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Gaussian asymptotics for a non-linear Langevin type equation driven by a symmetric α-stable Lévy noise

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Abstract

Consider the dynamics of a particle whose speed satisfies a one-dimensional stochastic differential equation driven by a small symmetric α-stable Lévy process in a potential of the form a power function of exponent $\beta+1$. Two cases are studied: the noise could be path continuous, namely a standard Brownian motion, if $\alpha = 2$, or pure jump Lévy process, if $\alpha \in (0,2)$. The main goal is to study a scaling limit of the position process with this speed, and one proves that the limit is Brownian in either case. This result is a generalization in some sense of the quadratic potential case studied recently by Hintze and Pavlyukevich [9].

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1 Introduction

In this paper we consider the one-dimensional and non-linear Langevin type equation driven by a symmetric α-stable Lévy process. Let us denote by x_t^ε the one-dimensional process describing the position of a particle at time $t \geq 0$, having the speed v_t^ε

$$x_t^\varepsilon = x_0 + \int_0^t v_s^\varepsilon \mathrm{d}s,\ t \geq 0, \tag{1.1}$$

and such that v_t^ε is a small symmetric α-stable Lévy process in a potential $U(x) := \frac{2}{\beta+1}|x|^{\beta+1}$,

$$\mathrm{d}v_t^\varepsilon = \varepsilon \mathrm{d}\ell_t - \frac{1}{2}U'(v_t^\varepsilon)\mathrm{d}t, \quad v_0^\varepsilon = v_0. \tag{1.2}$$

In other words v_t^ε verifies the following integral equation

$$v_t^\varepsilon = v_0 + \varepsilon\ell_t - \int_0^t \mathrm{sgn}(v_s^\varepsilon)|v_s^\varepsilon|^\beta \mathrm{d}s,\ t \geq 0. \tag{1.3}$$

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Here $\beta > -1$ and $\{\ell_t : t \geq 0\}$ is an α-stable Lévy process, $\alpha \in (0, 2]$. The 2-stable Lévy process is the standard Brownian motion $\{b_t : t \geq 0\}$ which is a path continuous process. If $\alpha \in (0, 2)$, the Lévy process is a pure jump process with càdlàg paths and the jump measure is given by

$$\nu(\mathrm{d}z) = |z|^{-1-\alpha}\big[a_- \mathbb{1}_{\{z<0\}} + a_+ \mathbb{1}_{\{z>0\}}\big]\,\mathrm{d}z.$$

A Lévy process ℓ is α-stable if and only if the property of self-similarity holds (see Proposition 13.5 in [14], p. 71). In fact this means that the processes $\{\ell_t : t \geq 0\}$ and $\{c^{-1/\alpha}\ell_{ct} : t \geq 0\}$ have the same law, for any $c > 0$. The α-stable Lévy process is said to be symmetric if $a_- = a_+$. In this paper, we will consider only symmetric α-stable Lévy processes.

The case of a harmonic potential when the speed is an Ornstein-Uhlenbeck process, was already considered by Hintze and Pavlyukevich [9]. The dynamics of the integrated Ornstein-Uhlenbeck process appears in some financial mathematics (volatility) models (see for instance Barndorff-Nielsen and Shephard [2]) or in models in physics of plasma (see for instance Chechkin, Gonchar and Szydlowski [5]). In the paper by Hintze and Pavlyukevich, the authors study the asymptotic behaviour of the integrated Ornstein-Uhlenbeck and prove that this process converges weakly, as $\varepsilon \to 0$, to the underlying α-stable Lévy process. In particular, when the driving process is a Brownian motion ($\alpha = 2$), the asymptotic behaviour is Gaussian. In [9], asymptotics of the first exit time from an interval are deduced. Several papers in physics pointed out that new interesting phenomena appear when one considers super-harmonic potentials (see for instance Metzler, Chechkin, Klafter [11]).

Our goal is to answer the same question in the situation of a super-harmonic potential: what is the asymptotic behaviour of the position process x_t^ε, as $\varepsilon \to 0$? On the one hand, the non-linear case introduces new technical difficulties, mainly since the solution is no longer explicit. Indeed, this fact was essential to prove weak convergence in the linear case. On the other hand, different conditions on the two parameters α and β will generate different asymptotics for the position process. The intuition suggests that the big jumps should be compensated by a over-damped negative drift, and that small jumps should have some regularizing effect. In the present paper, we answer the question by showing that for α and β in some unbounded domain, the position process x_t^ε will behave as a Brownian motion when ε goes to 0. In other words, we get Gaussian asymptotic behaviour even if α is smaller than 2, provided that β is not very small, more precisely if $\beta + \frac{\alpha}{2} > 2$. When α and β are somehow "small" the previous heuristic fails. To get convergence toward a stable process, one needs to change the approach and other technical difficulties appear. This case would be presented in a forthcoming work (see [8]).

2 Main result

To state the main result of the present paper, we will perform some scaling transformations. Without loss of generality, we can assume that the initial position is the origin $x_0 = 0$. Moreover we will assume that the initial speed vanishes $v_0 = 0$, contrary to the linear case. By using the self-similarity, it is clear that the process $\{\mathcal{L}_t^\varepsilon := \varepsilon\ell_{\varepsilon^{-\alpha}t} : t \geq 0\}$ is also an α-stable process. Let us denote, for $t \geq 0$,

$$\mathcal{X}_t^\varepsilon := x_{\varepsilon^{-\alpha}t}^\varepsilon \quad \text{and} \quad \mathcal{V}_t^\varepsilon := v_{\varepsilon^{-\alpha}t}^\varepsilon \tag{2.1}$$

satisfying, respectively,

$$\mathcal{X}_t^\varepsilon = \frac{1}{\varepsilon^\alpha}\int_0^t \mathcal{V}_s^\varepsilon \mathrm{d}s \quad \text{and} \quad \mathcal{V}_t^\varepsilon = \mathcal{L}_t^\varepsilon - \frac{1}{\varepsilon^\alpha}\int_0^t \mathrm{sgn}(\mathcal{V}_s^\varepsilon)|\mathcal{V}_s^\varepsilon|^\beta \mathrm{d}s. \tag{2.2}$$

To simplify the notations, all along the paper we will set

$$\theta = \theta_{\alpha,\beta} := \frac{\alpha}{\alpha + \beta - 1} > 0, \quad \text{provided that } \alpha + \beta - 1 > 0. \tag{2.3}$$

Moreover we introduce

$$L_t^\varepsilon := \frac{\mathcal{L}_{t\,\varepsilon^{\alpha\theta}}^\varepsilon}{\varepsilon^\theta} = \frac{\ell_{t\,\varepsilon^{-(\beta-1)\theta}}}{\varepsilon^{(\beta-1)\theta/\alpha}} \quad \text{and} \quad V_t^\varepsilon := \frac{\mathcal{V}_{t\,\varepsilon^{\alpha\theta}}^\varepsilon}{\varepsilon^\theta}, \tag{2.4}$$

By self-similarity again, L^ε is distributed as an α-stable Lévy process and we have

$$\mathcal{X}_t^\varepsilon = \varepsilon^{(2-\beta)\theta}\int_0^{t\varepsilon^{-\alpha\theta}} V_s^\varepsilon \mathrm{d}s \quad \text{and} \quad V_t^\varepsilon = L_t^\varepsilon - \int_0^t \operatorname{sgn}(V_s^\varepsilon)|V_s^\varepsilon|^\beta \mathrm{d}s. \tag{2.5}$$

Let us note that if $\alpha = 2$, all previous computations hold true with ℓ, $\mathcal{L}$ or L replaced respectively by b, $\mathcal{B}$ or B a standard Brownian motion.

Our main result is the following:

Theorem 2.1. *1. (Brownian driving noise) Assume that $\alpha = 2$, $\beta > -1$ and recall that $\theta = \frac{2}{\beta+1}$. There exists a positive constant $\kappa_{2,\beta}$ such that the process*

$$\{\varepsilon^{(\beta-1)\theta}x^\varepsilon_{\varepsilon^{-2}t} : t \geq 0\} = \{\varepsilon^{(\beta-1)\theta}\mathcal{X}_t^\varepsilon : t \geq 0\} \tag{2.6}$$

converges in distribution, in the space of continuous functions $\mathrm{C}([0,\infty))$ endowed with the uniform topology, to a Brownian motion process with variance $\kappa_{2,\beta}$, as $\varepsilon \to 0$. The constant $\kappa_{2,\beta}$ has the integral representation given in (3.9) *below.*

2. (symmetric stable driving noise) Assume that $\alpha \in (0,2)$, $\beta + \frac{\alpha}{2} > 2$ and recall that $\theta = \frac{\alpha}{\alpha+\beta-1}$. There exists a positive constant $\kappa_{\alpha,\beta}$ such that the process

$$\{\varepsilon^{(\beta+\frac{\alpha}{2}-2)\theta}x^\varepsilon_{\varepsilon^{-\alpha}t} : t \geq 0\} = \{\varepsilon^{(\beta+\frac{\alpha}{2}-2)\theta}\mathcal{X}_t^\varepsilon : t \geq 0\} \tag{2.7}$$

converges in distribution, in the space of continuous functions $\mathrm{C}([0,\infty))$ endowed with the uniform topology, to a Brownian motion process with variance $\kappa_{\alpha,\beta}$, as $\varepsilon \to 0$. The constant $\kappa_{\alpha,\beta}$ has the integral representation given in (4.30) *below.*

Remark 2.2. 1. Hypotheses $\alpha \in (0,2)$ and $\beta + \frac{\alpha}{2} > 2$ imply that $\beta > 1$, in other words the drift is over-damped. In particular we have $\theta \in (0,1)$.

2. If the driving noise is the Brownian motion $\alpha = 2$, the normalizing factor behaves differently following with the position of β with respect to 1. If $\beta = 1$, the position process X^ε converges in distribution to a standard Brownian motion, see also Remark 3.3 below.

3. The case when $\beta + \frac{\alpha}{2} = 2$ should be considered as critical for some phase transition from Gaussian to stable case. It should be reasonable that there is some continuity but the proof seems more delicate since natural integrability conditions are not fulfilled (see the method of proof described below).

4. As an application one can find asymptotics of the first exit time from an interval: Corollary 2.1, p. 269, in [9] applies.

Let us explain the method of the proof and the organization of the paper. It is a simple observation that

$$\varepsilon^{\theta(\beta+\frac{\alpha}{2}-2)}\mathcal{X}_t^\varepsilon = \varepsilon^{\frac{\alpha\theta}{2}}\int_0^{t\varepsilon^{-\alpha\theta}} V_s^\varepsilon \mathrm{d}s\,,$$

hence, since L is a symmetric α-stable process, V is zero mean and Theorem 2.1 is a second order type ergodic theorem. Let us only note that in the asymmetric case, a drift term appears in the Lévy-Itô decomposition of the driving noise. Consequently, the expression of the infinitesimal generator of V will be different and finally a first order term should appear in the limit, for an asymmetric situation. By using the stochastic calculus, we will show that the latter quantity is the sum of a square integrable martingale, provided that $\beta + \frac{\alpha}{2} > 2$ for the case $\alpha \in (0,2)$, and a term which tends in probability toward 0, as $\varepsilon \to 0$. The result is then obtained by using the functional central limit theorem for martingales and the continuous-mapping theorem. In the critical case $\beta + \frac{\alpha}{2} = 2$ and $\alpha \in (0,2)$, the L^2-integrability fails. We point out that the critical case for the Brownian noise is the case studied in [9].

In the next section, we consider the case when the driving noise is the Brownian motion: in this case computations are performed by using Itô's calculus and are more explicit. For instance, the constant $\kappa_{2,\beta}$ can be written in terms of the scale function and the speed measure. In Section 3, we follow the same structure of the proof for a pure jump driving noise. Computations are more technical and new ideas are employed: for instance, we need to find and use a Lyapunov function which allows to perform the same reasoning by using the Itô-Lévy calculus. We give in the appendix two technical proofs.

3 Brownian motion driving noise

Let us note that for the case, $\alpha = 2$, $\{b_t : t \geq 0\}$ is the standard one-dimensional Brownian motion. If $\beta > -1$, then $\theta = \theta_{2,\beta} = \frac{2}{\beta+1} > 0$ and we set $\mathcal{B}_t^\varepsilon := \varepsilon b_{\varepsilon^{-\alpha} t}$,

$$B_t^\varepsilon := \frac{\mathcal{B}^\varepsilon_{t\,\varepsilon^{2\theta}}}{\varepsilon^\theta} = \frac{b_{t\,\varepsilon^{(1-\beta)\theta}}}{\varepsilon^{(\beta-1)\theta/2}}, \quad \text{and} \quad V_t^\varepsilon := \frac{\mathcal{V}^\varepsilon_{t\,\varepsilon^{2\theta}}}{\varepsilon^\theta}. \tag{3.1}$$

Recall also that

$$\mathcal{X}_t^\varepsilon = \varepsilon^{\frac{2(2-\beta)}{(\beta+1)}} \int_0^{t\varepsilon^{-4/(\beta+1)}} V_s^\varepsilon \mathrm{d}s \quad \text{and} \quad V_t^\varepsilon = B_t^\varepsilon - \int_0^t \operatorname{sgn}(V_s^\varepsilon)|V_s^\varepsilon|^\beta \mathrm{d}s\,. \tag{3.2}$$

B^ε is distributed as the standard Brownian motion so, to simplify the notation, we will suppress the index ε, as well as for V^ε.

3.1 The scaled speed process

3.1.1 Existence and uniqueness

Thanks to (3.1_2), $\mathcal{V}^\varepsilon$ and V are connected to each other. If $\beta \geq 1$, the drift coefficient in (3.2_2) is a locally Lipschitz function hence by well known results (see, for instance, Theorem 12.1, p. 132 in [12]), we get a path-wise unique strong solution V to equation (3.2_2), whereas if $-1 < \beta < 1$, Girsanov's theorem gives the existence of a weak solution to equation (3.2_2). For both situations, the solution is defined up to an explosion time τ_e, but it is not difficult to prove that $\tau_e = \infty$ a.s. by using Theorem 10.2.1, p. 254, in [15] and a convenient Lyapunov function. For instance, we can choose as a Lyapunov function $h(x) = 1 + x^2$ for all $|x| \geq 1$, $h(x) = 1$ for all $|x| \leq 1/2$, and $h \geq 1$. Introduce the scale function and the speed measure associated to the diffusion V given by (3.2_2)

$$s_\beta(x) := \int_0^x \mathrm{e}^{-c_\beta(y)}\mathrm{d}y \quad \text{and} \quad m_\beta(\mathrm{d}x) := 2\mathrm{e}^{c_\beta(x)}\mathrm{d}x, \quad \text{where} \quad c_\beta(x) := -\frac{2}{\beta+1}|x|^{\beta+1}\,. \tag{3.3}$$

Since $\int_0^\infty m_\beta([0,x])\mathrm{e}^{-c_\beta(x)}\mathrm{d}x = \infty$, by Theorem 52.1, p. 297 in [12], the path-wise uniqueness holds for the equation (3.2_2). Finally, there exists a path-wise unique strong solution V of the equation (3.2_2).

3.1.2 Convergence in probability

The main result of this section is the following:

Proposition 3.1. *1. Fix $p \geq 4$ and $T > 0$. There exists a positive constant $C'_{p,\beta}$ such that*

$$\mathbb{E}\Big[\big(\sup_{0\leq t\leq T}|V_{t\,\varepsilon^{-2\theta}}|\big)^p\Big] \leq C'_{p,\beta}\, T^2\, \varepsilon^{-4\theta}. \tag{3.4}$$

2. As $\varepsilon \to 0$, $\{\mathcal{V}_t^\varepsilon : t \geq 0\}$ converges to 0 in probability uniformly on each compact time interval.

Proof. Before proving the first part we provide a simpler estimate than (3.4). Precisely, we show that if $p \geq 2$, there exists a positive constant $C_{p,\beta}$ such that, for any $t \geq 0$,

$$\mathbb{E}\Big(|V_t|^p\Big) \leq C_{p,\beta}\, t. \tag{3.5}$$

Indeed, by using Itô's formula and the equation (3.2$_2$), we can write

$$|V_t|^p = p\int_0^t \mathrm{sgn}(V_s)|V_s|^{p-1}\mathrm{d}B_s + p\int_0^t \Big(\frac{p-1}{2}|V_s|^{p-2} - |V_s|^{p-1+\beta}\Big)\mathrm{d}s$$

Since $\beta > -1$, there exists a constant $C_{p,\beta} > 0$ such that

$$p\Big((1/2)(p-1)|x|^{p-2} - |x|^{p-1+\beta}\Big) \le C_{p,\beta},\ \forall x \in \mathbb{R}.$$

We deduce that

$$|V_t|^p \le C_{p,\beta}t + p\int_0^t \mathrm{sgn}(V_s)|V_s|^{p-1}\mathrm{d}B_s \tag{3.6}$$

We show that $\int_0^t \mathrm{sgn}(V_s)|V_s|^{p-1}\mathrm{d}B_s$ is a martingale. Fix $T > 0$, for all $t \le T$, since $(a+b)^2 \le 2(a^2+b^2)$ and $|x|^{2p-2} \le 1 + |x|^{2p}$, by using the Burkholder-Davis-Gundy inequality, we can see that there exists a positive constant C_1' such that

$$\begin{aligned}\mathbb{E}\Big[(\sup_{0\le u\le t}|V_u|^p)^2\Big] &\le 2C_{p,\beta}^2\,T^2 + 2p^2\,\mathbb{E}\Big[(\sup_{0\le u\le t}\int_0^u \mathrm{sgn}(V_s)|V_s|^{p-1}\mathrm{d}B_s)^2\Big] \le 2C_{p,\beta}^2\,T^2\\ &+ 2p^2\,C_1'\int_0^t \mathbb{E}\big(|V_s|^{2p-2}\big)\mathrm{d}s \le 2p^2\,C_1'\,T + 2C_{p,\beta}^2\,T^2 + 2p^2\,C_1'\int_0^t \mathbb{E}\big(|V_s|^{2p}\big)\mathrm{d}s\\ &\le 2p^2\,C_1'\,T + 2C_{p,\beta}^2\,T^2 + 2p^2\,C_1'\int_0^t \mathbb{E}\Big[(\sup_{0\le u\le s}|V_u|^p)^2\Big]\mathrm{d}s.\end{aligned}$$

By Gronwall's lemma, we get, for all $t \le T$,

$$\mathbb{E}\Big[(\sup_{0\le u\le t}|V_u|^p)^2\Big] \le (2p^2\,C_1'\,T + 2C_{p,\beta}^2\,T^2)\exp(2p^2\,C_2'\,T)\,.$$

Hence $\int_0^t \mathrm{sgn}(V_s)|V_s|^{p-1}\mathrm{d}B_s$ is a martingale. By taking the expectation in (3.6) we get (3.5).

It is now possible to improve the inequality (3.5) and get the first part of the proposition. Indeed, by (3.5) we can see that

$$\begin{aligned}\mathbb{E}\Big[(\sup_{0\le t\le T}|V_{t\,\varepsilon^{-2\theta}}|)^p\Big] &= \mathbb{E}\Big[(\sup_{0\le t\le T}|V_{t\,\varepsilon^{-2\theta}}|^{\frac{p}{2}})^2\Big] \le \frac{p^2}{2}\,\mathbb{E}\Big[(\sup_{0\le t\le T}\int_0^{t\,\varepsilon^{-2\theta}}|V_s|^{\frac{p}{2}-1}\mathrm{d}B_s)^2\Big] + 2\,C_{\frac{p}{2},\beta}^2\,T^2\,\varepsilon^{-4\theta}\\ &\le \frac{p^2}{2}C_1'\int_0^{T\,\varepsilon^{-2\theta}}\mathbb{E}\big(|V_s|^{p-2}\big)\mathrm{d}s + 2C_{\frac{p}{2},\beta}^2\,T^2\,\varepsilon^{-4\theta} \le \frac{p^2}{4}C_1'\,C_{p-2,\beta}\,T^2\,\varepsilon^{-4\theta} + 2C_{\frac{p}{2},\beta}^2\,T^2\,\varepsilon^{-4\theta}.\end{aligned}$$

Therefore (3.4) follows by taking $C_{p,\beta}' := \frac{p^2}{4}C_1'\,C_{p-2,\beta} + 2C_{\frac{p}{2},\beta}^2$.

To prove the second part we note that from (3.1$_2$), the relation between $\mathcal{V}^\varepsilon$ and V is $\mathcal{V}_t^\varepsilon = \varepsilon^\theta V_{t\,\varepsilon^{-2\theta}}$. By taking $p > 4$ in the first part, we deduce that for any $T > 0$, as $\varepsilon \to 0$, $\sup_{0\le t\le T}|\mathcal{V}_t^\varepsilon|$ converges to 0 in $\mathrm{L}^p(\Omega)$. The conclusion follows. □

3.1.3 Ergodicity

Recall that the scale function and the speed measure were introduced in (3.3). Since $s_\beta(\infty) = \infty$ and $m_\beta(\mathbb{R}) < \infty$, the diffusion V is regular (see for instance (45.2) and (46.10) pp. 272-275 in [12]). Moreover, it is a recurrent and ergodic process with the invariant measure m_β (see for instance Theorem 53.1, p. 300 in [12]). Therefore, for all $f \in \mathrm{L}^1(m_\beta)$,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(V_s)\mathrm{d}s = \frac{1}{m_\beta(\mathbb{R})}\int_\mathbb{R} f(x)m_\beta(\mathrm{d}x),\ \text{ almost surely.} \tag{3.7}$$

3.2 The scaled position process

We recall that the infinitesimal generator of V is given by $\mathcal{A}_{2,\beta} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \mathrm{sgn}(x)|x|^\beta \frac{\mathrm{d}}{\mathrm{d}x}$. Introduce

$$g_\beta(x) := \int_0^x \left(\int_y^{+\infty} -2z\mathrm{e}^{c_\beta(z)}\mathrm{d}z \right) \mathrm{e}^{-c_\beta(y)}\mathrm{d}y,\ x \in \mathbb{R}, \tag{3.8}$$

and note that $(\mathcal{A}_{2,\beta}\, g_\beta)(x) = x$, for all $x \in \mathbb{R}$. Set

$$\kappa_{2,\beta} := \frac{1}{m_\beta(\mathbb{R})} \int_{\mathbb{R}} g_\beta'(x)^2 m_\beta(\mathrm{d}x) = -\frac{2}{m_\beta(\mathbb{R})} \int_{\mathbb{R}} x g_\beta(x) m_\beta(\mathrm{d}x), \tag{3.9}$$

where the latter equality is obtained by integration by parts. Now we can give the proof of the main result.

Proof of Theorem 2.1 for the case $\alpha = 2$. By applying Itô's formula, we can see that

$$g_\beta(V_t) = \int_0^t g_\beta'(V_s)\mathrm{d}B_s + \int_0^t (\mathcal{A}_{2,\beta}\, g_\beta(V_s)\mathrm{d}s = \int_0^t g_\beta'(V_s)\mathrm{d}B_s + \int_0^t V_s \mathrm{d}s,$$

and therefore

$$\varepsilon^{(\beta-1)\theta} X_t^\varepsilon = -\varepsilon^\theta \int_0^{t\,\varepsilon^{-2\theta}} g_\beta'(V_s)\mathrm{d}B_s + \varepsilon^\theta g_\beta(V_{t\,\varepsilon^{-2\theta}})\,.$$

The continuous local martingale

$$M_t^\varepsilon := -\varepsilon^\theta \int_0^{t\,\varepsilon^{-2\theta}} g_\beta'(V_s)\mathrm{d}B_s$$

has the quadratic variation

$$\langle M^\varepsilon \rangle_t = \varepsilon^{2\theta} \int_0^{t\,\varepsilon^{-2\theta}} g_\beta'(V_s)^2 \mathrm{d}s\,.$$

Thanks to (3.7), for all $t \geq 0$,

$$\lim_{\varepsilon \to 0} \langle M^\varepsilon \rangle_t = \kappa_{2,\beta} t, \quad \text{almost surely.}$$

Here $\kappa_{2,\beta}$ is given by (3.9), and it is the constant in the statement of first part of Theorem 2.1. By using Whitt's theorem (see Theorem 2.1(ii), p. 270 in [17]), we deduce that M^ε converges in distribution, as a process, toward $\kappa_{2,\beta}^{1/2} B$.

We will prove that the second term on the right hand side converges in probability, uniformly on compact sets, toward 0. In order, to prove this convergence we need a technical result:

Lemma 3.2. *There exist two positive constants a_0, a_1 such that for all $x \in \mathbb{R}$,*

$$|g_\beta(x)| \leq a_1 |x|^{(2-\beta)\vee 1} + a_0. \tag{3.10}$$

We postpone the proof of the lemma to the appendix and finish the proof of Theorem 2.1 for the case $\alpha = 2$. By using the classical inequality $(a+b)^{2m} \leq 2^{2m-1}(a^{2m} + b^{2m})$, $m \geq 1$ being an integer, we obtain

$$|\varepsilon^\theta g_\beta(V_{t\,\varepsilon^{-2\theta}})|^{2m} \leq 2^{2m-1} a_1^{2m} \varepsilon^{2m\theta} |V_{t\,\varepsilon^{-2\theta}}|^{2m((2-\beta)\vee 1)} + 2^{2m-1} a_0^{2m} \varepsilon^{2m\theta}\,.$$

By choosing the integer $m \geq 1$ such that $p := 2m((2-\beta)\vee 1) > 4$, the first part of Proposition 3.1 can be used, and we get, for all $T > 0$,

$$\lim_{\varepsilon \to 0} \mathbb{E}\Big[\sup_{0 \leq t \leq T} \varepsilon^{2m\theta} g_\beta^{2m}(V_{t\,\varepsilon^{-2\theta}}) \Big] = 0.$$

We can finish the proof of the theorem by employing the joint convergence theorem and the simple continuous-mapping theorem on the space of continuous functions $\mathrm{C}([0,\infty))$ endowed with the uniform topology (see Theorem 11.4.5 p. 379 and Theorem 3.4.1, p. 85 in [16]). □

Remark 3.3. Let us note that if $\beta = 1$ (Ornstein-Uhlenbeck case), $g_1(x) = -x$, $\kappa_{2,1} = 1$ and the result of Theorem 2.1 coincides with the result of Proposition 2.1, p. 268, in [9].

4 Symmetric stable driving noise

We recall that L^ε is distributed as a α-stable Lévy process (see (2.4_1)) so, to simplify the notation, we will suppress the index ε, as well as for V^ε (see (2.5_2)).

4.1 The scaled speed process

4.1.1 Existence and uniqueness

Once again, the processes $\mathcal{V}^\varepsilon$ and V satisfy relation (2.4). If $\beta > 1$, the drift coefficient in (2.5_2) is a locally Lipschitz function and it is well known (see, for instance, Theorem 6.2.11, p. 376 in [1]) that there exists a locally path-wise unique strong solution V for equation (2.5_2) defined up to an explosion random time τ. Moreover, it can be proved that $\tau = \infty$ almost surely, hence V is a global solution. For the sake of completeness, we give the proof of the latter statement (see also [13], p. 73) by following some ideas in [6], pp. 156-157.

Lemma 4.1. *For any $\alpha \in (0,2)$, any $\delta \in (0,\alpha)$ and any $T > 0$, $\mathbb{E}\Big[\sup_{t\in[0,T]} |V_t|^\delta\Big] < \infty$.*

Proof. By the Lévy-Itô decomposition, there exists a Poisson process N and its compensated $\tilde{N}$ such that

$$L_t = \int_0^t \int_{|z|\le 1} z\tilde{N}(\mathrm{d}s,\mathrm{d}z) + \int_0^t \int_{|z|>1} zN(\mathrm{d}s,\mathrm{d}z).$$

Therefore the equation satisfied by V, starting from any $x \in \mathbb{R}$, is

$$V_t = x + \int_0^t \int_{|z|\le 1} z\tilde{N}(\mathrm{d}s,\mathrm{d}z) + \int_0^t \int_{|z|>1} zN(\mathrm{d}s,\mathrm{d}z) - \int_0^t \mathrm{sgn}(V_s)|V_s|^\beta \mathrm{d}s. \tag{4.1}$$

Firstly, we skip the big jumps term and show that the resulting process Y has moments of any order. Secondly, we use an interlacing procedure to handle the process V. In fact, we consider the equation

$$Y_t = x + \int_0^t \int_{|z|\le 1} z\tilde{N}(\mathrm{d}s,\mathrm{d}z) - \int_0^t \mathrm{sgn}(Y_s)|Y_s|^\beta \mathrm{d}s, \tag{4.2}$$

and apply the Itô-Lévy formula. We obtain

$$\begin{aligned} Y_t^2 &= x^2 + \tilde{M}_t + \int_0^t \int_{|z|\le 1} [(Y_s+z)^2 - Y_s^2 - 2zY_s]\nu(\mathrm{d}z)\mathrm{d}s - 2\int_0^t |Y_s|^{\beta+1}\mathrm{d}s \\ &= x^2 + \tilde{M}_t + t\int_{|z|\le 1} z^2\nu(\mathrm{d}z) - 2\int_0^t |Y_s|^{\beta+1}\mathrm{d}s, \end{aligned} \tag{4.3}$$

where the local martingale term is given by

$$\tilde{M}_t := \int_0^t \int_{|z|\le 1} [(Y_s+z)^2 - Y_s^2]\tilde{N}(\mathrm{d}s,\mathrm{d}z).$$

In this proof, the constants depending only on α and β will be denoted c_α or $k_{\alpha,\beta}$ and could change from line to line. Let us denote the third term in (4.3) as $c_\alpha t$ and it is clear that $\lim_{|y|\to\infty}(c_\alpha - 2|y|^{\beta+1}) = -\infty$. We deduce that there exists a positive constant $k_{\alpha,\beta}$ such that, for all $t \ge 0$,

$$Y_t^2 \le x^2 + k_{\alpha,\beta}t + \tilde{M}_t. \tag{4.4}$$

By the Kunita-Watanabe inequality (see for instance [1], p. 265) and by our convention on constants,

$$\begin{aligned} \mathbb{E}\left[\sup_{0\le s\le t} Y_s^2\right] &\le x^2 + k_{\alpha,\beta}t + c_\alpha \int_0^t \int_{|z|\le 1} \mathbb{E}\left[(Y_s+z)^2 - Y_s^2\right]^2 \nu(\mathrm{d}z)\mathrm{d}s \\ &\le x^2 + k_{\alpha,\beta}t + c_\alpha \int_0^t \mathbb{E}[Y_s^2]\mathrm{d}s \le x^2 + k_{\alpha,\beta}t + c_\alpha \int_0^t \mathbb{E}\left[\sup_{0\le u\le s} Y_u^2\right]\mathrm{d}s. \end{aligned} \tag{4.5}$$

Applying Gronwall's inequality, we get

$$\mathbb{E}\left[\sup_{0\le u\le t} Y_u^2\right] \le (x^2+k_{\alpha,\beta}t)e^{c_\alpha t}. \tag{4.6}$$

Hence M is a square integrable martingale and, taking expectation in (4.4), we obtain

$$\mathbb{E}[Y_t^2] \le x^2 + k_{\alpha,\beta}t. \tag{4.7}$$

Re-injecting this in (4.5), we get that, for any $T>0$, there exists a positive constant $C_{\alpha,\beta,T}$ depending also on T, such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} Y_t^2\Big] \le C_{\alpha,\beta,T}(1+x^2). \tag{4.8}$$

We proceed with the study of (4.1). Denote by $0=T_0<T_1<T_2<\dots$ the jumping times of N restricted to $\{|z|>1\}$. The jumps (Z_n) are i.i.d. random variables with the distribution $\lambda^{-1}\mathbb{1}_{\{|z|>1\}}\nu(\mathrm{d}z)$, where $\lambda := \int_{\{|z|>1\}}\nu(\mathrm{d}z)$. Therefore $\int_0^t\int_{|z|>1} zN(\mathrm{d}s,\mathrm{d}z) = \sum_{n\in\mathbb{N}} Z_n\mathbb{1}_{\{T_n\le t\}}$ and (4.1) coincides with (4.2) on each time interval (T_n,T_{n+1}). Since V is a solution of (4.2) on $[0,T_1)$, by using (4.8),

$$\mathbb{E}\Big[\sup_{t\in[0,T_1\wedge T)} V_t^2\Big|\mathcal{G}\Big] \le C_{\alpha,\beta,T}(1+x^2), \quad \text{almost surely,} \quad \text{with } \mathcal{G}:=\sigma(T_1,T_2,\dots).$$

By using Jensen's inequality and the classical inequality $(|a|+|b|)^\delta \le c_\delta(|a|^\delta+|b|^\delta)$, we get

$$\mathbb{E}\Big[\sup_{t\in[0,T_1\wedge T)} |V_t|^\delta\Big|\mathcal{G}\Big] \le C_{\alpha,\beta,\delta,T}(1+|x|^\delta) \quad \text{almost surely.}$$

Furthermore, $V_{T_1} = V_{T_1-}+Z_1$, hence $|V_{T_1}|^\delta \le c_\delta\big(|V_{T_1-}|^\delta+|Z_1|^\delta\big)$. Since $\delta<\alpha$, $\mathbb{E}(|Z_1|^\delta)<\infty$. Consequently we obtain

$$\mathbb{E}\Big[\sup_{t\in[0,T_1\wedge T]} |V_t|^\delta\Big|\mathcal{G}\Big] \le C_{\alpha,\beta,\delta,T}(1+|x|^\delta) \quad \text{almost surely.}$$

By using the strong Markov property and the latter inequality on (T_n,T_{n+1}), but starting from V_{T_n}, we can show that, for any $n\ge 0$,

$$u_n := \mathbb{E}\Big[\sup_{t\in[T_n\wedge T,T_{n+1}\wedge T]} |V_t|^\delta\Big|\mathcal{G}\Big] \le C'_{T,\delta}(1+\mathbb{E}[|V_{T_n}|^\delta|\mathcal{G}]).$$

Then the sequence $(u_n)_{n\ge0}$ satisfies $u_0\le C'_{T,\delta}$ and $u_{n+1}\le C'_{T,\delta}(1+u_n)$, implying that there exists $C_{T,\delta,x}>1$ such that $u_n\le C_{T,\delta,x}^{n+1}$. We deduce that

$$\mathbb{E}\Big[\sup_{t\in[0,T_n\wedge T]} |V_t|^\delta\Big|\mathcal{G}\Big] \le u_0+\dots+u_{n-1} \le \frac{C_{T,\delta,x}^{n+1}}{C_{T,\delta,x}-1} \quad \text{almost surely.}$$

Finally,

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |V_t|^\delta\Big] \le \sum_{n\ge0}\mathbb{E}\Big[\mathbb{1}_{T_n<T<T_{n+1}}\mathbb{E}\Big(\sup_{t\in[0,T_n\wedge T]} V_t^\delta|\mathcal{G}\Big)\Big] \le \frac{1}{C_{T,\delta,x}-1}\sum_{n\ge0} C_{T,\delta,x}^{n+2}\frac{(\lambda T)^n}{n!}e^{-\lambda T}<\infty.$$

□

4.1.2 Ergodicity

The ergodic feature of the process V is a consequence of Proposition 0.1, p. 604 in [10]. Indeed, provided that $\beta>1$, the drift coefficient $b(x)=-\mathrm{sgn}(x)|x|^\beta$ and the jump measure $\nu(\mathrm{d}z) = |z|^{-1-\alpha}\mathbb{1}_{\mathbb{R}\setminus\{0\}}\mathrm{d}z$ clearly satisfy the conditions in the cited result. Hence V is an exponential ergodic and Harris recurrent process having an unique invariant distribution, denoted by $m_{\alpha,\beta}$. The measure $m_{\alpha,\beta}$ satisfies

$$m_{\alpha,\beta}([x,+\infty)) \underset{|x|\to\infty}{\sim} \int_{|x|}^{+\infty} \frac{\nu([u,+\infty))}{-b(x)} \mathrm{d}u = \frac{C}{|x|^{\alpha+\beta-1}}\,, \tag{4.9}$$

as it follows from Theorem 4.1, p. 92 in [13]. Let us note that in [13], p. 76, it is conjectured that the tail behaviour of $m_{\alpha,\beta}$ remains true in the asymmetric case, but the proof seems more technical.

Clearly, under the hypothesis of Theorem 2.1, $\beta + \frac{\alpha}{2} - 2 > 0$, the identity function id belongs to $\mathrm{L}^1(m_{\alpha,\beta})$. By the classical ergodic theorem, for all $f \in \mathrm{L}^1(m_{\alpha,\beta})$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(V_s)\mathrm{d}s = \int_{\mathbb{R}} f(x) m_{\alpha,\beta}(\mathrm{d}x), \text{ a.s.} \tag{4.10}$$

Recall that we are interested on the behaviour as $\varepsilon \to 0$ of

$$\varepsilon^{\theta(\beta+\frac{\alpha}{2}-2)} x^{\varepsilon}_{\varepsilon^{-\alpha}t} = \varepsilon^{\frac{\alpha\theta}{2}} \int_0^{t\varepsilon^{-\alpha\theta}} V_s \mathrm{d}s, \tag{4.11}$$

where θ is given by (2.3). In other words, we are studying a large time behaviour of a functional of V, hence it is quite natural to perform the study in steady state. This fact is contained in the following lemma (see also [3], Theorem 2.6, p. 194):

Lemma 4.2. *Suppose that $\beta + \frac{\alpha}{2} - 2 > 0$. Assume that the process $\left\{\varepsilon^{\alpha\theta/2} \int_0^{t\varepsilon^{-\alpha\theta}} V_s \mathrm{d}s \,:\, t \geq 0\right\}$ converges, as $\varepsilon \to 0$, in distribution to a Brownian motion, provided that V is starting with $m_{\alpha,\beta}$ as an initial distribution. Then the same process converges in distribution to a Brownian motion when $V_0 = 0$.*

Proof. In this proof we will denote the process in (4.11) by $Z_{\varepsilon,0}(t)$, and for $\Delta \geq 0$,

$$Z_{\varepsilon,\Delta}(t) := \varepsilon^{\frac{\alpha\theta}{2}} \int_{\Delta}^{t\varepsilon^{-\alpha\theta}+\Delta} V_s \mathrm{d}s.$$

Firstly, let us prove that $Z_{\varepsilon,\Delta}(\cdot)$ converges in distribution, as $\Delta \to \infty$ and $\varepsilon \to 0$, to a Brownian motion, when $V_0 = 0$. Denoting by μ_Δ the distribution of V_Δ, for each bounded continuous real function ψ on $\mathrm{C}([0,+\infty))$, by the Markov property, we have

$$\mathbb{E}\big[\psi(Z_{\varepsilon,\Delta}(\cdot))|V_0 = 0\big] = \mathbb{E}\big[\psi(Z_{\varepsilon,0}(\cdot))|V_0 \sim \mu_\Delta\big].$$

We can write, for all $\varepsilon > 0$,

$$\begin{aligned}\Big|\mathbb{E}\big[\psi(Z_{\varepsilon,0}(\cdot))|V_0 \sim \mu_\Delta\big] - \mathbb{E}\big[\psi(Z_{\varepsilon,0}(\cdot))|V_0 \sim m_{\alpha,\beta}\big]\Big| &= \Big| \int_{\mathbb{R}} \mathbb{E}\big[\psi(Z_{\varepsilon,0}(\cdot))|V_0 = y\big] \big(\mu_\Delta(\mathrm{d}y) - m_{\alpha,\beta}(\mathrm{d}y)\big)\Big| \\ &\leq ||\psi||_\infty \int_{\mathbb{R}} \big|p(\Delta,0,\mathrm{d}y) - m_{\alpha,\beta}(\mathrm{d}y)\big| \leq ||\psi||_\infty ||p(\Delta,0,\mathrm{d}y) - m_{\alpha,\beta}(\mathrm{d}y)||_{\mathrm{TV}}\,,\end{aligned}$$

where $p(t,x,\mathrm{d}y) = \mathbb{P}_x(V_t \in \mathrm{d}y)$ is the transition kernel of V (and therefore $p(\Delta,0,\mathrm{d}y) = \mu_\Delta(\mathrm{d}y)$), and $\|\cdot\|_{\mathrm{TV}}$ is the norm in total variation. Since V is exponentially ergodic, we get that

$$\lim_{\Delta\to\infty} \big|\mathbb{E}\big[\psi(Z_{\varepsilon,0}(\cdot))|V_0 \sim \mu_\Delta\big] - \mathbb{E}\big[\psi(Z_{\varepsilon,0}(\cdot))|V_0 \sim m_{\alpha,\beta}\big]\big| = 0, \quad \text{uniformly in } \varepsilon.$$

Secondly, by choosing $\Delta = \Delta(\varepsilon) = \varepsilon^{-\alpha\theta/4}$ we obtain

$$\sup_{t\geq 0} \left\{\Big|Z_{\varepsilon,\Delta(\varepsilon)}(t) - \varepsilon^{\frac{\alpha\theta}{2}} \int_0^{t\varepsilon^{-\alpha\theta}+\Delta(\varepsilon)} V_s \mathrm{d}s\Big|\right\} \leq \varepsilon^{\frac{\alpha\theta}{2}} \int_0^{\Delta(\varepsilon)} |V_s|\mathrm{d}s = \varepsilon^{\frac{\alpha\theta}{4}} \frac{1}{\Delta(\varepsilon)} \int_0^{\Delta(\varepsilon)} |V_s|\mathrm{d}s.$$

The right hand side term of the latter inequality tends to 0 almost surely, by using the ergodicity (4.10). Therefore $\varepsilon^{\alpha\theta/2} \int_0^{\bullet\varepsilon^{-\alpha\theta}+\Delta(\varepsilon)} V_s \mathrm{d}s$ converges in distribution, as $\varepsilon \to 0$, to a Brownian motion when $V_0 = 0$. Clearly, $\lim_{\varepsilon\to 0}(t - \Delta(\varepsilon)\varepsilon^{\alpha\theta}) = t$, and applying a consequence of the continuous mapping theorem for the composition function stated in Lemma p. 151 in [4], we can conclude. □

In the sequel, we will always assume that the initial distribution of V is $m_{\alpha,\beta}$. Let us recall that the infinitesimal generator of V is given by

$$(\mathcal{A}_{\alpha,\beta}\, g)(x) = -\mathrm{sgn}(x)|x|^{\beta} g'(x) + \int_{\mathbb{R}} \Big[g(x+y) - g(x) - y g'(x) \mathbb{1}_{|y|\leq 1} \Big] \nu(\mathrm{d}y)\,, \tag{4.12}$$

with the domain $D_{\mathcal{A}_{\alpha,\beta}}$. Also, denote by $(\mathcal{T}_t)_{t\geq 0}$ the semi-group associated to the operator $\mathcal{A}_{\alpha,\beta}$ or to the process V. We collect in the following lemma some useful properties of the process V.

Lemma 4.3.

1. *The domain $D_{\mathcal{A}_{\alpha,\beta}}$ contains the space of bounded twice differentiable functions $\mathrm{C}_b^2(\mathbb{R})$.*
2. *For all $p \geq 1$, $\mathcal{T}_t$ is a contraction semi-group on $\mathrm{L}^p(m_{\alpha,\beta})$ and for each $f \in \mathrm{L}^p(m_{\alpha,\beta})$,*

$$\lim_{t\to 0} \|\mathcal{T}_t f - f\|_{\mathrm{L}^p(m_{\alpha,\beta})} = 0. \tag{4.13}$$

Proof. To prove the first point, we fix $f \in \mathrm{C}_b^2(\mathbb{R})$ and we show that $(\mathcal{A}_{\alpha,\beta}\, f)(x) < \infty$. Let us note that, $-\mathrm{sgn}(x)|x|^{\beta} f'(x)$ is well defined for all $x \in \mathbb{R}$. Since $f \in \mathrm{C}_b^2(\mathbb{R})$, for any $y \in [-1,1]$,

$$\Big| f(x+y) - f(x) - y f'(x) \Big| \leq y^2 \sup_{z\in[x-1,x+1]} |f''(z)| < \infty,$$

and we find

$$\int_{|y|\leq 1} \Big[f(x+y) - f(x) - y f'(x) \Big] \nu(\mathrm{d}y) \leq \Big[\sup_{z\in[x-1,x+1]} |f''(z)| \Big] \int_{|y|\leq 1} y^2 \nu(\mathrm{d}y) < \infty.$$

Since f is bounded, we can see that

$$\int_{|y|>1} \Big[f(x+y) - f(x) \Big] \nu(\mathrm{d}y) \leq 2||f||_{\infty} \int_{|y|>1} \nu(\mathrm{d}y) < \infty\,,$$

hence $f \in \mathcal{D}_{\mathcal{A}_{\alpha,\beta}}$.

We proceed with the proof of the second point. Fix $f \in \mathrm{L}^p(m_{\alpha,\beta})$ and we show first that

$$\|\mathcal{T}_t f\|_{\mathrm{L}^p(m_{\alpha,\beta})} \leq \|f\|_{\mathrm{L}^p(m_{\alpha,\beta})}.$$

Since

$$\|\mathcal{T}_t f\|^p_{\mathrm{L}^p(m_{\alpha,\beta})} = \int_{\mathbb{R}} |\mathcal{T}_t f(x)|^p m_{\alpha,\beta}(\mathrm{d}x) = \int_{\mathbb{R}} |\mathbb{E}_x(f(V_t))|^p m_{\alpha,\beta}(\mathrm{d}x),$$

by Jensen's inequality $(p \geq 1)$, we get

$$\|\mathcal{T}_t f\|_p^p \leq \int_{\mathbb{R}} \mathbb{E}_x(|f(V_t)|^p) m_{\alpha,\beta}(\mathrm{d}x) = \mathbb{E}_{m_{\alpha,\beta}}(|f(V_t)|^p) = ||f||^p_{\mathrm{L}^p(m_{\alpha,\beta})}.$$

Finally, we prove (4.13). Since $\mathrm{C}_b^2(\mathbb{R})$ is dense in $\mathrm{L}^p(m_{\alpha,\beta})$, there exists $f_\eta \in \mathrm{C}_b^2(\mathbb{R})$ such that $||f - f_\eta||_{\mathrm{L}^p(m_{\alpha,\beta})} \leq \eta/3$. Since $\mathcal{T}_t$ is a contraction semi-group and $m_{\alpha,\beta}$ is a probability measure, we get

$$\|\mathcal{T}_t f - f\|_{\mathrm{L}^p(m_{\alpha,\beta})} \leq 2\|f - f_\eta\|_{\mathrm{L}^p(m_{\alpha,\beta})} + \|\mathcal{T}_t f_\eta - f_\eta\|_{\infty} \leq (2\eta)/3 + \|\mathcal{T}_t f_\eta - f_\eta\|_{\infty}.$$

Clearly $\mathcal{T}_t$ is a Feller semi-group (see for instance, [1], p. 151). Hence $\|\mathcal{T}_t f_\eta - f_\eta\|_{\infty} \leq \eta/3$, for t small enough, and we deduce (4.13). The proof is complete. □

4.1.3 Convergence in probability

One describes the behaviour of the speed process by using a Lyapunov function. The statement of this important result is given below.

Proposition 4.4. *Suppose that $\beta + \frac{\alpha}{2} > 2$ and let p and γ such that*

$$p > 1, \;\; p\gamma > 2, \;\; 2 - \beta < \gamma < \frac{\alpha}{2}. \tag{4.14}$$

Introduce the Lyapunov function

$$h_{p,\gamma}(x) := (1 + |x|^{p\gamma})^{1/p}. \tag{4.15}$$

Then, as $\varepsilon \to 0$, $\{\varepsilon^{\alpha\theta/2} h_{p,\gamma}(\varepsilon^{-\theta} V_t^\varepsilon) : t \geq 0\}$ converges to 0 in probability, uniformly on each compact time interval. More precisely, there exists $q > 2$ such that, for any fixed $T > 0$,

$$\lim_{\varepsilon \to 0} \mathbb{E}\Big[\Big(\sup_{t\in[0,T]} \varepsilon^{\frac{\alpha\theta}{2}} h_{p,\gamma}\Big(\varepsilon^{-\theta}\mathcal{V}_t^\varepsilon\Big)\Big)^q\Big] = \mathbb{E}\Big[\Big(\sup_{t\in[0,T]} \varepsilon^{\frac{\alpha\theta}{2}} h_{p,\gamma}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big)\Big)^q\Big] = 0. \tag{4.16}$$

In order to prove this result, we need the following lemma whose proof is postponed to the appendix. The first part of the lemma collects some regularity properties and the asymptotic behaviour of the Lyapunov function, while the second part contains the Foster-Lyapunov conditions which allows to solve Poisson's equations.

Lemma 4.5.

1. *If $p\gamma > 2$, $h_{p,\gamma}$ is a twice differentiable function and there exists a positive constant k such that for all $(x, y) \in \mathbb{R}^2$,*
   - *if $|x| < 1$ then*
   $$|h_{p,\gamma}(x+y) - h_{p,\gamma}(x)| \leq k(|y|\mathbb{1}_{\{|y|\leq 1\}} + |y|^\gamma \mathbb{1}_{\{|y|>1\}});$$
   - *if $|x| \geq 1$ then*
   $$|h_{p,\gamma}(x+y) - h_{p,\gamma}(x)| \leq k(|y||x|^{\gamma-1}\mathbb{1}_{\{|y|\leq i(x)\}} + |y|^\gamma \mathbb{1}_{\{i(x)<|y|\}}),$$

   where $i(x) := (2|x|^{p\gamma} + 1)^{1/p\gamma} - |x|$.

2. *Assume that $p\gamma > 2$ and $2 - \beta < \gamma < \alpha$. There exist a continuous function $f_{p,\alpha,\beta,\gamma}$, a compact set K and a constant d, depending only on p, α, β, γ, such that*
   $$\forall x \in \mathbb{R}, \;\; f_{p,\alpha,\beta,\gamma}(x) \geq 1 + |x|, \quad f_{p,\alpha,\beta,\gamma}(x) \underset{|x|\to\infty}{\sim} \gamma|x|^{\gamma+\beta-1}, \tag{4.17}$$
   and
   $$(\mathcal{A}_{\alpha,\beta}\, h_{p,\gamma})(x) \leq -f_{p,\alpha,\beta,\gamma}(x) + d\,\mathbb{1}_K. \tag{4.18}$$

Proof of Proposition 4.4. By (2.4_2), we can write

$$\varepsilon^{\frac{\alpha\theta}{2}} h_{p,\gamma}\Big(\frac{\mathcal{V}_t^\varepsilon}{\varepsilon^\theta}\Big) = \varepsilon^{\frac{\alpha\theta}{2}} h_{p,\gamma}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big) \tag{4.19}$$

and the first equality in (4.16) is clear. Since $2 - \beta < \frac{\alpha}{2}$ and $\beta > 1$, we can fix q such that $\frac{2}{p} \vee (2-\beta) < \gamma < 2\gamma < q\gamma < \alpha$ and $2 < q < \frac{\beta-1}{\alpha} + 2$. By noting that $h_{p,\gamma}(x)^q = h_{\frac{p}{q},q\gamma}(x)$, we can write

$$\mathbb{E}\Big[\Big(\sup_{t\in[0,T]} \varepsilon^{\frac{\alpha\theta}{2}} h_{p,\gamma}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big)\Big)^q\Big] = \varepsilon^{q\frac{\alpha\theta}{2}}\, \mathbb{E}\Big[\Big(\sup_{t\in[0,T]} h_{\frac{p}{q},q\gamma}\Big(V_{t\,\varepsilon^{-\alpha\theta}}\Big)\Big)\Big].$$

Employing Itô's formula with $h_{\frac{p}{q},q\gamma}$, we get

$$h_{\frac{p}{q},q\gamma}(V_t) - h_{\frac{p}{q},q\gamma}(V_0) = R_t + \int_0^t (\mathcal{A}_{\alpha,\beta}\, h_{\frac{p}{q},q\gamma})(V_s)\mathrm{d}s, \tag{4.20}$$

where

$$R_t := \int_0^t \int_{\mathbb{R}} \Big(h_{\frac{p}{q},q\gamma}(V_s + y) - h_{\frac{p}{q},q\gamma}(V_s) \Big) \tilde{N}(\mathrm{d}y, \mathrm{d}s).$$

By Lemma 4.5 applied to the function $h_{\frac{p}{q},q\gamma}$, we see that there exists $c > 0$ such that, for all $t \in [0,T]$,

$$\int_0^t (\mathcal{A}_{\alpha,\beta}\, h_{\frac{p}{q},q\gamma})(V_s)\mathrm{d}s \leq c\,t.$$

Moreover, let us note that $h_{\frac{p}{q},q\gamma}$ is continuous and that $h_{\frac{p}{q},q\gamma}(x) \sim |x|^{q\gamma}$, as $|x| \to \infty$. Hence, by the choice of q, we have $h_{\frac{p}{q},q\gamma} \in \mathrm{L}^1(m_{\alpha,\beta})$. Taking the expectation in (4.20), we obtain

$$\varepsilon^{q\frac{\alpha\theta}{2}}\, \mathbb{E}\Big[\Big(\sup_{t\in[0,T]} h_{\frac{p}{q},q\gamma}\Big(V_{t\,\varepsilon^{-\alpha\theta}}\Big)\Big)\Big] \leq \varepsilon^{q\frac{\alpha\theta}{2}}\, \|h_{\frac{p}{q},q\gamma}\|_{\mathrm{L}^1(m_{\alpha,\beta})} + \varepsilon^{(q-2)\frac{\alpha\theta}{2}}\, cT + \varepsilon^{q\frac{\alpha\theta}{2}}\, \mathbb{E}\Big(\sup_{t\in[0,T]} R_{t\,\varepsilon^{-\alpha\theta}}\Big).$$

Since $q > 2$, the first and the second term converge to 0. For the last term, we use the Kunita-Watanabe inequality (see for instance [1], p. 265). Since $V_0 \sim m_{\alpha,\beta}$, then for all t, $V_t \sim m_{\alpha,\beta}$ and there exists a positive constant C such that

$$\mathbb{E}\Big(\sup_{t\in[0,T]} R_{t\,\varepsilon^{-\alpha\theta}}\Big) \leq \mathbb{E}\Big(\sup_{t\in[0,T]} R^2_{t\,\varepsilon^{-\alpha\theta}}\Big)^{1/2} \leq C\,\sqrt{T}\,\varepsilon^{-\frac{\alpha\theta}{2}} \iint_{\mathbb{R}^2} \big(h_{\frac{p}{q},q\gamma}(x+y) - h_{\frac{p}{q},q\gamma}(x)\big)^2 \nu(\mathrm{d}y) m_{\alpha,\beta}(\mathrm{d}x).$$

It is sufficient to show that

$$\iint_{\mathbb{R}^2} \big(h_{\frac{p}{q},q\gamma}(x+y) - h_{\frac{p}{q},q\gamma}(x)\big)^2 \nu(\mathrm{d}y) m_{\alpha,\beta}(\mathrm{d}x) < \infty. \tag{4.21}$$

This fact is obtained by using Lemma 4.5. If $|x| \geq 1$,

$$\big(h_{\frac{p}{q},q\gamma}(x+y) - h_{\frac{p}{q},q\gamma}(x)\big)^2 \leq k^2 (|y|^2 |x|^{2q\gamma-2} \mathbb{1}_{\{|y| \leq i(x)\}} + |y|^{2q\gamma} \mathbb{1}_{\{i(x) < |y|\}}),$$

hence

$$\int_{\mathbb{R}} \Big(h_{\frac{p}{q},q\gamma}(x+y) - h_{\frac{p}{q},q\gamma}(x)\Big)^2 \nu(\mathrm{d}y) = O(|x|^{2q\gamma-\alpha}), \text{ as } |x| \to +\infty,$$

and, since $q < \frac{\beta-1}{\alpha} + 2$, we get (4.21). If $|x| < 1$,

$$\big(h_{\frac{p}{q},q\gamma}(x+y) - h_{\frac{p}{q},q\gamma}(x)\big)^2 \leq k^2 (|y|^2 \mathbb{1}_{\{|y| \leq 1\}} + |y|^{2q\gamma} \mathbb{1}_{\{|y| > 1\}})$$

and $\int_{\mathbb{R}^2} \big(h_{\frac{p}{q},q\gamma}(x+y) - h_{\frac{p}{q},q\gamma}(x)\big)^2 \nu(\mathrm{d}y)$ is finite independently of x. Since $m_{\alpha,\beta}$ is a probability measure, (4.21) is verified again. The proof is complete except for Lemma 4.5. □

4.2 The position process

We are ready to give the proof of our main result concerning the behaviour of the position process. Recall that, thanks to Lemma 4.2, we assume that the initial distribution of V is the measure $m_{\alpha,\beta}$.

Proof of Theorem 2.1 for the case $\alpha \in (0,2)$. Thanks to (4.17), Theorem 3.2, p. 924 in [7] applies and we deduce that the Poisson equation $\mathcal{A}_{\alpha,\beta} g = \mathrm{id}$ admits a solution $g_{\alpha,\beta}$ satisfying $|g_{\alpha,\beta}| \leq c(h_{p,\gamma} + 1)$, with c a positive constant. Applying the Itô-Levy formula with $g_{\alpha,\beta}$, we get

$$g_{\alpha,\beta}(V_t) - g_{\alpha,\beta}(V_0) = \int_0^t V_s \mathrm{d}s + M_t, \tag{4.22}$$

where

$$M_t := \int_0^t \int_{\mathbb{R}} [g_{\alpha,\beta}(z + V_s) - g_{\alpha,\beta}(V_s)] \tilde{N}(\mathrm{d}s, \mathrm{d}z). \tag{4.23}$$

Step 1) We prove that M, given by the latter formula, is a square integrable martingale. On one hand we have

$$\mathbb{E}[g_{\alpha,\beta}(V_t)^2] = \mathbb{E}[g_{\alpha,\beta}(V_0)^2] = \int_{\mathbb{R}} g_{\alpha,\beta}(x)^2 m_{\alpha,\beta}(\mathrm{d}x) < \infty.$$

Indeed, recall that $h_{p,\gamma}^2$ is continuous and it behaves as $|x|^{2\gamma}$, as $|x| \to \infty$. Recalling that γ was chosen such that $\frac{4}{p} \vee (4-2\beta) < 2\gamma < \alpha$, by using (4.9), we see that

$$\int_{\mathbb{R}} h_{p,\gamma}(x)^2 m_{\alpha,\beta}(\mathrm{d}x) < \infty. \tag{4.24}$$

We point out that the assumption $\beta + \frac{\alpha}{2} > 2$ is essential for the latter condition of integrability.

On the other hand, we can write

$$\mathbb{E}\left[\Big(\int_0^t V_s \mathrm{d}s\Big)^2\right] = \mathbb{E}\int_0^t\int_0^t V_u V_s \,\mathrm{d}u\mathrm{d}s = 2\mathbb{E}\int_0^t \mathrm{d}s\int_0^s \mathrm{d}u\, V_u V_s \leq 2\mathbb{E}\int_0^t \mathrm{d}s\int_0^s \mathrm{d}u\, |V_u||V_s|.$$

Using the Markov property and that V_u and V_0 follow the invariant law, we obtain, for $u < s$, $\mathbb{E}\big(|V_s||V_u|\big) = \mathbb{E}\big(|V_{s-u}||V_0|\big)$. Therefore

$$\begin{aligned}\mathbb{E}\Big[\Big(\int_0^t V_s \mathrm{d}s\Big)^2\Big] &\leq 2\int_0^t \mathrm{d}s\int_0^s \mathrm{d}u\,\mathbb{E}\big(|V_{s-u}||V_0|\big) = 2\int_0^t \mathrm{d}s\int_0^s \mathrm{d}u\,\mathbb{E}\big(|V_u||V_0|\big)\\ &= 2\int_0^t \mathrm{d}s\,\mathbb{E}\Big(|V_0|\int_0^s \mathcal{T}_u|\mathrm{id}|(V_0)\mathrm{d}u\Big).\end{aligned}$$

Applying again Theorem 3.2, p. 924 in [7], we deduce that the Poisson equation $\mathcal{A}_{\alpha,\beta}\, g = |\mathrm{id}|$ admits a solution $\tilde{g}_{\alpha,\beta}$, This solution satisfies $|\tilde{g}_{\alpha,\beta}| \leq c'(h_{p,\gamma}+1)$, with c' a positive constant. Moreover

$$\int_0^s \mathcal{T}_u|\mathrm{id}|(V_0)\mathrm{d}u = \mathcal{T}_s\tilde{g}_{\alpha,\beta}(V_0) - \tilde{g}_{\alpha,\beta}(V_0).$$

Replacing in the latter inequality

$$\mathbb{E}\Big[\Big(\int_0^t V_s \mathrm{d}s\Big)^2\Big] \leq 2\int_0^t \mathbb{E}\left(|V_0||\mathcal{T}_s\tilde{g}_{\alpha,\beta}(V_0) - \tilde{g}_{\alpha,\beta}(V_0)|\right)\mathrm{d}s = 2\int_0^t \mathrm{d}s\int_{\mathbb{R}} |x||\mathcal{T}_s\tilde{g}_{\alpha,\beta}(x) - \tilde{g}_{\alpha,\beta}(x)| m_{\alpha,\beta}(\mathrm{d}x).$$

At this level, we need to apply the Hölder inequality to conclude that

$$\mathbb{E}\Big[\int_0^t V_s \mathrm{d}s\Big]^2 < \infty. \tag{4.25}$$

Firstly, if $\beta < 2$ then we choose γ close enough to $2-\beta$ such that $\tilde{g}_{\alpha,\beta} \in \mathrm{L}^{(3-\beta)/(2-\beta)}(m_{\alpha,\beta})$. Since $\frac{3-\beta}{2-\beta} > 1$, using the second part of Lemma 4.3, we get

$$\|\mathcal{T}_s\tilde{g}_{\alpha,\beta} - \tilde{g}_{\alpha,\beta}\|_{\mathrm{L}^{(3-\beta)/(2-\beta)}(m_{\alpha,\beta})} \leq 2\|\tilde{g}_{\alpha,\beta}\|_{\mathrm{L}^{(3-\beta)/(2-\beta)}(m_{\alpha,\beta})}.$$

By the Hölder inequality and the fact that $|\mathrm{id}| \in \mathrm{L}^{3-\beta}(m_{\alpha,\beta})$, we get (4.25). Secondly, if $\beta \geq 2$, we choose $\gamma < 1$ close enough to 0 such that $|\mathrm{id}| \in \mathrm{L}^{1/(1-\gamma)}(m_{\alpha,\beta})$. Since $\tilde{g}_{\alpha,\beta} \in \mathrm{L}^{1/\gamma}(m_{\alpha,\beta})$, using again Lemma 4.3, we get

$$\|\mathcal{T}_t\tilde{g}_{\alpha,\beta} - \tilde{g}_{\alpha,\beta}\|_{\mathrm{L}^{1/\gamma}(m_{\alpha,\beta})} \leq 2\|\tilde{g}_{\alpha,\beta}\|_{\mathrm{L}^{1/\gamma}(m_{\alpha,\beta})}.$$

Since $|\mathrm{id}| \in \mathrm{L}^{1/(1-\gamma)}(m_{\alpha,\beta})$, we can apply the Hölder inequality and get (4.25) again.

We conclude that M given by (4.23) is a square integrable martingale. Moreover, we can compute its quadratic variation

$$\langle M\rangle_t = \int_0^t\int_{\mathbb{R}} [g_{\alpha,\beta}(y+V_s) - g_{\alpha,\beta}(V_s)]^2\nu(\mathrm{d}y)\mathrm{d}s. \tag{4.26}$$

Hence

$$\mathbb{E}[\langle M\rangle_t] = t\iint_{\mathbb{R}^2} [g_{\alpha,\beta}(x+y) - g_{\alpha,\beta}(x)]^2\nu(\mathrm{d}y)m_{\alpha,\beta}(\mathrm{d}x) < \infty. \tag{4.27}$$

Step 2) Performing a simple time change in (4.22), we see that the process in (2.7) can be written

$$\varepsilon^{\theta(\beta+\frac{\alpha}{2}-2)}\mathcal{X}_t^\varepsilon = \varepsilon^{\frac{\alpha\theta}{2}}\Big[\mathscr{g}_{\alpha,\beta}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big) - \mathscr{g}_{\alpha,\beta}(V_0)\Big] - \varepsilon^{\frac{\alpha\theta}{2}} M_{t\,\varepsilon^{-\alpha\theta}}. \tag{4.28}$$

In this step, we show that the martingale term on the right hand side of the latter equality converges to a Brownian motion by using Whitt's theorem (see Theorem 2.1 (ii) in [17], pp. 270-271). We need to verify the hypotheses of this result. Indeed, since the function

$$x \mapsto \int_{\mathbb{R}} [\mathscr{g}_{\alpha,\beta}(x+y) - \mathscr{g}_{\alpha,\beta}(x)]^2 \nu(\mathrm{d}y) \in \mathrm{L}^1(\mathscr{m}_{\alpha,\beta}),$$

by using (4.26) and the ergodic theorem (4.10), we deduce that

$$\begin{aligned}\lim_{\varepsilon\to 0}\langle \varepsilon^{\frac{\alpha\theta}{2}} M_{\bullet\,\varepsilon^{-\alpha\theta}}\rangle_t = \lim_{\varepsilon\to 0}\varepsilon^{\frac{\alpha\theta}{2}}\int_0^{t\varepsilon^{-\alpha\theta}}\int_{\mathbb{R}}[\mathscr{g}_{\alpha,\beta}(y+V_s) - \mathscr{g}_{\alpha,\beta}(V_s)]^2\nu(\mathrm{d}y)\mathrm{d}s \\ = t\iint_{\mathbb{R}^2}[\mathscr{g}_{\alpha,\beta}(x+y) - \mathscr{g}_{\alpha,\beta}(x)]^2\nu(\mathrm{d}y)\mathscr{m}_{\alpha,\beta}(\mathrm{d}x).\end{aligned}$$

The condition (6) in [17], p. 271 is fulfilled. Again by (4.26), we see that $\langle M\rangle$ has no jump, hence the condition (4) in [17], p. 270 is trivial. Let us note also that, by (4.22), the jumps of the martingale M_t are $J(M_t) := \mathscr{g}_{\alpha,\beta}(V_t) - \mathscr{g}_{\alpha,\beta}(V_{t-})$. Therefore we deduce that the jumps of the martingale term on the right hand side of (4.28) are

$$\begin{aligned}J\Big(\varepsilon^{\frac{\alpha\theta}{2}} M_{t\,\varepsilon^{-\alpha\theta}}\Big) := \varepsilon^{\frac{\alpha\theta}{2}}\Big[\mathscr{g}_{\alpha,\beta}\big(V_{\varepsilon^{-\alpha\theta}t}\big) - \mathscr{g}_{\alpha,\beta}\big(V_{\varepsilon^{-\alpha\theta}t-}\big)\Big] \le c\,\varepsilon^{\frac{\alpha\theta}{2}}\Big[\big|h_{p,\gamma}\big(V_{\varepsilon^{-\alpha\theta}t}\big)\big| + \big|h_{p,\gamma}\big(V_{\varepsilon^{-\alpha\theta}t-}\big)\big| + 2\Big] \\ \le 2c\,\varepsilon^{\frac{\alpha\theta}{2}}\Big[\sup_{t\in[0,T]}\Big|h_{p,\gamma}\Big(\varepsilon^{-\theta}V_t^\varepsilon\Big)\Big| + 1\Big],\end{aligned}$$

by using the fact that $|\mathscr{g}_{\alpha,\beta}| \le c(h_{p,\gamma}+1)$ and (4.19). By Proposition 4.4,

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\sup_{t\in[0,T]} J\Big(\varepsilon^{\frac{\alpha\theta}{2}} M_{t\,\varepsilon^{-\alpha\theta}}\Big)^2\Big] = 0.$$

Therefore we can apply Whitt's theorem to deduce that $\big\{\varepsilon^{(\alpha\theta)/2} M_{t\,\varepsilon^{-\alpha\theta}} : t \ge 0\big\}$ converges in distribution (as a process) to $\kappa_{\alpha,\beta}^{1/2}B$, where B is the standard Brownian motion and

$$\kappa_{\alpha,\beta} := \iint_{\mathbb{R}^2}[\mathscr{g}_{\alpha,\beta}(x+y) - \mathscr{g}_{\alpha,\beta}(x)]^2\nu(\mathrm{d}y)\mathscr{m}_{\alpha,\beta}(\mathrm{d}x) > 0. \tag{4.29}$$

Let us explain why the constant $\kappa_{\alpha,\beta}$ is positive. Indeed it suffices to note that ν is absolutely continuous with respect to the Lebesgue measure, that $\mathscr{m}_{\alpha,\beta}$ has a non-empty support, and that $\mathscr{g}_{\alpha,\beta}$ could not be a constant function, since $\mathscr{A}_{\alpha,\beta}\mathscr{g}_{\alpha,\beta} = \mathrm{id}$.

Step 3) By using that $|\mathscr{g}_{\alpha,\beta}| \le c(h_{p,\gamma}+1)$, we get

$$\Big|\mathscr{g}_{\alpha,\beta}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big) - \mathscr{g}_{\alpha,\beta}(V_0)\Big|^2 \le 4c^2\Big(\Big|h_{p,\gamma}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big)\Big|^2 + \Big|h_{p,\gamma}(V_0)\Big|^2 + 2\Big).$$

Hence, by using Proposition 4.4,

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\varepsilon^{\alpha\theta}\sup_{t\in[0,T]}\Big|\mathscr{g}_{\alpha,\beta}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big) - \mathscr{g}_{\alpha,\beta}(V_0)\Big|^2\Big] = 0.$$

Therefore, $\big\{\varepsilon^{(\alpha\theta)/2}\big[\mathscr{g}_{\alpha,\beta}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big) - \mathscr{g}_{\alpha,\beta}(V_0)\big] : t \ge 0\big\}$ converges in probability to 0, uniformly on compact sets.

Step 4) Our processes are valued in the Skorokhod space of càdlàg functions $\mathrm{D}([0,\infty))$ endowed with J_1 Skorokhod topology (see [16], §3.3). It is not difficult to see that a sequence which converges in probability to 0, uniformly on compact sets, is also convergent in probability for J_1 metric, hence in distribution in J_1 topology. Recall that in the Skorokhod space, the summation is not a continuous map (see for instance [16], p. 84). In our case, the limits of the terms on the right hand side of equality (4.28) are, respectively, 0 and a Brownian motion, and have continuous paths. By using the joint convergence theorem (Theorem 11.4.5, p. 379 in [16]) and the continuous-mapping theorem (Theorem 3.4.3, p. 86 in [16]), we obtain the conclusion of Theorem 2.1. More precisely, the convergence in the theorem holds in the space of continuous functions $\mathrm{C}([0,\infty))$ endowed with the uniform topology. Let us note that our situation is simpler than in [9] since the limit is a continuous paths process. □

Proposition 4.6. *Assume that $\alpha \in (0,2)$ and $\beta + \frac{\alpha}{2} > 2$. The constant $\kappa_{\alpha,\beta}$ of the second part of Theorem 2.1, given in (4.29), satisfies*

$$\kappa_{\alpha,\beta} = -2\int_{\mathbb{R}} x\mathscr{g}_{\alpha,\beta}(x)\mathscr{m}_{\alpha,\beta}(\mathrm{d}x) > 0. \tag{4.30}$$

Proof. Since, by (4.27) and (4.29), $\kappa_{\alpha,\beta} = \frac{1}{t}\mathbb{E}\big[M_t^2\big]$, for all $t>0$, by taking $t = \varepsilon^{\alpha\theta}$ and using Itô's formula, we get

$$\begin{aligned}\kappa_{\alpha,\beta} = \varepsilon^{-\alpha\theta}\mathbb{E}\Big[\Big(\mathscr{g}_{\alpha,\beta}\big(V_{\varepsilon^{\alpha\theta}}\big) - \mathscr{g}_{\alpha,\beta}(V_0) - \int_0^{\varepsilon^{\alpha\theta}} V_s\mathrm{d}s\Big)^2\Big] = \varepsilon^{-\alpha\theta}\Big\{\mathbb{E}\Big[\Big(\mathscr{g}_{\alpha,\beta}\big(V_{\varepsilon^{\alpha\theta}}\big) - \mathscr{g}_{\alpha,\beta}(V_0)\Big)^2\Big] \\ + \mathbb{E}\Big[\Big(\int_0^{\varepsilon^{\alpha\theta}} V_s\mathrm{d}s\Big)^2\Big] - 2\mathbb{E}\Big[\Big(\mathscr{g}_{\alpha,\beta}\big(V_{\varepsilon^{\alpha\theta}}\big) - \mathscr{g}_{\alpha,\beta}(V_0)\Big)\int_0^{\varepsilon^{\alpha\theta}} V_s\mathrm{d}s\Big]\Big\}.\end{aligned} \tag{4.31}$$

The first term on the right hand side of (4.31) can be written :

$$\begin{aligned}\mathbb{E}\Big[\Big(\mathscr{g}_{\alpha,\beta}\big(V_{\varepsilon^{\alpha\theta}}\big) - \mathscr{g}_{\alpha,\beta}(V_0)\Big)^2\Big] &= 2\int \mathscr{g}_{\alpha,\beta}(x)^2\mathscr{m}_{\alpha,\beta}(\mathrm{d}x) - 2\mathbb{E}\Big[\mathscr{g}_{\alpha,\beta}(V_0)\mathscr{g}_{\alpha,\beta}\big(V_{\varepsilon^{\alpha\theta}}\big)\Big] = 2\int \mathscr{g}_{\alpha,\beta}(x)^2\mathscr{m}_{\alpha,\beta}(\mathrm{d}x) \\ &- 2\mathbb{E}\Big[\mathscr{g}_{\alpha,\beta}(V_0)\mathbb{E}\Big(\mathscr{g}_{\alpha,\beta}\big(V_{\varepsilon^{\alpha\theta}}\big)|V_0\Big)\Big] = 2\int \mathscr{g}_{\alpha,\beta}(x)^2\mathscr{m}_{\alpha,\beta}(\mathrm{d}x) - 2\mathbb{E}\Big[\mathscr{g}_{\alpha,\beta}(V_0)\big(\mathcal{T}_{\varepsilon^{\alpha\theta}}\mathscr{g}_{\alpha,\beta}\big)(V_0)\Big] \\ &= 2\int \mathscr{g}_{\alpha,\beta}(x)^2\mathscr{m}_{\alpha,\beta}(\mathrm{d}x) - 2\mathbb{E}\Big[\mathscr{g}_{\alpha,\beta}(V_0)\Big(\mathscr{g}_{\alpha,\beta}(V_0) + \int_0^{\varepsilon^{\alpha\theta}} (\mathcal{T}_s\mathrm{id})(V_0)\mathrm{d}s\Big)\Big] \\ &= -2\mathbb{E}\Big[\mathscr{g}_{\alpha,\beta}(V_0)\int_0^{\varepsilon^{\alpha\theta}} (\mathcal{T}_s\mathrm{id})(V_0)\mathrm{d}s\Big] = -2\int \mathscr{g}_{\alpha,\beta}(x)\,\mathscr{m}_{\alpha,\beta}(\mathrm{d}x)\int_0^{\varepsilon^{\alpha\theta}} (\mathcal{T}_s\mathrm{id})(x)\mathrm{d}s \\ &= -2\,\varepsilon^{\alpha\theta}\int x\mathscr{g}_{\alpha,\beta}(x)\mathscr{m}_{\alpha,\beta}(\mathrm{d}x) - 2\int \mathscr{g}_{\alpha,\beta}(x)\,\mathscr{m}_{\alpha,\beta}(\mathrm{d}x)\int_0^{\varepsilon^{\alpha\theta}} \big((\mathcal{T}_s\mathrm{id}) - \mathrm{id}\big)(x)\mathrm{d}s.\end{aligned}$$

By using the Hölder inequality, we prove that,

$$\mathbb{E}\left[\Big(\mathscr{g}_{\alpha,\beta}\big(V_{\varepsilon^{\alpha\theta}}\big) - \mathscr{g}_{\alpha,\beta}(V_0)\Big)^2\right] \sim -2\,\varepsilon^{\alpha\theta}\int x\mathscr{g}_{\alpha,\beta}(x)\mathscr{m}_{\alpha,\beta}(\mathrm{d}x), \text{ as } \varepsilon \to 0. \tag{4.32}$$

We need to distinguish two cases following the position of β with respect to 2. Indeed, if $2-\frac{\alpha}{2} < \beta < 2$,

$$\mathscr{g}_{\alpha,\beta} \in \mathrm{L}^{(3-\beta)/(2-\beta)}(\mathscr{m}_{\alpha,\beta}) \quad \text{and} \quad \lim_{s\to 0}\|(\mathcal{T}_s\mathrm{id}) - \mathrm{id}\|_{\mathrm{L}^{3-\beta}(\mathscr{m}_{\alpha,\beta})} = 0.$$

If $\beta \geq 2$,

$$\mathscr{g}_{\alpha,\beta} \in \mathrm{L}^{\frac{1}{\gamma}}(\mathscr{m}_{\alpha,\beta}) \quad \text{and} \quad \lim_{s\to 0}\|(\mathcal{T}_s\mathrm{id}) - \mathrm{id}\|_{\mathrm{L}^{1/(1-\gamma)}(\mathscr{m}_{\alpha,\beta})} = 0.$$

By using (4.25) and Fubini's theorem, the second term on the right hand side of (4.31) can be written

$$\begin{aligned}\mathbb{E}\Big[\Big(\int_0^{\varepsilon^{\alpha\theta}} V_s\mathrm{d}s\Big)^2\Big] &= \int_0^{\varepsilon^{\alpha\theta}} \mathrm{d}s\int_0^s \mathbb{E}\Big(V_sV_u\Big)\mathrm{d}u = \int_0^{\varepsilon^{\alpha\theta}} \mathrm{d}s\int_0^s \mathbb{E}\Big(V_{s-u}V_0\Big)\mathrm{d}u \\ &= \int_0^{\varepsilon^{\alpha\theta}} \mathrm{d}s\int_0^s \mathbb{E}\Big(V_0\,(\mathcal{T}_{s-u}\mathrm{id})(V_0)\Big)\mathrm{d}u = \int_0^{\varepsilon^{\alpha\theta}} \mathrm{d}u\,\mathbb{E}\Big(V_0\int_u^{\varepsilon^{\alpha\theta}} (\mathcal{T}_{s-u}\mathrm{id})(V_0)\,\mathrm{d}s\Big) \\ &= \int_0^{\varepsilon^{\alpha\theta}} \mathrm{d}u\,\mathbb{E}\left[V_0\Big(\big(\mathcal{T}_{\varepsilon^{\alpha\theta}-u}\mathscr{g}_{\alpha,\beta}\big)(V_0) - \mathscr{g}_{\alpha,\beta}(V_0)\Big)\right] = \int_0^{\varepsilon^{\alpha\theta}} \mathrm{d}u\int x\Big(\big(\mathcal{T}_{\varepsilon^{\alpha\theta}-u}\mathscr{g}_{\alpha,\beta}\big) - \mathscr{g}_{\alpha,\beta}\Big)(x)\mathscr{m}_{\alpha,\beta}(\mathrm{d}x).\end{aligned}$$

Once again by the Hölder inequality, we prove that

$$\mathbb{E}\left[\Big(\int_0^{\varepsilon^{\alpha\theta}} V_s \mathrm{d}s\Big)^2\right] = o(\varepsilon^{\alpha\theta}), \text{ as } \varepsilon \to 0. \tag{4.33}$$

Indeed, if $2 - \frac{\alpha}{2} < \beta < 2$ then $\mathrm{id} \in \mathrm{L}^{3-\beta}(m_{\alpha,\beta})$, and we note that

$$\lim_{\varepsilon\to 0} \sup_{0\le u\le \varepsilon^{\alpha\theta}} \|(\mathcal{T}_{\varepsilon^{\alpha\theta}-u} g_{\alpha,\beta}) - g_{\alpha,\beta}\|_{\mathrm{L}^{3-\beta/2-\beta}(m_{\alpha,\beta})} = 0.$$

Similarly, if $\beta \ge 2$ then $\mathrm{id} \in \mathrm{L}^{1/(1-\gamma)}(m_{\alpha,\beta})$, and we see that

$$\lim_{\varepsilon\to 0} \sup_{0\le u\le \varepsilon^{\alpha\theta}} \|\mathcal{T}_{\varepsilon^{\alpha\theta}-u} g_{\alpha,\beta}) - g_{\alpha,\beta}\|_{\mathrm{L}^{\frac{1}{\gamma}}(m_{\alpha,\beta})} = 0.$$

Finally, the third term in (4.31) is analysed by using the Cauchy-Schwarz inequality and the behaviour of the other terms. We get that

$$-2\,\mathbb{E}\Big[\Big(g_{\alpha,\beta}\big(V_{\varepsilon^{\alpha\theta}}\big) - g_{\alpha,\beta}(V_0)\Big)\int_0^{\varepsilon^{\alpha\theta}} V_s \mathrm{d}s\Big] = o(\varepsilon^{\alpha\theta}), \text{ as } \varepsilon \to 0. \tag{4.34}$$

Putting together (4.31)-(4.33), we obtain that

$$\kappa_{\alpha,\beta} = -2\int x g_{\alpha,\beta}(x) m_{\alpha,\beta}(\mathrm{d}x) + o(1), \text{ as } \varepsilon \to 0.$$

and the result is proved. □

4.3 Appendix

Proof of Lemma 3.2. We note that g_β is an odd function. Let us introduce $\varphi_\beta(x) = -\int_x^{+\infty} 2y \mathrm{e}^{c_\beta(y)} \mathrm{d}y$. By the continuity of g_β on $[0,1]$, it is sufficient to prove (3.2) for $x > 1$. Assume that $\beta \in [1,\infty)$. Then, since $x > 1$,

$$\varphi_\beta(x) = \int_x^{+\infty} z^{1-\beta}\left(-2z^\beta \mathrm{e}^{-\frac{2}{\beta+1}z^{\beta+1}}\right)\mathrm{d}z \ge \int_x^{+\infty} -2z^\beta \mathrm{e}^{-\frac{2}{\beta+1}z^{\beta+1}}\mathrm{d}z = -\mathrm{e}^{-\frac{2}{\beta+1}x^{\beta+1}},$$

hence

$$\int_1^x \mathrm{e}^{\frac{2}{\beta+1}y^{\beta+1}} \varphi_\beta(y)\mathrm{d}y \ge 1 - x.$$

(3.2) is true in this case. If $\beta \in [0,1)$, by integration by parts,

$$\begin{aligned}\varphi_\beta(x) = \int_x^{+\infty} z^{1-\beta}\big(-2z^\beta \mathrm{e}^{-\frac{2}{\beta+1}z^{\beta+1}}\big)\mathrm{d}z &= -x^{1-\beta}\mathrm{e}^{-\frac{2}{\beta+1}x^{\beta+1}} + \frac{1-\beta}{2}\int_x^{+\infty} z^{-2\beta}\big(-2z^\beta \mathrm{e}^{-\frac{2}{\beta+1}z^{\beta+1}}\big)\mathrm{d}z \\ &\ge -x^{1-\beta}\mathrm{e}^{-\frac{2}{\beta+1}x^{\beta+1}} - \frac{1-\beta}{2}x^{-2\beta}\mathrm{e}^{-\frac{2}{\beta+1}x^{\beta+1}}.\end{aligned}$$

Hence,

$$\int_1^x \mathrm{e}^{\frac{2}{\beta+1}y^{\beta+1}}\varphi_\beta(y)\mathrm{d}y \ge \int_1^x \left(-y^{1-\beta} - \frac{1-\beta}{2}y^{-2\beta}\right)\mathrm{d}y,$$

and (3.2) follows. More generally, assume that $\beta \in [-\frac{n}{n+2}, \frac{1-n}{n+1})$, for an integer $n \ge 0$. Set $d_0 = 1$ and $d_k := 2^{-k}\prod_{j=0}^{k-1}((1-\beta) - j(1+\beta))$, for $k \ge 1$ integer. By the choice of n, we can see that $d_n > 0$. If we iterate n times the integration by parts, we get:

$$\varphi_\beta(x) = -\sum_{k=0}^{n} d_k x^{(1-\beta)-k(1+\beta)}\mathrm{e}^{-\frac{2}{\beta+1}x^{\beta+1}} + d_n\int_x^{+\infty} z^{(1-\beta)-(n+1)(\beta+1)}(-2z^\beta \mathrm{e}^{-\frac{2}{\beta+1}z^{\beta+1}})\mathrm{d}z.$$

Since $(1-\beta)-(n+1)(\beta+1)\leq 0$, we can write

$$\varphi_\beta(x)\geq -\left(\sum_{k=0}^n d_k x^{(1-\beta)-k(1+\beta)}+d_n x^{(1-\beta)-(n+1)(\beta+1)}\right)\mathrm{e}^{-\frac{2}{\beta+1}x^{\beta+1}}.$$

By integrating, we have

$$\int_1^x \mathrm{e}^{\frac{2}{\beta+1}y^{\beta+1}}\varphi_\beta(y)\mathrm{d}y\geq \int_1^x\left(\sum_{k=0}^n d_k y^{(1-\beta)-k(1+\beta)}+d_n y^{(1-\beta)-(n+1)(\beta+1)}\right)\mathrm{d}y,$$

and we easily deduce (3.2). The proof of (3.2) is complete for all $\beta\in(-1,\infty)$. □

Proof of Lemma 4.5. Recall that $h_{p,\gamma}(x)=(1+|x|^{p\gamma})^{1/p}$ and assume firstly that $|x|<1$. Since $h_{p,\gamma}$ is continuously differentiable and equivalent to $|x|^\gamma$, as $|x|\to\infty$, there exists $k>0$ such that

$$|h_{p,\gamma}(x+y)-h_{p,\gamma}(x)|\leq |y|\sup_{z\in[-2,2]}|h'_{p,\gamma}(z)|\mathbb{1}_{\{|y|\leq 1\}}+k|y|^\gamma\mathbb{1}_{\{|y|>1\}}.$$

The desired inequality is then clear. Secondly, assume that $|x|\geq 1$. It is a simple computation to see that for all $z\geq 0$ and $r>0$, there exists $c_r>0$, such that

$$(1+z)^r-1\leq c_r\big(z\mathbb{1}_{\{z\leq 1\}}+z^r\mathbb{1}_{\{z>1\}}\big).$$

We deduce that, for all $(u,v)\in[0,\infty)\times[0,\infty)$, there exist $k_r>0$ such that

$$(u+v)^r-u^r=u^r\left[\Big(1+\frac{v}{u}\Big)^r-1\right]\leq k_r\big(vu^{r-1}\mathbb{1}_{\{v\leq u\}}+v^r\mathbb{1}_{\{u<v\}}\big). \tag{4.35}$$

Since $x\neq 0$,

$$\begin{aligned}|h_{p,\gamma}(x+y)-h_{p,\gamma}(x)|&=|x|^\gamma\left|\Big(\frac{1}{|x|^{p\gamma}}+\Big|1+\frac{y}{x}\Big|^{p\gamma}\Big)^{1/p}-\Big(\frac{1}{|x|^{p\gamma}}+1\Big)^{1/p}\right|\\&\leq |x|^\gamma\left[\Big(\frac{1}{|x|^{p\gamma}}+\Big(1+\Big|\frac{y}{x}\Big|\Big)^{p\gamma}\Big)^{1/p}-\Big(\frac{1}{|x|^{p\gamma}}+1\Big)^{1/p}\right].\end{aligned}$$

Applying (4.35) with $u=\frac{1}{|x|^{p\gamma}}+1$, $v=\Big(1+\Big|\frac{y}{x}\Big|\Big)^{p\gamma}-1$ and $r=\frac{1}{p}$, we obtain

$$\begin{aligned}|h_{p,\gamma}(x+y)-h_{p,\gamma}(x)|\leq k_{1/p}|x|^\gamma&\left[\Big(\Big(1+\Big|\frac{y}{x}\Big|\Big)^{p\gamma}-1\Big)^{1/p}\mathbb{1}_{\{i(x)\leq|y|\}}\right.\\&\left.+\Big(\Big(1+\Big|\frac{y}{x}\Big|\Big)^{p\gamma}-1\Big)\Big(\frac{1}{|x|^{p\gamma}}+1\Big)^{\frac{1-p}{p}}\mathbb{1}_{\{|y|<i(x)\}}\right].\end{aligned}$$

Since $i(x)>|x|$, we can use again (4.35) to estimate the first term in the bracket on the right hand of the latter inequality. We let $u=1$, $v=\big|\frac{y}{x}\big|$ and $r=p\gamma$ and we get

$$|h_{p,\gamma}(x+y)-h_{p,\gamma}(x)|\leq k_{1/p}k_{p\gamma}|y|^\gamma\mathbb{1}_{\{i(x)\leq|y|\}}+k_{1/p}|x|^\gamma\Big(\Big(1+\Big|\frac{y}{x}\Big|\Big)^{p\gamma}-1\Big)\Big(\frac{1}{|x|^{p\gamma}}+1\Big)^{\frac{1-p}{p}}\mathbb{1}_{\{|y|<i(x)\}}.$$

Since $|x|\geq 1$, ${}^{i(x)}/_{|x|}$ is bounded, and since $p>1$, $({}^{1}/_{|x|^{p\gamma}}+1)^{(1-p)/p}\leq 1$. Using that $p\gamma>2$ and the fact that ${}^{|y|}/_{|x|}$ is bounded, we obtain the existence of a $k'>0$ such that

$$\Big(\Big(1+\frac{|y|}{|x|}\Big)^{p\gamma}-1\Big)\leq k'\frac{|y|}{|x|}.$$

Taking $k=\max(k_{1/p}k_{p\gamma},k_{1/p}k')$, we get the second inequality of the first part of Lemma 4.5.

We proceed with the second part and we note that, since $p\gamma > 2$, $h_{p,\gamma}$ is twice differentiable with

$$h''_{p,\gamma}(x) = \gamma\,|x|^{p\gamma-2}\Big[(\gamma-1)|x|^{p\gamma} + p\gamma - 1\Big](1+|x|^{p\gamma})^{1/p-2}.$$

Moreover, since $\gamma < \alpha < 2$, $h''_{p,\gamma} \in \mathrm{L}^\infty$. We split $(\mathcal{A}_{\alpha,\beta}\, h_{p,\gamma})(x)$ into three terms

$$\mathcal{A}_{\alpha,\beta} h_{p,\gamma}(x) = -\gamma \frac{|x|^{p\gamma+\beta-1}}{(1+|x|^{p\gamma})^{1-1/p}} + \int_{|y|\le 1} \Big[h_{p,\gamma}(x+y) - h_{p,\gamma}(x) - y h'_{p,\gamma}(x)\Big]\nu(\mathrm{d}y) \\ + \int_{|y|>1} \Big[h_{p,\gamma}(x+y) - h_{p,\gamma}(x)\Big]\nu(\mathrm{d}y).$$

The first term on the right hand side is equivalent to $-\gamma|x|^{\gamma+\beta-1}$ at infinity, while for the second term, since $|y| \le 1$, we have

$$\Big|h_{p,\gamma}(x+y) - h_{p,\gamma}(x) - y h'_{p,\gamma}(x)\Big| \le y^2 \sup_{|z|\le 1} |h''_{p,\gamma}(x+z)| \le y^2 \|h''_{p,\gamma}\|_\infty.$$

Hence

$$\left|\int_{|y|\le 1} \Big[h_{p,\gamma}(x+y) - h_{p,\gamma}(x) - y h'_{p,\gamma}(x)\Big]\nu(\mathrm{d}y)\right| \le c_\alpha \|h''_{p,\gamma}\|_\infty,$$

where $c_\alpha := \int_{|y|\le 1} y^2 \nu(\mathrm{d}y)$. We use the first part of the lemma to estimate the third term on the right hand side of the expression of $\mathcal{A}_{\alpha,\beta} h_{p,\gamma}(x)$. There are two situations: if $|x| \ge 1$, we get

$$\Big|h_{p,\gamma}(x+y) - h_{p,\gamma}(x)\Big| \le k(|y||x|^{\gamma-1}\mathbb{1}_{\{|y|\le i(x)\}} + |y|^\gamma \mathbb{1}_{\{i(x)<|y|\}}).$$

Hence

$$\Big|\int_{|y|>1} \Big[h_{p,\gamma}(x+y) - h_{p,\gamma}(x)\Big]\nu(\mathrm{d}y)\Big| \le k|x|^{\gamma-1} \int_{\{i(x)\ge|y|>1\}} |y|\nu(\mathrm{d}y) + k\int_{\{\max(1,i(x))\le|y|\}} |y|^\gamma \nu(\mathrm{d}y) \\ \le k|x|^{\gamma-1} \int_{\{i(x)\ge|y|>1\}} |y|\nu(\mathrm{d}y) + k c'_{\alpha,\gamma}\,,$$

where $c'_{\alpha,\gamma} := \int_{\{|y|>1\}} |y|^\gamma \nu(\mathrm{d}y)$. Since $i(x) = O(|x|)$, as $|x| \to \infty$,

$$k|x|^{\gamma-1} \int_{\{i(x)\ge|y|>1\}} |y|\nu(\mathrm{d}y) = O(|x|^{\gamma-1}) + O(|x|^{\gamma-\alpha}), \quad \text{as } |x| \to \infty.$$

If $|x| < 1$, since $|y| > 1$,

$$|h_{p,\gamma}(x+y) - h_{p,\gamma}(x)| \le k|y|^\gamma,$$

so

$$\Big|\int_{|y|>1} \Big[h_{p,\gamma}(x+y) - h_{p,\gamma}(x)\Big]\nu(\mathrm{d}y)\Big| \le \int_{|y|>1} |y|^\gamma \nu(\mathrm{d}y) < +\infty.$$

Denote by u the continuous function $-\mathcal{A}_{\alpha,\beta} h_{p,\gamma}$. Putting together the previous estimates we obtain that, since $\beta > 1$ and $\frac{2}{p} < \gamma < \alpha$,

$$u(x) \sim |x|^{\gamma+\beta-1}, \quad \text{as } |x| \to \infty,$$

and since $\gamma > 2 - \beta$,

$$1 + |x| = o(u(x)), \quad \text{as } |x| \to \infty.$$

Set $K = [k^-, k^+]$, with

$$k^+ := \inf\{x > 0 : y \ge x \Rightarrow u(y) > y + 1\}, \quad k^- := \sup\{x < 0 : y \le x \Rightarrow u(y) > -y + 1\}\,,$$

and

$$d := -\inf_{\{x\in K\}} (u(x) - 1 - |x|), \quad f_{p,\alpha,\beta,\gamma}(x) := u(x)\mathbb{1}_{K^c} + (1+|x|)\mathbb{1}_K\,.$$

Then relations (4.17) and (4.18) hold true and the proof is complete. □

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