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Fixed points of the multivariate smoothing transform: the critical case

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Abstract

Given a sequence $(\mathbf{T}_1, \mathbf{T}_2, \dots)$ of random $d \times d$ matrices with nonnegative entries, suppose there is a random vector X with nonnegative entries, such that $\sum_{i\geq 1} \mathbf{T}_i X_i$ has the same law as X, where $(X_1, X_2, \dots)$ are i.i.d. copies of X, independent of $(\mathbf{T}_1, \mathbf{T}_2, \dots)$. Then (the law of) X is called a fixed point of the multivariate smoothing transform. Similar to the well-studied one-dimensional case $d = 1$, a function m is introduced, such that the existence of $\alpha \in (0, 1]$ with $m(\alpha) = 1$ and $m'(\alpha) \leq 0$ guarantees the existence of nontrivial fixed points. We prove the uniqueness of fixed points in the critical case $m'(\alpha) = 0$ and describe their tail behavior. This complements recent results for the non-critical multivariate case. Moreover, we introduce the multivariate analogue of the derivative martingale and prove its convergence to a non-trivial limit.

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1 Introduction

Let $d \geq 2$ and $(\mathbf{T}_i)_{i\geq 1}$ be a sequence of random $d \times d$ matrices with nonnegative entries. Assume that

$$N := \#\{i : \mathbf{T}_i \neq 0\}$$

is finite a.s. We will presuppose throughout that the $(\mathbf{T}_i)_{i\geq 1}$ are ordered in such a way that $\mathbf{T}_i \neq 0$ if and only if $i \leq N$. Given a random variable $X \in \mathbb{R}^d_{\geq} := [0, \infty)^d$, let $(X_i)_{i\geq 1}$ be i.i.d. copies of X and independent of $(\mathbf{T}_i)_{i\geq 1}$. Then $\sum_{i=1}^N \mathbf{T}_i X_i$ defines a new random variable in $\mathbb{R}^d_{\geq}$. If it holds that

$$X \stackrel{d}{=} \sum_{i=1}^{N} \mathbf{T}_i X_i, \tag{1.1}$$

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where $\stackrel{d}{=}$ means same law, then we call the law $\mathcal{L}(X)$ of X a fixed point of the multivariate smoothing transform (associated with $(\mathbf{T}_i)_{i\geq 1}$). By a slight abuse of notation, we will also call X a fixed point.

This notion goes back to Durrett and Liggett [20]. For $d = 1$, it is known that properties of fixed points are encoded in the function $m(s) := \mathbb{E}\sum_{i=1}^{N} T_i^s$ (here $(T_i)_{i\geq 1}$ are nonnegative random numbers): If some non-lattice and moment assumptions are satisfied, then the existence of an $\alpha \in (0,1]$ with $m(\alpha) = 1$ and $m'(\alpha) \leq 0$ is equivalent to the existence of fixed points which then are unique up to scaling. See [30, Theorem 1.1] and [4, Theorem 6.1(a)] for more precise statements of necessary conditions for the existence of fixed points.

Moreover, if $\psi(r) = \mathbb{E}\left[e^{-rX}\right]$ is the Laplace transform of a fixed point, then there is a positive function L, slowly varying at 0, and $K > 0$ such that

$$\lim_{r\to 0} \frac{1-\psi(r)}{L(r)r^{\alpha}} = K. \tag{1.2}$$

The function L is constant if $m'(\alpha) < 0$ and $L(t) = (|\log t| \vee 1)$ if $m'(\alpha) = 0$, the latter being called the *critical case*. For $\alpha < 1$, the property (1.2) implies that the fixed points have Pareto-like tails with index α, i.e. $\lim_{t\to\infty} t^{-\alpha}\mathbb{P}(X > t)/L(1/t) \in (0,\infty)$, see [30] for details. Tail behavior in the case $\alpha = 1$, in which there is no such implication, is investigated in [23, 30, 16].

Existence and uniqueness results in the multivariate setting $d \geq 2$ for the non-critical case have been recently proved in [32]. The aim of this work is to provide the corresponding result for the multivariate critical case. In order to so, we will first review necessary notation and definitions from [32], in particular introducing the multivariate analogue of the function m, as well as a result about the existence of fixed points in the critical case. Following the approach in [8, 10, 28] we will then prove that a multivariate regular variation property similar to (1.2) holds for fixed points (with an essentially unique, but yet undetermined slowly varying function L), which we use in order to prove the uniqueness of fixed points, up to scalars. Under some extra (density) assumption, we identify the slowly varying function to be the logarithm also in the multivariate case, which allows us to introduce and prove convergence of the multivariate version of the so-called *derivative martingale*, a notion coined in [9]. It appears prominently in the limiting distribution of the minimal position in branching random walk, see [1, 2, 9, 14] for details and further references.

Our results can be interpreted in the setting of multi-type branching random walk as follows: Consider a particle positioned at $e_1 := (1, 0, \cdots, 0)^\top \in \mathbb{R}^d_{\geq}$, which produces a random number N of offspring, which are placed at positions $x_i := \mathbf{T}_i e_1$. This first generation produces offspring independently in the same manner: The j-th particle has N_j children, which are being placed at positions $x_{ji} := \mathbf{T}_i(j)x_j$, where $(N_j, (\mathbf{T}_i(j))_{i\in\mathbb{N}})$ are copies of $(N, (\mathbf{T}_i)_{i\in N})$. Denote by $\mathbb{S}_{\geq} := S^{d-1} \cap \mathbb{R}^d_{\geq}$ the intersection of the unit sphere with the cone of vectors with nonnegative entries. Now writing the particle positions $x_v = u_v e^{s_v}$ in logarithmic polar coordinates with $u_v \in \mathbb{S}_{\geq}$ and $s_v \in \mathbb{R}$, we observe that the logarithmic distances s_v of the particles from the origin perform a multitype branching random walk, with the law of the increments $s_{ji} - s_j = \log|\mathbf{T}_i(j)u_j|$ depending on the spherical position u_j of the ancestor, thus the type space is given by $\mathbb{S}_{\geq}$. Eq. (1.1) was studied for multitype branching random walks with finite type spaces in [11, 29] and in a two-type setup with type space $\{-1, 1\}$ in [24, Section 2.6]. Note that our setting is not as general as it seems, for the increment laws depend continuously on the ancestors position u_v. Nevertheless, to the best of our knowledge, [24] is the only other reference where the functional equation of the multitype branching random walk in the critical case is studied.

2 Statement of Results

In order to avoid repetition, we start by introducing the assumptions and the notation in full detail before stating the results.

Write $\mathcal{P}(\mathbb{R}^d_\geq)$ for the set of probability measures on $\mathbb{R}^d_\geq$ and $M(d\times d,\mathbb{R}_\geq)$ for the set of $d\times d$ matrices with nonnegative entries. Given a sequence $T:=(\mathbf{T}_i)_{i\geq 1}$ of random matrices from $M(d\times d,\mathbb{R}_\geq)$, only the first N of which are nonzero, with $N<\infty$ a.s., we aim to determine the set of fixed points of the mapping $\mathcal{S}:\mathcal{P}(\mathbb{R}^d_\geq)\to\mathcal{P}(\mathbb{R}^d_\geq)$,

$$\mathcal{S}\eta := \mathcal{L}\left(\sum_{i=1}^N \mathbf{T}_i X_i\right), \quad \text{for } (X_i)_{i\geq 1} \text{ i.i.d. with law } \eta \text{ and independent of } (\mathbf{T}_i)_{i\geq 1}.$$

Without further mention, we assume $(\Omega,\mathcal{B},\mathbb{P})$ to be a probability space which is rich enough to carry all the occurring random variables.

2.1 The weighted branching process and iterations of $\mathcal{S}$

Let $\mathbb{V}:=\bigcup_{n=0}^\infty \mathbb{N}^n$ be a tree with root $\emptyset$ and Ulam-Harris labeling. We write $|v|=n$ if $v=v_1\cdots v_n\in\{1,\dots,N\}^n$, $v|k=v_1\cdots v_k$ for the ancestor in the k-th generation and $vi=v_1\cdots v_n i$ for the i-th child of v, $i\in\mathbb{N}$.

To each node $v\in\mathbb{V}$ assign an independent copy $T(v)$ of T and, given a random variable $X\in\mathbb{R}^d_\geq$, as well an independent copy $X(v)$ of X, such that $(T(v))_{v\in\mathbb{V}}$ and $(X(v))_{v\in\mathbb{V}}$ are independent. Introduce a filtration by

$$\mathcal{B}_n \;:=\; \sigma\Big((T(v))_{|v|\leq n}\Big).$$

Upon defining recursively the product of weights along the path from $\emptyset$ to v by

$$\mathbf{L}(\emptyset):=\mathbf{Id}, \qquad \mathbf{L}(vi)=\mathbf{L}(v)\mathbf{T}_i(v),$$

we obtain the iteration formula

$$\mathcal{S}^n\mathcal{L}(X)=\mathcal{L}\left(\sum_{|v|=n}\mathbf{L}(v)X(v)\right),$$

which in terms of Laplace transforms $\phi(x)=\mathbb{E}\Big[e^{-\langle x,X\rangle}\Big]$ becomes

$$\mathcal{S}^n\phi(x)=\mathbb{E}\Big[\prod_{|v|=n}\phi(\mathbf{L}(v)^\top x)\Big], \qquad x\in\mathbb{R}^d_\geq. \tag{2.1}$$

2.2 Assumptions

As noted before, we assume

$$\text{the r.v. } N:=\#\{i\,:\,\mathbf{T}_i\neq 0\}=\sup\{i\,:\,\mathbf{T}_i\neq 0\} \text{ satisfies } 1<\mathbb{E}N<\infty\,. \tag{A1}$$

This assumption guarantees, that the underlying Galton-Watson tree (consisting of the nodes v with $\mathbf{L}(v)\neq 0$) is supercritical and we denote its survival set by $\mathcal{N}$. Moreover, (A1) allows to define a probability measure μ on $M(d\times d,\mathbb{R}_\geq)$ by

$$\int f(\mathbf{a})\,\mu(d\mathbf{a}) \;:=\; \frac{1}{\mathbb{E}N}\mathbb{E}\left[\sum_{i=1}^N f(\mathbf{T}_i)\right]. \tag{2.2}$$

We call a matrix $\mathbf{a} \in M(d\times d, \mathbb{R}_\geq)$ *allowable*, if it has no zero row or column and denote by $\mathcal{M} \subset M(d\times d, \mathbb{R}_\geq)$ the set of allowable matrices. Further, we write $\mathcal{M}_+ := M(d\times d, \mathbb{R}_>)$ for the set of (allowable) matrices with all entries positive. On the (support of the) measure μ, we will impose the following condition (C):

Definition 2.1. *A subsemigroup $\Gamma \subset M(d\times d, \mathbb{R}_\geq)$ satisfies condition (C), if*

1. *every $\mathbf{a}$ in Γ is allowable, i.e. $\Gamma \subset \mathcal{M}$*
2. *Γ contains a matrix with all entries positive, i.e. $\Gamma \cap \mathcal{M}_+ \neq \emptyset$.*

For the measure μ as defined in Eq. (2.2), we assume

$$\text{The subsemigroup } [\operatorname{supp}\mu] \text{ generated by } \operatorname{supp}\mu \text{ satisfies } (C). \tag{A2}$$

Note that if $\mathbf{a} \in \mathcal{M}$, then we can define its action on $\mathbb{S}_\geq$ by

$$\mathbf{a}.u \ := \ \frac{\mathbf{a}u}{|\mathbf{a}u|}, \qquad u \in \mathbb{S}_\geq.$$

Let $\mathbf{M}, (\mathbf{M}_n)_{n\in\mathbb{N}}$ be i.i.d. random matrices with law μ, and write $\mathbf{\Pi}_n := \prod_{i=1}^n \mathbf{M}_i$. Then it is shown in [32], that the multivariate analogue of the function m is given by

$$m(s) := \mathbb{E}[N] \lim_{n\to\infty} \left(\mathbb{E}\left\|\mathbf{\Pi}_n\right\|^s\right)^{1/n},$$

which is finite on some convex interval containing 0. Since m is log-convex the left derivatives $m'(s^-)$ exist.

We assume to be in the critical case, i.e.

$$\text{there is } \alpha \in (0,1] \text{ with } m(\alpha) = 1 \text{ and } m'(\alpha^-) = 0. \tag{A3}$$

For the multivariate case, the classical *T-log T* condition splits into an upper bound and a lower bound: Introducing $\iota(\mathbf{a}) := \inf_{u\in\mathbb{S}_\geq} |\mathbf{a}u|$, we observe that $\iota(\mathbf{a}) > 0$ for $\mathbf{a} \in \mathcal{M}$, and that for all $u \in \mathbb{S}_\geq$,

$$\iota(\mathbf{a}) \ \leq \ |\mathbf{a}u| \ \leq \ \|\mathbf{a}\|.$$

Note that if $\mathbf{a}$ is invertible, then $\left\|\mathbf{a}^{-1}\right\|^{-1} \leq \iota(\mathbf{a})$.

$$\mathbb{E}\left[\ \|\mathbf{M}\|^\alpha \log(1+\|\mathbf{M}\|)\right] < \infty, \qquad \mathbb{E}\left[(1+\|\mathbf{M}\|)^\alpha \left|\log\iota(\mathbf{M}^\top)\right|\ \right] < \infty \tag{A4}$$

Assumptions (A1) **–** (A4) **will be in force throughout the paper.** At one point, we will impose a stronger condition on the lower bound, namely

$$\text{There is } c>0 \text{ such that } \mathbb{P}\left(\iota(\mathbf{M}^\top) \geq c\right) = 1, \tag{A5}$$

which together with the first part of (A4) implies the second part of (A4). See Remark 5.4 for a discussion of (A5).

Furthermore, we need a multivariate analogue of a non-lattice condition: Recall that a matrix $\mathbf{a} \in \mathcal{M}_+$ has an algebraic simple dominant eigenvalue $\lambda_\mathbf{a} > 0$ with corresponding normalized eigenvector $v_\mathbf{a}$ the entries of which are all positive.

$$\{\log\lambda_\mathbf{a} \,:\, \mathbf{a} \in [\operatorname{supp}\,\mu] \cap \mathcal{M}_+\} \text{ generates a dense subgroup of } (\mathbb{R},+). \tag{A6}$$

In the second part of the paper, we will need stronger assumptions on μ, which guarantee that the associated Markov random walk (to be defined below) is Harris recurrent. We will consider the absolute continuity assumption

$$\exists\, \mathbf{a}_0 \in \mathcal{M}_+ \,\exists\, \gamma_0, c>0 \text{ s.t. } \mathbb{P}\left(\mathbf{M} \in \cdot\right) \ \geq \ \gamma_0\, l^{d\times d}(\cdot \cap B_c(\mathbf{a}_0)), \tag{A6c}$$

where $l^{d\times d}$ denotes the Lebesgue measure on the set of $d \times d$ matrices, seen as a subset of $\mathbb{R}^{d^2}$ and $B_c(\mathbf{a}_0)$ is the open ball with radius c around $\mathbf{a}_0$. A similar assumption for invertible matrices appears in [26, Theorem 6] and subsequently in [5]. It is easy to check that (A6c) implies (A6).

We will consider as well a quite degenerate case, namely

$$\operatorname{supp}\mu \text{ is finite and consists of rank-one matrices, and (A6) holds.} \tag{A6f}$$

Note that an allowable rank-one matrix $\mathbf{a}$ has all entries positive, the columns are multiples of a vector $v_{\mathbf{a}} \in \operatorname{int}(\mathbb{S}_\geq)$, and consequently, $\mathbf{a}.u = v_{\mathbf{a}}$ for all $u \in \mathbb{S}_\geq$.

We will also impose a stronger moment condition, namely there are $\varepsilon > 0$, $0 \leq \beta \leq \alpha + \varepsilon$ and $p > 1$ such that

$$m(\alpha+\varepsilon) = \mathbb{E}\Big[\sum_{i=1}^N \|\mathbf{T}_i\|^{\alpha+\varepsilon}\Big] < \infty \text{ and } \mathbb{E}\Big[\Big(\sum_{i=1}^N \|\mathbf{T}_i\|^{\beta}\Big)^p\Big] < \infty. \tag{A7}$$

2.3 Previous Results

We have the following existence result in the critical case.

Proposition 2.2. *Assume* (A1) – (A4). *Then Eq.* (1.1) *has a nontrivial fixed point.*

Source: Theorem 1.2 in [32]. The assumption $N \geq 1$ a.s. is imposed there for convenience, but the existence result can be obtained along the same lines without this assumption. The resulting fixed point then has an atom at zero with mass $1 - \mathbb{P}(\mathcal{N})$. ☐

The main contribution of this paper is to prove the uniqueness of this fixed point, and to give asymptotic properties of its Laplace transform.

A main technical tool used therefore is a so-called many-to-one lemma, which associates a *Markov random walk* $(U_n, S_n)_{n\in\mathbb{N}}$ to the multitype branching random walk given by

$$S^u(v) := -\log\left|\mathbf{L}(v)^\top u\right|, \qquad v \in \mathbb{V}$$

with type process

$$U^u(v) := \mathbf{L}(v)^\top.u, \qquad v \in \mathbb{V}$$

for an initial state $u \in \mathbb{S}_\geq$. We refer the reader to [17, 32] for details of the construction; the main point being the existence of a continuous function $H : \mathbb{S}_\geq \to (0,\infty)$ with the property

$$\frac{1}{\mathbb{E}N} H(u) \;=\; \mathbb{E}\left|\mathbf{M}u\right|^\alpha H(\mathbf{M}.u) \qquad \forall u \in \mathbb{S}_\geq,$$

see [17, Prop. 3.1].

Proposition 2.3. *Assume* (A1) – (A4). *Let* $(U_n, S_n)_{n\in\mathbb{N}}$ *be a sequence of random variables with values in* $(\mathbb{S}_\geq \times \mathbb{R})^{\mathbb{N}}$. *For* $u \in \mathbb{S}_\geq$ *and* $t \in \mathbb{R}$, *let* $\mathbb{P}_{u,t}$ *be a probability measure such that*

$$\begin{aligned}
&\mathbb{E}_{u,t}\big[f(U_0, S_0, \cdots, U_n, S_n)\big] \\
&= \frac{\mathbb{E}N}{H(u)}\, \mathbb{E}\left[f\Big((\mathbf{\Pi}_k^\top.u,\, t - \log|\mathbf{\Pi}_k^\top u|)_{k\leq n}\Big)\, |\mathbf{\Pi}_k^\top u|^\alpha H(\mathbf{\Pi}_k^\top.u)\right] \\
&= \frac{1}{H(u)}\, \mathbb{E}\left[\sum_{|v|=n} f\Big((U^u\big(v|k\big), t + S^u\big(v|k\big))_{k\leq n}\Big)\, e^{-\alpha S^u(v)} H(U^u(v))\right]
\end{aligned}$$

for all $n \in \mathbb{N}$ *and measurable* $f : (\mathbb{S}_\geq \times \mathbb{R})^{n+1} \to \mathbb{R}$*. Then* $(U_n, S_n)_{n\in\mathbb{N}}$ *constitutes a Markov random walk under* $\mathbb{P}_{u,t}$*, i.e.* $(U_n, S_n)_{n\in\mathbb{N}}$ *is a Markov chain and the increments* $S_n - S_{n-1}$ *are independent conditioned upon* $(U_n)_{n\in\mathbb{N}}$*.*

Source: Corollary 4.3 in [32]. □

We will use the shorthand $\mathbb{P}_u = \mathbb{P}_{u,0}$ and $\mathbb{P}_\eta = \int \mathbb{P}_{u,s}\, \eta(du, ds)$ if we are given a probability measure η on $\mathbb{S}_\geq \times \mathbb{R}$. The *associated Markov random walk* $(U_n, S_n)_{n\in\mathbb{N}}$ generalizes the concept of the associated random walk in [20, 30]. In particular, it holds for all $u \in \mathbb{S}_\geq$, that

$$\lim_{n\to\infty} \frac{S_n}{n} \;=\; m'(\alpha^-) \;=\; 0 \qquad \mathbb{P}_u\text{-a.s.},$$

see [17, Theorem 6.1]. Concerning the deviations from the mean drift, it is shown in [19, Lemma 7.1] that the function

$$b(u) := \lim_{n\to\infty} \mathbb{E}_u S_n$$

is well defined and continuous, and satisfies

$$\mathbb{E}_u[S_1 + b(U_1)] = b(u). \tag{2.3}$$

Using Eq. (2.3) together with Proposition 2.3, we obtain that

$$\mathcal{W}_n(u) := \sum_{|v|=n} \left[S^u(v) + b(U^u(v))\right] \, H(U(v)) e^{-\alpha S(v)}$$

defines a martingale with respect to the filtration $\mathcal{B}_n$, which we will show to be the multivariate analogue of the derivative martingale. In fact, b can be considered as the derivative of H, see [19, (7.9)].

2.4 Main Results

Our first result proves that, upon imposing the non-lattice condition (A6) and the boundedness assumption (A5), the fixed point given by Proposition 2.2 is unique up to scaling and satisfies a multivariate analogue of the regular variation property (1.2). See also Remark 5.4 for a discussion of assumption (A5).

Theorem 2.4. *Assume* (A1) – (A5)*. If* $\alpha \neq 1$*, assume* (A6) *in addition. Then there is a random measurable function* $Z : \mathbb{S}_\geq \to [0, \infty)$ *with* $\mathbb{P}(Z(u) > 0) = \mathbb{P}(\mathcal{N}) > 0$ *for all* $u \in \mathbb{S}_\geq$*, such that* X *is a nontrivial fixed point of* (1.1) *on* $\mathbb{R}^d_\geq$ *if and only if its Laplace transform satisfies*

$$\psi(ru) \;:=\; \mathbb{E}\left[e^{-r\langle u, X\rangle}\right] \;=\; \mathbb{E}\left[e^{-r^\alpha K Z(u)}\right] \qquad \forall u \in \mathbb{S}_\geq,\, r \in \mathbb{R}_\geq \tag{2.4}$$

for some $K > 0$*.*

There is a positive function L*, unique up to asymptotic equivalence and slowly varying at* 0 *with* $\limsup_{r\to0} L(r) = \infty$*, such that*

$$\lim_{r\to0} \frac{1 - \psi(ru)}{L(r)\, r^\alpha} \;=\; KH(u). \tag{2.5}$$

Remark 2.5. Asymptotically equivalent means that if L_1 and L_2 satisfy Eq. (2.5), then $\lim_{r\to0} L_1(r)/L_2(r) = 1$. Depending on the value of α, additional information can be extracted from Eq. (2.5).

1. If $\alpha < 1$, then a Tauberian theorem (see [21, XIII.(5.22)]) together with [7, Theorem 1.1] implies multivariate regular variation of the tail, see [32, Section 6] for details.

2. If $\alpha = 1$, then $\mathbb{E}\,|X| = \infty$ for every non-trivial fixed point, see Lemma 5.5. Moreover, the aperiodicity condition (A6) is not needed. This is in analogy with the one-dimensional situation, see e.g. [30, Corollary 1.5].

Upon imposing the additional assumptions (A6c) or (A6f) on μ, we will identify the function L as well as the random variable Z. Note that assumption (A5) is not needed here.

Theorem 2.6. *Assume* (A1) – (A4), (A7) *and either* (A6c) *or* (A6f). *Then* $\mathcal{W}_n(u)$ *converges a.s. to a nonnegative limit* $\mathcal{W}(u)$ *with* $\mathbb{P}\left(\mathcal{W}(u) > 0\right) = \mathbb{P}\left(\mathcal{N}\right) = 1$, *and a random variable* $X \in \mathbb{R}^d_{\geq}$ *is a nontrivial fixed point of* (1.1) *if and only if for some* $K > 0$,

$$\mathbb{E}\left[e^{-r\langle u,X\rangle}\right] \;=\; \mathbb{E}\left[e^{-r^{\alpha}K\mathcal{W}(u)}\right] \qquad \forall u \in \mathbb{S}_{\geq},\, r \in \mathbb{R}_{\geq}.$$

Moreover, the slowly varying function L *in Eq.* (2.5) *can be chosen as (a scalar multiple of)* $L(r) = |\log r| \vee 1$.

2.5 Structure of the Paper

The further organization is as follows: In Section 3, we study the associated Markov random walk, which is recurrent due to the criticality assumption. Under assumptions (A6c), a regeneration property known from the theory of Harris recurrent Markov chains will be shown to hold. In Section 4, we prove that each fixed point satisfies (2.5), which is a main ingredient in the proof of uniqueness in Section 5. In Section 6, we turn to the proof of Theorem 2.6 and study the behavior of the Laplace transform of the fixed point. We conclude with Section 7, where the convergence of the derivative martingale is proved.

Assumptions (A1) **–** (A4) **are standing assumptions throughout the paper and only additional assumptions will be mentioned.**

3 The Associated Markov Random Walk

In this section, we provide additional information about the associated Markov random walk, in particular about its stationary distribution and recurrence properties. Moreover, we show that it is Harris recurrent and satisfies a minorization condition under the additional assumption (A6c).

3.1 The Associated Markov Random Walk

Markov random walks such as $(U_n, S_n)_n$, which are generated by the action of nonegative matrices were first studied by Kesten in his seminal paper [26], and very detailed results are given in [17]. For the reader's convenience, we cite those which are important for what follows. Recall that we denote the Perron-Frobenius eigenvalue and the corresponding normalized eigenvector of a matrix $\mathbf{a} \in \mathcal{M}_+$ by $\lambda_{\mathbf{a}}$ resp. $v_{\mathbf{a}}$.

Proposition 3.1. *In the situation of Proposition 2.3, the following holds:*

1. *The Markov chain* $(U_n)_n$ *on* $\mathbb{S}_{\geq}$ *has a unique stationary distribution* π.
2. $\operatorname{supp}\pi = \overline{\{v_{\mathbf{a}} \,:\, \mathbf{a} \in [\operatorname{supp}\mu] \cap \mathcal{M}_+\}}$.
3. *For all* $u \in \mathbb{S}_{\geq}$,

$$\lim_{n\to\infty} \frac{S_n}{n} = \mathbb{E}_{\pi}[S_1] = \int_{\mathbb{S}_{\geq}} \mathbb{E}_u[S_1]\,\pi(du) = \frac{m'(\alpha^-)}{m(\alpha)} \qquad \mathbb{P}_u\text{-a.s.}$$

Source: Section 4 of [17]. □

3.2 Recurrence of Markov Random Walks

By Proposition 3.1 (3), in the critical case $m'(\alpha^-) = 0$ the Markov random walk $(S_n)_n$ is centered in the stationary regime and satisfies a strong law of large numbers. Alsmeyer [3] studied recurrence properties of such Markov random walks, which we will make use of.

Lemma 3.2. *Assume* (A6). *For any open set A with $\pi(A) > 0$ and any open interval $B \subset \mathbb{R}$, it holds that*

$$\mathbb{P}_\pi((U_n, S_n) \in A \times B \text{ infinitely often}) = 1. \tag{3.1}$$

If the aperiodicity condition (A6) *is not assumed, then still*

$$\liminf_{n\to\infty} S_{\tau_n} \;=\; -\infty, \qquad \limsup_{n\to\infty} S_{\tau_n} \;=\; \infty \qquad \mathbb{P}_\pi\text{-a.s.}, \tag{3.2}$$

where (τ_n) is the sequence of hitting times of the set A by U_n.

Proof. Let A be any open set A with $\pi(A) > 0$. By the strong law of large numbers for Markov chains (see [15]),

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(U_k) \;=\; \int f(x)\, \pi(dx) \qquad \mathbb{P}_\pi\text{-a.s.}, \tag{3.3}$$

thus, using $f = \mathbb{1}_A$, we obtain that $\mathbb{P}_\pi(U_n \in A \text{ infinitely often}) = 1$. This shows in particular that the successive hitting times of A, $(\tau_n)_{n\in\mathbb{N}}$, are finite a.s. Then (U_{τ_n}, S_{τ_n}) is again a Markov random walk, and $\pi_A := \pi(\cdot \cap A)/\pi(A)$ is the stationary probability measure for U_{τ_n}. The aperiodicity assumption (A6) implies that (U_n, S_n) are nonarithmetic in the sense of [3], see [19] for details. Lemma 1 in [3] gives that (U_{τ_n}, S_{τ_n}) is nonarithmetic as well. Using (3.3) with $f = \mathbb{1}_A$ again, this gives that $n/\tau_n \to \pi(A)$ a.s. Combining this with the strong law of large numbers (3) in Proposition 3.1, we deduce that

$$\lim_{n\to\infty} \frac{S_{\tau_n}}{n} = \lim_{n\to\infty} \frac{S_{\tau_n}}{\tau_n} \frac{\tau_n}{n} = \frac{1}{\pi(A)} \cdot 0 \qquad \mathbb{P}_\pi\text{-a.s.}.$$

Then Theorem 2 in [3] (for the nonarithmetic case) gives that the recurrence set

$$\{s \in \mathbb{R} \;:\; \text{for all } \varepsilon > 0,\ S_{\tau_n} \in (s-\varepsilon, s+\varepsilon) \text{ infinitely often } \}$$

is equal to $\mathbb{R}$, which shows that $\mathbb{P}_\pi(S_{\tau_n} \in B \text{ infinitely often}) = 1$.

In the arithmetic case, the recurrence set is still a closed subgroup of $\mathbb{R}$, which implies the oscillation property. □

Corollary 3.3. *There is $u_0 \in \operatorname{int}(\mathbb{S}_\geq) \cap (\operatorname{supp} \pi)$ such that*

$$\liminf_{n\to\infty} S_{\tau_n} \;=\; -\infty, \qquad \limsup_{n\to\infty} S_{\tau_n} \;=\; \infty \qquad \mathbb{P}_{u_0}\text{-a.s.} \tag{3.4}$$

If (A6) *holds, then moreover*

$$\mathbb{P}_{u_0}((U_n, S_n) \in A \times B \text{ infinitely often}) = 1 \tag{3.5}$$

for any open set A with $\pi(A) > 0$ and any open interval $B \subset \mathbb{R}$.

Proof. By Proposition 3.1, $\operatorname{supp} \pi$ consists of the (closure of the) set of normalized Perron-Frobenius eigenvectors of matrices $\mathbf{a} \in [\operatorname{supp} \mu] \cap \mathcal{M}_+$. By part (2) of (C), this set is nonempty, hence $\operatorname{int}(\mathbb{S}_\geq) \cap (\operatorname{supp} \pi) \neq \emptyset$ and even $\pi(\operatorname{int}(\mathbb{S}_\geq)) = 1$. On the other hand, Lemma 3.2 implies validity of (3.5) and (3.4) for π-a.e. $u \in \mathbb{S}_\geq$, hence we can find $u_0 \in \operatorname{int}(\mathbb{S}_\geq)$ satisfying the assertions. □

3.3 Implications of Assumptions (A6c) and (A6f)

In this subsection, we explain how Assumptions (A6c) and (A6f) imply that the Markov chain $(U_n)_{n\in\mathbb{N}}$ has an atom (possibly after redefining it on an extended probability space), which can be used to obtain a sequence $(\sigma_n)_{n\in\mathbb{N}}$ of *regeneration times* for the Markov random walk (U_n, S_n), i.e. stopping times such that$(U_{\sigma_n}, S_{\sigma_n} - S_{\sigma_{n-1}})_{n\in\mathbb{N}}$ becomes an i.i.d. sequence. Namely, we are going to prove the following lemma for the Markov chain $(U_n, Y_n) := (U_n, S_n - S_{n-1})$.

Lemma 3.4. *Assume* (A6c) *or* (A6f)*. On a possibly enlarged probability space, one can redefine $(U_n, Y_n)_{n\geq 0}$ together with an increasing sequence $(\sigma_n)_{n\geq 0}$ of random times such that the following conditions are fulfilled under any $\mathbb{P}_u$, $u \in \mathbb{S}_{\geq}$:*

- *(R1) There is a filtration $\mathcal{G} = (\mathcal{G}_n)_{n\geq 0}$ such that $(U_n, Y_n)_{n\geq 0}$ is Markov adapted and each σ_n a stopping time with respect to $\mathcal{G}$, moreover, $\{\sigma_n = k\} \in \mathcal{G}_{k-1}$ for all $n, k \geq 0$.*
- *(R2) Then there is an open subset $\mathcal{R} \subset \mathbb{S}_{\geq} \times \mathbb{R}$ and a probability measure η, supported on $\mathcal{R}$, such that the sequence $(\sigma_{n+1} - \sigma_n)_{n\geq 1}$ is i.i.d. with law $\mathbb{P}_\eta(\sigma_1 \in \cdot)$ and is independent of σ_1.*
- *(R3) For each $k \geq 1$, $(U_{\sigma_k+n}, Y_{\sigma_k+n})_{n\geq 0}$ is independent of $(U_j, Y_j)_{0\leq j\leq \sigma_k-1}$ with distribution $\mathbb{P}_\eta((U_n, Y_n)_{n\geq 0} \in \cdot)$.*
- *(R4) There is $q \in (0, 1)$ and $l \in \mathbb{N}$ such that $\sup_{u\in\mathbb{S}_{\geq}} \mathbb{P}_u(\sigma_1 > ln) \leq q^n$.*

This lemma is quite immediate under condition (A6f), for Proposition 3.1, (2) shows that the unique stationary measure π for (U_n) under $\mathbb{P}_u$ is supported on the *finite* set $\mathbb{F} := \{v_{\mathbf{a}} : \mathbf{a} \in \operatorname{supp}\mu\}$ (note that $v_{\mathbf{ab}} = v_{\mathbf{a}}$ if $\mathbf{a}$ has rank one, thus the semigroup $[\operatorname{supp}\mu]$ can be replaced by $\operatorname{supp}\mu$.) Moreover, independent of the initial value $u \in \mathbb{S}_{\geq}$, $U_1 \in \mathbb{F}$ $\mathbb{P}_u$-f.s., i.e. $\mathbb{S}_{\geq} \setminus \mathbb{F}$ is uniformly transient for $(U_n)_{n\in\mathbb{N}}$, and thus we can study $(U_n)_{n\in\mathbb{N}}$ on the finite state space $\mathbb{F}$. Then, if $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence of successive hitting times of a point $u_0 \in \mathbb{F}$, the assertions of the lemma follow from the theory of Markov chains with finite state space.

Remark 3.5. A crucial point is that we also obtain the independence of Y_{σ_k} from $(U_j, Y_j)_{0\leq j\leq\sigma_k-1}$, thereby strengthening analogous results for invertible matrices, obtained in [5, 31].

From now on, until the end of the section, assume (A6c).

We are going to prove that the chain (U_n, Y_n) satisfies a minorization condition as in [6, Definition 2.2] resp. [34, (M)]. If $v_{\mathbf{a}_0} \in \mathbb{S}_{\geq}$ is the Perron-Frobenius eigenvalue of the matrix $\mathbf{a}_0$ from (A6c), then we have the following result:

Lemma 3.6. *For each $u \in \mathbb{S}_{\geq}$, $\delta > 0$,*

$$\mathbb{P}_u(U_n \in B_\delta(v_{\mathbf{a}_0}) \text{ infintely often }) = 1,$$

moreover, if τ denotes the first hitting time of $B_\delta(v_{\mathbf{a}_0})$, then there are $l \geq 1$ and $q_0 \in (0, 1)$ such that

$$\sup_{u\in\mathbb{S}_{\geq}} \mathbb{P}_u(\tau > ln) \leq q_0^n,$$

i.e. τ/l is stochastically bounded by a random variable with geometric distribution.

Source: This is proved in [26, p.218-220, proof of I.1], the crucial point being that $v_{\mathbf{a}_0}$ is a strict contraction on $\mathbb{S}_{\geq}$ with attractive fixed point $v_{\mathbf{a}_0}$, and small perturbations of $\mathbf{a}_0$ still attract to a neighborhood of $v_{\mathbf{a}_0}$, and such matrices are realized with positive probability. □

Lemma 3.7. *There are $\delta > 0$, $\gamma > 0$ and a probability measure η on $\mathcal{R} := B_\delta(v_{\mathbf{a}_0}) \times \mathbb{R}$ such that for all $u \in B_\delta(v_{\mathbf{a}_0})$ and all measurable subsets $A \subset B_\delta(v_{\mathbf{a}_0})$, $B \subset \mathbb{R}$*

$$\mathbb{P}_u(U_1 \in A, Y_1 \in B) \ \geq \ \gamma\, \eta(A \times B).$$

Proof. We follow the approach in [5, 31].

Step 1: Given $c > 0$, $\mathbf{a}_0 \in \mathcal{M}_+$, there is $\varepsilon > 0$ such that for all orthogonal matrices $\mathbf{O}$, satisfying $\|\mathbf{O} - \mathbf{Id}\| < \varepsilon$, $B_{c/2}(\mathbf{a}_0)\mathbf{O} \subset B_c(\mathbf{a}_0)$. *Proof*: Let $\mathbf{b} \in B_{c/2}\mathbf{a}_0$, then, since $\mathbf{O}$ is an isometry,

$$\|\mathbf{b}\mathbf{O} - \mathbf{a}_0\| \leq \|\mathbf{b}\mathbf{O} - \mathbf{a}_0\mathbf{O}\| + \|\mathbf{a}_0\mathbf{O} - \mathbf{a}_0\| \leq \|\mathbf{b} - \mathbf{a}_0\| - \|\mathbf{a}_0\|\,\|\mathbf{O} - \mathbf{Id}\| \leq c/2 + \varepsilon\,\|\mathbf{a}_0\|\,.$$

Step 2: For all $\varepsilon > 0$ there is $\delta > 0$ such that for each $u \in B_\delta(v_{\mathbf{a}_0})$ there exists an orthogonal matrix $\mathbf{O}_u$ with $u = \mathbf{O}_u v_{\mathbf{a}_0}$ and $\|\mathbf{O}_u - \mathbf{Id}\| < \varepsilon$. *Source*: [31, Lemma 15.1].
Step 3: Introduce the finite measure

$$\tilde{\eta}(A \times B) \ := \ \int_{B_{c/2}(\mathbf{a}_0)} \mathbb{1}_A(\mathbf{a}.v_{\mathbf{a}_0})\, \mathbb{1}_B(-\log|\mathbf{a}v_{\mathbf{a}_0}|)\, l^{d\times d}(d\mathbf{a}).$$

Combining Steps 1 and 2 and Assumption (A6c), there is $\delta > 0$, such that for all $u \in B_\delta(v_{\mathbf{a}_0})$ there exists an orthogonal matrix $\mathbf{O}_u$ with $u = \mathbf{O}_u v_{\mathbf{a}_0}$ and $B_{c/2}(\mathbf{a}_0)\mathbf{O}_u \subset B_c(\mathbf{a}_0)$. Hence for all $u \in B_\delta(v_{\mathbf{a}_0})$, by Assumption (A6c) and using that $l^{d\times d}$ is invariant under transformations by a matrix with determinant 1 (see [31, proof of Prop. 15.2, Step 1] for more details, using the Kronecker product)

$$\begin{aligned}
\mathbb{P}(\mathbf{M}^\top.u \in A, -\log\left|\mathbf{M}^\top u\right| \in B) \ &\geq \gamma_0 \int_{B_{c/2}(\mathbf{a}_0)\mathbf{O}_u} \mathbb{1}_A(\mathbf{a}.u)\, \mathbb{1}_B(-\log|\mathbf{a}u|)\ l^{d\times d}(d\mathbf{a}) \\
&= \gamma_0 \int_{B_{c/2}(\mathbf{a}_0)} \mathbb{1}_A(\mathbf{a}\mathbf{O}_u^{-1}.u)\, \mathbb{1}_B(-\log\left|\mathbf{a}\mathbf{O}_u^{-1}u\right|)\ l^{d\times d}(d\mathbf{a}) \\
&= \gamma_0\, \tilde{\eta}(A \times B).
\end{aligned}$$

Step 4: To obtain a minorization for the shifted measure $\mathbb{P}_u$, recall that H is bounded from below and above, to obtain that

$$\begin{aligned}
\mathbb{P}_u(U_1 \in A, Y_1 \in B) \ &\geq \ \int_{A\cap B_\delta(v_{\mathbf{a}_0})} \int_B \frac{H(w)}{H(u)} e^{-\alpha y}\ \mathbb{P}(\mathbf{M}^\top.u \in dw, -\log\left|\mathbf{M}^\top u\right| \in dy) \\
&\geq \gamma_1 \int_{A\cap B_\delta(v_{\mathbf{a}_0})} \int_B H(w) e^{-\alpha y}\ \tilde{\eta}(dw, dy) \ =: \ \eta(A \times B)
\end{aligned}$$

Upon renormalizing η to a probability measure, and thereby determining γ, we obtain the assertion. □

Now we are ready to prove Lemma 3.4 under Assumption (A6c):

Proof of Lemma 3.4. Lemmata 3.6 and 3.7 imply that the Markov chain $(U_n, Y_n)_{n\geq 0}$ is $(\mathcal{R}, \gamma, \eta, 1)$-recurrent in the sense of [6, Definition 2.2]. Then the lemma follows from [6, Lemma 3.1 and Corollary 3.4]. The *regeneration times* σ_n are constructed as follows: Let $(\xi_n)_{n\geq 0}$ be a sequence of i.i.d. Bernoulli(1,γ) random variables, independent of $(U_n, Y_n)_{n\geq 0}$. Whenever (U_n, Y_n) enters the set $\mathcal{R}$, (U_{n+1}, Y_{n+1}) is generated according to η if $\xi_n = 1$, and according to $(1-\gamma)^{-1}(P - \gamma\eta)$ if $\xi = 0$. The total transition probability thus remains $P = \mathbb{P}_u((U_1, Y_1) \in \cdot)$. Together with Lemma 3.6, this construction immediately gives that σ_1 can be bounded stochastically by a random variable with geometric distribution. □

4 Regular Variation of Fixed Points

In this section, we show that every fixed point of $\mathcal{S}$, the existence of which is provided by Proposition 2.2, satisfies the regular variation property (2.5).

Let ψ be the Laplace transform of a fixed point of $\mathcal{S}$ in the critical case $m'(\alpha) = 0$. Introduce

$$D(u,t) := \frac{1-\psi(e^{-t}u)}{e^{-\alpha t}H(u)}, \qquad u \in \mathbb{S}_{\geq},\, t \in \mathbb{R}. \tag{4.1}$$

Our aim is to study the behavior of D as t goes to infinity. Let u_0 be given by Corollary 3.3. Following the approach in [28], we are going to show that

$$h_t(u,s) := \frac{D(u,s+t)}{D(u_0,t)} = \frac{1-\psi(e^{-(s+t)}u)}{e^{-\alpha s}(1-\psi(e^{-t}u_0))}\frac{H(u_0)}{H(u)}$$

converges to 1 as t tends to infinity. This shows in particular, that $D(u_0,t)$ is slowly varying as $t \to \infty$. We then use the results of [32] to deduce that this already implies that $D(u,t)$ is slowly varying for all $u \in \mathbb{S}_{\geq}$.

Lemma 4.1. *For every sequence $(t_k)_{k\in\mathbb{N}}$, tending to infinity, there is a subsequence $(t_n)_{n\in\mathbb{N}}$ such that $h_{t_n}(u,s)$ converges pointwise to a continuous function $h : \mathbb{S}_{\geq} \times \mathbb{R} \to [0,\infty)$.*

Proof. Introduce for $t \in \mathbb{R}$ the function $f_t : \mathbb{R}^d_{\geq} \to [0,\infty)$

$$f_t(x) := \frac{1-\psi(e^{-t}x)}{1-\psi(e^{-t}u_0)}.$$

Since ψ is a Laplace transform and t is fixed, it follows (using the multivariate version of the Bernstein theorem, [13, Theorem 4.2.1]), that the derivative of f_t is completely monotone in the multivariate sense, and hence,

$$\varphi_t(x) := \exp(-f_t(x))$$

is the Laplace transform of a probability measure on $\mathbb{R}^d_{\geq}$, due to [21, Criterion XIII.4.2]. Note $\varphi_t(0) = 1$, while the limit as $|x| \to \infty$ may be positive, so the corresponding probability measure might have some mass in zero.

Since the set of probability measures is vaguely compact, we deduce that for any sequence t_k, tending to infinity, there is a subsequence t_n such that φ_{t_n} converges pointwise to the Laplace transform φ of a (sub-)probability measure on $\mathbb{R}^d_{\geq}$, which is continuous on $\mathbb{R}^d_{\geq} \setminus \{0\}$. Since $\varphi_{t_n}(u_0) = e^{-1} > 0$ for all n, it follows that $\varphi > 0$ on $\mathbb{R}^d_{\geq}$, and hence, we obtain that

$$\lim_{n\to\infty} f_{t_n}(x) \;=\; f(x) \;:=\; -\log\varphi(x)$$

exists for all $x \in \mathbb{R}^d_{\geq}$ with f being continuous on $\mathbb{R}^d_{\geq} \setminus \{0\}$.

This implies the pointwise convergence

$$\lim_{n\to\infty} h_{t_n}(u,s) \;=\; h(u,s) \;:=\; \frac{f(e^{-s}u)}{e^{-\alpha s}}\,\frac{H(u_0)}{H(u)},$$

where the function h is continuous on $\mathbb{R} \times \mathbb{S}_{\geq}$. ☐

Lemma 4.2. *Let t_n be a sequence such that h_{t_n} converges to a limit h. Then h is superharmonic for (U_n, S_n) under $\mathbb{P}_u$, i.e.*

$$h(u,s) \geq \mathbb{E}_u\, h(U_1, s+S_1).$$

Proof. Using Eq. (2.1) and a telescoping sum, we obtain (since ψ is a fixed point),

$$\begin{aligned} D(u,s+t) = & \frac{1-\psi(e^{-(s+t)}u)}{e^{-\alpha(s+t)}H(u)} \\ = & \mathbb{E}\left[\frac{1-\prod_{i=1}^N \psi(\mathbf{T}_i^\top e^{-(s+t)}u)}{e^{-\alpha(s+t)}H(u)}\right] \\ = & \mathbb{E}\left[\sum_{i=1}^N \frac{1-\psi(\mathbf{T}_i^\top e^{-(s+t)}u)}{e^{-\alpha(s+t)}H(u)} \prod_{1\le j<i} \psi(\mathbf{T}_i^\top e^{-(s+t)}u)\right]. \end{aligned}$$

Now divide by $e^{\alpha t}(1-\psi(e^{-t}u_0))/H(u_0)$ to obtain

$$\begin{aligned} h_t(u,s) = & \frac{H(u_0)}{H(u)}\mathbb{E}\Bigg[\sum_{i=1}^N \frac{1-\psi(e^{-S^u(i)-(s+t)}U^u(i))}{(1-\psi(e^{-t}u_0))H(U^u(i))e^{-\alpha S^u(i)}e^{-\alpha s}} e^{-\alpha S^u(i)}H(U^u(i)) \\ & \times \prod_{1\le j<i} \psi(e^{-S^u(i)-(s+t)}U^u(i))\Bigg] \\ = & \frac{H(u_0)}{H(u)}\mathbb{E}\Bigg[\sum_{i=1}^N \frac{f_t\Big(e^{-S^u(i)-s}, U^u(i)\Big)}{(H(U^u(i))e^{-\alpha(S^u(i)+s)}} e^{-\alpha S^u(i)}H(U^u(i)) \\ & \times \prod_{1\le j<i} \psi(e^{-S^u(i)-(s+t)}U^u(i))\Bigg] \\ = & \frac{1}{H(u)}\mathbb{E}\Bigg[\sum_{i=1}^N h_t\Big(U^u(i), s+S^u(i)\Big)\, e^{-\alpha S^u(i)}H(U^u(i)) \\ & \times \prod_{1\le j<i} \psi(e^{-S^u(i)-(s+t)}U^u(i))\Bigg]. \end{aligned}$$

Now consider the subsequential limit $t_n \to \infty$, then the LHS converges by assumption to h, while for the RHS, we use Fatou's lemma and observe that the product tends to 1, so that we obtain:

$$\begin{aligned} h(u,s) \ge & \frac{1}{H(u)}\mathbb{E}\left[\sum_{i=1}^N h\Big(U^u(i), s+S^u(i)\Big)\, e^{-\alpha S^u(i)}H(U^u(i))\right] \\ = & \mathbb{E}_u\, h(U_1, s+S_1). \end{aligned}$$

□

Lemma 4.3. *Assume* (A6) *if* $\alpha \neq 1$. *The (subsequential limit) function* h *is constant and equal to 1 on* $\operatorname{supp}\pi \times \mathbb{R}$.

Proof. Suppose first that $\alpha \neq 1$ and (A6) holds. Let $u \in \operatorname{supp}\pi$ and $t \in \mathbb{R}$. By Corollary 3.3, under $\mathbb{P}_{u_0}$, (U_n, S_n) visits every neighborhood of $(u_0, 0)$ and (u,t) infinitely often. By the continuity of h, $h(U_n,S_n)$ converges to $h(u_0,0) = 1$ and to $h(u,t)$ along subsequences. Refering to Lemma 4.2, (U_n,S_n) is a nonnegative supermartingale, which hence converges $\mathbb{P}_{u_0}$-a.s. This implies $h(u,t) = h(u,0) = 1$.

Condition (A6) is not needed if $\alpha = 1$, because then $s \mapsto h(u, \log s)$ is a Laplace transform for each u, which is in particular monotone. Let $u \in \operatorname{supp}\pi$. By (3.2), $h(U_n,S_n)$ converges to $h(u,\infty)$, $h(u_0,\infty)$, $h(u,-\infty)$ and to $h(u_0,\infty)$ along subsequences. Using as before the a.s. convergence of $h(U_n, s+S_n)$, it follows first that $h(u,\cdot)$ is constant for any $u \in \operatorname{supp}\pi$, and subsequently, that $h(u,0) = h(u_0) = 1$, hence h is equal to 1 on $\operatorname{supp}\pi \times \mathbb{R}$. □

Lemma 4.4. *Assume* (A6) *if* $\alpha \neq 1$. *It holds that*

$$\lim_{t\to\infty} \frac{1-\psi(e^{-(s+t)}u)}{e^{-\alpha s}(1-\psi(e^{-t}u_0))} \frac{H(u_0)}{H(u)} \;=\; 1 \qquad \forall u \in \mathbb{S}_\geq,\, s \in \mathbb{R}, \tag{4.2}$$

and the convergence is uniform on compact subsets of $\mathbb{S}_\geq \times \mathbb{R}$. *In particular, the positive function*

$$L(r) \;:=\; \frac{1-\psi(ru_0)}{r^\alpha H(u_0)} \qquad \Big(=\; D(u_0, -\log r)\Big) \tag{4.3}$$

is slowly varying at 0, and

$$\lim_{r\to 0} \sup_{u\in\mathbb{S}_\geq} \left| \frac{1-\psi(ru)}{L(r)\, r^\alpha} - H(u) \right| \;=\; 0. \tag{4.4}$$

Proof. Combining Lemmata 4.2 and 4.3, we obtain that for every sequence $t_k \to \infty$ there is a subsequence $t_n \to \infty$ such that for each $s \in \mathbb{R}$,

$$1 \;=\; \lim_{n\to\infty} h_{t_n}(u_0, s) \;=\; \lim_{n\to\infty} \frac{1-\psi(e^{-(s+t_n)}u_0)}{e^{-\alpha s}(1-\psi(e^{-t_n}u_0))}.$$

Since all subsequential limits are the same, we infer that $\lim_{t\to\infty} h_t(u_0, s) = 1$ for all $s \in \mathbb{R}$, which in particular proves the slow variation assertion about the function $L(r)$, for $L(sr)/L(r) = h_{-\log r}(u_0, -\log s)$. Using the estimate

$$\left(\min_{1\le i\le d}(u_0)_i\right)(1-\psi(r\mathbf{1})) \;\le\; (1-\psi(ru_0)) \;\le\; (1-\psi(r\mathbf{1}))$$

(see [32, Lemma A.1]), we deduce further that

$$0 \;<\; \liminf_{r\to\infty} \frac{1-\psi(r\mathbf{1})}{L(r)r^\alpha} \;\le\; \limsup_{r\to\infty} \frac{1-\psi(r\mathbf{1})}{L(r)r^\alpha} \;<\; \infty,$$

i.e., ψ is *L-α-regular* in the sense of [32, Definition 2.1]. Then [32, Theorem 8.2] provides us with the first assertion, i.e. the (uniform) convergence in Eq. (4.2). Then Eq. (4.4) is a direct consequence when considering the compact set $\mathbb{S}_\geq \times \{0\}$. □

5 Uniqueness of Fixed Points

In this section, we are going to finish the proof of Theorem 2.4. Therefore, we show that the slowly varying function appearing in (2.5) is unique up to asymptotic equivalence, and that this property then identifies the fixed points. The approach is the multivariate analogue of [8, Theorem 8.6].

We start with the following lemma, the proof of which we postpone to the end of this section for a better stream of arguments.

Lemma 5.1. *It holds that*

$$\lim_{n\to\infty} \max_{|v|=n} \|\mathbf{L}(v)\| = \lim_{n\to\infty} \sum_{|v|=n} \|\mathbf{L}(v)\|^\alpha = 0 \qquad \mathbb{P}\text{-a.s.} \tag{5.1}$$

For $u \in \mathbb{S}_\geq$, we can introduce for $t \in \mathbb{R}$ the *homogeneous stopping line*

$$\mathcal{I}_t^u \;:=\; \{v \in \mathbb{V} \,:\, S^u(v) > t,\; S^u(v|k) \le t,\, \forall k < |v|\}\,.$$

Since $\max_{|v|=n} \|\mathbf{L}(v)\| \to 0$ $\mathbb{P}$-a.s. by Lemma 5.1, this stopping line is finite $\mathbb{P}$-a.s. and *dissecting*, see [27, Section 2] for a definition.

Let ψ be a fixed point of $\mathcal{S}$. Define

$$M_n(x) := \prod_{|v|=n} \psi(\mathbf{L}(v)^\top x), \qquad x \in \mathbb{R}^d_\geq.$$

By Eq. (2.1), this constitutes a bounded martingale w.r.t. $\mathcal{B}_n$ for every x and we call its $\mathbb{P}$-a.s. limit $M(x) \in [0,\infty)$ the *disintegration* of the fixed point ψ. Setting

$$Z(x) \ := \ -\log M(x),$$

the martingale property together with boundedness implies that $\psi(x) = \mathbb{E}\exp(-Z(x))$ for all $x \in \mathbb{R}^d_\geq$. Following the proof of [4, Lemma 4.1], one can show that $M(\cdot,\omega)$ is a Laplace transform for $\mathbb{P}$-a.e. $\omega \in \Omega$, and that M is jointly measurable on $\mathbb{S}_\geq \times \Omega$. This implies the measurability of Z as well.

Proposition 5.2. *Assume* (A6) *if* $\alpha \neq 1$. *Let* ψ *be a nontrivial fixed point of* $\mathcal{S}$ *with disintegration* M. *Let* $F : \mathbb{R}^d_\geq \to [0,\infty)$ *be a nonnegative measurable function with* $\lim_{s\to 0}\sup_{u\in\mathbb{S}_\geq} |F(su)-\gamma| = 0$ *for some* $\gamma \geq 0$. *Then the following holds:*

1. $\lim_{n\to\infty} \sum_{|v|=n} F(\mathbf{L}(v)^\top x)(1-\psi(\mathbf{L}(v)^\top x)) = \gamma Z(x)$ $\mathbb{P}$*-a.s.*
2. *For all* $u \in \mathbb{S}_\geq$, $r \in \mathbb{R}_>$, $Z(ru) = r^\alpha Z(u)$.
3. $\psi(ru) = \mathbb{E}e^{-r^\alpha Z(u)}$ *for all* $u \in \mathbb{S}_\geq$, $r \geq 0$.
4. $\mathbb{P}\left(Z(u) \in (0,\infty) \,|\, \mathcal{N}\right) = 1$.
5. $\lim_{t\to\infty} \sum_{v\in\mathcal{I}^u_t} \left(1-\psi(e^{-S^u(v)}U^u(v))\right) \ = \ Z(u)$ $\mathbb{P}$*-a.s. for all* $u \in \mathbb{S}_\geq$.

Proof. Using Lemma 5.1, the proof of Assertion (1) is the same as for [32, Lemma 7.3] and therefore omitted. By Lemma 4.4, for all $r \in \mathbb{R}_>$ and $u \in \mathbb{S}_\geq$, the function $F(su) := \frac{1-\psi(rsu)}{1-\psi(ru)}$ converges uniformly to r^α. Thus we obtain (2) by an application of (1). Then (3) is an immediate consequence of $\psi(x) = \mathbb{E}\exp(-Z(x))$.

Reasoning as in the proof of [20, Theorem 3.2], we see that for any nontrivial fixed point ψ of $\mathcal{S}$, $\psi(\infty) = 1 - \mathbb{P}(\mathcal{N})$. Moreover, $Z(u) = 0$ on $\mathcal{N}^c$ and consequently $Z(u) > 0$ $\mathbb{P}$-a.s. on $\mathcal{N}$. On the other hand, since ψ is the Laplace transform of a random variable on $\mathbb{R}^d_\geq$, $Z(u) < \infty$ $\mathbb{P}$-a.s. ☐

The subsequent lemma is where we use assumption (A5). Using the definition of μ, it implies that with $c' := -\log c$

$$\begin{aligned}
\mathbb{P}\left(S^u(i) > c' \ \forall\, 1 \leq i \leq N\right) \ &\leq \mathbb{E}\Bigg[\sum_{i=1}^N \mathbb{1}\left(S^u(i) > c'\right)\Bigg] \\
&= \mathbb{E}N\mathbb{E}\Big[\mathbb{1}\left(-\log\left|\mathbf{M}^\top u\right| > c'\right)\Big] \ = \ \mathbb{E}N\mathbb{P}\left(\left|\mathbf{M}^\top u\right| < c\right) \\
&\leq \mathbb{E}N\mathbb{P}\left(\iota(\mathbf{M}^\top) < c\right) \ = \ 0.
\end{aligned}$$

In other words, the increments of $S(vi) - S(v)$ are $\mathbb{P}$-a.s. bounded from below by c'.

Lemma 5.3. *Assume* (A5)*; and* (A6) *if* $\alpha \neq 1$. *Let* ψ *be a nontrivial fixed point of* $\mathcal{S}$ *with associated slowly varying function* L *given by Eq.* (4.3)*. Then*

$$\lim_{t\to\infty} L(e^{-t}) \sum_{v\in\mathcal{I}^u_t} H(U^u(v))e^{-\alpha S^u(v)} \ = \ Z(u) \quad \mathbb{P}_u\text{-a.s.} \tag{5.2}$$

Proof. By Lemma 4.4,

$$\lim_{t\to\infty} \frac{1-\psi(e^{-s-t}y)}{H(y)e^{-\alpha(s+t)}L(e^{-t})} = 1,$$

and the convergence is uniform on compact sets for (y,s). In particular, it is uniform on the set $\mathbb{S}_\geq \times [0,c']$. Now applying this result with $s = S^u(v) - t$ and $y = U^u(v)$ with $v \in \mathcal{I}_t^u$ and using that

$$0 \ < \ S^u(v) - t \ \leq \ S^u(v) - S^u(v|(|v|-1)) \in [0,c']$$

by Assumption (A5), we deduce from Proposition 5.2, (5) that

$$\begin{aligned} Z(u) \ &= \ \lim_{t\to\infty} \sum_{v\in\mathcal{I}_t} L(e^{-t})H(U^u(v))e^{-\alpha S^u(v)} \frac{1-\psi(e^{-(S^u(v)-t)-t}U^u(v))}{H(U^u(v))e^{-\alpha(S^u(v)-t+t)}L(e^{-t})} \\ &= \ \lim_{t\to\infty} L(e^{-t}) \sum_{v\in\mathcal{I}_t} H(U^u(v))e^{-\alpha S^u(v)} \qquad \mathbb{P}\text{-a.s.} \end{aligned}$$

□

Remark 5.4. The idea of this proof follows that of [8, Theorem 8.6]. There an assumption similar to (A5) is avoided by first reducing to an embedded non-critical smoothing transform (with weights bounded by 1) via stopping lines, and subsequently using the theory of general branching processes developed in [25, 33] to show that the fraction of particles with overshoot $S(v) - t$ being large becomes small.

In the multivariate case, the embedding procedure necessitates to formulate (A1), (A2) and (A6) also in terms of the sequence

$$(\tilde{N}, (\tilde{\mathbf{T}}_i)_{i\geq 1}) \ := \ (\#\mathcal{I}, (\mathbf{L}(v))_{v\in\mathcal{I}}),$$

with

$$\mathcal{I} \ := \ \left\{ v \in \mathbb{V} \ : \ \|\mathbf{L}(v)\| < 1, \ \|\mathbf{L}(v|k)\| \geq 1 \, \forall k < |v| \right\}.$$

However, validity of (A1), (A2) and (A6) for $(N, (\mathbf{T}_i)_{i\geq 1})$ does not imply that these assumptions hold for $(\tilde{N}, (\tilde{\mathbf{T}}_i)_{i\geq 1})$, too. In order to avoid getting off track too much and introducing assumptions on objects different from $(\mathbf{T}_i)_{i\geq 1}$, we decided to impose (A6).

Nevertheless, once the embedded smoothing transform is shown to satisfy all the stated assumptions, one can use results from [32, Section 4] to bound the number of particles with $S(v) - t \geq c'$ as in the one-dimensional case.

Now we are ready to prove our main result.

Proof of Theorem 2.4. *Step 1*: By Proposition 2.2, there is a nontrivial fixed point of $\mathcal{S}$ with LT ψ, say. By Proposition 5.2, for each $u \in \mathbb{S}_\geq$, there is a random variable $Z(u)$ with $\mathbb{P}(Z(u) > 0 \,|\, \mathcal{N}) = 1$ and and such that $\psi(ru) = \mathbb{E}[\exp(-r^\alpha Z(u))]$ for all $r \in [0,\infty)$. Define $L(r)$ by (4.3), choosing a suitable u_0.

Step 2: Let now ψ_2 be the Laplace transform of a different nontrivial fixed point, with corresponding disintegration M_2 and Z_2, and slowly varying function L_2, defined by (4.3), using the same u_0 as before. Recall that $Z(u)$ and $Z_2(u)$ are $\mathbb{P}$-a.s. positive and finite by by Proposition 5.2, (4) for each $u \in \mathbb{S}_\geq$. Then we have by Lemma 5.3 that $\mathbb{P}$-a.s. on $\mathcal{N}$,

$$\frac{Z_2(u)}{Z(u)} = \lim_{t\to\infty} \frac{L_2(e^{-t}) \sum_{v\in\mathcal{I}_t^u} H(U^u(v))e^{-\alpha S^u(v)}}{L(e^{-t}) \sum_{v\in\mathcal{I}_t^u} H(U^u(v))e^{-\alpha S^u(v)}} = \lim_{t\to\infty} \frac{L_2(e^{-t})}{L(e^{-t})}.$$

Since the right hand side is deterministic and does not depend on u it is constant i.e. $Z_2(u) = KZ(u)$ on $\mathcal{N}$. Moreover, $Z = Z_2 = 0$ on $\mathcal{N}^c$, hence $Z_2(u) = KZ(u)$ $\mathbb{P}$-a.s. Consequently,

$$\psi_2(ru) = \mathbb{E}\left[e^{-r^\alpha Z_2(u)}\right] = \mathbb{E}\left[e^{-r^\alpha KZ(u)}\right] = \psi(K^{1/\alpha}ru),$$

which proves Eq. 2.4.

Step 3: Fix L to be the slowly varying function corresponding to ψ. Then Eq. (2.5) follows from Eq. (4.4) for this particular ψ, and moreover,

$$\lim_{r\to 0} \frac{1-\psi_2(ru)}{r^\alpha L(r)} \;=\; \lim_{r\to 0} \frac{K(1-\psi(K^{1/\alpha}ru))}{Kr^\alpha L(K^{1/\alpha}r)} \frac{L(K^{1/\alpha}r)}{L(r)} \;=\; KH(u)$$

The final assertion about $\limsup_{r\to 0} L(r)$ will be proved in Lemma 5.5. ☐

Proof of Lemma 5.1. For $u_0 \in \mathrm{int}(\mathbb{S}_\geq)$ observe that

$$W_n(u_0) \;:=\; \sum_{|v|=n} H(\mathbf{L}(v)^\top u_0) = \sum_{|v|=n} H(\mathbf{L}(v)^\top . u_0)|\mathbf{L}(v)u_0|^\alpha,$$

by Proposition 2.3, defines a nonnegative martingale w.r.t. the filtration $\mathcal{B}_n$ having $\mathbb{P}$-a.s. limit $W(u_0)$. By [9, Theorem 2.1 (iii)] and (3.4) it follows that $\mathbb{E}W(u_0) = 0$. Consequently, $W_n(u_0)$ converge to 0.

Since all entries of u_0 are positive, there is a constant C such that $\|\mathbf{a}\| \leq C\,|\mathbf{a}u_0|$ for all $\mathbf{a} \in \mathfrak{M}$. Moreover, the function H is bounded from below on $\mathbb{S}_\geq$, hence there is a constant C' such that

$$\sum_{|v|=n} \|\mathbf{L}(v)\|^\alpha \leq C' W_n(u_0),$$

which proves the assertion. ☐

Lemma 5.5. *Assume* (A5)*; and* (A6) *if* $\alpha \neq 1$*. Then* $\limsup_{r\to 0} L(r) = \infty$*. If* $\alpha = 1$*, then* $\mathbb{E}\,|X| = \infty$ *for every nontrivial fixed point* X*.*

Proof. Suppose that $\limsup_{r\to 0} L(r) \leq C < \infty$. By an extension of Prop. 5.2, (1),

$$\begin{aligned} Z(u) \;&=\; \lim_{n\to\infty} \sum_{|v|=n} L(\big|\mathbf{L}(v)^\top u\big|)\, H(\mathbf{L}(v)^\top u) \frac{1-\psi(\mathbf{L}(v)^\top u)}{L(|\mathbf{L}(v)^\top u|)\, H(\mathbf{L}(v)^\top u)} \\ &\leq C \lim_{n\to\infty} \sum_{|v|=n} H(\mathbf{L}(v)^\top u) \;=\; C \lim_{n\to\infty} \sum_{|v|=n} \|\mathbf{L}(v)\|^\alpha = 0 \end{aligned}$$

by Proposition 5.1, which gives a contradiction.

If now $\alpha = 1$, then

$$\lim_{r\to 0} \frac{1-\psi(ru)}{r} \;=\; \langle u, \mathbb{E}X \rangle,$$

being finite or not. Combining this with Eq. (2.5) implies that

$$\lim_{r\to 0} L(r) \;=\; \frac{\langle u, \mathbb{E}X\rangle}{KH(u)},$$

hence $\mathbb{E}\,|X| = \infty$, since $\limsup_{r\to 0} L(r) = \infty$. ☐

6 Determining the Slowly Varying Function

In this section, we want to identify the slowly varying function L, which was defined in Eq. (4.3) to be

$$L(r) \;=\; \frac{1-\psi(ru_0)}{r^\alpha H(u_0)} \;=\; D(u_0, -\log r)$$

(recall the definition of D in (4.1)). We are going to show that

$$\lim_{t\to\infty} \frac{D(u_0,t)}{t} = K' \in (0,\infty), \tag{6.1}$$

which gives that $\lim_{r\to 0} L(r)/\left|\log r\right| = K'$, i.e. we may choose the slowly varying function to be a scalar multiple of $\left|\log r\right| \vee 1$.

The basic idea to prove Eq. (6.1) comes from [20] and is by using a renewal equation satisfied by (the one-dimensional analogue of) $D(u_0,t)$. In the present multivariate situation, we obtain a Markov renewal equation for a drift-less Markov random walk. By a clever application of the regeneration lemma, we can reduce this again to a (one-dimensional) renewal equation for a drift-less random walk, for which enough theory is known to solve it.

Throughout Section 6, we assume that (in addition to the standing assumptions (A1) – (A4)) either (A6c) or (A6f) holds.

6.1 The Renewal Equation

In this subsection we present the Markov renewal equation for $D(u,t)$ and show how, using Lemma 3.4, it can be replaced by a one-dimensional renewal equation.

Lemma 6.1. *The following renewal equation holds*

$$D(u,t) = \mathbb{E}_u D(U_1, t+S_1) - G(u,t), \tag{6.2}$$

where

$$G(u,t) := \frac{e^{\alpha t}}{H(u)} \mathbb{E}\left[\prod_{i=1}^N \phi(e^{-t}\mathbf{T}_i^\top u) + \sum_{i=1}^N \big(1-\phi(e^{-t}\mathbf{T}_i^\top u)\big) - 1\right]. \tag{6.3}$$

Source: Lemma 9.6 in [31], note there the different notation $V_1 = -S_1$. □

Lemma 6.2. *We have*

1. $G(u,t) \ge 0$ *for all* $(u,t) \in \mathbb{S}_\ge \times \mathbb{R}$.
2. *For all* $u \in \mathbb{S}_\ge$, $t \mapsto e^{-\alpha t}G(u,t)$ *is decreasing.*

Source: Lemma 9.7 in [31], being a straightforward generalization of [20, Lemma 2.4]. □

Since the assumptions of the Regeneration Lemma (Lemma 3.4) are satisfied, we know that there is a sequence of stopping times $(\sigma_n)_{n\in\mathbb{N}}$ and a probability measure η on $\mathbb{S}_\ge \times \mathbb{R}$ such that (R3) holds.

For any nonnegative measurable function F on $\mathbb{S}_\ge \times \mathbb{R}$ we define $\hat{F} : \mathbb{R} \mapsto \mathbb{R}$ by

$$\hat{F}(t) \;:=\; \mathbb{E}_\eta\, F(U_{\sigma_1 - 1}, t + S_{\sigma_1 - 1}). \tag{6.4}$$

Moreover, under each $\mathbb{P}_u$, let $(V_n)_{n\in\mathbb{N}}$ be a zero-delayed random walk with increment distribution $\mathbb{P}_\eta(S_{\sigma_1 - 1} \in \cdot)$, independent of all other occurring random variables. Note that V_n is a drift-less random walk.

Lemma 6.3. *For any nonnegative measurable function F on $\mathbb{S}_{\geq} \times \mathbb{R}$ and $k \geq 0$, the following equation holds*

$$\mathbb{E}_\eta \left[F(U_{\sigma_{k+1}-1}, S_{\sigma_{k+1}-1}) \right] = \mathbb{E}_\eta \hat{F}(V_k)$$

Proof. We give the proof using induction on k. By the definition of $\hat{F}$, the equation holds for $k=0$. Suppose now that it holds for some $k \geq 0$. Then

$$\begin{aligned}
\mathbb{E}_\eta \left[F(U_{\sigma_{k+2}-1}, S_{\sigma_{k+2}-1}) \right] &= \mathbb{E}_\eta \left[\mathbb{E}_\eta \left[F(U_{\sigma_{k+2}-1}, S_{\sigma_1 - 1} + (S_{\sigma_{k+2}-1} - S_{\sigma_1-1})) | \mathcal{F}_{\sigma_1 - 1} \right] \right] \\
&= \mathbb{E}_\eta \left[\mathbb{E}_\eta{}' \left[F(U'_{\sigma_{k+1}-1}, S_{\sigma_1-1} + S'_{\sigma_{k+1}-1}) \right] \right] \\
&= \mathbb{E}_\eta \left[\mathbb{E}_\eta{}' \left[\hat{F}(S_{\sigma_1 - 1} + V'_k) \right] \right] = \mathbb{E}_\eta \left[\hat{F}(V_{k+1}) \right],
\end{aligned}$$

where (R3) from Lemma 3.4 is used in the second equality and we denote by $(U'_n, S'_n), V_k$ an independent copy of $(U_n, S_n), V_k$ with corresponding expectation $\mathbb{E}_\eta{}'$. ☐

Now we can formulate the univariate renewal equation, corresponding to Eq. (6.2), in terms of the function $\hat{D}$, obtained from D by the formula (6.4).

Lemma 6.4. *For $g(t) = \mathbb{E}_\eta \left[\sum_{i=0}^{\sigma_1 - 2} G(U_i, t + V_1 + S_i) \right]$ we have*

$$\hat{D}(t) = \mathbb{E}_\eta \hat{D}(t + V_1) - g(t). \tag{6.5}$$

Proof. Let

$$M_n = D(U_n, t + S_n) - \sum_{i=0}^{n-1} G(U_i, t + S_i).$$

Since (U_n, S_n) is a Markov chain, the Markov renewal equation (6.2) implies that M_n is a $\mathbb{P}_u$-martingale (with respect to the filtration $\mathcal{G}_n$) for each u. Since $\tau = \sigma_1 - 1$ is a stopping time by (3.4), the optional stopping theorem implies that

$$D(u, t+s) = \mathbb{E}_{u,s} \left[D(U_{\sigma_1 - 1}, t + S_{\sigma_1 - 1}) - \sum_{i=0}^{\sigma_1 - 2} G(U_i, t + S_i) \right] \tag{6.6}$$

and

$$D(u, t+s) = \mathbb{E}_{u,s} \left[D(U_{\sigma_2 - 1}, t + S_{\sigma_2 - 1}) - \sum_{i=0}^{\sigma_2 - 2} G(U_i, t + S_i) \right]. \tag{6.7}$$

Equating the right hand sides of (6.6) and (6.7) and integrating with respect to η, we obtain

$$\begin{aligned}
\hat{D}(t) = \mathbb{E}_\eta \left[D(U_{\sigma_1 - 1}, t + S_{\sigma_1 - 1}) \right] &= \mathbb{E}_\eta \left[D(U_{\sigma_2 - 1}, t + S_{\sigma_2 - 1}) - \sum_{i=\sigma_1 - 1}^{\sigma_2 - 2} G(U_i, t + S_i) \right] \\
&= \mathbb{E}_\eta \hat{D}(t + V_1) - \mathbb{E}_\eta \left[\sum_{i=0}^{\sigma_1 - 2} G(U_i, t + V_1 + S_i) \right].
\end{aligned}$$

☐

6.2 Solving the Renewal Equation

In this subsection, we will show that $\lim_{t\to\infty} D(u_0,t)/t = 1$. Before we can use the renewal equation, we first have to consider some technicalities, e.g. direct Riemann integrability of g. We start by considering moments of V_1.

Lemma 6.5. *Assume* (A7). *There exists $\delta > 0$ such that $\mathbb{E}_\eta e^{\delta|V_1|} < \infty$.*

Proof. We proof the boundedness of $\mathbb{E}_\eta e^{-\delta V_1}$ and $\mathbb{E}_\eta e^{\delta V_1}$ separately, starting with the first one.

Property (R4) implies that there exists δ_0 such that $\sup_u \mathbb{E}_u\left[e^{\delta_0(\sigma-1)}\right] < \infty$.

Due to Assumption (A7), there is $\varepsilon > 0$ such that $m(\alpha+\varepsilon) \le e^{\delta_0}$. By [17, Proposition 3.1] there exists a continuous function $\tilde{H} : \mathbb{S}_\ge \to (0,\infty)$, such that

$$\mathbb{E}\,|\mathbf{M}u|^{\alpha+\varepsilon}\,\tilde{H}(\mathbf{M}.u) \;=\; \frac{m(\alpha+\varepsilon)}{\mathbb{E}N\tilde{H}(u)}$$

for all $u \in \mathbb{S}_\ge$, and consequently

$$\mathbb{E}_{u,t}\left[\frac{\tilde{H}(U_1)}{H(U_1)}e^{-\varepsilon S_1}\right] = m(\alpha+\varepsilon).$$

As a consequence, there is $C_\varepsilon < \infty$ such that

$$\frac{e^{-\varepsilon S_n}}{m(\alpha+\varepsilon)^n} \;\le\; C_\varepsilon \frac{H(u)}{\tilde{H}(u)}\frac{\tilde{H}(U_n)}{H(U_n)}\frac{e^{-\varepsilon S_n}}{m(\alpha+\varepsilon)^n},$$

and the right hand side is a martingale under $\mathbb{P}_u$ with expectation C_ε. Therefore, the optional stopping theorem and the Fatou lemma imply

$$\mathbb{E}_u\left[\frac{e^{-\varepsilon S_{\sigma-1}}}{m(\alpha+\varepsilon)^{\sigma-1}}\right] \le \lim_{n\to\infty}\mathbb{E}_u\left[\frac{e^{-\varepsilon S_{(\sigma-1)\wedge n}}}{m(\alpha+\varepsilon)^{(\sigma-1)\wedge n}}\right] \le C_\varepsilon.$$

The choice of ε gives us $\sup_u \mathbb{E}_u\left[m(\alpha+\varepsilon)^{\sigma-1}\right] < \infty$, hence by the Cauchy-Schwartz inequality,

$$(\mathbb{E}_u\left[e^{-\frac{\varepsilon}{2}S_{\sigma-1}}\right])^2 \le \mathbb{E}_u\left[e^{-\varepsilon S_{\sigma-1}}/m(\alpha+\varepsilon)^{\sigma-1}\right]\mathbb{E}_u\left[m(\alpha+\varepsilon)^{\sigma-1}\right]$$

is bounded uniformly in u. Choose $\delta = \min\{\delta_0, \varepsilon/2\}$.

The proof of the second part is along the same lines, using the finiteness of $m(\alpha-\varepsilon)$ (this follows from the convexity of m and the assumption (A1), which gives $m(0) = \mathbb{E}N < \infty$.) $\square$

Before proving that $g(t)$ is dRi, we need the following consequence of the slow variation of $D(u_0,t)$ (for $t\to\infty$).

Lemma 6.6. *Let $d^*(t) = \sup_{u\in\mathbb{S}_\ge} D(t,u)$. Then for all $0<\varepsilon<\alpha$, there is $C>0$, such that for $t\ge 0$ and any s*

$$d^*(s) \le Ce^{\varepsilon s}, \tag{6.8}$$

$$\frac{d^*(t+s)}{L(e^{-t})} \le Ce^{\varepsilon|s|}. \tag{6.9}$$

Proof. Since the ratio $D(t,u)/L(e^{-t})$ is bounded it suffice to show the above inequalities with $L(e^{-t})$ instead of $d^*(t)$. Potter's theorem [12, Theorem 1.5.6], applied to the slowly varying function L proves that

$$\frac{L(e^{-x})}{L(e^{-y})} \le Ce^{\varepsilon|x-y|}, \tag{6.10}$$

for any positive x, y. Using also the trivial bound $L(e^{-t}) \le Ce^{\alpha t}$ we get (6.8). In order to show (6.9) we use (6.10) in the case when $t + s \ge 0$. When $t + s \le 0$ we have

$$\frac{L(e^{-t-s})}{L(e^{-t})} = \frac{L(e^{-t-s})}{L(1)} \frac{L(1)}{L(e^{-t})} \le Ce^{\alpha(t+s)}e^{\varepsilon t} \le Ce^{\varepsilon|s|}.$$

□

Below, we will need the following moment estimate, which is a consequence of (A7).

Lemma 6.7. *Assume* (A7). *There is $\delta > 0$ such that*

$$\mathbb{E}\bigg[\bigg(\sum_{i=1}^{N} \|\mathbf{T}_i\|^{\alpha/(1+\delta)}\bigg)^{1+\delta}\bigg] < \infty. \tag{6.11}$$

Proof. First, observe that if there are $0 < \theta_0 < \theta_1$ and $p_0 \ge 1$, $p_1 \ge 1$ satisfying

$$\mathbb{E}\bigg[\bigg(\sum_{i=1}^{N} \|\mathbf{T}_i\|^{\theta_k}\bigg)^{p_k}\bigg] < \infty, \qquad k \in \{0, 1\},$$

then Hölder's inequality implies that for any $\theta \in [\theta_0, \theta_1]$

$$\mathbb{E}\bigg[\bigg(\sum_{i=1}^{N} \|\mathbf{T}_i\|^{\theta}\bigg)^{p(\theta)}\bigg] < \infty,$$

with

$$\frac{1}{p(\theta)} = \frac{\theta_1 - \theta}{\theta_1 - \theta_0} \cdot \frac{1}{p_0} + \frac{\theta - \theta_0}{\theta_1 - \theta_0} \cdot \frac{1}{p_1}. \tag{6.12}$$

Indeed, we have

$$\begin{aligned}
\mathbb{E}\bigg[\bigg(\sum_{i=1}^{N} \|\mathbf{T}_i\|^{\theta}\bigg)^{p(\theta)}\bigg] &= \mathbb{E}\bigg[\bigg(\sum_{i=1}^{N} \|\mathbf{T}_i\|^{\theta_0 \frac{\theta_1-\theta}{\theta_1-\theta_0} + \theta_1 \frac{\theta-\theta_0}{\theta_1-\theta_0}}\bigg)^{p(\theta)}\bigg] \\
&\le \mathbb{E}\bigg[\bigg(\sum_{i=1}^{N} \|\mathbf{T}_i\|^{\theta_0}\bigg)^{\frac{\theta_1-\theta}{\theta_1-\theta_0}p(\theta)} \bigg(\sum_{i=1}^{N} \|\mathbf{T}_i\|^{\theta_1}\bigg)^{\frac{\theta-\theta_0}{\theta_1-\theta_0}p(\theta)}\bigg] \\
&\le \mathbb{E}\bigg[\bigg(\sum_{i=1}^{N} \|\mathbf{T}_i\|^{\theta_0}\bigg)^{p_0}\bigg]^{\frac{\theta_1-\theta}{\theta_1-\theta_0}\frac{p(\theta)}{p_0}} \mathbb{E}\bigg[\bigg(\sum_{i=1}^{N} \|\mathbf{T}_i\|^{\theta_1}\bigg)^{p_1}\bigg]^{\frac{\theta-\theta_0}{\theta_1-\theta_0}\frac{p(\theta)}{p_1}} < \infty.
\end{aligned}$$

Applying the observation from above for $\theta_0 = 0, p_0 = 1, \theta_1 = \beta, p_1 = p$ and for $\theta_0 = \beta, p_0 = p, \theta_1 = \alpha + \varepsilon, p_1 = 1$ we obtain, by (A1), (A7) and (6.12), that for all $\theta \in (0, \alpha + \varepsilon)$ there exists $p(\theta) > 1$ such that

$$\mathbb{E}\bigg[\bigg(\sum_{i=1}^{N} \|\mathbf{T}_i\|^{\theta}\bigg)^{p(\theta)}\bigg] < \infty,$$

and now the conclusion easily follows. □

Lemma 6.8. *Assume* (A7). *The function g, defined in Lemma 6.4, is nonnegative and directly Riemann integrable.*

Proof. Referring to Lemma 6.2, G is nonnegative and $t \mapsto e^{-\alpha t}G(t)$ is decreasing, hence the same holds for g. For such functions, a sufficient condition for direct Riemann integrability is that $g \in L^1(\mathbb{R})$, see [22, Lemma 9.1]. Since moreover, by Lemma 3.4, $\mathbb{E}\,\sigma_1 < \infty$, it suffices to show the integrability of $g^* : t \mapsto \sup_{u \in \mathbb{S}_\geq} G(u,t)$.

Introduce the function $h(x) := e^{-x} + x - 1$. Since h is positive for $x \geq 0$, we have $\phi(e^{-t}\mathbf{T}_i^\top u) \leq e^{(1-\phi(e^{-t}\mathbf{T}_i^\top u))}$. Therefore

$$\begin{aligned}
\int g^*(t)dt &= \int \sup_{u\in\mathbb{S}_\geq} \frac{e^{\alpha t}}{H(u)} \mathbb{E}\left[\prod_{i=1}^N \phi(e^{-t}\mathbf{T}_i^\top u) + \sum_{i=1}^N \big(1-\phi(e^{-t}\mathbf{T}_i^\top u)\big) - 1\right] dt \\
&\leq C \int \sup_{u\in\mathbb{S}_\geq} e^{\alpha t}\mathbb{E}\left[e^{\sum_{i=1}^N(1-\phi(e^{-t}\mathbf{T}_i^\top u))} + \sum_{i=1}^N \big(1-\phi(e^{-t}\mathbf{T}_i^\top u)\big) - 1\right] dt \\
&= C \int \sup_{u\in\mathbb{S}_\geq} e^{\alpha t}\mathbb{E}\left[h\left(\sum_{i=1}^N (1-\phi(e^{-t}\mathbf{T}_i^\top u))\right)\right] dt.
\end{aligned}$$

Using Lemma 6.6, boundedness of H and fact that $h(x)$ is increasing, comparable with $\min(x,x^2)$ on the positive half line, the later can be bounded by

$$\begin{aligned}
\int \sup_{u\in\mathbb{S}_\geq} e^{\alpha t}\mathbb{E}\left[h\left(\sum_{i=1}^N e^{(\varepsilon-\alpha)t}\|\mathbf{T}_i^\top u\|^{\alpha-\varepsilon}\right)\right] dt \leq\ & C\,\mathbb{E}\left[\int e^{\alpha t} h\left(e^{(\varepsilon-\alpha)t}\sum_{i=1}^N \|\mathbf{T}_i\|^{\alpha-\varepsilon}\right) dt\right] \\
\leq\ & C\,\mathbb{E}\left[\left(\sum_{i=1}^N \|\mathbf{T}_i\|^{\alpha-\varepsilon}\right)^{\frac{\alpha}{\alpha-\varepsilon}} \int e^{\frac{\alpha}{\alpha-\varepsilon}s} h\left(e^{-s}\right) ds\right] < \infty,
\end{aligned}$$

by (6.11), provided $\frac{\alpha}{\alpha-\varepsilon} < 1+\delta < 2$.

□

Now we show that $\hat{D}$ and $D(u_0,\cdot)$ are asymptotically equivalent.

Lemma 6.9. *Assume* (A7). *We have*

$$\lim_{t\to\infty} \hat{D}(t)/D(u_0,t) = 1.$$

In particular, $\hat{D}(t+s)/\hat{D}(t)$ converge to 1 as t goes to infinity.

Proof. Recalling the definition of h_t from Section 4, we have that

$$\hat{D}(t)/D(u_0,t) = \mathbb{E}_\eta\left[\frac{D(U_{\sigma_1-1}, t+S_{\sigma_1-1})}{D(u_0,t)}\right] \ = \ \mathbb{E}_\eta\left[h_t(U_{\sigma_1-1}, S_{\sigma_1-1})\right]$$

Using Lemma 4.4, $\lim_{t\to\infty} h_t \equiv 1$. Lemmata 6.5 and 6.6 allow us to apply the dominated convergence theorem to obtain the assertion. □

Proposition 6.10. *Let $\hat{D}$ be a function, such that $\hat{D}(t+s)/\hat{D}(t) \to 1$ as $t\to\infty$. Assume that $\hat{D}$ satisfies the renewal equation* (6.5) *with a directly Riemann integrable function g and a nonarithmetic random variable V_1 such that $\mathbb{E}_\eta\left[e^{\delta|V_1|}\right] < \infty$ for some positive δ. Then $\lim_{t\to\infty}\hat{D}(t)/t$ exists and is positive.*

Source: The proof is almost the same as the proof of Theorem 2.18 in [20]. Note that, although in [20] the derivative of $\hat{D}$ is used this can be easily avoided (for details, see the proof of Proposition 3.22 in [18]). □

Now we can identify the slowly varying function L.

Theorem 6.11. *Assume* (A7)*; and* (A6c) *or* (A6f)*. Then there exists* $K > 0$ *such that*

$$\lim_{r\to 0} \frac{1-\psi(ru)}{r^\alpha \, |\log r|} \;=\; KH(u).$$

Proof. Combining Lemmata 6.5 and 6.8 with Proposition (6.10), we infer that

$$\lim_{r\to 0} \frac{\hat{D}(-\log r)}{|\log r|} \;=\; K'$$

for some $K' > 0$. Using Lemma 6.9, it follows that

$$\lim_{r\to 0} \frac{D(u_0, -\log r)}{|\log r|} \;=\; K'.$$

Referring to the uniform convergence of $D(u, -\log r) = (1-\Psi(ru))/(r^\alpha \, |\log r|)$, proved in (4.4), we infer the assertion with $K = K'/H(u_0)$. □

7 The Derivative Martingale

In this section, we finish the proof of Theorem 2.6, by proving the convergence of

$$\mathcal{W}_n(u) = \sum_{|v|=n} [S(v)+b(U(v))] \; H(U(v)) e^{-\alpha S(v)}$$

to a nontrivial limit, which constitutes the exponent of fixed points. The assertions of Theorem 2.6 are contained in the Theorem below, except for the identification of the slowly varying function, which was given in Section 6, in particular in Theorem 6.11.

Theorem 7.1. *Assume* (A7)*; and* (A6c) *or* (A6f)*. Then for each* $u \in \mathbb{S}_\geq$*, the martingale* $\mathcal{W}_n(u)$ *has a nonnegative, nontrivial limit* $\mathcal{W}(u)$*, and* $\psi(ru) := \mathbb{E}\left[e^{-r^\alpha \mathcal{W}(u)}\right]$ *is a fixed point of* $\mathcal{S}$*.*

Proof. Let $M(u)$ be the disintegration of the (up to scaling) unique fixed point of $\mathcal{S}$ (described in Theorem 2.4). By Theorem 6.11, combined with Eq. (4.4) from Lemma 4.4, there is $K' \in (0,\infty)$ such that

$$\lim_{r\to 0} \sup_{u\in\mathbb{S}_\geq} \left| \frac{1-\psi(ru)}{r^\alpha H(u) K' \, |\log(r)|} - 1 \right| \;=\; 0.$$

Then by (1) from Proposition 5.2,

$$\lim_{n\to\infty} \sum_{|v|=n} K' S^u(v) H(U^u(v)) e^{-\alpha S^u(v)} \frac{1-\psi(e^{-S^u(v)}U^u(v))}{K' S^u(v) H(U^u(v)) e^{-\alpha S^u(v)}} \;=\; Z(u) \quad \mathbb{P}\text{-a.s.}$$

As a continuous function on $\mathbb{S}_\geq$, $u \mapsto b(u)$ is bounded, and by Lemma 5.1,

$$\lim_{n\to\infty} \sup_{|v|=n} \left| \frac{S^u(v) + b(U^u(v))}{S^u(v)} - 1 \right| = 0.$$

Therefore, we can replace $S^u(v)$ by $S^u(v) + b(U^u(v))$, and obtain

$$\lim_{n\to\infty} \sum_{|v|=n} \left[S^u(v) + b(U^u(v))\right] H(U^u(v)) e^{-\alpha S^u(v)} \;=\; K' Z(u) \quad \mathbb{P}\text{-a.s.}$$

This shows the $\mathbb{P}$-a.s. convergence of $\mathcal{W}_n(u)$ to $\mathcal{W}(u) := K'Z(u)$. Then $\mathbb{P}\left(\mathcal{W}(u) > 0\right) = 1$ by (4) of Proposition 5.2. That $\psi(ru) = \mathbb{E}\left[e^{-r^\alpha \mathcal{W}(u)}\right]$ is a fixed point follows immediately, since $\mathbb{E}\left[e^{-r^\alpha K' Z(u)}\right]$ is a fixed point for any $K' > 0$. □

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