functions u_x and u_1 imply that

$$\mathcal{P}_{\mathcal{L}_1}(2\omega_1/3 + \lambda_x x) = \frac{1}{6\lambda_x^2}. \tag{5.12}$$

Since $(6\lambda_x^2)^{-1} \to \infty$ as $x \to \infty$, it follows that $2\omega_1/3 + \lambda_x x$ converges to ω_1, as this is the smallest positive pole of $\mathcal{P}_{\mathcal{L}_1}$. The result now follows from the equation (5.9). $\qquad\square$

Remark 5.5. By exploiting the fact that $\mathcal{P}_{\mathcal{L}_1}(\omega_1 - z) \sim 1/z^2$ as $z \to 0$, one can obtain from the equation (5.12) the sharper approximation

$$\omega_x = 3x + 3/\sqrt{6} + o(1) \quad \text{as} \ \ x \to \infty. \tag{5.13}$$

The scaling laws (5.10) and the period asymptotics (5.11) now combine to provide the large$-x$ asymptotic behavior of the hitting probability function $u_x(y)$.

Theorem 5.6. *For branching Brownian motion with Moranian offspring distribution and killing at* 0*, the tail distribution function* $u_x(y) = \mathbb{P}^y(M \geq x)$ *of the maximum attained position* M *satisfies*

$$\lim_{x \to \infty} x^3 u_x(y) = C_1 \cdot y \quad \text{for each} \ \ y > 0, \tag{5.14}$$

where $C_1 = 6c_1^3 \mathcal{P}'_{\mathcal{L}_1}(2\omega_1/3) = 33.0822 \cdots$ *and* $c_1 = \omega_1/3 = 3.29428 \cdots$ *are constants that do not depend on* x *or* y*. Furthermore, for each fixed* $0 < s < 1$*,*

$$\lim_{x \to \infty} x^2 u_x(sx) = C_2(s) \tag{5.15}$$

where $C_2(s) = 6c_1^2 \mathcal{P}_{\mathcal{L}_1}(2\omega_1/3 + sc_1)$*.*

Proof of Theorem 5.6. By equations (5.3) and (5.2), the function $\mathcal{P}_{\mathcal{L}_1}$ satisfies the second-order differential equation

$$\mathcal{P}''_{\mathcal{L}_1}(z) = 6\mathcal{P}_{\mathcal{L}_1}(z)^2. \tag{5.16}$$

Using the abbreviation $\alpha_x = 2\omega_x/3$ and the fact that $\mathcal{P}_{\mathcal{L}_1}(\alpha_1) = 0$ (cf. Proposition 4.2), it follows by taking successive derivatives that

$$
\begin{aligned}
\mathcal{P}''_{\mathcal{L}_1}(\alpha_1) &= 0, \\
\mathcal{P}'''_{\mathcal{L}_1}(\alpha_1) &= 0, \quad \text{and} \\
\mathcal{P}^{(4)}_{\mathcal{L}_1}(\alpha_1) &= 12\left(\mathcal{P}'_{\mathcal{L}_1}(\alpha_1)\right)^2.
\end{aligned}
$$

Lemma 5.3 implies that $\mathcal{P}'_{\mathcal{L}_1}(\alpha_1) > 0$, since this is proportional to $u'_1(0)$. Consequently, since $\lambda_x \to 0$ as $x \to \infty$, Taylor expansion around the point α_1 yields

$$
\begin{aligned}
u_x(y) = 6\mathcal{P}_{\mathcal{L}_x}(\alpha_x + y) &= 6\lambda_x^2 \mathcal{P}_{\mathcal{L}_1}(\alpha_1 + \lambda_x y) \\
&= 6\lambda_x^2 \left(\mathcal{P}'_{\mathcal{L}_1}(\alpha_1)\lambda_x y + \mathcal{P}^{(4)}_{\mathcal{L}_1}(\alpha_1)\lambda_x^4 y^4 + \cdots \right) \\
&= 6\mathcal{P}'_{\mathcal{L}_1}(\alpha_1)\lambda_x^3 y + O(\lambda_x^6),
\end{aligned}
$$

Therefore, by Corollary 5.4,

$$\lim_{x \to \infty} x^3 u_x(y) = 6c_1^3 \mathcal{P}'_{\mathcal{L}_1}(\alpha_1)y = C_1 y.$$

and (5.14) follows.

To prove (5.15), notice that for $y = sx$, we have

$$x^2 u_x(y) = 6x^2 \mathcal{P}_{\mathcal{L}_x}(\alpha_x + y) = 6(\lambda_x x)^2 \mathcal{P}_{\mathcal{L}_1}(\alpha_1 + s(\lambda_x x))$$

By the continuity of $\mathcal{P}_{\mathcal{L}_1}(z)$ on the interval (α_1, ω_1), it follows that

$$\lim_{x \to \infty} x^2 u_x(y) = 6c_1^2 \mathcal{P}_{\mathcal{L}_1}(\alpha_1 + sc_1) = C_2(s).$$

$\square$

6 Distribution of M: General Case

In this section we will show that the asymptotic formulas (5.14) and (5.15) extend to branching Brownian motions with arbitrary mean 1 offspring distributions with finite third moments. The main result is as follows.

Theorem 6.1. *Assume that the offspring distribution has mean* 1, *positive variance* σ^2, *and finite third moment. Then for each* $y > 0$, *the probability* $u_x(y) = \mathbb{P}^y\{M \geq x\}$ *that a particle of the branching Brownian motion with initial particle at location* y *reaches location* x *satisfies*

$$\lim_{x \to \infty} x^3 u_x(y) = C_3 \cdot y, \tag{6.1}$$

where $C_3 = C_1/\sigma^2$. *In addition, for each fixed* $0 < s < 1$,

$$\lim_{x \to \infty} x^2 u_x(sx) = C_4, \tag{6.2}$$

where $C_4 = C_2/\sigma^2$. *Here* C_1, C_2 *are the constants in Theorem 5.6.*

This theorem will be deduced from Theorem 5.6 by comparison arguments for differential equations. The strategy is similar to that used by Lee [L90-1]: the key is that for small values of $u_x(y)$, the forcing term $h(u_x(y))$ in the differential equation

$$u_x'' = h(u_x) \tag{6.3}$$

(which follows from Proposition 3.1 as in the Moranian case) is well-approximated by the quadratic function $\sigma^2 u_x(y)^2$. To see this, let $\Psi(z)$ be the probability generating function of the offspring distribution. If the offspring distribution has finite third moment, then by Taylor expansion

$$\Psi(1-z) = \Psi(1) - \Psi'(1)z + \frac{1}{2}\Psi''(1)z^2 + O(z^3) \quad \text{as } z \to 0.$$

By hypothesis, the offspring distribution has mean 1 and positive variance $0 < \sigma^2 < \infty$, so $\Psi'(1) = 1$ and $\Psi''(1) = \sigma^2$. Consequently, as $z \to 0$,

$$h(z) = 2\left[\Psi(1-z) - (1-z)\right] = \sigma^2 z^2 + O(z^3). \tag{6.4}$$

Our arguments will use the following comparison principle for solutions to differential equations and inequalities. This is a minor modification of the Comparison Lemma in [L90-1]; because the result is standard and its proof involves only elementary calculus, we shall omit it.

Lemma 6.2 (Comparison Principle). *Let* v_1, v_2 *be positive functions on an interval* $[y_1, y_2]$ *such that for some constant* $a > 0$,

$$v_1''(y) - av_1(y)^2 \leq 0 \quad and \tag{6.5}$$
$$v_2''(y) - av_2(y)^2 \geq 0$$

for all $y_1 < y < y_2$. *If*

$$v_1(y_1) \geq v_2(y_1) \quad and \tag{6.6}$$
$$v_1(y_2) \geq v_2(y_2), \tag{6.7}$$

then

$$v_1(y) \geq v_2(y) \quad for\ all\ \ y_1 \leq y \leq y_2. \tag{6.8}$$

Next, we record several monotonicity properties of the functions $u_x(y) = \mathbb{P}^y\{M \geq x\}$.

Proposition 6.3. *The function $u_x(y)$ is strictly decreasing in x and strictly increasing in y, and $u_x'(0)$ is strictly decreasing in x.*

Proof. The monotonicity of $u_x(y)$ in y and in x follow directly from the construction of the branching Brownian motion outlined in section 2.

To prove that $u_x'(0)$ is strictly decreasing in x, recall that for each $x > 0$ the function u_x satisfies the differential equation $u'' = h(u)$, together with the boundary conditions $u_x(0) = 0$ and $u_x(x) = 1$. Hence, by the uniqueness theorem for differential equations, if $0 < x_1 < x_2$ then $u_{x_1}'(0) \neq u_{x_2}'(0)$, because otherwise the functions $u_{x_i}(y)$ would be equal for all $y \in [0, x_1]$, which is impossible because $u_{x_2}(x_1) < 1$.

Thus, to complete the proof it suffices to show that if $0 < x_1 < x_2$ then $u_{x_1}'(0) < u_{x_2}'(0)$ is impossible. But since $u_{x_1}(0) = u_{x_2}(0) = 0$, if $u_{x_1}'(0) < u_{x_2}'(0)$ then for all y in some neighborhood $(0, \varepsilon)$ we would have $u_{x_1}(y) < u_{x_2}(y)$. This would contradict the monotonicity of $u_x(y)$ in x. $\qquad\square$

To study the behavior of $u_x(y)$ near $y = x$, we introduce the function

$$w_x(t) = u_x(x - t) \tag{6.9}$$

Observe that $w_x(t)$ is the probability that a branching Brownian motion with killing at $-(x - t)$ and initial particle at 0 will produce a particle that reaches location t. The construction in section 2 shows that $w_x(t)$ is *strictly* monotone in both x and t, and that

$$\lim_{x \to \infty} w_x(t) = w_\infty(t) \tag{6.10}$$

where $w_\infty(t)$ is the probability that a branching Brownian motion with *no* killing started at location 0 will produce a particle that reaches location t (cf. equation (2.2)). The convergence (6.10) holds uniformly for t in any finite interval $[0, t_*]$. The function $w_\infty(t)$ is the same as the function $p(x)$ in the $d = 1$ case studied in [SF79], who proved that

$$w_\infty(t) = \frac{6}{\sigma^2 t^2} + O\left(\frac{1}{t^3}\right) \quad \text{as } t \to \infty. \tag{6.11}$$

To prove Theorem 6.1, we must determine the behavior of $w_x(t)$ for *large* t, and in particular for t within distance $O(1)$ of x. The basic strategy will be as follows. For any $\varepsilon > 0$ there exists $t_* < \infty$ so large that $w_\infty(t_*) < \varepsilon$. This implies that $w_x(t_*) < \varepsilon$, or equivalently $u_x(x - t_*) < \varepsilon$, for all large x. Thus, in the interval $[0, x - t_*]$ the function u_x will be bounded above by ε, and so $h(u_x)$ will be well-approximated by the quadratic function $\sigma^2 u_x^2$. The analysis of sections 4–5 shows that the differential equation (3.2) with $h(u) = Cu^2$ admits an exact solution in terms of a Weierstrass $\mathcal{P}-$function, so it will follow from the comparison principle above that in the interval $[0, x - t_*]$ the function u_x will be trapped between two such $\mathcal{P}-$functions. By taking $\varepsilon \to 0$, we will obtain sharp asymptotic approximations to u_x.

Taylor expansion of $h(z)$ shows that for all $0 < \delta < 1$ there exists $\varepsilon = \varepsilon(\delta) > 0$ such that if $0 \leq u_x(y) \leq \varepsilon$, then

$$\sigma^2(1 - \delta)u_x^2(y) \leq h\left(u_x(y)\right) \leq \sigma^2(1 + \delta)u_x^2(y). \tag{6.12}$$

On the other hand, (6.11) and the monotonicity of $w_\infty(t)$ imply that for each $\varepsilon \in (0, 1)$, there exists t_ε such that $w_\infty(t_\varepsilon) = \varepsilon$ and $w_\infty(t) \leq \varepsilon$ for all $t \geq t_\varepsilon$. By (6.10), for all $0 \leq y \leq x - t_\varepsilon$,

$$u_x(y) = w_x(x - y) \leq w_\infty(x - y).$$

Therefore, (6.12) applies for all $0 \leq y \leq x - t_\varepsilon$. Define

$$\eta(x, \varepsilon) = u_x(x - t_\varepsilon);$$

then for each $\varepsilon > 0$ the function $\eta(x, \varepsilon)$ is increasing in x, and so

$$\eta(x, \varepsilon) = w_x(t_\varepsilon) \uparrow w_\infty(t_\varepsilon) = \varepsilon \quad \text{as } x \to \infty. \tag{6.13}$$

Corollary 6.4 (Pinching)**.** *Let t_ε and $\eta(x, \varepsilon)$ be as above, and define*

$$\begin{cases} a_+ = \sigma^2(1 + \delta) > 0 \\ a_- = \sigma^2(1 - \delta) > 0. \end{cases} \tag{6.14}$$

If $u_x(y), u_x^+(y)$ and $u_x^-(y)$ satisfy the boundary value problems

$$\begin{cases} u_x''(y) = h(u_x(y)) \\ u_x(0) = 0 \\ u_x(x - t_\varepsilon) = \eta(x, \varepsilon), \end{cases} \qquad \begin{cases} u_x^{\pm''}(y) = a_\pm u_x^{\pm 2}(y) \\ u_x^\pm(0) = 0 \\ u_x^\pm(x - t_\varepsilon) = \eta(x, \varepsilon), \end{cases} \tag{6.15}$$

then

$$u_x^+(y) \leq u_x(y) \leq u_x^-(y) \quad \text{for all } 0 \leq y \leq x - t_\varepsilon. \tag{6.16}$$

Proof. This is an immediate consequence of the comparison principle (Lemma 6.2). □

The differential equations (6.15) for the functions $u_x^\pm$ are, except for the constants $a_\pm$, identical to the differential equation (4.8) for the $\mathcal{P}-$function. Consequently, they are related by a simple scaling law.

Lemma 6.5. *Suppose that the functions $u_x^\pm(y)$ satisfy (6.15), and set*

$$\widehat{u}_x^\pm(y) = u_x^\pm\left(\frac{y}{\sqrt{a_\pm}}\right). \tag{6.17}$$

Then the functions $\widehat{u}_x^\pm$ satisfy the boundary value problems

$$\begin{cases} \widehat{u}_x^{\pm''}(y) = \widehat{u}_x^{\pm 2}(y) \\ \widehat{u}_x^\pm(0) = 0 \\ \widehat{u}_x(\sqrt{a_\pm}(x - t_\varepsilon)) = \eta(x, \varepsilon). \end{cases} \tag{6.18}$$

Proof. This is an immediate consequence of the definition (6.17) and equation (6.15). □

Proof of Theorem 6.1. The differential equation in (6.18) is the same as in the Moranian case, so

$$\widehat{u}_x^\pm(y) = 6\mathcal{P}_{\mathcal{L}_x^\pm}\left(\alpha_x^\pm + y\right),$$

where the period lattice $\mathcal{L}_x^\pm$ has fundamental periods $\omega_x^\pm$ and $\omega_x^\pm e^{2\pi i/3}$. The centering constant $\alpha_x^\pm$ is the larger zero of $\mathcal{P}_{\mathcal{L}_x^\pm}(z)$ on $(0, \omega_x^\pm)$. By the scaling laws for the $\mathcal{P}-$functions,

$$\widehat{u}_x^\pm(y) = 6\lambda_x^{\pm 2}\mathcal{P}_{\mathcal{L}_1}\left(\alpha_1 + \lambda_x^\pm y\right). \tag{6.19}$$

where $\lambda_x^\pm = \omega_1/\omega_x^\pm$. The boundary conditions at $y = \sqrt{a_\pm}(x - t_\varepsilon)$ in (6.18) and (6.19) imply that

$$\mathcal{P}_{\mathcal{L}_1}(\alpha_1 + \sqrt{a_\pm}\lambda_x^\pm(x - t_\varepsilon)) = \frac{\eta(x, \varepsilon)}{6\lambda_x^{\pm 2}}. \tag{6.20}$$

By (6.13), $\lim_{x \to \infty} \eta(x, \varepsilon) = \varepsilon$, and furthermore $x - t_\varepsilon \sim x$, since $t_\varepsilon = O(1)$ as $x \to \infty$. Hence, $\lambda_x^\pm \to 0$, by (5.9). Consequently, the right hand side of (6.20) goes to ∞ as $x \to \infty$. It follows that $\alpha_1 + \sqrt{a_\pm}\lambda_x^\pm x \to \omega_1$, as $x \to \infty$. Therefore,

$$\lim_{x \to \infty} \lambda_x^\pm x = \frac{\omega_1 - \alpha_1}{\sqrt{a_\pm}} = \frac{c_1}{\sqrt{a_\pm}}. \tag{6.21}$$

Equations (6.17) and (6.19) imply that

$$u_x^\pm(y) = \widehat{u}_x^\pm(\sqrt{a_\pm}\, y) = 6\lambda_x^{\pm 2} \mathcal{P}_{\mathcal{L}_1}(\alpha_1 + \lambda_x^\pm \sqrt{a_\pm}\, y). \tag{6.22}$$

Consequently, when $0 \le y \le x - t_\varepsilon$ is fixed, Taylor expansion of $\mathcal{P}_{\mathcal{L}_1}(z)$ around $z = \alpha_1$ as in the Moranian case yields

$$
\begin{aligned}
\lim_{x\to\infty} x^3 u_x^\pm(y) &= \lim_{x\to\infty} 6x^3 \lambda_x^{\pm 2} \mathcal{P}_{\mathcal{L}_1}(\alpha_1 + \sqrt{a_\pm}\lambda_x^\pm y) \\
&= \lim_{x\to\infty} 6(\lambda_x^\pm x)^3 \sqrt{a_\pm} \mathcal{P}'_{L_1}(\alpha_1) y + O(\lambda_x^{\pm 3}) \\
&\overset{(6.21)}{=} \frac{6c_1^3 \mathcal{P}'_{\mathcal{L}_1}(\alpha_1) y}{a_\pm} = \frac{C_1 y}{a_\pm}.
\end{aligned}
$$

Similarly, if $y = sx$ for some fixed $0 < s < 1$, then by the continuity of $\mathcal{P}_{\mathcal{L}_1}(z)$ on (α_1, ω_1) and (6.21),

$$
\begin{aligned}
\lim_{x\to\infty} x^2 u_x^\pm(sx) &= \lim_{x\to\infty} 6(x\lambda_x^\pm)^2 \mathcal{P}_{\mathcal{L}_1}(\alpha_1 + s\sqrt{a_\pm}\lambda_x^\pm x) \\
&\overset{(6.21)}{=} \frac{6c_1^2 \mathcal{P}_{\mathcal{L}_1}(\alpha_1 + sc_1)}{a_\pm} = \frac{C_2(s)}{a_\pm}.
\end{aligned}
$$

Here C_1 and $C_2(s)$ are as in Theorem 5.6.

Finally, by Corollary 6.4,

$$\text{for each } y \text{ fixed }, \quad \frac{C_1 y}{a_+} \le \lim_{x\to\infty} x^3 u_x(y) \le \frac{C_1 y}{a_-}; \quad \text{and}$$

$$\text{for each } 0 < s < 1 \text{ fixed }, \quad \frac{C_2(s)}{a_+} \le \lim_{x\to\infty} x^2 u_x(sx) \le \frac{C_2(s)}{a_-}.$$

Letting $\delta \to 0$, so that $a_\pm \to \sigma^2$, we obtain (6.1) and (6.2). $\qquad \square$

7 The Number of Killed Particles

In this section we discuss the distribution of the number $N = N_0$ of particles killed during the course of a critical branching Brownian motion with killing at 0 initiated by a single particle at position $y > 0$. Our primary interest is in the tail of the distribution, that is, in the large-k behavior of the probabilities $\mathbb{P}^y\{N \ge k\}$.

Theorem 7.1. *If the offspring distribution has mean* 1*, positive variance* σ^2*, and finite third moment then*

$$\sum_{k=1}^m k\mathbb{P}^y(N \ge k) \sim 2C_7 y\sqrt{m} \quad \text{where} \quad C_7 = \frac{\sigma}{\sqrt{6\pi}}. \tag{7.1}$$

The proof, which will use a form of Karamata's Tauberian theorem, will be given in section 7.4.

If we knew that the sequence $k\mathbb{P}^y(N \ge k)$ were monotone then we could conclude from (7.1) that $\mathbb{P}^y(N \ge k) \sim C_7 y/k^{3/2}$. However, it seems unlikely that monotonicity of the sequence $k\mathbb{P}^y(N \ge k)$ holds in general. Thus, to obtain sharp asymptotic results about the individual probabilities $\mathbb{P}^y(N = k)$, we will impose more restrictive hypotheses on the offspring distribution that will allow us to avoid the use of Karamata's theorem. In its place, we will use a result of Flajolet and Odlyzko [FO90] that allows one to extract information about the asymptotic behavior of the coefficients of a power series from information about its behavior on the circle Γ of convergence. Our hypotheses are most conveniently formulated in terms of the functions

$$h(s) = 2[\Psi(1-s) - (1-s)] \quad \text{and} \quad \kappa(s) = \int_0^s h(s')\, ds', \tag{7.2}$$

where Ψ is the probability generating function of the offspring distribution. Since the power series for a probability generating function has radius of convergence 1, the function h extends to an analytic function $h(z)$ in the disk of radius 1 centered at $z = 1$, as does its integral κ. If the offspring distribution has finite support then the functions h and κ are polynomials, and consequently are well-defined and analytic in the entire plane $\mathbb{C}$. Observe that κ has a zero of degree 3 at $s = 0$, since $h(s) = s^2 + O(|s|^3)$.

Theorem 7.2. *Assume that the offspring distribution has mean 1, positive variance σ^2, and that the function $h(z)$ extends analytically to a disk of radius $2 + \varepsilon$ centered at 0. If the indefinite integral $\kappa(s)$ has no zeros in the punctured disk $0 < |s| \leq 2$, then for each $y > 0$,*

$$\mathbb{P}^y(N \geq k) \sim \frac{C_7 y}{k^{3/2}} \quad \text{where} \quad C_7 = \frac{\sigma}{\sqrt{6\pi}}. \tag{7.3}$$

and

$$\mathbb{P}^y(N = k) \sim \frac{C_8 y}{k^{5/2}} \quad \text{where} \quad C_8 = \frac{3\sigma}{2\sqrt{6\pi}}. \tag{7.4}$$

For the double-or-nothing (Moranian) offspring distribution, the functions $h(z)$ and $\kappa(z)$ are given by $h(z) = z^2$ and $\kappa(z) = z^3/3$, and so the conclusions of Theorem 7.2 hold. (In this case, we will exhibit an explicit closed-form representation of the distribution, in Theorem 7.9 below.) Consequently, by Rouche's theorem, the hypotheses of Theorem 7.2 hold for all finitely-supported offspring distributions in a neighborhood of the double-or-nothing distribution, in the following sense: for any integer $m \geq 3$ there exists $\alpha_m \in (0, 1/2)$ such that the hypotheses of Theorem 7.2 hold for any probability distribution $\{q_n\}_{0 \leq n \leq m}$ such that $\min(q_0, q_2) > \alpha_m$.

Theorem 7.2 should be compared with recent results of Maillard [M13] and Berestycki *et al.* [BBHM15], which give sharp tail probability estimates for the number of killed particles in the somewhat different context of supercritical branching Brownian motion with particle drift. Both [M13] and [BBHM15] also use the Flajolet-Odlyzko theorem, and so must also contend with the issue of analytic continuation of the generating function. In [BBHM15], the reproduction mechanism is simple binary fission, and so there is no need to impose additional conditions. In [M13], the offspring distribution is arbitrary, but must have exponentially decaying tails and mean greater than 1. In all three cases, it would be of interest to determine optimal hypotheses on the offspring distribution.

The proof of Theorem 7.2 will be given in section 7.5. In sections 7.1, 7.2, and 7.4, we shall assume only that the hypotheses of Theorem 7.1 are in force; in section 7.3 we shall assume that the offspring distribution is the double-or-nothing distribution; and in section 7.5 we shall assume that the offspring distribution satisfies the hypotheses of Theorem 7.2.

7.1 Expected number of killed particles

Proposition 7.3.
$$\mathbb{E}^y[N] = 1. \tag{7.5}$$

The proof will use the following *a priori* bound on the expectation.

Lemma 7.4.
$$\mathbb{E}^y[N] \leq 1$$

Proof. For each time t, let N_t denote the total number of particles frozen at 0 by time t, and Z_t the total number of particles in the enveloping branching Brownian motion with no particle freezing (see the construction in section 2). The counting process $\{N_t\}_{t>0}$ is clearly increasing in t, and

$$\lim_{t \to \infty} N_t = N.$$

Consequently, by the monotone convergence theorem, it will suffice to show that $\mathbb{E}^y[N_t] \leq 1$ for any $t > 0$.

Let $\widetilde{Z}_t$ be the total number of particles at time t in the branching Brownian motion with freezing (including those particles frozen at 0), and W_t the initial particle's location at time $t \geq 0$. Define T to be the time of the first reproduction event (recall that this is a unit exponential random variable independent of the branching Brownian motion) and τ_0 the first time that a particle reaches the origin. Since the offspring distribution has mean 1,

$$E^y Z_{T \wedge \tau_0} = 1.$$

Now a particle that reaches zero will, in the enveloping branching Brownian motion, engender a *critical* descendant branching Brownian motion, and so by the strong Markov property,

$$\mathbb{E}^0[Z_{t-(T \wedge \tau_0)}] = 1.$$

This implies that $E^y[\widetilde{Z}_t] = \mathbb{E}^y[Z_t] = 1$. Clearly $N_t \leq \widetilde{Z}_t$, so $\mathbb{E}^y[N_t] \leq \mathbb{E}^y[\widetilde{Z}_t] = 1$ for any $t > 0$. $\qquad\square$

Proof of Proposition 7.3. Let

$$Z_t^* = \widetilde{Z}_t - N_t \tag{7.6}$$

be the number of particles of the branching Brownian motion alive at time t that are not frozen. Since $\mathbb{E}^y[\widetilde{Z}_t] = 1$ for any $t > 0$, it is enough to show that $E Z_t^* \to 0$ as $t \to \infty$.

Clearly, $Z_t^* \leq Z_t$, because the particles counted in Z_t^* are contained in the set of particles counted by Z_t. In fact,

$$Z_t^* = \sum_{i=1}^{Z_t} \mathbf{1}\{\text{particle } i \text{ trajectory } \subset (0, \infty)\} \implies$$

$$E^y Z_t^* = E^y \sum_{i=1}^{Z_t} \mathbf{1}\{\text{particle } i \text{ trajectory } \subset (0, \infty)\}.$$

To evaluate the last expectation, we use the discrete Brownian snake representation of branching Brownian motion described in section 2. Recall that in this construction particles are represented by vertices of a Galton-Watson tree, and their locations are obtained by running conditionally independent Wiener processes along the edges. Thus, for any particle i counted in Z_t, the conditional distribution of the trajectory $\{W_s^i\}_{s \leq t}$ of particle i up to time t, given the realization of the skeletal branching process, is that of Brownian motion started at y. Hence,

$$E^y Z_t^* = E^y \sum_{i=1}^{Z_t} \mathbf{1}\{\text{particle } i \text{ trajectory } \subset (0, \infty)\}$$

$$= E Z_t P^y \{W_s \text{ does not hit } 0 \text{ by time } t\}$$

$$= P^y \{W_s \text{ does not hit } 0 \text{ by time } t\}$$

$$\longrightarrow 0 \quad \text{as } t \to \infty.$$

$\qquad\square$

7.2 Probability Generating Function of N

Define

$$\varphi(y, s) = \mathbb{E}^y[s^N] = \sum_{k=0}^{\infty} \mathbb{P}^y\{N = k\} s^k \tag{7.7}$$

to be the probability generating function of the random variable N under the probability measure $\mathbb{P}^y$. Because the sequence s^n is multiplicative, Proposition 3.1 implies that for each complex number s in the disk $|s| < 1$ the function $y \mapsto \varphi(y, s)$ is C^2 and satisfies the differential equation

$$\frac{1}{2}\partial_{yy}\varphi(y, s) = \varphi(y, s) - \Psi(\varphi(y, s)) \quad \text{for} \ \ y > 0. \tag{7.8}$$

Furthermore, φ satisfies the boundary conditions

$$\varphi(0, s) = s \quad \text{and} \tag{7.9}$$
$$\varphi(\infty, s) = 1.$$

Because we are interested in the tail of the distribution, we will find it useful to reformulate the boundary value problem for φ as an equivalent problem for the generating function

$$H(y, s) = \sum_{k=1}^{\infty} \mathbb{P}^y(N \geq k)s^k. \tag{7.10}$$

Proposition 7.5. *For each s such that $|s| < 1$, the function $H(y, s)$ satisfies the differential equation*

$$\partial_{yy}H(y, s) = \frac{s}{1-s}h\left(\frac{1-s}{s}H(y, s)\right). \tag{7.11}$$

where $h(z) = 2[\Psi(1 - z) - (1 - z)]$. In addition, $H(y, s)$ satisfies the following boundary conditions:

$$H(0, s) = s, \tag{7.12}$$
$$H(\infty, s) = 0, \quad \text{and} \tag{7.13}$$
$$\lim_{s \to 1} H(y, s) = 1. \tag{7.14}$$

Proof. The generating functions H and φ are related by

$$\begin{aligned}
\varphi(y, s) &= \sum_{k=0}^{\infty} \mathbb{P}^y(N = k)s^k \\
&= \sum_{k=0}^{\infty} \mathbb{P}^y(N \geq k)s^k - \sum_{k=0}^{\infty} \mathbb{P}^y(N \geq k+1)s^k \\
&= 1 + H(y, s)\left(\frac{s-1}{s}\right).
\end{aligned} \tag{7.15}$$

Thus, the differential equation (7.11) follows directly from that for φ, as do boundary conditions (7.12) and (7.13). Finally, the additional boundary condition (7.14) follows from the hypothesis that the branching Brownian motion is critical, as this makes $E^y N = 1$, by Proposition 7.3, and

$$\lim_{s \to 1} H(y, s) = \sum_{k=1}^{\infty} \mathbb{P}^y(N \geq k) = \mathbb{E}^y[N].$$

$\square$

7.3 The Moranian Case

Consider now the Moranian case, where the number of offspring is either 0 or 2, each with probability $\frac{1}{2}$. In this case the function h in the differential equation (7.11) reduces to $h(z) = z^2$, and so (7.11) becomes

$$\partial_{yy} H(y, s) = \frac{1 - s}{s} H^2(y, s). \tag{7.16}$$

The boundary conditions (7.12) and (7.14) uniquely determine the solution, which can be written explicitly as

$$H(y, s) = s \left(\frac{1}{\sqrt{6}} y \sqrt{1 - s} + 1 \right)^{-2} \tag{7.17}$$

Remark 7.6. The function

$$\widetilde{H}(y, s) = s \left(\frac{1}{\sqrt{6}} y \sqrt{1 - s} - 1 \right)^{-2}$$

also satisfies the boundary value problem, but since it has a pole at $y = \sqrt{6/(1 - s)}$, it cannot be a probability generating function for all $y > 0$.

Similarly, let

$$u(y, s) = 1 - \varphi(y, 1 - s), \tag{7.18}$$

then the differential equation (7.8) and the boundary conditions (7.9) become

$$\begin{aligned} \partial_{yy} u(y, s) &= u^2(y, s) \\ u(0, s) &= s \\ u(\infty, s) &= 0, \end{aligned} \tag{7.19}$$

which has the solution

$$u(y, s) = \frac{6s}{(y\sqrt{s} + \sqrt{6})^2}.$$

so by (7.18), the generating function $\varphi(y, s)$ is

$$\varphi(y, s) = 1 - u(y, 1 - s) = 1 - \frac{6(1 - s)}{(y\sqrt{1 - s} + \sqrt{6})^2}. \tag{7.20}$$

The equation (7.20) completely determines the distribution of N under $\mathbb{P}^y$. In Theorem 7.9 below, we will use (7.20) to provide explicit formulas for the probabilities $\mathbb{P}^y(N \geq k)$. But before doing so, we will show that (7.20) leads to the asymptotic formulas (7.3) and (7.4).

Theorem 7.7. *In the Moranian case,*

$$\lim_{k \to \infty} k^{\frac{3}{2}} \mathbb{P}^y(N \geq k) = \frac{y}{\sqrt{6\pi}} := C_5 y, \tag{7.21}$$

and

$$\lim_{k \to \infty} k^{\frac{5}{2}} \mathbb{P}^y(N = k) = \frac{3y}{2\sqrt{6\pi}} := C_6 y. \tag{7.22}$$

The proof will rely on the following theorem of Flajolet and Odlyzko [FO90].

Theorem 7.8 (Corollary 2, [FO90])**.** *Assume that the power series* $A(z) = \sum_{n=1}^{\infty} a_n z^n$ *defines an analytic function in* $|z| < 1$ *that has an analytic continuation to a* Pacman *domain*

$$D_{\circ b, \delta} := \{|z| < 1 + \delta\} \cap \{|\arg(z - 1)| > \beta\}$$

for some $\delta > 0$ and $0 \le \beta < \pi/2$. If

$$A(z) \sim K(1-z)^{\alpha} \quad \text{as } z \to 1 \text{ in } D_{\alpha,\delta}, \tag{7.23}$$

then as $n \to \infty$,

$$a_n \sim \frac{K}{\Gamma(-\alpha)} n^{-\alpha-1}. \tag{7.24}$$

provided $\alpha \notin \{0, 1, 2, \cdots\}$.

Proof of Theorem 7.7. The functions H and φ given by equations (7.17) and (7.20) have algebraic singularities at $s = 1$, but for each y have unique analytic continuations to the slit plane $\mathbb{C} \setminus \{1 < s < \infty\}$. Expansion around $s = 1$ yields

$$
\begin{aligned}
H(y, s) &= s / \left(\frac{1}{\sqrt{6}} y \sqrt{1-s} + 1 \right)^2 \\
&= 1 - \frac{2y}{\sqrt{6}} (1-s)^{\frac{1}{2}} + O(|1-s|).
\end{aligned}
\tag{7.25}
$$

and

$$
\begin{aligned}
\varphi(y, s) &= 1 - (1-s) / \left(1 + \frac{2y}{\sqrt{6}} (1-s)^{\frac{1}{2}} + \frac{y^2}{6} (1-s) \right) \\
&= s + \frac{2y}{\sqrt{6}} (1-s)^{\frac{3}{2}} + O(|1-s|^2).
\end{aligned}
\tag{7.26}
$$

Thus, the hypotheses of Theorem 7.8 are satisfied, and so the relations (7.21) and (7.22) follow. $\qquad\square$

Because H is a simple algebraic function of the argument s, its power series coefficients can be determined exactly. These provide an explicit formula for the distribution of N in the Moranian case.

Theorem 7.9. *In the Moranian case,*

$$\mathbb{P}^y(N \ge k) = A_k + B_k, \tag{7.27}$$

where

$$A_k = k(k + c^2) \frac{c^{2k-2}}{(c^2 - 1)^{k+1}}, \tag{7.28}$$

$$B_k = 2c \sum_{i+j=k, i,j=0,1,\cdots} \frac{(2i-3)!!}{2^i i!} \frac{jc^{2j-2}}{(c^2 - 1)^{j+1}}, \tag{7.29}$$

and

$$c = \frac{y}{\sqrt{6}}. \tag{7.30}$$

Proof. The equation (7.17) for the generating function $H(y, s)$ can be rewritten as

$$H(y, s) = s(1 + c\sqrt{1-s})^{-2}.$$

Expanding around $s = 0$ yields

$$
\begin{aligned}
\frac{1}{\left[1 + c\sqrt{1-s}\right]^2} &= \frac{\left[1 - c\sqrt{1-s}\right]^2}{\left[1 + c\sqrt{1-s}\right]^2 \left[1 - c\sqrt{1-s}\right]^2} \\
&= \frac{1 - 2c\sqrt{1-s} + c^2(1-s)}{\left[1 - c^2(1-s)\right]^2} \\
&= \frac{1}{\left[(1-c^2) + c^2 s\right]^2} - \frac{2c\sqrt{1-s}}{\left[(1-c^2) + c^2 s\right]^2} + c^2 \frac{(1-s)}{\left[(1-c^2) + c^2 s\right]^2} \\
&= I + II + III.
\end{aligned}
$$

For I, we have

$$I = \left[\frac{1}{(1-c^2) + c^2 s} \right]^2$$

$$= \frac{1}{(1-c^2)^2} \left(\frac{1}{1 - \frac{c^2}{c^2-1}s} \right)^2$$

$$= \frac{1}{(1-c^2)^2} \left[\left(\sum_{i=0}^{\infty} \left(\frac{c^2}{c^2-1} \right)^i s^i \right) \cdot \left(\sum_{j=0}^{\infty} \left(\frac{c^2}{c^2-1} \right)^j s^j \right) \right]$$

$$= \frac{1}{(1-c^2)^2} \sum_{k=0}^{\infty} (k+1) \left(\frac{c^2}{c^2-1} \right)^k s^k$$

$$= \sum_{k=0}^{\infty} (k+1) \frac{c^{2k}}{(c^2-1)^{k+2}} s^k.$$

(Here s must be taken to lie in a neighborhood of 0, to ensure that the geometric series converges. Since we are only interested in establishing formulas for the coefficients of the power series for $H(y, s)$, however, the radius of convergence is irrelevant, as long as it is positive.) Now let

$$\alpha_k = (k+1) \frac{c^{2k}}{(c^2-1)^{k+2}}; \tag{7.31}$$

then for III, we have

$$III = c^2(1-s) \cdot I$$

$$= c^2 \left(\sum_{k=0}^{\infty} \alpha_k s^k - \sum_{k=0}^{\infty} \alpha_k s^{k+1} \right)$$

$$= c^2 \left(\sum_{k=0}^{\infty} \alpha_k s^k - \sum_{k=1}^{\infty} \alpha_{k-1} s^k \right)$$

$$= c^2 \left(\alpha_0 + \sum_{k=1}^{\infty} (\alpha_k - \alpha_{k-1}) s^k \right)$$

$$\overset{(7.31)}{=} c^2 \left(\frac{1}{(c^2-1)^2} + \sum_{k=1}^{\infty} \frac{c^{2k-2}}{(c^2-1)^{k+1}} \left[\frac{c^2}{c^2-1}(k+1) - k \right] s^k \right)$$

$$= \sum_{k=0}^{\infty} \frac{c^{2k}}{(c^2-1)^{k+1}} \left[\frac{c^2}{c^2-1}(k+1) - k \right] s^k.$$

Next, set

$$\beta_k = \frac{c^{2k}}{(c^2-1)^{k+1}} \left[\frac{c^2}{c^2-1}(k+1) - k \right]; \tag{7.32}$$

then we find that $A_{k+1} = \alpha_k + \beta_k$ by direct computation.

It remains to show that

$$II = \sum_{k=0}^{\infty} B_{k+1} s^k.$$

By Newton's binomial formula,

$$(1-s)^{1/2} = \sum_{i=0}^{\infty} \binom{1/2}{i} (-s)^i. \tag{7.33}$$

Consequently,

$$
\begin{aligned}
II &= -2c \left(\sum_{i=0}^{\infty} \binom{1/2}{i} (-s)^i \right) \left(\sum_{j=0}^{\infty} \alpha_j s^j \right) \\
&= -2c \left(\sum_{i=0}^{\infty} \frac{(-1)^{i-1}(2i-3)!!}{2^i i!} (-1)^i s^i \right) \left(\sum_{j=0}^{\infty} \alpha_j s^j \right) \\
&= 2c \left(\sum_{i=0}^{\infty} \frac{(2i-3)!!}{2^i i!} s^i \right) \left(\sum_{j=0}^{\infty} \alpha_j s^j \right) \\
&= 2c \sum_{k=0}^{\infty} \left(\sum_{i+j=k, i,j=0,1\cdots} \frac{(2i-3)!!}{2^i i!} \alpha_j \right) s^k \\
&\overset{(7.31)}{=} \sum_{k=0}^{\infty} B_{k+1} s^k.
\end{aligned}
$$

$\square$

7.4 Proof of Theorem 7.1

In the general case, where the offspring distribution is assumed only to have mean 1, variance $\sigma^2 > 0$, and finite third moment, the behavior of the generating functions $\varphi(y,s)$ and $H(y,s)$ as $s \to 1-$ can be deduced from that in the Moranian case by comparison arguments, as in section 6. This derivation exploits the fact that for s near 1 the differential equation for H looks like that for the Moranian case, for which we have exact solutions. The result, the details of whose proof we defer to section 7.6, is as follows.

Lemma 7.10. *For each $y > 0$, the generating functions $\varphi(y,s)$ and $H(y,s)$ satisfy*

$$
\varphi(y,s) - s \sim \frac{2\sigma y}{\sqrt{6}} (1-s)^{3/2} \quad \text{and} \tag{7.34}
$$

$$
H(y,s) - 1 \sim \frac{-2\sigma y}{\sqrt{6}} (1-s)^{1/2} \tag{7.35}
$$

as $s \uparrow 1$.

Because the generating functions $\varphi(y,s)$ and $H(y,s)$ are defined by power series with nonnegative coefficients, the singular behavior of their derivatives can be deduced from Lemma 7.10, by the following elementary fact.

Lemma 7.11. *Let $A : [0,1] \to \mathbb{R}_+$ be an absolutely continuous, nonnegative, increasing function whose derivative A' is non-decreasing on $(0,1)$. If for some constants $C > 0$ and $\alpha \in (0,1)$,*

$$
A(1) - A(s) \sim C(1-s)^\alpha \quad \text{as } s \uparrow 1, \tag{7.36}
$$

then

$$
A'(s) \sim C\alpha (1-s)^{\alpha-1} \quad \text{as } s \uparrow 1. \tag{7.37}
$$

Proof. Since A is absolutely continuous,

$$
A(s_1) - A(s_0) = \int_{s_0}^{s_1} A'(t)\, dt \quad \text{for all } 0 < s_0 < s_1 \leq 1.
$$

Suppose that for some $\delta > 0$ there were a sequence $s_n \to 1-$ along which $A'(s_n) < C\alpha(1-\delta)(1-s_n)^{\alpha-1}$. Since A' is non-decreasing, it would then follow that $A'(s) <$

$C\alpha(1-\delta)(1-s_n)^{\alpha-1}$ for all $s < s_n$, and so for any $\varepsilon > 0$,

$$A(s_n) - A(s_n(1-\varepsilon)) \le C\alpha(1-\delta)(1-s_n)^{\alpha-1}\varepsilon.$$

But this would lead to a contradiction of the hypothesis (7.36) provided ε is sufficiently small relative to δ. A similar argument shows that it is impossible for $A'(s_n) > C\alpha(1+\delta)(1-s_n)^{\alpha-1}$ along a sequence $s_n \to 1-$. $\qquad\square$

Corollary 7.12. *For each $y > 0$, as $s \to 1-$,*

$$\frac{d}{ds}H(y,s) = \sum_{k=1}^{\infty} k\mathbb{P}^y(N \ge k)s^{k-1} \sim \frac{\sigma y}{\sqrt{6}}(1-s)^{-1/2}. \tag{7.38}$$

Theorem 7.1 follows directly from Corollary 7.12 and Karamata's Tauberian theorem (cf. [BGT87], Corollary 1.7.3), which we now recall.

Theorem 7.13. *Let $A(z) = \sum a_n z^n$ be a power series with nonnegative coefficients a_n and radius of convergence 1. If, for some constants $C, \beta > 0$,*

$$A(s) \sim C/(1-s)^{\beta} \quad \text{as } s \uparrow 1, \tag{7.39}$$

then as $n \to \infty$.

$$\sum_{k=1}^{n} a_k \sim Cn^{\beta}/\Gamma(1+\beta). \tag{7.40}$$

$\qquad\square$

7.5 Proof of Theorem 7.2

In this section we assume that the offspring distribution satisfies the conditions enumerated in Theorem 7.2, in particular, that the function $\kappa(z)$ defined by (7.2) has no zeros z such that $0 < |z| \le 2$. We will once again make use of the Flajolet–Odlyzko theorem (Theorem 7.8), which requires (a) that the function defined by the power series in question should vary regularly as functions of s near the singularity $s = 1$, and (b) that this function has an analytic continuation to a Pacman domain. Lemma 7.10 implies that $\varphi(y,s)$ and $H(y,s)$ vary regularly as $s \to 1-$ along the real axis from below. The following lemma ensures that they have analytic continuations to a Pacman domain, and that the regular variation persists in this region.

Lemma 7.14. *If the offspring distribution satisfies the hypotheses of Theorem 7.2, then for each $y > 0$ the generating functions $H(y,s)$ and $\varphi(y,s)$ have analytic continuations $H(y,z)$ and $\varphi(y,z)$ to a slit disk $\{|z| < 1+\delta\} \setminus \{1 \le z < 1+\delta\}$, and the relations (7.34) and (7.35) hold as $s \to 1$ in the slit domain.*

Proof. In view of the relation (7.15), to prove that the function $H(y,s)$ has an analytic continuation it suffices to show that the probability generating function $\varphi(y,s)$ can be analytically continued, or alternatively that the function

$$u(y,z) := 1 - \varphi(y, 1-z)$$

has an analytic continuation to a slit disk $\{|1-z| < 1+\delta\} \setminus \{1 \le 1-z < 1+\delta\}$. Recall (Proposition 3.1) that for all *real* z in the interval $|1-z| < 1$ the function $u(y,z)$ satisfies the boundary value problem

$$\begin{aligned}
\partial_{yy}u(y,z) &= h(u(y,z)), \\
u(0,z) &= z, \\
u(\infty,z) &= 0.
\end{aligned} \tag{7.41}$$

Integration of this differential equation , as in section 3 of [SF79], leads to the equation

$$\int_{u(x,z)}^{z} \frac{dy}{\sqrt{\kappa(y)}} = x \quad \text{where} \quad \kappa(y) := 2 \int_{0}^{y} h(y') \, dy'. \tag{7.42}$$

(The upper limit of integration is z because $u(0,z) = z$.)

It is easily checked that in any region of the $z-$plane where the equation (7.42) has a solution $u(x,z)$, the solution will satisfy the boundary value problem (7.41). Thus, to prove the first assertion of the lemma it will suffice to show that the integral equation (7.42) implicitly defines $u(x,z)$ as an analytic function of z for z in a slit disk.

Define a function of two complex variables z, w by

$$G(w,z) := \int_{w}^{z} \frac{dy}{\sqrt{\kappa(y)}}.$$

This function is analytic in z and w in any domain $\hat{D} \subset \mathbb{C}^2$ such that there exists a simply connected domain $D \subset \mathbb{C}$ in which $1/\sqrt{\kappa}$ is analytic and such that for any $z, w \in \mathbb{C}$ there is a path from z to w in D. Moreover,

$$\frac{\partial G}{\partial z} = 1/\sqrt{\kappa(z)} \quad \text{and} \quad \frac{\partial G}{\partial w} = 1/\sqrt{\kappa(w)}.$$

Therefore, by the complex implicit function theorem, the equation $G(w(z), z) = x$ defines $w(z)$ as an analytic function of z in a neighborhood of any solution $G(w_0, z_0) = x$ where $1/\sqrt{\kappa(w_0)} \neq 0$.

The function $\varphi(x,z)$ is analytic in the unit disk and, since it is a probability generating function, satisfies $|\varphi| < 1$. Consequently, the function $u(x,z) = 1 - \varphi(x, 1-z)$ is analytic in the disk $D_1 := \{|1 - z| < 1\}$ and satisfies $u(x,z) \in D_1$ for all $z \in D_1$. By hypothesis, the functions $h(z)$ and $\kappa(z)$ have analytic continuations to a disk of radius $2 + \delta > 2$ centered at 0, and the only zero of $\kappa(z)$ in this disk is at $z = 0$. Therefore, the functional equation

$$G(u(x,z), z) = x \tag{7.43}$$

holds for all $z \in D_1$.

We claim that for any point $\xi \in \partial D_1$ *except* $\xi = 0$, the function $u(x,z)$ converges as $z \to \xi$ to a value $u(x,\xi)$ such that $u(x,\xi) \in D_1$. This is clearly equivalent to the assertion that for any point $e^{i\theta} \neq 1$ of the unit circle, the function $\varphi(x,z)$ converges as $z \to e^{i\theta}$ to a value $\varphi(x, e^{i\theta})$ of absolute value less than 1. To see that this is so, recall that $\varphi(x,z)$ is the probability generating function of the random variable N under P^x. It is easily seen (for instance, using the discrete Brownian snake construction) that for any $x > 0$,

$$P^x\{N = k\} > 0 \quad \text{for every} \quad k = 0, 1, 2, \ldots.$$

But this implies that $|E^x e^{i\theta N}|$ is less than 1 for every $\theta \in [-\pi, \pi] \setminus \{0\}$.

It now follows that the functional equation (7.43) extends by continuity to all $z \in \partial D_1$ except $z = 0$, and that at any such boundary point, $u(x,z) \neq 0$. Hence, the function $1/\sqrt{\kappa(w)}$ is analytic and nonzero in a neighborhood of $w = u(x,z)$. This implies that $u(x,z)$ has an analytic continuation to a neighborhood of every $z \in \partial D_1$ except the singular point $z = 0$.

Next, we must prove that $u(x,z)$ has an analytic continuation to a slit neighborhood of $z = 0$. Since $\varphi(x,1) = 1$, the function $u(x,z)$ converges to 0 as $z \to 0$ in the disk $\{|1 - z| < 1\}$; thus, the function $\kappa(u)$ approaches 0. Our assumptions on the offspring

distribution imply that

$$h(z) = \sigma^2 z^2 + \sum_{j=3}^{\infty} h_j z^j \quad \text{and hence}$$

$$\kappa(z) = \frac{2}{3}\sigma^2 z^3 + \sum_{j=4}^{\infty} k_j z^j;$$

consequently,

$$\frac{1}{\sqrt{\kappa(z)}} = K z^{-3/2}(1 + R(z))$$

where $K = \sqrt{3/2\sigma^2}$ and $R(z)$ is an analytic function in some neighborhood of $z = 0$ and satisfies $R(0) = 0$. Now the integral equation (7.42) and the implicit function theorem imply that, for fixed $x > 0$, the function $u = u(x, z)$ satisfies the differential equation

$$\frac{du}{dz} = \frac{\sqrt{\kappa(u)}}{\sqrt{\kappa(z)}} = \frac{u^{3/2}}{z^{3/2}} \frac{(1 + R(z))}{(1 + R(u))}$$

for all z in a domain $\{|1 - z| < 1\} \cap \{|z| < \delta\}$. Using the analyticity of R, we conclude that

$$u^{-1/2}\left(1 + \sum_{k=1}^{\infty} b_k u^k\right) = z^{-1/2}\left(1 + \sum_{k=1}^{\infty} b_k z^k\right) + C,$$

where C is a constant of integration. Squaring both sides exhibits u as a meromorphic function of $\sqrt{z}$. This shows that u has an analytic continuation to a slit disk $\{|z| < \delta\} \setminus (-\delta, 0]$, and so it follows, by the relations $u(x, z) = 1 - \varphi(x, 1 - z)$ and equation (7.15), that $\varphi(x, z)$ and $H(x, z)$ admit analytic continuations to a slit disk $\{|1 - z| < \delta'\} \setminus [1, 1 + \delta']$.

Finally, we must prove that the relations (7.34) and (7.35) hold as $z \to 1$ in the extended domain of definition of the functions $\varphi(x, z)$ and $H(x, z)$. But the analytic continuation argument above shows that, in a slit disk centered at $z = 1$, the functions $\varphi(x, z) - z$ and $H(x, z) - 1$ are meromorphic functions of $\sqrt{1 - z}$, and hence have Puiseux expansions in powers of $(1 - z)^{1/2}$. Since (7.34) and (7.35) hold as $s \uparrow 1$, it follows that they persist in the slit disk.

$$\square$$

Proof of Theorem 7.2. Lemma 7.14 implies that the generating functions $\varphi(y, s)$ and $H(y, s)$ meet the requirements of the Flajolet-Odlyzko theorem (Corollary 7.8). Therefore, Theorem 7.2 follows from relation (7.24) (since $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$ and $\Gamma(-\frac{3}{2}) = \frac{4}{3}\sqrt{\pi}$). $\quad \square$

7.6 Proof of Lemma 7.10

The strategy is similar to that of section 6. As $s \to 1$,

$$u(y, 1 - s) \to 0$$

$$\frac{1 - s}{s} H(y, s) \to 0,$$

and so for s near 1 the differential equations (7.41) and (7.11) for $u(y, 1 - s)$ and $H(y, s)$ have forcing terms that are nearly quadratic. Taylor expansion of h shows that for any $\delta > 0$ there exists $\varepsilon > 0$ such that for $1 - \varepsilon < s < 1$,

$$a_- u^2(y, 1 - s) \leq h(u(y, 1 - s)) \leq a_+ u^2(y, 1 - s) \quad \text{and}$$

$$a_-\left[\frac{1 - s}{s} H(y, s)\right]^2 \leq h\left(\frac{1 - s}{s} H(y, s)\right) \leq a_+\left[\frac{1 - s}{s} H(y, s)\right]^2,$$

where $a_\pm$ are defined in (6.14). Let $u_\pm(y, 1-s)$ and $H_\pm(y,s)$ satisfy the boundary value problems

$$\begin{cases} \partial_{yy} u_\pm(y, 1-s) = a_\pm u_\pm^2(y, 1-s) \\ \quad u_\pm(0, s) = s \\ \quad u_\pm(\infty, s) = 0, \end{cases} \quad \begin{cases} \partial_{yy} H_\pm(y,s) = a_\pm \dfrac{1-s}{s} H_\pm^2(y,s) \\ \quad H_\pm(0, s) = s \\ \quad H_\pm(\infty, s) = 0, \end{cases}$$

and set $\varphi_\pm(y,s) = 1 - u_\pm(y, 1-s)$. By the same argument as in Corollary 6.4,

$$\begin{aligned} u_+(y, 1-s) &\leq u(y, 1-s) \leq u_-(y, 1-s) \\ \varphi_-(y,s) &\leq \varphi(y,s) \leq \varphi_+(y,s) \\ H_+(y,s) &\leq H(y,s) \leq H_-(y,s). \end{aligned} \tag{7.44}$$

Define re-scaled versions

$$\begin{aligned} \widehat{u}_\pm(y, 1-s) &= u_\pm\left(\frac{y}{\sqrt{a_\pm}}, 1-s\right) \\ \widehat{H}_\pm(y,s) &= H_\pm\left(\frac{y}{\sqrt{a_\pm}}, s\right); \end{aligned} \tag{7.45}$$

these satisfy the boundary value problems

$$\begin{cases} \partial_{yy} \widehat{u}_\pm(y, 1-s) = \widehat{u}_\pm^2(y, 1-s) \\ \quad \widehat{u}_\pm(0, s) = s \\ \quad \widehat{u}_\pm(\infty, s) = 0, \end{cases} \quad \begin{cases} \partial_{yy} \widehat{H}_\pm(y,s) = \dfrac{1-s}{s} \widehat{H}_\pm^2(y,s) \\ \quad \widehat{H}_\pm(0, s) = s \\ \quad \widehat{H}_\pm(\infty, s) = 0, \end{cases}$$

which are the same as in the Moranian case. Hence, $\widehat{\varphi}_\pm(y,s) = 1 - \widehat{u}_\pm(y, 1-s)$ and $\widehat{H}_\pm(y,s)$ have the same asymptotics as (7.26) and (7.25):

$$\begin{aligned} \widehat{\varphi}_\pm(y,s) &= s + \frac{2y}{\sqrt{6}}(1-s)^{\frac{3}{2}} + O\left(|1-s|^2\right) \\ \widehat{H}_\pm(y,s) &= 1 - \frac{2y}{\sqrt{6}}(1-s)^{\frac{1}{2}} + O\left(|1-s|\right) \end{aligned}$$

as $s \to 1$. Hence, applying (7.45) yields

$$\begin{aligned} \varphi_\pm(y,s) &= s + \frac{2y\sqrt{a_\pm}}{\sqrt{6}}(1-s)^{\frac{3}{2}} + O\left(|1-s|^2\right) \\ H_\pm(y,s) &= 1 - \frac{2y\sqrt{a_\pm}}{\sqrt{6}}(1-s)^{\frac{1}{2}} + O\left(|1-s|\right). \end{aligned}$$

Finally, as $\delta \to 0$ we have $a_\pm \to \sigma^2$, and so (7.44) implies

$$\begin{aligned} \varphi(y,s) - s &\sim \frac{2\sigma y}{\sqrt{6}}(1-s)^{\frac{3}{2}} \\ H(y,s) - 1 &\sim \frac{-2\sigma y}{\sqrt{6}}(1-s)^{\frac{1}{2}}. \end{aligned}$$

as $s \to 1$.

$\square$

References

[AS72] , M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, New York, Dover Publications, ISBN 978-0-486-61272-0, 1972.

[A] D. Aldous, Power laws and killed branching random walks, Available at `http://www.stat.berkeley.edu/~aldous/Research/OP/brw.html`

[AB11] L. Addario-Berry and N. Broutin, Total progeny in killed branching random walk, *Probab. Theory Relat. Fields*, Vol 151, No. 1-2, pp. 265–295, 2011. MR-2834719

[AHZ13] E. Aidekon, Y. Hu and O. Zindy, The precise tail behavior of the total progeny of a killed branching random walk, *Ann. Probab.*, Vol 41, No. 6, pp. 3786–3878. 2013. MR-3161464

[AN72] K. Athreya and P. Ney, *Branching processes*, Springer-Verlag, New York, 1972. Die Grundlehren der mathematischen Wissenschaften, Band 196. MR-0373040

[BBS13] J. Berestycki, N. Berestycki and J. Schweinsberg, The genealogy of branching Brownian motion with absorption, *Ann. Probab.*, Vol. 41, No. 2, 2013. MR-3077519

[BBHM15] J. Berestycki, E. Brunet, S. Harris, and P. Milos, Branching Brownian motion with absorption and the all-time minimum of branching Brownian motion with drift. Preprint, available at arXiv:1506.01429.

[BGT87] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1987. MR-0898871

[B78] M. Bramson, Maximal displacement of branching Brownian motion, *Comm. Pure. Appl. Math.*, Vol 31, No. 5, pp. 531–581, 1978. MR-0494541

[CG76] K. Crump and J. Gillespie, The dispersion of a neutural allele considered as a branching process. *J. Appl. Probability*, Vol. 13, No. 2, pp. 208–218, 1976. MR-0408877

[FO90] P. Flajolet and A. Odlyzko, Singularity analysis of generating functions, *SIAM J. Disc. Math.*, 1990. MR-1039294

[K78] H. Kesten, Branching brownian motion with absorbption, *Stochastic Processes Appl.*, Vol. 7, No. 1, pp. 9–47, 1978. MR-0494543

[K84] N. Koblitz, *Introduction to elliptic curves and modular forms*, Springer-Verlag, 1984. MR-0766911

[KPP37] A. Kolmogorov, I. Petrovskii and N. Piscounov, A study of the diffusion equation with increase in the amount of substance, and its application to a biological problem, *Selected Works of A.N. Kolmogorov I*, pp. 248–270, Kluwer, 1991, Translated by V.M. Volosov from *Bull. Moscow Univ. Math. Mech.*, Vol. 1, pp. 1–25, 1937.

[KK03] A. Korostelev and O. Korosteleva, Limit theorem for the spread of branching diffusion with stabilizing drift, *Stochastic Anal. Appl.*, Vol. 21, No. 3, 611–641, 2003. MR-1978237

[KK04] A. Korostelev and O. Korosteleva, Spread of branching diffusion: sharp asymptotics, *Stoch. Stoch. Rep.*, Vol. 76, No. 6, 509–516, 2004. MR-2100019

[K04] O. Korosteleva, The spread of branching diffusion under stabilizing drift, *Stoch. Stoch. Rep.*, Vol. 76, No. 3, 179–189, 2004. MR-2072379

[L09] S. Lalley, Spatial epidemics: critical behavior in one dimension, *Probab. Theorey Related Fields*, Vol. 144, No. 3–4, pp. 429–469, 2009. MR-2496439

[LS87] S. Lalley and T. Sellke, A conditional limit theorem for the frontier of a branching Brownian motion. *Ann. Probab.*, Vol. 15, No. 3, pp. 1052–1061, 1987. MR-0893913

[LS88] S. Lalley and T. Sellke, Traveling waves in inhomogeneous branching Brownian motions. I, *Ann. Probab.*, Vol. 16, No. 3, 1051–1062, 1988. MR-0942755

[LS89] S. Lalley and T. Sellke, Traveling waves in inhomogeneous branching Brownian motions. II, *Ann. Probab.*, Vol. 17, No. 1, 116–127, 1989. MR-0972775

[LS92] S. Lalley and T. Sellke, Limit theorems for the frontier of a one-dimensional branching diffusion, *Ann. Probab.*, Vol. 20, No. 3, 1310–1340, 1992. MR-1175264

[L90-1] T. Lee, Conditioned limit theorems of stopped critical branching bessel processes, *The Annals of Probability*, Vol. 18, No. 1, 272–289, 1990. MR-1043948

[L90-2] T. Lee, Some Limit Theorems for Critical Branching Bessel Processes, and Related Semilinear Differential Equations, *Probab. Th. Rel. Fields*, Vol. 84, No. 4, 505–520, 1990. MR-1042063

[L99] J. LeGall, *Spatial branching processes, random snakes and partial differential equations*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1999. MR-1714707

[M13] P. Maillard, The number of absorbed individuals in branching Brownian Motion with a barrier, *Annales de l'Institut Henri Poincaré - Probabilités et Statistiques*, Vol. 49, No. 2, 428–455, 2013. MR-3088376

[M69] H. McKean, *Stochastic integrals*, Probability and Mathematical Statistics, No. 5, Academic Press, New York-London, 1969. MR-0247684

[M75] H. McKean, Application of Brownian Motion to the equation of Kolmogorov-Petrovskii-Piskunov, *Comm. Pure. Appl. Math*, Vol. 28, No. 3, pp. 323–331, 1975 MR-0400428

[MM99] H. McKean and V. Moll, *Elliptic curves*, Cambridge University Press, 1999. MR-1471703

[N88] J. Neveu, Multiplicative martingales for spatial branching processes, *Seminar on Stochastic Processes, 1987*, 223–242, 1988. MR-1046418

[S76] S. Sawyer, Branching diffusion processes in population genetics, *Adv. Appl. Prob.*, Vol. 8, No. 4, pp. 659–689, 1976. MR-0432250

[SF79] S. Sawyer and J. Fleischman, Maximum geographic range of a mutant alle considered as a subtype of branching random field, *Proc. Natl. Acad. Sci.*, Vol. 76, No. 2, pp 872–875, 1979.

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