Electronic Journal of Probability

Electron. J. Probab. **20** (2015), no. 86, 1–17.
ISSN: 1083-6489 DOI: 10.1214/EJP.v20-4060

A stochastic particles model of fragmentation process with shattering*

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Abstract

A stochastic particle model for fragmentation process is considered. Evolution of the system of particles is described by a stochastic process on a space of discrete measures on a Polish space. A phenomenon of shattering into dust is studied and some criteria for mass conservation and loss of mass in our model are proven.

Keywords: fragmentation; stochastic particle systems; loss of mass; shattering; martingale problem.
AMS MSC 2010: Primary 60J35; 60J80, Secondary 70F45.
Submitted to EJP on January 19, 2015, final version accepted on August 9, 2015.

1 Introduction

The fragmentation phenomenon can be observed commonly in many physical, industrial and biological processes, including grinding and crashing of such materials as ore, stone or flour, polymer degradation, dissolving, fragmentation of organisms, or proliferation of cells, etc. Fragmentation has been mathematically described with various methods both stochastic and deterministic. There is a vast literature on fragmentation process, originating from physics of polymer degradation [26]. Some of them use deterministic description by means of transport equations [28, 33, 4, 29, 2], while the other ones use probabilistic approach, *e.g.* [23, 18, 10, 14, 21, 8]. The interesting review is given also in [31].

It has been observed that sufficiently rapid fragmentation may result in the decrease of total mass of the system, even though the mass is conserved in every breakup of a single particle. Probably the first stochastic model that provides conditions for loss of mass into zero-size particles is due to Filippov [18]. McGrady and Ziff [27] observed that, if the breakup is fast for small particles, some solutions for fragmentation equation, which are formally conservative, do not in fact preserve the total mass of particles. They called this phenomenon “shattering”. The loss of mass in similar equations was intensively investigated by means of differential equations [15, 1] and semigroups [4, 3, 5]. There are also several approaches to stochastic modeling of the fragmentation with shattering on

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the microscopic individual-based level. The first one is the *homogeneous fragmentation process* introduced by Bertoin [10], extended later for more general cases [12, 20] and for multitype particles [13]. Some other methods are used in the papers of Jeon [22] and Fournier and Giet [19]. The next possibility is the stochastic particle systems approach [7, 30] which will be used also here.

In [30] Wagner considers fragmentation-coagulation models in the framework of jump processes on the space of measures consisted of finite number of Dirac deltas and provides some criteria for explosion of such processes. Since in that case emergence of infinite number of particles implies explosion, his model does not allow for shattering.

The aim of this paper is to present a general stochastic individual-based fragmentation model with infinitely many particles, in which shattering may appear. We consider a number (possibly infinite) of particles with states in some space X. Usually, in fragmentation models particles are structured by their size or mass (either discrete, $X = \{1, 2, \dots\}$, or continuous, $X = (0, \infty)$), but we use more general setting here. Namely, the state space X is an arbitrary locally compact Polish space, which covers such examples as size and position of a particle or many types of particles. Each particle may split up into some number of particles or jump to another place. All the events happen randomly in time with some probability rates. We assume that each particle is described by its mass, which is given by a continuous positive function $h : X \to \mathbb{R}_+$. Total mass is preserved by every event. Preservation of mass means that after a fragmentation of a particle x into $x_1, x_2, \dots$ we have $\sum_i h(x_i) = h(x)$. We describe a particle in state x by the Dirac delta measure δ_x, so the state of the system is described by a measure $\mu = \sum_{i=1}^{k_\mu} \delta_{x_i}$,with a number of particles $k_\mu \in \{0, 1, 2, \dots\} \cup \{\infty\}$ and a finite total mass, *i.e.* $\sum_{i=1}^{k_\mu} h(x_i) < \infty$. Thus we define a phase space of the system as

$$\mathcal{N}_h = \left\{ \nu = \sum_{i=1}^{k_\nu} \delta_{x_i} : k_\nu \in \mathbb{N} \cup \{\infty\}, x_i \in X \text{ and } \sum_{i=1}^{k_\nu} h(x_i) < \infty \right\}. \tag{1.1}$$

We may imagine a jump process that satisfies the description above and describe it in the following way: let q be a probabilistic kernel from X into $\mathcal{N}_h$ that describe the fragmentations and jumps of particles, and a the fragmentation rate. In particular $a(x)q(x, \{\nu \in \mathcal{N}_h : \|\nu\| = 1\})$ is the probability rate of jump of a particle x, $a(x)q(x, \{\nu \in \mathcal{N}_h : \|\nu\| = n\})$ is the probability rate of fragmentation into n particles, whereas $a(x)q(x, \{\nu \in \mathcal{N}_h : \|\nu\| = \infty\})$ is the probability rate of fragmentation into the infinite number of particles. We can write the jump kernel of such a process in the form

$$\kappa(\nu, B) = \sum_{i=1}^{k_\nu} a(x_i) \int_{\mathcal{N}_h} \mathbb{1}_B\big(\nu - \delta_{x_i} + \eta\big) q(x_i, \mathrm{d}\eta), \tag{1.2}$$

for

$$\nu = \sum_{i=1}^{k_\nu} \delta_{x_i} \text{ and } B \in \mathcal{B}(\mathcal{N}_h).$$

If a number of particles in ν where always finite, κ would be indeed a kernel of some jump process (c.f. [30]). However, in our case $\kappa(\nu, \mathcal{N}_h)$ is generally infinite and, there does not exist a jump process governed by this kernel, because it is possible here to have infinite number of particles, and therefore the infinite number of events ('jumps') in each time interval. That is why we use the martingale problem approach and define the *infinitely-many-particles fragmentation process* as a solution to the martingale problem with the operator

$$\mathcal{L}f(\nu) = \int_{\mathcal{N}_h} \big(f(\mu) - f(\nu)\big)\kappa(\nu, \mathrm{d}\mu) \\ = \sum_{i=1}^{k_\nu} a(x_i) \int_{\mathcal{N}_h} \big[f\big(\nu - \delta_{x_i} + \eta\big) - f(\nu)\big] q(x_i, \mathrm{d}\eta). \tag{1.3}$$

In the paper we prove the existence of such a process and give some criteria for mass conservation and shattering. The IPF process defined in our paper can be thought of as a microscopic realisation of the Filippov's idea [18]. The setting of the IPF is quite general thanks to the fact that phase space is a locally compact Polish space. This generality allows for the description of moving particles or multitype processes. Moreover, this setting may be generalized to describe *e.g.* continuous movement or to include coagulation.

Section 2 provides some notations, the main results and some examples of applications. The next section contains some properties of considered spaces and proofs of the main results. Some auxiliary definitions and results are stated in the appendix.

2 Main results

2.1 Notation

If E is a Polish space, we write $\mathbf{C}_b(E)$, and $\mathbf{C}_c(E)$ for spaces of bounded continuous, and continuous with compact support functions, respectively. The sets of positive Radon measures and probabilistic Radon measures on E are denoted by $\mathcal{M}(E)$ and $\mathcal{M}_1(E)$, respectively. Note that Radon measure is an inner regular and locally finite measure on σ-algebra of Borel sets, see *e.g.* [6].

Throughout the paper, X is a locally compact Polish space. We write $\mathbf{C}_0(X)$ for the space of continuous functions on X vanishing at infinity. For $\mu \in \mathcal{M}(X)$ and a Borel measurable function f we use the notation $\langle f, \mu\rangle = \int_E f(x)\mu(\mathrm{d}x)$ and $\|\mu\| = \langle 1, \mu\rangle$. Let $\mathcal{N} \subset \mathcal{M}(X)$ denote the space of integer-valued measures:

$$\mathcal{N} = \left\{ \nu = \sum_{i=1}^{k_\nu} \delta_{x_i} : k_\nu \in \mathbb{N} \cup \{\infty\},\ x_i \in X \right\}, \tag{2.1}$$

where δ_x is a Dirac delta measure concentrated at x. Moreover, for a function $h \in C_0(X)$ let us define $\mathcal{M}_h = \{\mu \in \mathcal{M} : \langle h, \mu\rangle < \infty\}$ and $\mathcal{N}_h = \mathcal{N} \cap \mathcal{M}_h$. If $\mu \in \mathcal{N}$ is a measure of the form $\mu = \sum_{i=1}^{k} \delta_{x_i}$, we will sometimes write for shortness $\sum_{x_i \in \mu}$ meaning that sum extends over all $x_1, x_2, \ldots, x_k$ (even if some of them are equal).

We say that an E-valued càdlàg process $(\xi(t))_{t\geq 0}$ (*i.e.* with trajectories in the Skorochod space $D([0,\infty), E)$) solves a martingale problem for an operator $\mathcal{L}$ and initial value $\xi_0 \in E$ (or, equivalently, an $(\mathcal{L}, \xi_0)$-martingale problem) if

$$\mathrm{Prob}(\xi(0) = \xi_0) = 1$$

and

$$f(\xi(t)) - f(\xi(0)) - \int_0^t \mathcal{L}f(\xi(s))\mathrm{d}s \tag{2.2}$$

is a martingale with respect to the filtration generated by ξ for each f from the domain of $\mathcal{L}$.

By a *kernel* from one measurable space $(E_1, \mathcal{B}_1)$ to another $(E_2, \mathcal{B}_2)$ we mean a function $\kappa : E_1 \times \mathcal{B}_2 \to \mathbb{R}$ such that $\kappa(\cdot, B)$ is measurable for any $B \in \mathcal{B}_2$ and $\kappa(e, \cdot) \in \mathcal{M}_f(E)$ for any $e \in E_1$. κ is a *probabilistic kernel* if $\kappa(e, \cdot)$ is a probabilistic measure for any $e \in E_1$.

2.2 Infinitely-many-particles fragmentation process

Let X be a locally compact Polish space and fix a positive function $h \in \mathbf{C}_0(X)$. We define *infinitely-many-particles fragmentation process* as a solution to the martingale problem with the operator $\mathcal{L}$ given by (1.3) with the domain

$$\mathcal{D}(\mathcal{L}) = \left\{ F_g \in \mathbf{C}_b(\mathcal{N}) : F_g(\nu) = e^{-\langle g,\nu\rangle} \text{ with } g \in \mathbf{C}_c(X) \text{ and } \mathcal{L}F_g \in \mathbf{C}_b(\mathcal{N}) \} \right\}. \quad (2.3)$$

Theorem 2.1. *Let $h \in \mathbf{C}_0(X)$ be a positive function. Let us assume that a is a continuous function and q is such a probabilistic kernel from X to $\mathcal{N}_h$ that the function $x \mapsto \int_{\mathcal{N}_h} e^{\langle g,\eta\rangle} q(x, \mathrm{d}\eta)$ is continuous for $g \in \mathbf{C}_c(X)$. Moreover, assume that the mass h is conserved by q, namely:*

$$q(x, \{\nu \in \mathcal{N}_h : \langle h, \nu\rangle \neq h(x)\}) = 0 \ \text{ for all } x \in X. \quad (2.4)$$

Then, for every $\nu \in \mathcal{N}_h$ there exists a unique solution to the martingale problem with the operator $\big(\mathcal{L}, \mathcal{D}(\mathcal{L})\big)$ and the initial state ν with values in $\mathcal{N}$.

Definition 2.2. *The unique solution to the martingale problem with the operator $\big(\mathcal{L}, \mathcal{D}(\mathcal{L})\big)$ will be called the infinitely-many-particles fragmentation process or, in short, the IPF process.*

2.3 Criteria for the conservation of mass

We assume that in any particular fragmentation event the mass h is conserved, which is ensured by the assumption (2.4). Nevertheless, it occurs that this condition not always guarantees that the total mass of the system $\langle h, \xi_t\rangle$ is conserved. In this section we give firstly some conditions when the mass is really conserved and then some conditions when it is not.

Theorem 2.3. *If there exist a positive function $g \in \mathbf{C}_0(X)$ and a number $C > 0$ such that*

$$\frac{g(x)}{h(x)} \to \infty \text{ as } h(x) \to 0 \quad (2.5)$$

and

$$a(x) \int_{\mathcal{N}_h} \big[\langle g, \eta\rangle - g(x)\big] q(x, \mathrm{d}\eta) \le C g(x), \quad (2.6)$$

for all $x \in X$ and $\langle g, \xi_0\rangle < \infty$ then then the IPF process starting from ξ_0 is mass conserving, i.e. $\langle h, \xi_t\rangle = \langle h, \xi_0\rangle$ for all $t > 0$ a.s.

This theorem allows to state the following criterion in a more specific situation. We will call an IPF *homogeneous* if there exists a measure $\tilde{q}$ on $(0, 1]$ such that

$$\int_{\mathcal{N}_h} \langle h, \eta|_{X_\alpha}\rangle q(x, \mathrm{d}\eta) = h(x) \int_{\alpha/h(x)}^{1} s \, \tilde{q}(\mathrm{d}s),$$

for all $x \in X$ and $\alpha \le h(x)$, where $X_\alpha = \{x \in X : h(x) \ge \alpha\}$.

Corollary 2.4. *If the fragmentation rate $a(x)$ is bounded and the fragmentation is homogeneous, then the IPF process is mass conserving.*

On the other side, if the fragmentation rate grows too fast for small h then decay of mass is possible.

Theorem 2.5. *If the fragmentation rate a satisfies*

$$a(x) > C\, h(x)^{-\beta},$$

for some positive constants β and C, and there exists such a $\delta > 0$ that

$$\frac{1}{h(x)^{1+\beta}} \int_{\mathcal{N}_h} \langle h^{1+\beta}, \eta \rangle q(x, \mathrm{d}\eta) < 1 - \delta,$$

for all $x \in X$, then the IPF process is shattering, i.e. some mass is lost with positive probability.

The second condition assures that fragmentation does not stop for x with small mass. In particular, it is trivially satisfied for the homogeneous case. Note that the loss of mass is connected to the situation when some particles tend to infinity (where $h = 0$) and the corresponding Dirac masses converge vaguely to zero measure.

2.4 Applications

Let us firstly show the relation to some already-known models. Many of the available results on stochastic fragmentation are (or can be) formulated in terms of *partition fragmentation*, which is a process that lives in the space of partitions of $\mathbb{N}$. Most of them may be equivalently stated by means of the *interval representation* [11, 20], which is a process whose state space is the set of open subsets of the interval $(0, 1)$. Sometimes the so called *ranked fragmentation* approach is used [10, 9, 20] which lives in the space $\mathcal{S}^{\downarrow} = \{\mathbf{s} = (s_1, s_2, \dots) : s_1 \geq s_2 \geq \dots \geq 0, \sum_i s_i \leq 1\}$. See [9] for the proof of equivalence of those approaches. We will show that the special case of our IPF with $X = (0, 1]$, $h(x) = x$ and constant $a(x) = a$ is equivalent to the Bertoin's homogeneous fragmentation with finite splitting measure.

Proposition 2.6. *Let λ be an $\mathcal{S}^{\downarrow}$-valued ranked homogeneous fragmentation with no erosion and with a finite Lévy measure ν. Let $\Xi_x : \mathcal{S}^{\downarrow} \to \mathcal{N}_h$ for any $x \in (0, 1]$ be given by $\Xi_x(\mathbf{s}) = \sum_{\substack{i \in \mathbb{N} \\ s_i > 0}} \delta_{x\, s_i}$. Then $\xi(t) = \Xi_1(\lambda(t))$ is the IPF process with $a = \|\nu\|$ and $q(x, \mathrm{d}\eta) = \nu(\Xi_x^{-1}(\mathrm{d}\eta))$.*

Proof. As proved by Berestycki in [9], there exists a Poisson point process $K = (\mathbf{s}(t), k(t))$ with values in $\mathcal{S}^{\downarrow} \times \mathbb{N}$ and intensity measure $\nu \times \#$, such that λ only jumps at times at which $(\mathbf{s}(t); k(t))$ has an atom, and at such a time $\lambda(t)$ is obtained from $\lambda(t^-)$ by dislocating the $k(t)$-th component of $\lambda(t^-)$ by $\mathbf{s}(t)$ (i.e. replacing $\lambda_{k(t)}(t^-)$ by the sequence $\lambda_{k(t)}(t^-)\mathbf{s}(t)$) and reordering the new sequence of fragments. Therefore, for any $F_g \in \mathcal{D}(\mathcal{L})$ one can write

$$\tilde{F}_g(\lambda(t)) = \tilde{F}_g(\lambda(s)) + \int_{[s,t]} \left(\tilde{F}_g(\lambda^{(k(r), \mathbf{s}(r))}(r^-)) - \tilde{F}_g(\lambda(r^-)) \right) \mathrm{d}K_r,$$

where $\tilde{F}_g = F_g \circ \Xi_1$ and $\lambda^{(k(r), \mathbf{s}(r))}(r^-)$ is $\lambda(r^-)$ with $k(r)$-th component replaced by the sequence $\lambda_{k(r)}(r^-)\mathbf{s}(r))$ and with reordered fragments. Note that F_g depends on a finite number of points (since g has a compact support) and does not depend on the order of points. Taking expectations, we obtain

$$\mathrm{E}[\tilde{F}_g(\lambda(t))] = \mathrm{E}[\tilde{F}_g(\lambda(s))] + \int_s^t \sum_k \tilde{F}_g(\lambda(r^-)) \left(e^{g(\lambda_k(r^-)) - \sum_i g(\lambda_k(r^-) s_i)} - 1 \right) \nu(\mathrm{d}\mathbf{s}),$$

which means that

$$F_g(\xi(t)) - F_g(\xi(0)) - \int_0^t \mathcal{L} F_g(\xi(r)) \mathrm{d}r$$

is a martingale, and thereby ξ is an IPF process. ☐

It is intuitively clear that if we take $a(x) = \|\nu\|\, e^{\alpha x}$ for some $\alpha \in \mathbb{R}$ in the construction above, then we obtain the Bertoin's self-similar fragmentation, *cf.* [11]. However, the proof of this fact exceeds the scope of this paper. Taking $\alpha \in \mathbb{R}$ and a finite Lévy measure ν, we may think about the IPF process on $X = (0,1]$ with $a = \|\nu\| e^{\alpha x}$ and $q(x, \mathrm{d}\eta) = \nu(\Xi_x^{-1}(\mathrm{d}\eta))$ as the self-similar fragmentation with index α. Then Corollary 2.4 and Theorem 2.5 imply the following dichotomy: if $\alpha \geq 0$ then self-similar fragmentation is mass-conserving; if $\alpha < 0$ then self-similar fragmentation is shattering.

Let us look at another possible application of the model: the alternative description of the multitype fragmentation processes, *cf.* [13]. Consider a finite set of types $\{1, \dots, k\}$ and take the state space $X = (0,1] \times \{1, \dots, k\}$, so that $x = (s,i)$ denotes a particle of size $s \in (0,1]$ and type i; the mass function is just $h(s,i) = s$. Then one can define any rule of splitting of such a particle into smaller ones of any type by specifying a and q (as long as the assumptions of Theorem 2.1 are satisfied). In particular, taking a family of finite dislocation measures $(\nu_i)_{i \in \{1,\dots,k\}}$ from [13] with no erosion ($c_i = 0$), one can construct as before the splitting kernel q that describes the Bertoin's homogeneous multitype fragmentations. Note also, that the set of types does not need to be finite — it only has to be compact. We may take any compact set of types, say Y, and define $X = (0,1] \times Y$.

In a very similar way we can describe particles that, besides of splitting, move in some compact space according to a jump process possibly depending on size. Namely, let a compact set Y denote the space, let $X = (0,1] \times Y$ and $h(s,y) = s$. Consider a family $\kappa(s; y, \mathrm{d}z)$ of jump kernels on Y parametrized by size, that is for any $s \in (0,1]$ operator $L_s f(y) = \int_E [f(z) - f(y)] \kappa(s; y, \mathrm{d}z)$ generates a jump process on Y. Moreover, consider a family of splitting kernels $q_0(y; s, \mathrm{d}\eta)$ parametrized by y and splitting rate $a_0(s,y)$. Then we can define an IPF process by $a(s,y) = a_0(s,y) + \kappa(s; y, Y)$ and

$$q((s,y), B_s \times B_y) = q_0(y; s, B_y) + \frac{1}{\kappa(s; y, Y)} \kappa(s; y, B_s),$$

for $B_s \in \mathcal{B}((0,1])$ and $B_y \in \mathcal{B}(Y)$, if the resulting a and q satisfy the continuity conditions of Theorem 2.1.

3 Proofs

3.1 Properties of spaces

As before, X is a locally compact Polish space and $h \in \mathbf{C}_0(X)$ is a positive function. For any $\alpha > 0$ let us denote

$$X_\alpha = \{x \in X : h(x) \geq \alpha\}.$$

Note that X_α is a compact set and that for any compact set $C \subset X$ there exist $\alpha > 0$ such that $C \subset X_\alpha$. In the present section we provide some necessary information on the spaces of measures $\mathcal{N}$, $\mathcal{M}_h$, and $\mathcal{N}_h$. The spaces $\mathcal{M} = \mathcal{M}(X)$ and $\mathcal{N}$ (and sometimes $\mathcal{M}_h$ and $\mathcal{N}_h$) are equipped with the metrics ρ_0 which is a metrization of vague convergence of measures according to the proof of Theorem 31.5 from [6]:

Definition 3.1. *Let D_0 be a dense subset of $\mathbf{C}_c(X)$. Let $(H_n)_{n \in \mathbb{N}}$ be a sequence of compact subsets of X and let $(G_n)_{n \in \mathbb{N}}$ be a sequence of open subsets such that $H_n \subset G_n \subset H_{n+1}$ for $n \in \mathbb{N}$ and $H_n \uparrow X$. Thus let $e_n : X \to [0,1]$ be such functions from $\mathbf{C}_c(X)$ that $e_n(H_n) = \{1\}$ and $e_n(X \setminus G_n) = \{0\}$ (one can take e.g. $e_n = \chi_n$ given by* (3.3)*). Let now $(d_n)_{n \in \mathbb{N}}$ be an enumeration of the elements of the countable set*

$$D = D_0 \cup \{f \cdot e_n : f \in D_0, n \in \mathbb{N}\} \cup \{e_n : n \in \mathbb{N}\},$$

namely $D = \{d_n : n \in \mathbb{N}\}$. Using this enumeration we define a metric

$$\rho_0(\mu, \nu) = \sum_{n=1}^{\infty} 2^{-n} \min\{1, |\langle d_n, \mu - \nu \rangle|\}. \tag{3.1}$$

Remark 3.2. Note that $(\mathcal{M}, \rho_0)$ is a Polish space and $(\mathcal{N}, \rho_0)$ as well, since $\mathcal{N}$ is closed in $(\mathcal{M}, \rho_0)$.

Spaces $\mathcal{M}_h$ and $\mathcal{N}_h$ (unless otherwise stated) are provided with the metrics ρ_h defined by the formula

$$\rho_h(\mu, \nu) = \rho_0(\mu, \nu) + \sup_{n \in \mathbb{N}} |\langle h \cdot (1 - \chi_n), \mu - \nu \rangle|, \tag{3.2}$$

where $\chi_n \in \mathbf{C}_c(X)$ are defined in (3.3).

Lemma 3.3. *For every $\alpha > 0$ the set $\mathcal{K}_{h,\alpha} = \{\nu \in \mathcal{N}_h : \langle h, \nu \rangle \le \alpha\}$ is vaguely compact.*

Proof. According to Theorem 31.2 of [6], a set $\mathcal{K}_{h,\alpha} \subset \mathcal{M}$ is vaguely relatively compact if $\sup_{\mu \in \mathcal{K}_{h,\alpha}} |\int f \, \mathrm{d}\mu| < \infty$ for every $f \in \mathbf{C}_c(X)$. But, since $h > 0$, for every $f \in \mathbf{C}_c(X)$ there exists such an $M > 0$ that $\|f\| \le Mh$. Thus $|\int f \, \mathrm{d}\mu| \le M \int h \, \mathrm{d}\mu \le M\alpha$ for $\mu \in \mathcal{K}_{h,\alpha}$. To check that $\mathcal{K}_{h,\alpha}$ is compact, take a sequence $f_n \in \mathbf{C}_c(X)$, $f_n \uparrow h$ and note that for any convergent sequence μ_n of measures from $\mathcal{K}_{h,\alpha}$ we have $\langle h, \lim_{n\to\infty} \mu_n \rangle = \lim_{k\to\infty} \langle f_k, \lim_{n\to\infty} \mu_n \rangle = \lim_{k\to\infty} \lim_{n\to\infty} \langle f_k, \mu_n \rangle \le \alpha$. □

Lemma 3.4. *For every $\varepsilon > 0$ there exists such a number $\alpha > 0$ that for all $\mu, \nu \in \mathcal{M}$ if $\mu|_{X_\alpha} = \nu|_{X_\alpha}$ then $\rho_0(\mu, \nu) < \varepsilon$.*

Proof. Take such an $n_0 \in \mathbb{N}$ that $\frac{1}{2^{n_0}} < \varepsilon$. Let $C = \bigcup_{n=1}^{n_0} \operatorname{supp} d_n$ and $\alpha = \min_{x \in C} h(x)$. Then

$$\rho_0(\mu, \nu) < \sum_{n=1}^{n_0} 2^{-n} \min\{1, \big|\langle d_n, \mu - \nu \rangle\big|\} + \sum_{n=n_0+1}^{\infty} 2^{-n} < 0 + \varepsilon.$$

□

Lemma 3.5. *The space $(\mathcal{M}_h, \rho_h)$ is complete.*

Proof. Take a Cauchy sequence $(\mu_n)_{n\in\mathbb{N}}$ of measures from $\mathcal{M}_h$ and fix $\varepsilon > 0$. Note that $\rho_0 \le \rho_h$, and $\mathcal{M}$ is complete, so there exists a vague limit of (μ_n), say μ_∞. Take such an n_0 that $\rho_n(\mu_m, \mu_n) \le \varepsilon$ for $m, n \ge n_0$. By approximating a function $h \cdot (1 - \chi_n)$ from below by functions with compact support we obtain that $\langle h \cdot (1 - \chi_n), \mu_\infty \rangle \le \langle h(1 - \chi_n), \mu_{n_0} \rangle + \varepsilon$. Take such an $\alpha = 1/k_0$ that $\int_{X \setminus X_\alpha} h(x) \mu_{n_0}(\mathrm{d}x) \le \varepsilon$. Then $\langle h(1 - \chi_k), \mu_{n_0} \rangle \le \varepsilon$ and $\langle h(1 - \chi_k), \mu_\infty \rangle \le 2\varepsilon$ for any $k \ge k_0$. For $k < k_0$ and $n \ge n_0$ we have

$$\langle h(1 - \chi_{k_0}), \mu_\infty - \mu_n \rangle \le 3\varepsilon + |\langle h(\chi_k - \chi_{k_0}), \mu_\infty - \mu_n \rangle|.$$

The function $h(\chi_k - \chi_{k_0})$ has a compact support and can be approximated by the functions from the set D in the definition of ρ_0, so the last integral is arbitrarily small by the vague convergence of μ_n. □

Remark 3.6. The set $\mathcal{N}_h$ is closed in $(\mathcal{M}_h, \rho_h)$ and thereby complete by Lemma 3.5.

Lemma 3.7. *For every $\beta > 0$ and a continuous positive function g such that*

$$\frac{g(x)}{h(x)} \to \infty \text{ as } h(x) \to 0,$$

the set $\mathcal{K}_{g,\beta} = \{\nu \in \mathcal{N} : \langle g, \nu \rangle \le \beta\}$ is relatively compact in $(\mathcal{N}_h, \rho_h)$.

Proof. Note that $\mathcal{K}_{g,\beta} \subset \mathcal{K}_{h,\alpha}$ for some $\alpha > 0$, thus $\mathcal{K}_{g,\beta}$ is relatively compact in ρ_0 by Lemma 3.3 and that $\rho_0 < \rho_h$. Therefore, it suffices to check that if μ_∞ is a vague limit of a sequence of measures $\mu_n \in \mathcal{K}_{g,\beta}$ then the convergence holds also in ρ_h. Since $\langle g, \mu_n \rangle \le \beta$ and $g/h \to \infty$ as $h \to 0$, for any $\varepsilon > 0$, we can find such a k_0 that $\langle h(1 - \chi_k), \mu_n \rangle \le \varepsilon$ for all μ_n and $k \ge k_0$, and thereby $\langle h(1 - \chi_k), \mu_\infty \rangle \le \varepsilon$. For $k < k_0$ the integral $\langle h(1 - \chi_k), \mu_\infty \rangle$ is small by the same argument as in the proof of Lemma 3.5. □

Lemma 3.8. *Let g be such a continuous positive function that*

$$\frac{g(x)}{h(x)} \to \infty \text{ as } h(x) \to 0.$$

For every $\varepsilon > 0$ and $\beta > 0$ there exists such a number $\alpha > 0$ that for all $\mu_1, \mu_2 \in \mathcal{M}$ if $\mu_1|_{X_\alpha} = \mu_2|_{X_\alpha}$ and $\langle g, \mu_i \rangle \leq \beta$, $i = 1, 2$, then $\rho_h(\mu_1, \mu_2) < \varepsilon$.

Proof. Take $n_0 \in \mathbb{N}$ such that $\frac{1}{2^{n_0}} < \varepsilon/3$ and let $\alpha_1 = \min_{x \in C} h(x)$ where $C = \bigcup_{n=1}^{n_0} \operatorname{supp} d_n$. Take now α_2 such that $g(x)/h(x) > 3\beta/\varepsilon$ for $x \notin X_{\alpha_2}$ and let $\alpha = \min\{\alpha_1, \alpha_2\}$. Then $\int_{X \setminus X_\alpha} h(1 - \chi_k)\mu_i(\mathrm{d}x) < \varepsilon/3$ for $k > k_0 = [1/\alpha] + 1$ and for $k \leq 1/\alpha$ we have $\langle h(1 - \chi_k), \mu_1 - \mu_2 \rangle \leq 2\varepsilon/3 + |\langle h(\chi_{k_0} - \chi_k), \mu_1 - \mu_2 \rangle| = 2\varepsilon/3$. So $\rho_h(\mu_1, \mu_2) < \varepsilon$. □

3.2 Proof of theorem 2.1

To construct our target process which solves the martingale problem for the operator $\mathcal{L}$ defined by the formula (1.3), we construct a sequence of approximating models in which only finitely many particles move. To this end let a and q satisfy the assumptions of Theorem 2.1 and let κ and $\mathcal{L}$ be given by (1.2) and (1.3), respectively. Consider the sequence of sets $X_{1/N} = \left\{x \in X \,:\, h(x) \geq \frac{1}{N}\right\}$ and let

$$\chi_N(x) = \begin{cases} 1, & \text{for } x \in X_{1/N}, \\ N\left[(N+1)h(x) - 1\right], & \text{for } x \in X_{1/(N+1)} \setminus X_{1/N}, \\ 0, & \text{for } x \notin X_{1/(N+1)}. \end{cases} \tag{3.3}$$

Note that $\chi_N \in \mathbf{C}_c(X)$ and $0 \leq \chi_N \leq 1$.

Let us now define approximating transition kernels as

$$\kappa_N(\nu, B) = \sum_{x_i \in \nu} \chi_N(x_i) a(x_i) \int_{\mathcal{N}_h} \mathbb{1}_B\big(\nu - \delta_{x_i} + \eta\big) q(x_i, \mathrm{d}\eta), \tag{3.4}$$

and a sequence of approximating operators

$$\begin{aligned} \mathcal{L}_N f(\nu) &= \int_{\mathcal{N}_h} \big(f(\mu) - f(\nu)\big) \kappa_N(\nu, \mathrm{d}\mu) \\ &= \sum_{x_i \in \nu} \chi_N(x_i) a(x_i) \int_{\mathcal{N}_h} \big[f\big(\nu - \delta_{x_i} + \eta\big) - f(\nu)\big] q(x_i, \mathrm{d}\eta) \end{aligned} \tag{3.5}$$

with the domain $\mathcal{D}(\mathcal{L}_N) = \{f \in \mathbf{C}_b(\mathcal{N}) : \sup_{\nu \in \mathcal{N}} |\mathcal{L}_N f(\nu)| < \infty\}$.

Lemma 3.9. *Let $\nu_0 \in \mathcal{N}$. For every $N \in \mathbb{N}$ there exists a unique solution ξ_N to the martingale problem for operator $\mathcal{L}_N$ and initial point ν_0.*

Proof. We use Proposition A.1. Let us define a function

$$\psi(\nu) = \langle h\, \chi_N, \nu \rangle. \tag{3.6}$$

Notice that ψ is continuous on $\mathcal{N}$ and

$$\kappa_N(\nu, \mathcal{N}_h) = \sum_{x_i \in \nu} \chi_N(x_i) a(x_i) \leq (N+1)\psi(\nu) \max_{x \in X_{1/(N+1)}} a(x).$$

Note that $h\chi_N = h$ on $X_{\frac{1}{N}}$ and check that

$$\mathcal{L}_N\psi(\nu) = \sum_{x_i\in\nu} \chi_N(x_i)a(x_i)\int_{\mathcal{N}_h} \big[\langle \chi_N h, \nu - \delta_{x_i} + \eta\rangle - \langle \chi_N h, \nu\rangle\big] q(x_i, \mathrm{d}\eta) \leq$$
$$\sum_{\substack{x_i\in\nu\\ x_i\in X_{1/N}}} \chi_N(x_i)a(x_i)\int_{\mathcal{N}_h} \big[\langle h, \eta\rangle - h(x_i)\big] q(x_i, \mathrm{d}\eta) +$$
$$\sum_{\substack{x_i\in\nu\\ x_i\in X_{1/(N+1)}\setminus X_{1/N}}} \chi_N(x_i)a(x_i)\int_{\mathcal{N}_h} \big[\chi_N(x_i)\langle h, \eta\rangle - \chi_N(x_i)h(x_i)\big] q(x_i, \mathrm{d}\eta),$$

which is less then 0. Notice moreover that $\mathcal{L}_N\psi(\nu) \geq -\sum_{x_i\in\nu}\chi_N(x_i)a(x_i)h(x_i)$, which is finite for every $\nu\in\mathcal{N}$. Thereby $\mathcal{L}_N\psi(\nu)$ is well defined for every $\nu\in\mathcal{N}$.

Let us now check that all bounded continuous functions with supports in sets $U_{\psi\leq\alpha}$ belong to the domain of $\mathcal{L}_N$. To that end take $F\in\mathbb{C}_b(\mathcal{N})$ such that $\operatorname{supp}F\subset\{\nu : \langle\chi_N h, \nu\rangle\leq\alpha\}$. Let us estimate

$$\begin{aligned}
\mathcal{L}_N F(\nu) &= \sum_{x_i\in\nu} \chi_N(x_i)a(x_i)\int_{\mathcal{N}_h} \big[F(\nu - \delta_{x_i} + \eta) - F(\nu)\big] q(x_i, \mathrm{d}\eta)\\
&\geq -\sum_{x_i\in\nu} \chi_N(x_i)a(x_i)F(\nu)\\
&\geq \begin{cases} 0, & \text{if } \nu\notin\{\mu : \langle\chi_N h, \mu\rangle\leq\alpha\}\\ -\|F\|(N+1)\psi(\nu)\max_{x\in X_{1/(N+1)}} a(x), & \text{if } \nu\in\{\mu : \langle\chi_N h, \mu\rangle\leq\alpha\}\end{cases}\\
&\geq -\|F\|(N+1)\alpha\max_{x\in X_{1/(N+1)}} a(x),
\end{aligned}$$

and on the other side

$$\mathcal{L}_N F(\nu) \leq \begin{cases} 0, & \text{if } \nu\notin\{\mu : \langle\chi_N h, \mu\rangle\leq\alpha\},\\ \|F\|(N+1)\alpha\max_{x\in X_{1/(N+1)}} a(x), & \text{if } \nu\in\{\mu : \langle\chi_N h, \mu\rangle\leq\alpha\}.\end{cases}$$

□

Remark 3.10. Let ξ_N be a solution to the martingale problem for $(\mathcal{L}_N, \xi_0)$, $\xi_0\in\mathcal{N}_h$. Then the same argument as in Lemma 3.13 gives $\mathrm{E}\langle h, \xi_N(t)\rangle\leq\mathrm{E}\langle h, \xi_0\rangle$. But we have more here, namely let us notice that ξ_N is a continuous-time Markov process with the jump kernel κ_N defined by the formula (3.4) and, thanks to the mass conservation assumption of Theorem 2.1, jumps cannot change the value of $\langle h, \xi_N(t)\rangle$. Therefore, $\langle h, \xi_N(t)\rangle = \langle h, \xi_0\rangle$ a.s.

Remark 3.11. Take $\nu_0\in\mathcal{N}_h$. Then we can replace the space $(\mathcal{N}, \rho_0)$ by $(\mathcal{N}_h, \rho_h)$ everywhere in Lemma 3.9 and in its proof. The process defined in such a way is actually the same process as the one defined by using Lemma 3.9 directly, but we will not prove nor use that fact here.

Lemma 3.12. *Take any $\xi_0\in\mathcal{N}_h$ and let ξ_N be a solution to the martingale problem for $(\mathcal{L}_N, \xi_0)$. The set $\{\xi_N\}_{N\in\mathbb{N}}$ is relatively compact in $D([0,\infty), \mathcal{N})$.*

Proof. Let us check the assumptions of Theorem A.2. By Remark 3.10 and Lemma 3.3 the assumption (a) is satisfied in a trivial way — all processes live in a compact set $\{\nu\in\mathcal{N} : \langle h, \nu\rangle\leq\langle h, \xi_0\rangle\}$. To check the second assumption fix $T > 0$ and $\varepsilon > 0$ and we have to prove that there exists an $r > 0$ such that

$$\sup_{N\in\mathbb{N}} \mathrm{Prob}\big\{w'(\xi_N, r, T)\geq\varepsilon\big\} < \varepsilon. \tag{3.7}$$

Using Lemma 3.4 chose a number $\alpha > 0$ for our ε. For each N consider the sequence of all stopping times τ_n^N such that there is a jump of process ξ_N that changes the measure on the set X_α, namely: $\xi_N\left((\tau_n^N)^-\right)(X_\alpha) \neq \xi_N(\tau_n^N)(X_\alpha)$ and $\xi_N(s)(X_\alpha) = \xi_N(\tau_n^N)(X_\alpha)$ for $\tau_n^N \leq s < \tau_{n+1}^N$. Note that for a given trajectory, if the distance between all subsequent times τ_n^N is bigger then r then

$$w'(\xi_N, r, T) \leq \max_{1\leq n\leq k} \sup_{s,t\in[\tau_{n-1}^N, \tau_n^N)} \rho_0(\xi_N(s), \xi_N(t)) < \varepsilon,$$

where k is such that $\tau_{k-1}^N \leq T < \tau_k^N$. So now it is sufficient to prove that

$$\mathrm{Prob}\big(\min\{\tau_n^N - \tau_{n-1}^N : \tau_n^N \leq T\} > r\big) < \varepsilon$$

for all $N \in \mathbb{N}$. To this end, note that the jump of process ξ_N changes a measure on the set X_α only when a particle of mass bigger then α (*i.e.* a particle being in X_α) fragmentizes. So, the probability rate of such a jump is

$$\sum_{\substack{x_i\in\xi_N(t)\\ h(x_i)>\alpha}} \chi_N(x_i)a(x_i) \leq \frac{\langle h, \xi_N(t)\rangle}{\alpha} \sup_{x\in X_\alpha} a(x) \leq \frac{\langle h, \xi_0\rangle}{\alpha} \sup_{x\in X_\alpha} a(x). \tag{3.8}$$

Therefore, the frequency of times τ_n^N is not bigger then frequency of jumps of Poison process with the intensity given by the right hand side of (3.8). So the probability that the minimal distance of two jumps in $[0, T]$ is less then r goes to zero as $r \to 0$. □

Lemma 3.13. *If ξ is a solution to the $(\mathcal{L}, \xi_0)$-martingale problem then*

$$\mathrm{E}\langle h, \xi_t\rangle \leq \mathrm{E}\langle h, \xi_0\rangle. \tag{3.9}$$

Proof. Let $h_N(x) = \chi_N(x)h(x)$ where χ_N is given by (3.3). Notice that $F(\nu) = e^{-\langle h_N,\nu\rangle}$ belongs to the domain of $\mathcal{L}$ and $\mathcal{L}F \geq 0$. Thus

$$\mathrm{E}\left[e^{-\lambda\langle h_N,\xi(t)\rangle}\right] = e^{-\lambda\langle h_N,\xi_0\rangle} + \int_0^t \mathrm{E}\,\mathcal{L}F(\xi_s)\mathrm{d}s \geq e^{-\lambda\langle h_N,\xi_0\rangle},$$

and using $s\,e^{-s} \leq 1 - e^{-s}$ we get

$$\mathrm{E}\left[\lambda\langle h_N, \xi(t)\rangle\, e^{-\lambda\langle h_N,\xi(t)\rangle}\right] \leq \mathrm{E}\left[1 - e^{-\lambda\langle h_N,\xi_0\rangle}\right].$$

Going with λ to zero we obtain $\mathrm{E}\langle h_N, \xi_t\rangle \leq \mathrm{E}\langle h_N, \xi_0\rangle$, and by the monotone convergence theorem the lemma is proved. □

Lemma 3.14. *For any ξ_0 such that $\mathrm{E}\langle h, \xi_0\rangle < \infty$ there exists at most one solution to the martingale problem for $(\mathcal{L}, \xi_0)$.*

Proof. We adapt here the method of the proof of Theorem 8.4.2 of [17]. By Theorem 4.2 of Chapter 4 in [17] it suffices to prove the uniqueness of one dimensional distributions. Let ξ_t be any solution to the $(\mathcal{L}, \xi_0)$-martingale problem, and write

$$\mathbf{v}_t(g) = \mathrm{E}[\exp(-\langle g, \xi_t\rangle)] \tag{3.10}$$

for $g \in E_h$, where

$$\begin{aligned} E_h = \{g \in \mathbf{C}_c(X) \,:\, &\text{there exist } 0 < \alpha < \beta \text{ and } c > 0 \text{ such that} \\ &g(x) = 0 \text{ for } x \in X \setminus X_\alpha,\ g(x) = c(h(x) - \alpha) \text{ for } x \in X_\alpha \setminus X_\beta, \\ &\text{and } \alpha \leq g(x) \leq 1 \text{ for } x \in X_\beta\}. \end{aligned} \tag{3.11}$$

Let us now define

$$E_{h,N} = \Big\{ g \in E_h \,:\, g(x) = 0 \text{ for } x \notin X_{1/N} \Big\}, \tag{3.12}$$

and notice that for any $g \in E_{h,N}$ the function $\nu \mapsto \exp(-\langle g, \nu \rangle)$ belongs to $\mathcal{D}(\mathcal{L})$. Moreover, $\mathbf{v}$ is bounded and continuous on $E_{h,N}$. Let us define an operator

$$A\mathbf{u}(g) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} [\mathbf{u}(\pi_\varepsilon g) - \mathbf{u}(g)] \tag{3.13}$$

for $\mathbf{u} \in \mathbf{C}(E_{h,N})$ if the convergence is uniform in g, where

$$\pi_\varepsilon g(x) = -\log\Big[e^{\varepsilon a(x)} e^{-g(x)} + \big(1 - e^{\varepsilon a(x)}\big) \int_{\mathcal{N}_h} e^{\langle -g, \eta \rangle} q(x, \mathrm{d}\eta) \Big]. \tag{3.14}$$

The operator A is dissipative. Notice that for $\mathbf{v}$ defined by (3.10) we have

$$\begin{aligned} &\frac{1}{\varepsilon}[\mathbf{v}_t(\pi_\varepsilon g) - \mathbf{v}_t(g)] \\ &\quad = \frac{1}{\varepsilon} \int_0^\varepsilon \mathrm{E}\Big[\exp(-\langle \pi_s g, \xi_t \rangle) \Big\langle \frac{\Phi g(\cdot)}{1 + \big(e^{-s\, a(\cdot)} - 1\big) \int_{\mathcal{N}_h} e^{g(\cdot) - \langle g, \eta \rangle} q(\cdot, \mathrm{d}\eta)}, \xi_t \Big\rangle \Big] \mathrm{d}s, \end{aligned} \tag{3.15}$$

where

$$\Phi g(x) = a(x) \left(\int_{\mathcal{N}_h} e^{g(x) - \langle g, \eta \rangle} q(x, \mathrm{d}\eta) - 1 \right), \tag{3.16}$$

so if the convergence of (3.15) for $\varepsilon \to 0$ is uniform then we have

$$A\mathbf{v}_t(g) = \mathrm{E}\Big[\exp(-\langle g, \xi_t \rangle) \langle \Phi g, \xi_t \rangle \Big]. \tag{3.17}$$

Indeed, one can check that for every $N \in \mathbb{N}$ the convergence is uniform on $E_{h,N}$ and thereby $\mathbf{v}_t$ for every t belongs to the domain of A.

Since ξ_t satisfies the martingale problem for $\mathcal{L}$, we have

$$\mathbf{v}_t(g) = \mathbf{v}_s(g) + \int_s^t A\mathbf{v}_r(g) \mathrm{d}r. \tag{3.18}$$

Note also that $A\mathbf{v}_t(g)$ is uniformly continuous. Thus by Proposition A.3 equation (3.18) has a unique solution in $\mathbf{C}(E_{h,N})$ for every $N \in \mathbb{N}$. Since $E_h = \bigcup_{N \in \mathbb{N}} E_{h,N}$, the term $\mathbf{v}_t(g)$ given by (3.10) is uniquely determined for every $g \in E_h$. But E_h separates points in $\mathcal{N}$, therefore the uniqueness for one dimensional distributions holds. □

Proof of Theorem 2.1. To prove the well-posedness of the martingale problem for the operator $\mathcal{L}$ and the initial state ξ_0 we use Theorem A.4. Let us check the assumptions of this theorem.

We have already checked in Lemma 3.14 the uniqueness condition for the martingale problem for $(\mathcal{L}, \xi_0)$. Let us consider a sequence of processes ξ_N which are the solutions to the martingale problems for $(\mathcal{L}_N, \xi_0)$. Thanks to Lemma 3.12 the sequence $\{\xi_N\}$ is relatively compact in $D([0,\infty), \mathcal{N})$.

It suffices to provide a sequence of sets U_N and a bounded sequence F_N of functions from $\mathcal{D}_{\mathcal{L}_N}$ satisfying (A.6) for every function $F_g \in \mathcal{D}_{\mathcal{L}}$. To that end, let us take an arbitrary $F_g \in \mathcal{D}_{\mathcal{L}}$. We set $U_N = \mathcal{N}_h$ and $F_N = F_g$, since $F_g \in \mathcal{D}_{\mathcal{L}_N}$ for every N. Then both $\sup_N \|F_N\| < \infty$, and $\lim_{N\to\infty} \sup_{y \in U_N} |F(y) - F_N(y)| = 0$ are trivial. To check the last estimate, let us calculate $|\mathcal{L}F_g(\mu) - \mathcal{L}_N F_g(\mu)|$ for $\mu \in U_N$.

$$\mathcal{L}F_g(\mu) - \mathcal{L}_N F_g(\mu) = e^{-\langle g, \mu \rangle} \sum_{x_i \in \mu} (1 - \chi_N(x_i)) a(x_i) \int_{\mathcal{N}_h} \big[e^{-\langle g, -\delta_{x_i} + \eta \rangle} - 1 \big] q(x_i, \mathrm{d}\eta),$$

which is 0 for N large enough to have $\operatorname{supp} g \subset X_{1/N}$, because $1 - \chi_N(x_i) = 0$ for $x_i \in X_{1/N}$ and $e^{-\langle g, -\delta_{x_i} + \eta \rangle} = 1$ for $x_i \notin X_{1/N}$. □

3.3 Conservation of mass

Firstly, we need to prove a new version of Lemma 3.12, namely

Lemma 3.15. *Take any $\xi_0 \in \mathcal{N}_h$ and let ξ_N be a solution to the martingale problem for $\mathcal{L}_N$ and the initial point ξ_0. Under the assumptions of Theorem 2.3 the set $\{\xi_N\}_{N\in\mathbb{N}}$ is relatively compact in $(\mathcal{N}_h, \rho_h)$.*

Proof. We will use Theorem A.2 again, but now the set $\{\nu \in \mathcal{N}_h : \langle h, \nu\rangle \leq \langle h, \xi_0\rangle\}$ is not compact. Therefore, to check the assumption (a) of Theorem A.2 we will show that for every $\varepsilon > 0$, $t \geq 0$ there exists $\beta > 0$ such that

$$\mathrm{Prob}(\xi_N(t) \in \mathcal{K}_{g,\beta}) \geq 1 - \varepsilon \text{ for all } N \in \mathbb{N}, \tag{3.19}$$

where $\mathcal{K}_{g,\beta} = \{\nu : \langle g, \nu\rangle \leq \beta\}$ is a compact set by Lemma 3.7. To this end, note that the function $\nu \mapsto \exp[-\lambda\langle \min\{nh, g\}, \nu\rangle]$ belongs to the domain of $\mathcal{L}_N$, so

$$e^{-\lambda\langle \min\{nh,g\},\xi_N(t)\rangle} - e^{-\lambda\langle \min\{nh,g\},\xi_0\rangle} - \int_0^t \mathcal{L}_N e^{-\lambda\langle \min\{nh,g\},\cdot\rangle}(\xi_N(s))\mathrm{d}s$$

is a martingale. Going to infinity with with n, by (2.6) and the dominated convergence theorem we also obtain that

$$e^{-\lambda\langle g,\xi_N(t)\rangle} - e^{-\lambda\langle g,\xi_N(0)\rangle} - \int_0^t \mathcal{L}_N e^{-\lambda\langle g,\cdot\rangle}(\xi_N(s))\mathrm{d}s \tag{3.20}$$

is a martingale. Thus

$$\mathrm{E}\left[e^{-\lambda\langle g,\xi_N(t)\rangle}\right] = e^{-\lambda\langle g,\xi_0\rangle} + \int_0^t \mathrm{E}\left[\mathcal{L}_N e^{-\lambda\langle g,\cdot\rangle}(\xi_N(s))\right]\mathrm{d}s \tag{3.21}$$

and

$$\begin{aligned}\mathrm{E}\left[\lambda\langle g, \xi_N(t)\rangle\, e^{-\lambda\langle g,\xi_N(t)\rangle}\right] &\leq \mathrm{E}\left[1 - e^{-\lambda\langle g,\xi_N(t)\rangle}\right] \\ &\leq 1 - e^{-\lambda\langle g,\xi_0\rangle} - \int_0^t \mathrm{E}\left[\mathcal{L}_N e^{-\lambda\langle g,\cdot\rangle}(\xi_N(s))\right]\mathrm{d}s.\end{aligned}$$

But from (2.6) it follows that

$$\mathrm{E}\left[\lambda\langle g, \xi_N(t)\rangle\, e^{-\lambda\langle g,\xi_N(t)\rangle}\right] \leq 1 - e^{-\lambda\langle g,\xi_0\rangle} - C\int_0^t \mathrm{E}\left[e^{-\lambda\langle g,\xi_N(s)\rangle}\lambda\langle g, \xi_N(s)\rangle\right]\mathrm{d}s,$$

and by Gronwall Lemma we have

$$\mathrm{E}\left[\lambda\langle g, \xi_N(t)\rangle\, e^{-\lambda\langle g,\xi_N(t)\rangle}\right] \leq \left(1 - e^{-\lambda\langle g,\xi_0\rangle}\right)e^{Ct}.$$

Going with λ to zero we obtain

$$\mathrm{E}\langle g, \xi_N(t)\rangle \leq \langle g, \xi_0\rangle e^{Ct}. \tag{3.22}$$

Thereby, there exists $\beta > 0$ such that

$$\mathrm{Prob}\{\langle g, \xi_N(t)\rangle > \beta\} < \varepsilon \text{ for all } N \in \mathbb{N},$$

which gives (3.19). Using (3.20) and (3.22), and some Doob-Kolmogorov martingale inequalities we obtain that

$$\mathrm{Prob}\left\{\sup_{t\in[0,T]}\langle g, \xi_N(t)\rangle > \beta\right\} < \frac{1}{\beta}\langle g, \xi_0\rangle e^{Ct} \quad \text{for all } N \in \mathbb{N}.$$

Knowing that ξ_t for $t \in [0,T]$ are in $\mathcal{K}_{g,\beta}$ with probability bigger then $1-\varepsilon$, we use Lemma 3.8 to find such an $\alpha > 0$ that $\rho_h(\mu,\nu) < \varepsilon$ if $\mu|_{X_\alpha} = \nu|_{X_\alpha}$. As in the proof of Lemma 3.10 we claim that the jumps on the set X_α are sufficiently rare and assumption (b) of Theorem A.2 is satisfied. □

Proof of Theorem 2.3. To prove the mass conservation we show that under the assumptions of Theorem 2.3 the convergence in the proof of Theorem 2.1 holds also in the space $\mathcal{N}_h$ with the metric ρ_h. We consider the same sequence of approximating processes ξ_N given by Lemma 3.9 as a sequence of unique solutions to the martingale problem with operators $\mathcal{L}_N$ defined in (3.5). Exactly in the same way as in the proof of Theorem 2.1, we check assumptions of Theorem A.4, using now Lemma 3.15 for the proof of relative compactness of the sequence $\{\xi_N\}$. Notice that Lemma 3.14 works in this case, as well as the rest of calculations in the proof of Theorem 2.1.

Observe now, that the function $\nu \mapsto \langle h,\nu\rangle$ is continuous in the space $(\mathcal{N}_h,\rho_h)$, so if $\langle h,\xi_N(t)\rangle = \langle h,\xi_0\rangle$ for each $N \in \mathbb{N}$ and ξ_N converge to ξ, then also $\langle h,\xi(t)\rangle = \langle h,\xi_0\rangle$.

Formally, the process obtained as the limit in $D([0,\infty),\mathcal{N}_h)$ is different from the process given by Theorem 2.1. However, $D([0,\infty),\mathcal{N}_h) \subset D([0,\infty),\mathcal{N})$, the σ-algebras of Borel sets on (N_h,ρ_h) and (N_h,ρ_0) coincide and the uniqueness Lemma 3.14 does not rely on the metrics. Therefore, the distributions of both processes are the same. □

For the proof of Corollary 2.4 we need the following auxiliary fact:

Lemma 3.16. *For any finite measure Borel μ on the interval $(0,1]$ there exists such a decreasing function $f : (0,1] \to \mathbb{R}_+$ that $\lim_{s\downarrow 0} f(s) = \infty$, $\int_0^1 f(s)\mu(\mathrm{d}s) < \infty$ and $f(st) \le f(s)f(t)$ for any $s,t \in (0,1]$.*

Proof. If there exists such an $\varepsilon > 0$ that $\mu((0,\varepsilon]) = 0$, take $f(x) = 1/x$. Otherwise, let us define the following sequences:

$$a_0 = 1, \quad a_n = \left(\frac{1}{2}\right)^{2^{n-1}}, \quad \text{for } n \ge 1,$$
$$b_0 = 1, \quad b_n = \sup\{s : \mu((0,s)) < 1/2^n\}, \quad \text{for } n \ge 1,$$
$$c_0 = 1, \quad c_{n+1} = \max\{a_k : \exists_{l\in\mathbb{N}}\ b_l \in [a_k,c_n)\}, \text{ for } n \ge 1,$$

and let $f(s) = 2+n$ for $s \in [c_{n+1},c_n)$. Note that $b_n \to 0$, so c_n is well defined, strictly decreasing and $c_n \to 0$ as well, and thereby $f(0+) = \infty$. Moreover, since $c_n \le b_n$, we have $f(s) \le 2+n$ for $s \ge b_{n+1}$, thus $\int_0^1 f(s)\mu(ds) \le \sum_{n=0}^\infty (2+n)\mu([b_{n+1},b_n)) \le \sum_{n=0}^\infty \frac{2+n}{2^n} < \infty$ (in the last sum one should be careful when an atom occurs at b_n). To complete the proof, take $0 < s \le t \le 1$ and such an n that $s \in [c_n,c_{n-1})$. Thus $t \ge c_n$ and $st \ge c_{n+1}$, so $f(st) \le n+2 < (n+1)^2 \le f(s)f(t)$. □

Proof of Corollary 2.4. By the previous Lemma, take a such function $f : (0,1] \to \mathbb{R}_+$ that $f(0^+) = \infty$, $\int_0^1 f(s)s\,\tilde{q}(\mathrm{d}s) = C < \infty$ and $f(st) \le f(s)f(t)$ for $s,t \in (0,1]$ and define $g(x) = f(h(x))h(x)$. Then

$$a(x)\int_{\mathcal{N}_h}\big[\langle g,\eta\rangle - g(x)\big]q(x,d\eta)$$
$$\le \|a\|\left[h(x)f(h(x))\int_0^1 f(s)s\,\tilde{q}(\mathrm{d}s) - f(h(x))h(x)\right] = \|a\|(C-1)g(x).$$

Therefore, by Theorem 2.3 the mass is conserved. □

Proof of Theorem 2.5. Let ξ be the IPF starting from ξ_0. Since ξ is the solution to the $(\mathcal{L}, \xi_0)$-martingale problem, for any $n \in \mathbb{N}$ we have

$$\mathrm{E}\, e^{-\langle \chi_n h^{1+\beta}, \xi(t)\rangle} = e^{-\langle \chi_n h^{1+\beta}, \xi_0\rangle} + \int_0^t \mathrm{E}\, \mathcal{L} e^{-\langle \chi_n h^{1+\beta}, \cdot\rangle}(\xi(s)) \mathrm{d}s. \tag{3.23}$$

Moreover, for any $\nu \in \mathcal{N}_h$ we have

$$\begin{aligned}
&\mathcal{L} e^{-\langle \chi_n h^{1+\beta}, \cdot\rangle}(\nu) \\
&\geq e^{-\langle \chi_n h^{1+\beta}, \nu\rangle} \sum_{x_i \in \nu} C\, h(x_i)^{-\beta} \int_{\mathcal{N}_h} \left[\chi_n(x_i) h(x_i)^{1+\beta} - \langle \chi_n h^{1+\beta}, \eta\rangle\right] q(x_i, \mathrm{d}\eta) \\
&\geq C\, e^{-\langle \chi_n h^{1+\beta}, \nu\rangle} \sum_{x_i \in \nu} h(x_i)^{-\beta} \chi_n(x_i) h(x_i)^{1+\beta} \int_{\mathcal{N}_h} \left[1 - \frac{1}{h(x_i)^{1+\beta}} \langle h^{1+\beta}, \eta\rangle\right] q(x_i, \mathrm{d}\eta) \\
&\geq C\, \delta\, e^{-\langle \lambda \chi_n h^{1+\beta}, \nu\rangle} \langle \lambda \chi_n h, \nu\rangle.
\end{aligned}$$

Using the above inequality in (3.23) and going with n to ∞ yields

$$\mathrm{E}\, e^{-\langle h^{1+\beta}, \xi(t)\rangle} \geq e^{-\langle h^{1+\beta}, \xi_0\rangle} + C\, \delta \int_0^t \mathrm{E}\left[e^{-\langle h^{1+\beta}, \xi(s)\rangle} \langle h, \xi(s)\rangle\right] \mathrm{d}s. \tag{3.24}$$

To obtain a contradiction, let us now assume that $\langle h, \xi(t)\rangle = \langle h, \xi(0)\rangle$ for all $t \geq 0$ a.s. Then

$$\mathrm{E}\, e^{-\langle h^{1+\beta}, \xi(t)\rangle} \geq e^{-\langle h^{1+\beta}, \xi_0\rangle} + C\, \delta \langle h, \xi(0)\rangle \int_0^t \mathrm{E}\, e^{-\langle h^{1+\beta}, \xi(s)\rangle} \mathrm{d}s.$$

This implies $\mathrm{E}\, e^{-\langle h^{1+\beta}, \xi(t)\rangle} \geq e^{-\langle h^{1+\beta}, \xi_0\rangle} e^{C\, \delta \langle h, \xi(0)\rangle t}$, so $\mathrm{E}\, e^{-\langle h^{1+\beta}, \xi(t)\rangle}$ becomes greater then 1 at some time, which is impossible. □

A Appendix

For easy reference we state here some external facts that we use in the proofs.

Proposition A.1. *Let E be a Polish space. Let κ be a kernel on E and consider an operator of the form*

$$\mathcal{L} f(e) = \int_E [f(e_1) - f(e)] \kappa(e, \mathrm{d}e_1) \tag{A.1}$$

with some domain $\mathcal{D}_{\mathcal{L}} \subset \mathbf{C}_b(E)$. Suppose that there exists a continuous function $\psi : E \to [0, \infty)$ such that

$$\mathcal{L}\psi(e) \leq c_1 + c_2 \psi(e) \text{ for } e \in E \tag{A.2}$$

for some $c_1, c_2 \geq 0$, and $\kappa(\cdot, E)$ is bounded on all sets $U_{\psi \leq \alpha} = \{e \in E : \psi(e) \leq \alpha\}$ (i.e. $\sup_{e \in U_{\psi \leq \alpha}} \kappa(e, E) < \infty$). Moreover, suppose that all bounded continuous functions with support in sets $U_{\psi \leq \alpha}$ belong to $\mathcal{D}_{\mathcal{L}}$.

Then the martingale problem for $\mathcal{L}$ has a unique solution for every initial point $e_0 \in E$.

One can find full the proof of the above proposition in [32]. *Cf.* also Proposition 2.2.(ii) of [24] and the books [16, 25]. We use the following criterion for relative compactness of a family of processes.

Theorem A.2 (Theorem 3.7.2 of [17])**.** *Let (E, r_E) be a Polish space and let $\{\xi_n\}$ be a family of càdlàg processes. The $\{\xi_n\}$ is relatively compact iff*

1. *for all $\varepsilon > 0$ and $t \in \mathbb{Q} \cap [0, \infty)$ there exists a compact set $\Gamma_{\varepsilon,t}$ such that*

$$\inf_n \mathrm{Prob}\big(\xi_n(t) \in \Gamma^{\varepsilon}_{\varepsilon,t}\big) \geq 1 - \varepsilon$$

where $\Gamma^{\varepsilon}_{\varepsilon,t} = \{e \in E : r_E(e, \Gamma_{\varepsilon,t}) < \varepsilon\}$, *and*

2. *for all $\varepsilon > 0$ and $T > 0$ there exists $r > 0$ such that*

$$\sup_n \operatorname{Prob}\big(w'(\xi_n, r, T) \geq \varepsilon\big) \leq \varepsilon$$

where

$$w'(\xi, r, T) = \inf_{\{t_i\}} \max_{1 \leq i \leq n} \sup_{s,t \in [t_{i-1}, t_i)} r_E(\xi(s), \xi(t)) \tag{A.3}$$

the infimum is over all partitions $0 = t_0 < t_1 < \cdots < t_{n-1} < T \leq t_n$ with $\min_{1 \leq i \leq n}(t_i - t_{i-1}) > r$ and $n \geq 1$.

Proposition A.3 (Proposition 1.2.10 of [17])**.** *Let L be a dissipative linear operator on a real Banach space E with the norm $\|\cdot\|$. Suppose that $u : [0, \infty) \to E$ is continuous, $u(t) \in \mathcal{D}(L)$ for all $t > 0$, $Lu : (0, \infty) \to E$ is continuous, and*

$$u(t) = u(\varepsilon) + \int_\varepsilon^t Lu(s)\mathrm{d}s, \tag{A.4}$$

for all $t > \varepsilon > 0$. Then

$$\|u(t)\| \leq \|u(0)\| \tag{A.5}$$

for all $t > 0$.

One can use Theorem 4.8.10 or Corollary 4.8.12 from [17] in proofs of Theorems 2.1 and 2.3, but we formulate here a fact that follows from Theorem 4.8.10 in [17] and is more convenient in our case:

Theorem A.4. *Let (E, ρ) be a Polish space. Let $L : \mathbf{C}_b(E) \supset \mathcal{D}_L \to \mathbf{C}_b(E)$ be a linear operator and $e \in E$, and suppose that the $D_E[0, \infty)$ martingale problem for (L, e) has at most one solution. For each $N \in \mathbb{N}$, suppose that ξ_N is a càdlàg solution to the (L_N, e)-martingale problem. Assume that $\{\xi_N\}$ is relatively compact and that $\xi_N(0) = e$. If for each $F \in \mathcal{D}_L$ and $T > 0$ there exist $F_N \in \mathcal{D}_{L_N}$ and $U_N \subset E$ such that $\xi_N(t) \in U_N$ for $t \leq T$ a.s., $\sup_N \|F_N\| < \infty$, and*

$$\lim_{N \to \infty} \sup_{y \in U_N} |F(y) - F_N(y)| = \lim_{N \to \infty} \sup_{y \in U_N} |LF(y) - L_N F_N(y)| = 0, \tag{A.6}$$

then there exists a solution ξ of the $D_E[0, \infty)$ martingale problem for (L, e) and the processes ξ_N converge in distribution on $D_E[0, \infty)$ to ξ.

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