

Existence of mark functions in marked metric measure spaces

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Abstract

We give criteria on the existence of a so-called mark function in the context of marked metric measure spaces (mmm-spaces). If an mmm-space admits a mark function, we call it functionally-marked metric measure space (fmm-space). This is not a closed property in the usual marked Gromov-weak topology, and thus we put particular emphasis on the question under which conditions it carries over to a limit. We obtain criteria for deterministic mmm-spaces as well as random mmm-spaces and mmm-space-valued processes. As an example, our criteria are applied to prove that the tree-valued Fleming-Viot dynamics with mutation and selection from [5] admits a mark function at all times, almost surely. Thereby, we fill a gap in a former proof of this fact, which used a wrong criterion.

Furthermore, the subspace of fmm-spaces, which is dense and not closed, is investigated in detail. We show that there exists a metric that induces the marked Gromov-weak topology on this subspace and is complete. Therefore, the space of fmm-spaces is a Polish space. We also construct a decomposition into closed sets which are related to the case of uniformly equicontinuous mark functions.

Keywords: mark function ; tree-valued Fleming-Viot process ; mutation ; marked metric measure space ; Gromov-weak topology ; Prohorov metric ; Lusin's theorem.

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1 Introduction

A metric (finite) measure spaces (*mm-space*) is a complete, separable metric space (X, r) together with a finite measure ν on it. Considering the space of (equivalence classes of) mm-spaces itself as a metric space dates back to Gromov's invention of the \square_λ -metric in [11, Chapter 3 $\frac{1}{2}$]. Motivated by Aldous' work on the Brownian continuum random tree ([1]), it was realised in [9] that the space of mm-spaces is a useful state space for tree-valued stochastic processes, and Polish when equipped with the Gromov-weak topology. That the Gromov-weak topology actually coincides with the one induced by the \square_λ -metric was shown in [15].

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Important examples for the use of mm-spaces within probability theory are given by individual-based populations X with mutual genealogical distances r between individuals. Here, r can for instance measure the time to the most recent common ancestor (MRCA) (cf. [5, (2.7), Remark 3.3]), where the resulting metric space is ultrametric. Another possibility is the number of mutations back to the MRCA (cf. [14]), where the resulting space is not ultrametric. Finally, there is a sampling measure ν on the space X which models population density. This means that the state of the process is an mm-space (X, r, ν) . Such individual-based models are often formulated for infinite population size (with diffuse measures ν) but obtained as the high-density limit of approximating models with finite populations (where ν is typically the uniform distribution on all individuals).

For encoding more information about the individuals, such as an (allelic) type or location (which may change over time), marked metric measure spaces (mmm-spaces) and the corresponding marked Gromov-weak topology (mGw-topology) have been introduced in [4]. For a fixed complete, separable metric space (I, d) of marks, the sampling measure ν is replaced by a measure μ on $X \times I$, which models population density in combination with mark distribution.

A natural question in this context is whether or not every point of the limiting population X has a single mark almost surely, that is, does genetic distance zero imply the same type/location? Put differently, we ask ourselves if μ factorizes into a “population density” measure ν on X and a mark function $\kappa: X \rightarrow I$ assigning each individual its mark. If this is the case, we call the mmm-space functionally-marked (fmm-space). This property is often desirable, and one might want to consider the space of fmm-spaces, rather than mmm-spaces, as the state space. Unfortunately, the subspace of fmm-spaces is not closed in the mGw-topology, which means that limits of finite-population models that are constructed as fmm-spaces might not admit mark functions themselves. It is therefore of interest, if the space of fmm-spaces with marked Gromov-weak topology is a Polish space (that is a “good” state space). Here, we show in Theorem 2.2 that this is indeed the case. We also produce criteria to enable one to check if an mmm-space admits a mark function. For limiting populations, they are given in terms of the approximating mmm-spaces. We derive such criteria for deterministic spaces (Theorem 3.1), random spaces (Theorem 3.7) and mmm-space-valued processes (Theorem 3.9 and Theorem 3.11).

An important example of such a high-density limit of approximating models with finite populations is the tree-valued Fleming-Viot dynamics. In the neutral case, it is constructed in [10] using the formalism of mm-spaces. In [5], (allelic) types – encoded as marks of mmm-spaces – are included, in order to model mutation and selection. For this process, the question of existence of a mark function has already been posed. In [5, Remark 3.11] and [6, Theorem 6] it is stated that the tree-valued Fleming-Viot process admits a mark function at all times, almost surely. The given proof, however, contains a gap, because it relies on the criterion claimed in [6, Lemma 7.1], which is wrong in general, as we show in Example 4.1. We fill this gap by applying our criteria and showing in Theorem 4.3 that the claim is indeed true and the tree-valued Fleming-Viot process with mutation and selection (TFVMS) admits a mark function at all times, almost surely. We also show in Theorem 4.4 that the same arguments apply to the Λ -version of the TFVMS in the neutral case, that is where selection is not present.

Intuitively, the existence of a mark function in the case of the TFVMS holds because mutations are large but rare in the approximating sequence of tree-valued Moran models. Hence, as genealogical distance becomes small, the probability that any mutation happened at all in the close past becomes small as well (recall that distance equals time to the MRCA). In contrast, in [14], where evolving phylogenies of trait-dependent branching with mutation and competition are under investigation, mutations happen at

a high rate but are small which justifies the hope for the existence of a mark function also for the limiting model. Our criteria are also suited for this kind of situation.

Outline. The paper is organized as follows. In the subsections of the introduction we first introduce notations and basic results for the Prohorov metric for finite measures. Then, we give a short introduction to the space \mathbb{M}_I of marked metric measure spaces (mmm-spaces) with the marked Gromov-weak topology, as well as the marked Gromov-Prohorov metric d_{mGP} on it. We continue with defining the so-called functionally-marked metric measure spaces (fmm-spaces) $\mathbb{M}_I^{\text{fct}} \subseteq \mathbb{M}_I$, and finally investigate the case of equicontinuous mark functions as an illustrative example. We emphasize that the restriction of the marked Gromov-Prohorov metric d_{mGP} to $\mathbb{M}_I^{\text{fct}}$ is not complete.

In Section 2, we therefore show that there exists another metric on $\mathbb{M}_I^{\text{fct}}$ that induces the marked Gromov-weak topology and is complete. As one sees in Subsection 1.4, the situation becomes easy if we restrict to a subspace of \mathbb{M}_I containing spaces with uniformly equicontinuous mark functions. We introduce in Subsection 2.2 several related subspaces capturing some aspect of equicontinuity, and obtain a decomposition of $\mathbb{M}_I^{\text{fct}}$ into closed sets. This decomposition is used to prove Polishness of $\mathbb{M}_I^{\text{fct}}$, and in Section 3 to formulate criteria for the existence of mark functions.

Section 3 gives criteria for the existence of mark functions. Based on the construction of the complete metric and the decomposition of $\mathbb{M}_I^{\text{fct}}$, we derive in Subsection 3.1 criteria to check if an mmm-space admits a mark function, especially in the case where it is given as a marked Gromov-weak limit. We then transfer the results in Subsection 3.2 to random mmm-spaces and in Subsection 3.3 to \mathbb{M}_I -valued stochastic processes.

To conclude, Section 4 gives examples. We first show that the criterion in [6] is wrong in general by means of counterexamples. Our criteria are then applied in Subsection 4.1 to prove the existence of a mark function for the tree-valued Fleming-Viot dynamics with mutation and selection. To this goal, we verify the necessary assumptions for a sequence of approximating tree-valued Moran models. In Subsection 4.2 we show that a similar strategy applies if we replace the tree-valued Moran models by so-called tree-valued Λ -Cannings models. Finally, in Subsection 4.3, a future application to evolving phylogenies of trait-dependent branching with mutation and competition is indicated.

1.1 Notations and prerequisites

In this paper, let all topological spaces be equipped with their Borel σ -algebras. We use the following notation throughout the article.

Notation 1.1. For a Polish space E , let $\mathcal{M}_1(E)$ respectively $\mathcal{M}_f(E)$ denote the space of probability respectively finite measures on the Borel σ -algebra $\mathfrak{B}(E)$ on E . The space $\mathcal{M}_f(E)$ is always equipped with the topology of weak convergence, which is denoted by \xrightarrow{w} . We also use the distance in variational norm of $\mu, \nu \in \mathcal{M}_f(E)$, which is

$$\|\mu - \nu\| := \sup_{B \in \mathfrak{B}(E)} |\mu(B) - \nu(B)|. \quad (1.1)$$

In particular, $\|\mu\| = \mu(E)$, and $\|\mu - \nu\| = \nu(E) - \mu(E)$ if $\mu \leq \nu$, that is $\mu(A) \leq \nu(A)$ for all $A \in \mathfrak{B}(E)$.

For $Y \in \mathfrak{B}(E)$ and $\mu \in \mathcal{M}_f(E)$, denote by $\mu|_Y \in \mathcal{M}_f(E)$ the restriction of μ to Y , that is $\mu|_Y(B) := \mu(B \cap Y)$ for all $B \in \mathfrak{B}(E)$. Because $\mu|_Y \leq \mu$, we have $\|\mu|_Y - \mu\| = \mu(E \setminus Y)$.

For $\varphi: E \rightarrow F$ measurable, with F some other Polish space, denote the image measure of μ under φ by $\varphi_*\mu := \mu \circ \varphi^{-1}$. Finally, for the product space $X := E \times F$, the canonical projection operators from X onto E and F are denoted by π_E and π_F , respectively.

Definition 1.2 (Prohorov metric). For finite measures μ_0, μ_1 on a metric space (E, r) ,

the Prohorov metric is defined as

$$d_{Pr}(\mu_0, \mu_1) := \inf\{\varepsilon > 0 : \mu_i(A) \leq \mu_{1-i}(A^\varepsilon) + \varepsilon \quad \forall A \in \mathfrak{B}(E), i \in \{0, 1\}\}, \quad (1.2)$$

where $A^\varepsilon := \{x \in E : r(A, x) < \varepsilon\}$ is the ε -neighbourhood of A .

It is well-known that the Prohorov metric metrizes the weak convergence of measures if and only if the underlying metric space is separable. The following equivalent expression for the Prohorov metric turns out to be useful.

Remark 1.3 (coupling representation of the Prohorov metric). Let (E, r) be a separable metric space and $\mu_1, \mu_2 \in \mathcal{M}_1(E)$. For a finite measure ξ on E^2 , we denote the marginals as $\xi_1 := \xi(\cdot \times E)$ and $\xi_2 := \xi(E \times \cdot)$. It is well-known (see, e.g., [7, Theorem III.1.2]) that

$$d_{Pr}(\mu_1, \mu_2) = \inf\{\varepsilon > 0 : \exists \xi \in \mathcal{M}_1(E^2) \text{ with } \xi(N_\varepsilon) \leq \varepsilon, \xi_i = \mu_i, i = 1, 2\}, \quad (1.3)$$

where $N_\varepsilon := \{(x, y) \in E^2 : r(x, y) \geq \varepsilon\}$. We obtain from this equation

$$d_{Pr}(\mu_1, \mu_2) = \inf\{\varepsilon > 0 : \exists \xi' \in \mathcal{M}_f(E^2) \text{ with } \xi'(N_\varepsilon) = 0, \xi'_i \leq \mu_i, \|\mu_i - \xi'_i\| \leq \varepsilon, i = 1, 2\}. \quad (1.4)$$

Indeed, consider $\xi' := \xi|_{E^2 \setminus N_\varepsilon}$ respectively $\xi := \xi' + (1 - \|\xi'\|)^{-1}((\mu_1 - \xi'_1) \otimes (\mu_2 - \xi'_2))$ to obtain equality in the above. Following the ideas of the proof of the representation (1.3) in [7], the representation (1.4) for the Prohorov metric $d_{Pr}(\mu_0, \mu_1)$ is easily seen to hold true for measures $\mu_1, \mu_2 \in \mathcal{M}_f(E)$ as well, which are not necessarily probability measures.

From (1.4), we can easily deduce the following lemma, which we use below.

Lemma 1.4 (rectangular lemma). Let (E, r) be a separable, metric space, $\varepsilon, \delta > 0$, and $\mu_1, \mu_2 \in \mathcal{M}_f(E)$. Assume that $d_{Pr}(\mu_1, \mu_2) < \delta$ and there is $\mu'_1 \leq \mu_1$ with $\|\mu_1 - \mu'_1\| \leq \varepsilon$. Then

$$\exists \mu'_2 \leq \mu_2 : d_{Pr}(\mu'_1, \mu'_2) < \delta, \|\mu_2 - \mu'_2\| \leq \varepsilon. \quad (1.5)$$

Proof. According to (1.4), we find $\xi \in \mathcal{M}_f(E^2)$ with marginals $\xi_i \leq \mu_i, i = 1, 2, \|\mu_i - \xi_i\| < \delta$, and $\xi(\{r \geq \delta\}) = 0$. Let L be a probability kernel from E to E (for existence see [12, Theorems 8.36–8.38]) with $\xi = \mu_1 \otimes L$ and define $\xi' := (\mu'_1 \wedge \xi_1) \otimes L$. Obviously, $\xi'_1 \leq \mu'_1$ and $\|\mu'_1 - \xi'_1\| \leq \|\mu_1 - \xi_1\| < \delta$. Now set

$$\mu'_2 := \xi'_2 + \mu_2 - \xi_2. \quad (1.6)$$

Then $\xi'_2 \leq \mu'_2, \|\mu'_2 - \xi'_2\| = \|\mu_2 - \xi_2\| < \delta$ and thus $d_{Pr}(\mu_2, \mu'_2) < \delta$ by (1.4). Furthermore, $\mu'_2 \leq \mu_2$ and $\|\mu_2 - \mu'_2\| = \|\xi_2 - \xi'_2\| \leq \|\mu_1 - \mu'_1\| \leq \varepsilon$. \square

1.2 The space of marked metric measure spaces (mmm-spaces)

In this subsection, we recall the space \mathbb{M}_I of marked metric measure spaces, and the marked Gromov-Prohorov metric d_{mGP} , which induces the marked Gromov-weak topology on it. This space, (\mathbb{M}_I, d_{mGP}) , will be the basic space used in the rest of the paper. These concepts have been introduced in [4], and are based on the corresponding non-marked versions introduced in [9]. In contrast to [4], we allow the measures of the marked metric measure spaces to be finite, that is do not restrict ourselves to probability measures only. Because a sequence of finite measures converges weakly if and only if their total masses and the normalized measures converge, or the masses converge to zero, this straight-forward generalization requires only minor modifications (compare [16, Section 2.1], where this generalization is done for metric measure spaces without marks).

In what follows, fix a complete, separable metric space (I, d) , called the *mark space*. It is the same for all marked metric measure spaces in \mathbb{M}_I .

Definition 1.5 (mmm-spaces, \mathbb{M}_I). (i) An (I)-marked metric measure space (mmm-space) is a triple (X, r, μ) such that (X, r) is a complete, separable metric space, and $\mu \in \mathcal{M}_f(X \times I)$, where $X \times I$ is equipped with the product topology.

(ii) Let $\mathcal{x}_i = (X_i, r_i, \mu_i)$, $i = 1, 2$, be two mmm-spaces, and $\nu_i := \mu_i(\cdot \times I)$ the marginal of μ_i on X_i . For a map $\varphi: X_1 \rightarrow X_2$ we use the notation

$$\tilde{\varphi}: X_1 \times I \rightarrow X_2 \times I, \quad (x, u) \mapsto \tilde{\varphi}(x, u) := (\varphi(x), u). \tag{1.7}$$

We call \mathcal{x}_1 and \mathcal{x}_2 equivalent if they are measure- and mark-preserving isometric, that is there is an isometry $\varphi: \text{supp}(\nu_1) \rightarrow \text{supp}(\nu_2)$, such that

$$\tilde{\varphi}_* \mu_1 = \mu_2. \tag{1.8}$$

(iii) Finally, define

$$\mathbb{M}_I := \{ \text{equivalence classes of mmm-spaces} \}. \tag{1.9}$$

With a slight abuse of notation, we identify an mmm-space with its equivalence class and write $\mathcal{x} = (X, r, \mu) \in \mathbb{M}_I$ for both mmm-spaces and equivalence classes thereof.

Next, we recall the marked Gromov-weak topology from [4, Section 2.2] that turns \mathbb{M}_I into a Polish space (cf. [4, Theorem 2]). To this goal, we first recall

Definition 1.6 (marked distance matrix distribution). Let $\mathcal{x} := (X, r, \mu) \in \mathbb{M}_I$ and

$$R^{(X,r)} := \begin{cases} (X \times I)^{\mathbb{N}} & \rightarrow \mathbb{R}_+^{\binom{\mathbb{N}}{2}} \times I^{\mathbb{N}}, \\ ((x_k, u_k)_{k \geq 1}) & \mapsto ((r(x_k, x_l))_{1 \leq k < l}, (u_k)_{k \geq 1}). \end{cases} \tag{1.10}$$

The marked distance matrix distribution of \mathcal{x} is defined as

$$\nu^{\mathcal{x}} := \|\mu\| \cdot (R^{(X,r)})_* \left(\frac{\mu}{\|\mu\|} \right)^{\mathbb{N}} \in \mathcal{M}_f(\mathbb{R}_+^{\binom{\mathbb{N}}{2}} \times I^{\mathbb{N}}). \tag{1.11}$$

The marked Gromov-weak topology is the one induced by the map $\mathcal{x} \mapsto \nu^{\mathcal{x}}$.

Definition 1.7 (marked Gromov-weak topology). Let $\mathcal{x}, \mathcal{x}_1, \mathcal{x}_2, \dots \in \mathbb{M}_I$. We say that $(\mathcal{x}_n)_{n \in \mathbb{N}}$ converges to \mathcal{x} in the marked Gromov-weak topology, $\mathcal{x}_n \xrightarrow[n \rightarrow \infty]{\text{mGw}} \mathcal{x}$, if and only if

$$\nu^{\mathcal{x}_n} \xrightarrow[n \rightarrow \infty]{w} \nu^{\mathcal{x}} \tag{1.12}$$

in the weak topology on $\mathcal{M}_f(\mathbb{R}_+^{\binom{\mathbb{N}}{2}} \times I^{\mathbb{N}})$.

Finally, let us recall the Gromov-Prohorov metric from [4, Section 3.2]. It is complete and metrizes the marked Gromov-weak topology, as shown in [4, Proposition 3.7].

Definition 1.8 (marked Gromov-Prohorov metric, d_{mGP}). For $\mathcal{x}_i = (X_i, r_i, \mu_i) \in \mathbb{M}_I$, $i = 1, 2$, set

$$d_{\text{mGP}}(\mathcal{x}_1, \mathcal{x}_2) := \inf_{(E, \varphi_1, \varphi_2)} d_{\text{Pr}}((\tilde{\varphi}_1)_* \mu_1, (\tilde{\varphi}_2)_* \mu_2), \tag{1.13}$$

where the infimum is taken over all complete, separable metric spaces (E, r) and isometric embeddings $\varphi_i: X_i \rightarrow E$, and $\tilde{\varphi}_i$ is as in (1.7), $i = 1, 2$. The Prohorov metric d_{Pr} is the one on $\mathcal{M}_f(E \times I)$, based on the metric $\tilde{r} = r + d$ on $E \times I$, metrizing the product topology. The metric d_{mGP} is called the marked Gromov-Prohorov metric.

A direct consequence of the fact that d_{mGP} induces the marked Gromov-weak topology is the following characterization of marked Gromov-weak convergence obtained in [4, Lemma 3.4].

Lemma 1.9 (embedding of marked Gromov-weakly converging sequences). *Let $x_n = (X_n, r_n, \mu_n) \in \mathbb{M}_I$ for $n \in \mathbb{N} \cup \{\infty\}$. Then $(x_n)_{n \in \mathbb{N}}$ converges to x_∞ Gromov-weakly if and only if there is a complete, separable metric space (E, r) , and isometric embeddings $\varphi_n: X_n \rightarrow E$, such that for $\tilde{\varphi}_n$ as in (1.7),*

$$(\tilde{\varphi}_n)_* \mu_n \xrightarrow{w} \mu. \tag{1.14}$$

1.3 Functionally-marked metric measure spaces (fmm-spaces)

Consider an I -marked metric measure space $x = (X, r, \mu) \in \mathbb{M}_I$. Since μ is a finite measure on the Polish space $X \times I$, regular conditional measures exist (cf. [12, Theorems 8.36–8.38]), and we write

$$\mu(dx, du) = \nu(dx) \cdot K_x(du), \tag{1.15}$$

in short $\mu = \nu \otimes K$, for the marginal $\nu := \mu(\cdot \times I) \in \mathcal{M}_f(X)$, and a (ν -a.s. unique) probability kernel K from X to I .

In the present article we investigate criteria for the existence of a mark function for x , that is (cf. [6, Section 3.3]) a measurable function $\kappa: X \rightarrow I$ such that

$$\mu(dx, du) = \nu(dx) \cdot \delta_{\kappa(x)}(du), \tag{1.16}$$

or equivalently, $K_x = \delta_{\kappa(x)}$ for ν -almost every x . Obviously, x admits a mark function if and only if K_x is a Dirac measure for ν -almost every x . Recall that the complete, separable mark space (I, d) is fixed once and for all.

Definition 1.10 (fmm-spaces, $\mathbb{M}_I^{\text{fct}}$). *We call $x = (X, r, \nu, \kappa)$ an (I -)functionally-marked metric measure space (fmm-space) if (X, r) is a complete, separable metric space, $\nu \in \mathcal{M}_f(X)$, and $\kappa: X \rightarrow I$ is measurable. We identify x with the marked metric measure space $(X, r, \mu) \in \mathbb{M}_I$, where μ satisfies (1.16). With a slight abuse of notation, we write $(X, r, \nu, \kappa) = (X, r, \mu)$ if (1.16) is satisfied. Denote by $\mathbb{M}_I^{\text{fct}} \subseteq \mathbb{M}_I$ the space of (equivalence classes of) fmm-spaces.*

A first, simple observation is that $\mathbb{M}_I^{\text{fct}}$ is a dense subspace of \mathbb{M}_I .

Lemma 1.11. *The subspace $\mathbb{M}_I^{\text{fct}}$ is dense in \mathbb{M}_I with marked Gromov-weak topology.*

Proof. For $x = (X, r, \mu) \in \mathbb{M}_I$, define $x_n = (X \times I, r_n, \nu_n, \kappa_n) \in \mathbb{M}_I^{\text{fct}}$ with $\nu_n = \mu$, $\kappa_n(x, u) = u$, and $r_n((x, u), (y, v)) := r(x, y) + e^{-n} \wedge d(u, v)$, for $x, y \in X$, $u, v \in I$. It is easy to see that $x_n \rightarrow x$ in the marked Gromov-weak topology. \square

1.4 The equicontinuous case

It directly follows from Lemma 1.11 that the subspace $\mathbb{M}_I^{\text{fct}}$ is not closed in \mathbb{M}_I , meaning that if $x_n \xrightarrow{\text{mGw}} x$ is a marked Gromov-weakly converging sequence in \mathbb{M}_I , and all x_n admit a mark function, this need not be the case for x . In applications, however, the limit x is often not known explicitly, and it would be important to have (sufficient) criteria for the existence of a mark function in terms of the x_n alone. An easy possibility is Lipschitz equicontinuity: if all x_n admit a mark function that is Lipschitz continuous with a common Lipschitz constant $L > 0$, the same is true for x (see [17]). More generally, this holds for uniformly equicontinuous mark functions as introduced below. We briefly discuss the equicontinuous case in this subsection, because it is straightforward and illustrates the main ideas.

Recall that a *modulus of continuity* is a function $h: \mathbb{R}_+ \rightarrow \mathbb{R}_+ \cup \{\infty\}$ that is continuous in 0 and satisfies $h(0) = 0$. A function $f: X \rightarrow I$, where (X, r) is a metric space, is *h-uniformly continuous* if $d(f(x), f(y)) \leq h(r(x, y))$ for all $x, y \in X$. Note that for every

modulus of continuity h , there exists another modulus of continuity $h' \geq h$ which is increasing and continuous with respect to the topology of the one-point compactification of \mathbb{R}_+ . Therefore, we can restrict ourselves without loss of generality to moduli of continuity from

$$\mathcal{H} := \{ h: \mathbb{R}_+ \rightarrow \mathbb{R}_+ \cup \{\infty\} \mid h(0) = 0, h \text{ is continuous and increasing} \}. \quad (1.17)$$

For $h \in \mathcal{H}$ and a metric space (X, r) , we define

$$A_h^X := A_h^{(X,r)} := \{ (x_i, u_i)_{i=1,2} \in (X \times I)^2 : d(u_1, u_2) \leq h(r(x_1, x_2)) \} \subseteq (X \times I)^2. \quad (1.18)$$

Note that $f: X \rightarrow I$ is h -uniformly continuous if and only if $((x, f(x)), (y, f(y))) \in A_h^X$ for all $x, y \in X$, and that A_h^X is a closed set in $(X \times I)^2$ with product topology.

Definition 1.12 (\mathfrak{M}_I^h). For $h \in \mathcal{H}$, let $\mathfrak{M}_I^h \subseteq \mathbb{M}_I^{\text{fct}}$ be the space of marked metric measure spaces admitting an h -uniformly continuous mark function.

The next lemma states that a marked metric measure space (X, r, μ) admits an h -uniformly continuous mark function if and only if a pair of independent samples from μ is almost surely in A_h^X . Furthermore, if a sequence with h -uniformly continuous mark functions converges marked Gromov-weakly, the limit space also admits an h -uniformly continuous mark function.

Lemma 1.13 (uniform equicontinuity). Fix a modulus of continuity $h \in \mathcal{H}$.

- (i) $\mathfrak{M}_I^h = \{ (X, r, \mu) \in \mathbb{M}_I : \mu^{\otimes 2}(A_h^X) = \|\mu^{\otimes 2}\| \}$.
- (ii) \mathfrak{M}_I^h is closed in the marked Gromov-weak topology.

Proof. The mmm-space $x = (X, r, \mu)$ is in \mathfrak{M}_I^h if and only if $\text{supp}(\mu)$ is the graph of an h -uniformly continuous function. This is clearly equivalent to $\mu^{\otimes 2}((X \times I)^2 \setminus A_h^X) = 0$. Item (ii) is obvious from (i), because A_h^X is a closed set. \square

This preliminary result is quite restrictive because of the condition to have the same modulus of continuity for all occurring spaces. In fact, the mark function of the tree-valued Fleming Viot dynamic considered in Subsection 4.1 is not even continuous.

At the heart of the following generalisation to measurable mark functions lies the fact that measurable functions are “almost continuous” by Lusin’s celebrated theorem (see for instance [3, Theorem 7.1.13]). Here, we give a version tailored to our setup:

Lusin’s theorem. Let X, Y be Polish spaces, μ a finite measure on X , and $f: X \rightarrow Y$ a measurable function. Then, for every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subseteq X$ such that $\mu(X \setminus K_\varepsilon) < \varepsilon$ and $f|_{K_\varepsilon}$ is continuous.

2 The space of fmm-spaces is Polish

The subspace $\mathbb{M}_I^{\text{fct}}$ is not closed in \mathbb{M}_I in the marked Gromov-weak topology, and hence the restriction of the marked Gromov-Prohorov metric d_{mGP} to $\mathbb{M}_I^{\text{fct}}$ is not complete. In this section, we show that there exists another metric on $\mathbb{M}_I^{\text{fct}}$ that induces the marked Gromov-weak topology and is complete. This shows that $\mathbb{M}_I^{\text{fct}}$ is a Polish space in its own right.

2.1 A complete metric on the space of fmm-spaces

For a measure ξ on I , we define

$$\beta_\xi := \int_I \int_I (1 \wedge d(u, v)) \xi(du) \xi(dv). \quad (2.1)$$

Note that $\beta_\xi = 0$ if and only if ξ is a Dirac measure. For $x = (X, r, \mu) \in \mathbb{M}_I$, with $\mu = \nu \otimes K$ as in (1.15), we define

$$\beta(x) := \int_X \beta_{K_x} \nu(dx) = \int_{X \times I} \int_I (1 \wedge d(u, v)) K_x(dv) \mu(d(x, u)). \tag{2.2}$$

Proposition 2.1 (characterization of $\mathbb{M}_I^{\text{fct}}$ as continuity points). *Let $\text{cont}(\beta) \subseteq \mathbb{M}_I$ be the set of continuity points of $\beta: \mathbb{M}_I \rightarrow \mathbb{R}_+$, where \mathbb{M}_I carries the marked Gromov-weak topology. Then*

$$\text{cont}(\beta) = \beta^{-1}(0) = \mathbb{M}_I^{\text{fct}}. \tag{2.3}$$

Proof (first part). As seen before, $x = (X, r, \nu \otimes K) \in \mathbb{M}_I$ admits a mark function if and only if K_x is a Dirac measure for ν -almost every $x \in X$, which is the case if and only if $\beta(x) = 0$. Hence $\beta^{-1}(0) = \mathbb{M}_I^{\text{fct}}$. Because $\mathbb{M}_I^{\text{fct}}$ is dense in \mathbb{M}_I by Lemma 1.11, no $x \in \mathbb{M}_I \setminus \beta^{-1}(0)$ can be a continuity point of β . Thus $\text{cont}(\beta) \subseteq \beta^{-1}(0)$.

We defer the proof of the inclusion $\beta^{-1}(0) \subseteq \text{cont}(\beta)$ to Subsection 2.2, because it requires a technical estimate on β derived in Proposition 2.7. \square

In view of (2.3), we can use standard arguments to construct a complete metric on $\mathbb{M}_I^{\text{fct}}$ that metrizes marked Gromov-weak topology. Namely consider the sets

$$F_m := \overline{\beta^{-1}([\frac{1}{m}, \infty))} \subseteq \mathbb{M}_I, \quad m \in \mathbb{N}, \tag{2.4}$$

where the closure is in the marked Gromov-weak topology. Then, due to Proposition 2.1, F_m is disjoint from $\mathbb{M}_I^{\text{fct}}$, and $\mathbb{M}_I^{\text{fct}} = \mathbb{M}_I \setminus \bigcup_{m \in \mathbb{N}} F_m$. Because F_m is also closed by definition, we obtain

$$\mathbb{M}_I^{\text{fct}} = \bigcap_{m \in \mathbb{N}} \{x \in \mathbb{M}_I : d_{\text{mGP}}(x, F_m) > 0\}. \tag{2.5}$$

We consider the metric d_{fGP} on $\mathbb{M}_I^{\text{fct}}$ defined for $x, y \in \mathbb{M}_I^{\text{fct}}$ by

$$d_{\text{fGP}}(x, y) := d_{\text{mGP}}(x, y) + \sup_{m \in \mathbb{N}} 2^{-m} \wedge \left| \frac{1}{d_{\text{mGP}}(x, F_m)} - \frac{1}{d_{\text{mGP}}(y, F_m)} \right|. \tag{2.6}$$

Theorem 2.2 ($\mathbb{M}_I^{\text{fct}}$ is Polish). *The space $\mathbb{M}_I^{\text{fct}}$ of I -functionally-marked metric measure spaces with marked Gromov-weak topology is a Polish space. Namely, d_{fGP} is a complete metric on $\mathbb{M}_I^{\text{fct}}$ inducing the marked Gromov-weak topology.*

Proof. First, we show that d_{fGP} induces the marked Gromov-weak topology on $\mathbb{M}_I^{\text{fct}}$. For $m \in \mathbb{N}$, $x \in \mathbb{M}_I$, define

$$\rho_m(x) := d_{\text{mGP}}(x, F_m), \tag{2.7}$$

with F_m defined in (2.4). Note that ρ_m is a continuous function on \mathbb{M}_I . Let $x_n, x \in \mathbb{M}_I^{\text{fct}}$. Then $\rho_m(x) > 0$ for all $m \in \mathbb{N}$ because of (2.5). Therefore, by definition, $d_{\text{fGP}}(x_n, x) \xrightarrow{n \rightarrow \infty} 0$ if and only if the two conditions $d_{\text{mGP}}(x_n, x) \xrightarrow{n \rightarrow \infty} 0$ and

$$\rho_m(x_n) \xrightarrow{n \rightarrow \infty} \rho_m(x) \quad \forall m \in \mathbb{N} \tag{2.8}$$

hold. We have to show that the marked Gromov-weak convergence already implies (2.8). This, however, follows from the continuity of the ρ_m .

It remains to show that d_{fGP} is a complete metric on $\mathbb{M}_I^{\text{fct}}$. Consider a d_{fGP} -Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathbb{M}_I^{\text{fct}}$. By completeness of d_{mGP} on \mathbb{M}_I , it converges marked Gromov-weakly to some $x = (X, r, \mu) \in \mathbb{M}_I$. Furthermore, for every fixed $m \in \mathbb{N}$, (2.6) implies that $1/\rho_m(x_n)$ converges as $n \rightarrow \infty$, and hence $d_{\text{mGP}}(x_n, F_m)$ is bounded away from zero. Thus $x \notin F_m$. Because $\mathbb{M}_I^{\text{fct}} = \mathbb{M}_I \setminus \bigcup_{m \in \mathbb{N}} F_m$, this means that $x \in \mathbb{M}_I^{\text{fct}}$, and by the first part of the proof $d_{\text{fGP}}(x_n, x) \xrightarrow{n \rightarrow \infty} 0$. \square

With $B_\delta^{\mathbb{M}_I}(x) := \{y \in \mathbb{M}_I : d_{\text{mGP}}(x, y) < \delta\}$ we denote the open δ -ball in \mathbb{M}_I with respect to d_{mGP} . The following corollary gives formal criteria for a limiting space to admit a mark function, which are useful only together with estimates on β .

Corollary 2.3. *Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in \mathbb{M}_I which converges marked Gromov-weakly to x . Then the following four conditions are equivalent:*

- (i) $x \in \mathbb{M}_I^{\text{fct}}$.
- (ii) $\limsup_{n \rightarrow \infty} \rho_m(x_n) > 0$ for all $m \in \mathbb{N}$, with ρ_m defined in (2.7).
- (iii) For every $\delta > 0$,

$$\limsup_{n \rightarrow \infty} \inf_{y \in \beta^{-1}([\delta, \infty])} d_{\text{mGP}}(x_n, y) > 0. \tag{2.9}$$

- (iv)

$$\lim_{\delta \downarrow 0} \liminf_{n \rightarrow \infty} \sup_{y \in B_\delta^{\mathbb{M}_I}(x_n)} \beta(y) = 0. \tag{2.10}$$

Proof. “(i) \Leftrightarrow (ii)”: We have $\rho_m(x) = \lim_{n \rightarrow \infty} \rho_m(x_n)$, and $\rho_m(x) > 0$ for all $m \in \mathbb{N}$ if and only if $x \in \mathbb{M}_I^{\text{fct}}$.

“(ii) \Leftrightarrow (iii)”: follows directly from the definition of ρ_m .

“(iii) \Leftrightarrow (iv)”: Using monotonicity in δ we obtain

$$(iii) \iff \forall \delta > 0 \exists \varepsilon > 0 \forall (y_n)_{n \in \mathbb{N}} \subseteq \mathbb{M}_I \text{ with } \beta(y_n) \geq \delta : \limsup_{n \rightarrow \infty} d_{\text{mGP}}(x_n, y_n) \geq \varepsilon \tag{2.11}$$

$$\iff \forall \delta > 0 \exists \varepsilon > 0 \forall (y_n)_{n \in \mathbb{N}} \subseteq \mathbb{M}_I : \liminf_{n \rightarrow \infty} \beta(y_n) < \delta \text{ or } \limsup_{n \rightarrow \infty} d_{\text{mGP}}(x_n, y_n) \geq \varepsilon$$

$$\iff \forall \varepsilon > 0 \exists \delta > 0 \forall (y_n)_{n \in \mathbb{N}} \subseteq \mathbb{M}_I \text{ with } y_n \in B_\delta^{\mathbb{M}_I}(x_n) : \liminf_{n \rightarrow \infty} \beta(y_n) < \varepsilon \iff (iv),$$

where, in the third equivalence, we renamed δ to ε and ε to δ . □

2.2 A decomposition of $\mathbb{M}_I^{\text{fct}}$ into closed sets and estimates on β

In this subsection, we derive some estimates on β and use them to complete the proof of Proposition 2.1. Furthermore, we construct a decomposition of $\mathbb{M}_I^{\text{fct}}$ into closed sets which are related to the sets \mathfrak{M}_I^h .

As we have seen in Subsection 1.4, the situation becomes easy if we restrict to the uniformly equicontinuous case, that is to the subspace \mathfrak{M}_I^h for some $h \in \mathcal{H}$ as in Definition 1.12. We introduce in what follows several related subspaces capturing some aspect of equicontinuity. In analogy to the definition of A_h^X in (1.18), we use for a metric space (X, r) , and $\delta, \varepsilon > 0$, the notation

$$A_{\delta, \varepsilon}^X := A_{\delta, \varepsilon}^{(X, r)} := \{(x_i, u_i)_{i=1,2} \in (X \times I)^2 : r(x_1, x_2) \geq \delta \text{ or } d(u_1, u_2) \leq \varepsilon\} \subseteq (X \times I)^2. \tag{2.12}$$

Note that $A_{\delta, \varepsilon}^X$ is a closed set. For every $h \in \mathcal{H}$, using monotonicity and continuity of h , we observe that

$$A_h^X = \bigcap_{\delta > 0} A_{\delta, h(\delta)}^X. \tag{2.13}$$

Definition 2.4 ($\mathfrak{M}_I^{\delta, \varepsilon}, \mathbb{M}_I^{\delta, \varepsilon}, \mathbb{M}_I^h$). *Let $\delta, \varepsilon > 0$ and $h \in \mathcal{H}$. We define*

$$\mathfrak{M}_I^{\delta, \varepsilon} := \{(X, r, \mu) \in \mathbb{M}_I : \mu^{\otimes 2}(A_{\delta, \varepsilon}^X) = \|\mu^{\otimes 2}\|\}, \tag{2.14}$$

$$\mathbb{M}_I^{\delta, \varepsilon} := \{(X, r, \mu) \in \mathbb{M}_I : \exists \mu' \in \mathcal{M}_f(X \times I) : \mu' \leq \mu, \|\mu - \mu'\| \leq \varepsilon, (X, r, \mu') \in \mathfrak{M}_I^{\delta, \varepsilon}\}, \tag{2.15}$$

and $\mathbb{M}_I^h := \bigcap_{\delta > 0} \mathbb{M}_I^{\delta, h(\delta)}$.

The intuition is that for spaces in $\mathfrak{M}_I^{\delta, h(\delta)}$, the measure behaves as if it admitted an h -uniformly continuous mark function when distances of order δ are observed. The same holds for the spaces in $\mathbb{M}_I^{\delta, h(\delta)}$ if we are additionally allowed to neglect a portion $h(\delta)$ of mass.

Remark 2.5. (i) Clearly $\mathfrak{M}_I^h \subseteq \mathbb{M}_I^h$. We will see in Lemma 2.8 that $\mathbb{M}_I^h \subseteq \mathbb{M}_I^{\text{fct}}$.

(ii) The space \mathbb{M}_I^h is *much* larger than \mathfrak{M}_I^h : while $\bigcup_{h \in \mathcal{H}} \mathfrak{M}_I^h$ contains only mmm-spaces admitting a uniformly *continuous* mark function, we will see in Lemma 2.8 that every element of $\mathbb{M}_I^{\text{fct}}$ is in some \mathbb{M}_I^h .

(iii) The spaces $\mathfrak{M}_I^{\delta, \varepsilon}$ and $\mathbb{M}_I^{\delta, \varepsilon}$ are not contained in $\mathbb{M}_I^{\text{fct}}$. For instance, consider $I = \mathbb{R}$ and $x = (\{0\}, 0, \delta_{(0,0)} + \delta_{(0,\varepsilon)}) \in \mathfrak{M}_I^{\delta, \varepsilon} \subseteq \mathbb{M}_I^{\delta, \varepsilon}$.

We have the following stability of $\mathbb{M}_I^{\delta, \varepsilon}$ with respect to small perturbations in the marked Gromov-Prohorov metric.

Lemma 2.6 (perturbation of $\mathbb{M}_I^{\delta, \varepsilon}$). *Let $\delta, \varepsilon > 0$, $x \in \mathbb{M}_I^{\delta, \varepsilon}$ and $\hat{x} \in \mathbb{M}_I$. Then*

$$\delta' := d_{\text{mGP}}(x, \hat{x}) < \frac{1}{2}\delta \implies \hat{x} \in \mathbb{M}_I^{\delta-2\delta', \varepsilon+2\delta'}. \quad (2.16)$$

Proof. Let $x = (X, r, \mu)$, $\hat{x} = (\hat{X}, \hat{r}, \hat{\mu})$. We may assume that X, \hat{X} are subspaces of some separable, metric space (E, r_E) such that $d_{\text{Pr}}(\mu, \hat{\mu}) < \delta'$. By definition of $\mathbb{M}_I^{\delta, \varepsilon}$, there is $\mu' \leq \mu$ with $\|\mu - \mu'\| \leq \varepsilon$ and $x' := (X, r, \mu') \in \mathfrak{M}_I^{\delta, \varepsilon}$. Due to Lemma 1.4, we find $\hat{\mu}' \leq \hat{\mu}$ with $\|\hat{\mu} - \hat{\mu}'\| \leq \varepsilon$ and $d_{\text{Pr}}(\mu', \hat{\mu}') < \delta'$, where $\hat{x}' = (\hat{X}, \hat{r}, \hat{\mu}')$. By the coupling representation of the Prohorov metric, (1.4), we obtain a measure ξ on $(E \times I)^2$ with marginals $\xi_1 \leq \mu'$ and $\xi_2 \leq \hat{\mu}'$ such that $\|\hat{\mu}' - \xi_2\| \leq \delta'$ and

$$\xi(\{(x, u), (\hat{x}, \hat{u}) \in (X \times I) \times (\hat{X} \times I) : r_E(x, \hat{x}) + d(u, \hat{u}) \geq \delta'\}) = 0. \quad (2.17)$$

By definition, $(\mu')^{\otimes 2}$ is supported by $A_{\delta, \varepsilon}^X$. Therefore, the same is true for $\xi_1^{\otimes 2}$ and we obtain

$$\begin{aligned} \|\xi_2^{\otimes 2}\| &= \|\xi^{\otimes 2}\| = \xi^{\otimes 2}(\{(x_i, u_i, \hat{x}_i, \hat{u}_i)_{i=1,2} \in ((X \times I) \times (\hat{X} \times I))^2 : (x_i, u_i)_{i=1,2} \in A_{\delta, \varepsilon}^X\}) \\ &\leq \xi_2^{\otimes 2}(\{(\hat{x}_i, \hat{u}_i)_{i=1,2} \in (\hat{X} \times I)^2 : r_E(\hat{x}_1, \hat{x}_2) \geq \delta - 2\delta' \text{ or } d(\hat{u}_1, \hat{u}_2) \leq \varepsilon + 2\delta'\}) \\ &= \xi_2^{\otimes 2}(A_{\delta-2\delta', \varepsilon+2\delta'}^{\hat{X}}), \end{aligned} \quad (2.18)$$

where the inequality follows from (2.17) together with the triangle-inequality. Therefore, $(\hat{X}, \hat{r}, \xi_2) \in \mathfrak{M}_I^{\delta-2\delta', \varepsilon+2\delta'}$. Now the claim follows from $\|\hat{\mu} - \xi_2\| \leq \|\hat{\mu} - \hat{\mu}'\| + \|\hat{\mu}' - \xi_2\| \leq \varepsilon + \delta'$. \square

Proposition 2.7 (estimates on β). *Let $\delta, \varepsilon > 0$ and consider $x = (X, r, \mu) \in \mathbb{M}_I$. Then the following hold:*

- (i) *If $\mu' \in \mathcal{M}_f(X \times I)$, then $\beta(x) \leq \beta((X, r, \mu')) + 2\|\mu - \mu'\|$.*
- (ii) *If $x \in \mathfrak{M}_I^{2\delta, \varepsilon}$, then $\beta(x) \leq \varepsilon\|\mu\|$.*
- (iii) *If $x \in \mathbb{M}_I^{2\delta, \varepsilon}$ and $\hat{x} \in \mathbb{M}_I$ with $d_{\text{mGP}}(x, \hat{x}) < \delta$, then $\beta(x) \leq \varepsilon(\|\mu\| + 2)$ and*

$$\beta(\hat{x}) \leq (\varepsilon + 2\delta)(2 + \|\mu\| + \delta). \quad (2.19)$$

Proof. (i) follows directly from the definition.

(ii) If $x \in X$ and $u, v \in I$ satisfy $((x, u), (x, v)) \in A_{2\delta, \varepsilon}^X$, then $d(u, v) \leq \varepsilon$ by definition of $A_{2\delta, \varepsilon}^X$. Thus $\beta(x) = \int_{X \times I} \int_I (1 \wedge d(u, v)) K_x(dv) \mu(d(x, u)) \leq \varepsilon\|\mu\|$.

(iii) Combining (i) and (ii) yields $\beta(x) \leq 2\varepsilon + \varepsilon\|\mu\|$. Let $\delta' = d_{\text{mGP}}(x, \hat{x})$. By Lemma 2.6, we have $\hat{x} \in \mathbb{M}_I^{2\delta-2\delta', \varepsilon+2\delta'}$ and thus $\beta(\hat{x}) \leq (2 + \|\hat{\mu}\|)(\varepsilon + 2\delta') \leq (2 + \|\mu\| + \delta)(\varepsilon + 2\delta)$. \square

In order to complete the proof of Proposition 2.1 with the help of Proposition 2.7, we first observe that, as a consequence of Lusin’s theorem, every functionally marked metric measure space is an element of \mathbb{M}_I^h for some $h \in \mathcal{H}$. Together with Lemma 2.9 below, this means that we have a nice (though uncountable) decomposition of $\mathbb{M}_I^{\text{fct}}$ into closed sets.

Lemma 2.8 (decomposition of $\mathbb{M}_I^{\text{fct}}$). *The following equality holds: $\mathbb{M}_I^{\text{fct}} = \bigcup_{h \in \mathcal{H}} \mathbb{M}_I^h$.*

Proof. We have $\mathbb{M}_I^h \subseteq \beta^{-1}(0) = \mathbb{M}_I^{\text{fct}}$ for every $h \in \mathcal{H}$. Indeed, the equality was shown in the first part of the proof of Proposition 2.1. To obtain the inclusion, that is $\beta(x) = 0$ for all $x \in \mathbb{M}_I^h$, recall \mathbb{M}_I^h from Definition 2.4 and choose $\varepsilon = h(2\delta)$ in Proposition 2.7(iii).

Conversely, let $x = (X, r, \nu, \kappa) \in \mathbb{M}_I^{\text{fct}}$. According to Lusin’s theorem, we find for every $\varepsilon > 0$ a compact set $K_\varepsilon \subseteq X$, and a modulus of continuity $h_\varepsilon \in \mathcal{H}$, such that $\nu(X \setminus K_\varepsilon) \leq \varepsilon$ and $\kappa|_{K_\varepsilon}$ is h_ε -uniformly continuous. In particular,

$$x \in \mathbb{M}_I^{\delta, h_\varepsilon(\delta) \vee \varepsilon} \quad \forall \varepsilon, \delta > 0. \tag{2.20}$$

We may assume without loss of generality that $\varepsilon \mapsto h_\varepsilon(\delta)$ is decreasing and right-continuous for every $\delta > 0$. We define

$$h(\delta) := \inf \{ \varepsilon > 0 : h_\varepsilon(\delta) < \varepsilon \} \in \mathbb{R}_+ \cup \{ \infty \}. \tag{2.21}$$

Clearly, $h(\delta)$ converges to 0 as $\delta \downarrow 0$ because $h_\varepsilon \in \mathcal{H}$. Furthermore, $h_{h(\delta)}(\delta) \leq h(\delta)$, and hence (2.20) with $\varepsilon = h(\delta)$ implies $x \in \mathbb{M}_I^h$. \square

Proof of Proposition 2.1 (completion). We still have to show continuity of β in every point $x \in \beta^{-1}(0)$. Due to Lemma 2.8, there is $h \in \mathcal{H}$ with $x \in \mathbb{M}_I^h$. Now Proposition 2.7 yields for $\delta > 0$ the estimate $\sup \{ \beta(\hat{x}) : \hat{x} \in B_\delta^{\mathbb{M}_I}(x) \} \leq (h(2\delta) + 2\delta)(2 + \|\mu\| + \delta)$, which converges to 0 as $\delta \downarrow 0$. \square

It directly follows from Proposition 2.7(iii) that the marked Gromov-weak closure of \mathbb{M}_I^h is contained in $\mathbb{M}_I^{\text{fct}}$. In fact, \mathbb{M}_I^h is even Gromov-weakly closed, which will be used in the proof of Theorem 3.11 below.

Lemma 2.9 (closedness of \mathbb{M}_I^h). *For every $\delta, \varepsilon > 0$, $\mathbb{M}_I^{\delta, \varepsilon}$ is marked Gromov-weakly closed in \mathbb{M}_I . In particular, \mathbb{M}_I^h is closed for every $h \in \mathcal{H}$.*

Proof. Fix $\varepsilon, \delta > 0$ and let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{M}_I^{\delta, \varepsilon}$ converging marked Gromov-weakly to some $x = (X, r, \mu) \in \mathbb{M}_I$. Using Lemma 1.9, we may assume that $X_n, n \in \mathbb{N}$, and X are subspaces of a common separable, metric space (E, r_E) , such that $\mu_n \xrightarrow{w} \mu$ on $E \times I$. By definition of $\mathbb{M}_I^{\delta, \varepsilon}$, we find $\mu'_n \leq \mu_n$, $\|\mu'_n - \mu_n\| \leq \varepsilon$, such that $(\mu'_n)^{\otimes 2}$ is supported by $A_{\delta, \varepsilon}^E$ for all $n \in \mathbb{N}$. Since $(\mu'_n)_{n \in \mathbb{N}}$ is tight, we may assume, by passing to a subsequence, that $\mu'_n \xrightarrow{w} \mu'$ for some $\mu' \in \mathcal{M}_f(E)$. Obviously, $\mu' \leq \mu$ and $\|\mu - \mu'\| = \lim_{n \rightarrow \infty} \|\mu_n\| - \|\mu'_n\| \leq \varepsilon$. Because $A_{\delta, \varepsilon}^E$ is closed, $(\mu')^{\otimes 2}$ is supported by $A_{\delta, \varepsilon}^E$ and hence $x \in \mathbb{M}_I^{\delta, \varepsilon}$. \square

3 Criteria for the existence of mark functions

Based on the construction of the complete metric and the decomposition $\mathbb{M}_I^{\text{fct}} = \bigcup_{h \in \mathcal{H}} \mathbb{M}_I^h$ into closed sets obtained in Section 2, we now derive criteria to check if a marked metric measure space admits a mark function, especially in the case where it is given as a marked Gromov-weak limit. We then transfer the results to random mmm-spaces and \mathbb{M}_I -valued stochastic processes.

3.1 Deterministic criteria

Our main criterion for deterministic spaces is a direct consequence of the results in Section 2. Recall that \mathcal{H} is the set of moduli of continuity defined in (1.17).

Theorem 3.1 (characterization of existence of a mark function in the limit). *Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in \mathbb{M}_I with $x_n \xrightarrow{\text{mGw}} x \in \mathbb{M}_I$. Then $x \in \mathbb{M}_I^{\text{fct}}$ if and only if there exists $h \in \mathcal{H}$ such that for every $\delta > 0$*

$$x_n \in \mathbb{M}_I^{\delta, h(\delta)} \quad \text{for infinitely many } n \in \mathbb{N}. \tag{3.1}$$

In this case, $x \in \mathbb{M}_I^h$.

Proof. First assume there is $h \in \mathcal{H}$ such that (3.1) is satisfied. Since $\mathbb{M}_I^{\delta, h(\delta)}$ is closed by Lemma 2.9, (3.1) implies that $x \in \mathbb{M}_I^{\delta, h(\delta)}$ for every δ , that is $x \in \mathbb{M}_I^h$. By Lemma 2.8, $\mathbb{M}_I^h \subseteq \mathbb{M}_I^{\text{fct}}$.

Conversely, assume $x \in \mathbb{M}_I^{\text{fct}}$. Then, by Lemma 2.8, we find $h \in \mathcal{H}$ with $x \in \mathbb{M}_I^h$. We claim that (3.1) holds with h replaced by $\hat{h}(\delta) := h(3\delta) + 2\delta$. Indeed, fix $\delta > 0$ and observe that $x \in \mathbb{M}_I^h \subseteq \mathbb{M}_I^{3\delta, h(3\delta)}$. Lemma 2.6 yields $x_n \in \mathbb{M}_I^{\delta, \hat{h}(\delta)}$ for all n with $d_{\text{mGP}}(x, x_n) < \delta$. \square

We will use Theorem 3.1 in the following form.

Corollary 3.2. *Let $x_n = (X_n, r_n, \nu_n, \kappa_n) \in \mathbb{M}_I^{\text{fct}}$, $x_n \xrightarrow{\text{mGw}} x \in \mathbb{M}_I$. Let $Y_{n,\delta} \subseteq X_n$ measurable for $n \in \mathbb{N}$, $\delta > 0$, and $h \in \mathcal{H}$. Then $x \in \mathbb{M}_I^{\text{fct}}$ if the following two conditions hold for every $\delta > 0$:*

$$\liminf_{n \rightarrow \infty} \nu_n(X_n \setminus Y_{n,\delta}) \leq h(\delta), \tag{3.2}$$

$$\forall n \in \mathbb{N}, x, y \in Y_{n,\delta} : r_n(x, y) < \delta \implies d(\kappa_n(x), \kappa_n(y)) \leq h(\delta). \tag{3.3}$$

Proof. Let $\mu'_n := \mu_n|_{Y_{n,\delta} \times I}$, where $\mu_n = \nu_n \otimes \delta_{\kappa_n}$. Then (3.3) implies $(X_n, r_n, \mu'_n) \in \mathfrak{M}_I^{\delta, h(\delta)}$ and (3.2) yields $\|\mu'_n - \mu_n\| \leq h(\delta)$ for infinitely many n . Hence we can apply Theorem 3.1. \square

Remark 3.3. To obtain $x \in \mathbb{M}_I^{\text{fct}}$, it is clearly enough to show in Theorem 3.1 and Corollary 3.2, (3.1) respectively (3.2) and (3.3) only for $\delta = \delta_m$ for a sequence $(\delta_m)_{m \in \mathbb{N}}$ with $\delta_m \downarrow 0$ as $m \rightarrow \infty$.

We illustrate the rôle of the exceptional set $X_n \setminus Y_{n,\delta}$, and the importance of its dependence on δ , with a simple example.

Example 3.4. Consider $X = [0, 1]$ with Euclidean metric r , $\nu = \lambda + \delta_0$, where λ is Lebesgue-measure, and $\kappa_n(x) = (nx) \wedge 1$. Obviously, $x_n = (X, r, \nu, \kappa_n)$ converges marked Gromov-weakly and the limit admits the mark function $\mathbb{1}_{(0,1]}$. To see this from Corollary 3.2, we choose $h(\delta) = \delta$ and $Y_{n,\delta} = \{0\} \cup [\delta \vee \frac{1}{n}, 1]$. Note that we cannot choose $Y_{n,\delta}$ independent of δ .

Remark 3.5 (equicontinuous case). If, in Corollary 3.2, $Y_{n,\delta} = Y_n$ does not depend on δ , then (3.3) means that κ_n is h -uniformly continuous on Y_n . Consequently, the mark function of x is in this case h -uniformly continuous. If we restrict to $Y_n = X_n$ for all n , we recover part (ii) of Lemma 1.13.

Corollary 3.6. *Let $x_n = (X_n, r_n, \nu_n, \kappa_n) \in \mathbb{M}_I^{\text{fct}}$ and assume that x_n converges to $x = (X, r, \mu) \in \mathbb{M}_I$ marked Gromov-weakly. Further assume that for $n \in \mathbb{N}$, $\delta > 0$, there are measurable sets $Z_{n,\delta} \subseteq X_n$, such that*

$$\lim_{\delta \downarrow 0} \liminf_{n \rightarrow \infty} \left(\nu_n(X_n \setminus Z_{n,\delta}) + \int_{Z_{n,\delta}} \left(1 \wedge \text{diam}(\kappa_n(B_\delta^{X_n}(x) \cap Z_{n,\delta})) \right) \nu_n(dx) \right) = 0, \tag{3.4}$$

where diam is the diameter of a set. Then \mathcal{X} admits a mark function, that is $\mathcal{X} \in \mathbb{M}_I^{\text{fct}}$.

Proof. For $\delta > 0$ let

$$g(\delta) := \sup_{0 < \delta' \leq \delta} \liminf_{n \rightarrow \infty} \left(\nu_n(X_n \setminus Z_{n,\delta'}) + \int_{Z_{n,\delta'}} \left(1 \wedge \text{diam}(\kappa_n(B_{\delta'}^{X_n}(x) \cap Z_{n,\delta'})) \right) \nu_n(dx) \right). \tag{3.5}$$

By (3.4), $\lim_{\delta \downarrow 0} g(\delta) = 0$ and g is increasing with $\|g\|_\infty \leq \|\mu\|$. Let $h \in \mathcal{H}$ be such that $g(\delta) \leq \frac{h(\delta)}{2} (1 \wedge h(\delta))$ for all $\delta > 0$. Then

$$\nu_n(\{x \in Z_{n,\delta} : \text{diam}(\kappa_n(B_\delta^{X_n}(x) \cap Z_{n,\delta})) > h(\delta)\}) \leq \frac{g(\delta)}{1 \wedge h(\delta)} \leq h(\delta)/2. \tag{3.6}$$

Now apply Corollary 3.2 with

$$Y_{n,\delta} := \{x \in Z_{n,\delta} : \text{diam}(\kappa_n(B_\delta^{X_n}(x) \cap Z_{n,\delta})) \leq h(\delta)\}. \tag{3.7}$$

Then (3.3) follows from the definition of $Y_{n,\delta}$ in (3.7), and $\nu_n(X_n \setminus Y_{n,\delta}) \leq \nu_n(X_n \setminus Z_{n,\delta}) + h(\delta)/2 \leq g(\delta) + h(\delta)/2 \leq h(\delta)$ holds by (3.6) and (3.5). \square

3.2 Random fmm-spaces

The following theorem is a randomized version of Theorem 3.1. It is our main criterion for \mathbb{M}_I -valued random variables.

Theorem 3.7 (random fmm-spaces as limits in distribution). *Let $(\mathcal{X}_n)_{n \in \mathbb{N}}$ be a sequence of \mathbb{M}_I -valued random variables which converges in distribution (w.r.t. marked Gromov-weak topology) to an \mathbb{M}_I -valued random variable \mathcal{X} . Further assume that for every $\varepsilon > 0$, there exists a modulus of continuity $h_\varepsilon \in \mathcal{H}$ such that*

$$\limsup_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}(\{\mathcal{X}_n \in \mathbb{M}_I^{\delta, h_\varepsilon(\delta)}\}) \geq 1 - \varepsilon. \tag{3.8}$$

Then \mathcal{X} admits almost surely a mark function, that is $\mathcal{X} \in \mathbb{M}_I^{\text{fct}}$ almost surely.

If additionally $\mathcal{X}_n = (X_n, r_n, \nu_n, \kappa_n) \in \mathbb{M}_I^{\text{fct}}$ almost surely for all $n \in \mathbb{N}$, we can replace (3.8) by existence of random measurable sets $Y_{n,\delta}^\varepsilon \subseteq X_n$, $n \in \mathbb{N}$, $\delta > 0$, in addition to the $h_\varepsilon \in \mathcal{H}$, such that the following two conditions hold for every $\varepsilon > 0$:

$$\limsup_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}(\{\nu_n(X_n \setminus Y_{n,\delta}^\varepsilon) \leq h_\varepsilon(\delta)\}) \geq 1 - \varepsilon. \tag{3.9}$$

$$\forall n \in \mathbb{N}, \delta > 0, x, y \in Y_{n,\delta}^\varepsilon : r_n(x, y) < \delta \implies d(\kappa_n(x), \kappa_n(y)) \leq h_\varepsilon(\delta). \tag{3.10}$$

Remark 3.8. In (3.9), we need not worry about measurability of the “event” $B_{n,\delta} := \{\nu_n(X_n \setminus Y_{n,\delta}^\varepsilon) \leq h_\varepsilon(\delta)\}$ due to the choice of $Y_{n,\delta}^\varepsilon$. The inequality (3.9) is to be understood in the sense of inner measure, that is we require that there are measurable sets $C_{n,\delta} \subseteq B_{n,\delta}$ with $\limsup_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}(C_{n,\delta}) \geq 1 - \varepsilon$.

Proof. The second statement follows in the same way as Corollary 3.2. We divide the proof of the main part in two steps. First, we show $\mathcal{X} \in \mathbb{M}_I^{\text{fct}}$ if, instead of (3.8), even

$$\mathbb{P}\left(\bigcap_{m \in \mathbb{N}} \{\mathcal{X}_n \in \mathbb{M}_I^{\delta_m, h_\varepsilon(\delta_m)} \text{ for infinitely many } n\}\right) \geq 1 - \varepsilon \tag{3.11}$$

holds for a sequence $\delta_m = \delta_m(\varepsilon) \downarrow 0$ as $m \rightarrow \infty$. In the second step, we show that, given (3.8), we can modify h_ε to $\hat{h}_\varepsilon \in \mathcal{H}$ such that (3.11) holds with h_ε replaced by \hat{h}_ε .

Step 1. By Skorohod’s representation theorem, we may assume that the \mathcal{X}_n are coupled such that they converge almost surely to \mathcal{X} in the marked Gromov-weak topology. The

inequality (3.11) implies that with probability at least $1 - \varepsilon$, for all $m \in \mathbb{N}$, $\mathcal{X}_n \in \mathbb{M}_I^{\delta_m, h_\varepsilon(\delta_m)}$ for infinitely many n . By Theorem 3.1 and Remark 3.3, this means that the probability that \mathcal{X} admits a mark function is at least $1 - \varepsilon$. Because ε is arbitrary, this implies $\mathcal{X} \in \mathbb{M}_I^{\text{fct}}$ almost surely.

Step 2. Let $T(\varepsilon, \delta) := \limsup_{n \rightarrow \infty} \mathbb{P}(\{\mathcal{X}_n \in \mathbb{M}_I^{\delta, h_\varepsilon(\delta)}\})$ in (3.8). Set

$$\delta_1 := \sup\{\delta \in [0, 1] : T(\varepsilon/4, \delta) \geq 1 - \varepsilon/2 \text{ and } h_{\varepsilon/4}(\delta) < 1\}. \quad (3.12)$$

By (3.8) and as $h_{\varepsilon/4} \in \mathcal{H}$, the set inside the supremum is non-empty. Next define recursively

$$\delta_m := \sup\{\delta \in [0, \delta_{m-1}/2] : T(\varepsilon 2^{-(m+1)}, \delta) \geq 1 - \varepsilon 2^{-m} \text{ and } h_{\varepsilon 2^{-(m+1)}}(\delta) < 1/m\} \quad (3.13)$$

for $m \in \mathbb{N}, m \geq 2$. Again, the set inside the supremum is non-empty by (3.8) and as $h_{\varepsilon 2^{-(m+1)}} \in \mathcal{H}$. Moreover, $\delta_m = \delta_m(\varepsilon) > 0$, $\delta_m \downarrow 0$ for $m \rightarrow \infty$ and $h_{\varepsilon 2^{-(m+1)}}(\delta_m) \leq 1/m$ follows. We can therefore set

$$\hat{h}_\varepsilon(\delta_m) := h_{\varepsilon 2^{-(m+1)}}(\delta_m) \quad (3.14)$$

and extend this to $\hat{h}_\varepsilon \in \mathcal{H}$. Using Fatou's lemma, we obtain

$$\begin{aligned} \mathbb{P}\left(\bigcup_{m \in \mathbb{N}} \{\mathcal{X}_n \notin \mathbb{M}_I^{\delta_m, \hat{h}_\varepsilon(\delta_m)} \text{ eventually}\}\right) &\leq \sum_{m \in \mathbb{N}} \mathbb{E}(\liminf_{n \rightarrow \infty} \mathbb{1}_{\mathbb{M}_I \setminus \mathbb{M}_I^{\delta_m, \hat{h}_\varepsilon(\delta_m)}}(\mathcal{X}_n)) \quad (3.15) \\ &\leq \sum_{m \in \mathbb{N}} \liminf_{n \rightarrow \infty} \mathbb{P}(\{\mathcal{X}_n \notin \mathbb{M}_I^{\delta_m, \hat{h}_\varepsilon(\delta_m)}\}) \\ &= \sum_{m \in \mathbb{N}} (1 - T(\varepsilon 2^{-(m+1)}, \delta_m)) \\ &\leq \sum_{m \in \mathbb{N}} \varepsilon 2^{-m} = \varepsilon. \end{aligned}$$

Thus (3.11) holds with h_ε replaced by \hat{h}_ε . □

3.3 Fmm-space-valued processes

Let $J \subseteq \mathbb{R}_+$ be a (closed, open or half-open) interval and consider a stochastic process $\mathcal{X} = (\mathcal{X}_t)_{t \in J}$ with values in \mathbb{M}_I and càdlàg paths, where \mathbb{M}_I is equipped with the marked Gromov-weak topology. We say that \mathcal{X} is an $\mathbb{M}_I^{\text{fct}}$ -valued càdlàg process if

$$\mathbb{P}(\{\mathcal{X}_t, \mathcal{X}_{t-} \in \mathbb{M}_I^{\text{fct}} \text{ for all } t \in J\}) = 1, \quad (3.16)$$

where \mathcal{X}_{t-} is the left limit of \mathcal{X} at t ($\mathcal{X}_{\ell-} := \mathcal{X}_\ell$ if ℓ is the left endpoint of J). In the following, we give sufficient criteria for \mathcal{X} to be an $\mathbb{M}_I^{\text{fct}}$ -valued càdlàg process. We are particularly interested in the situation where \mathcal{X} is the limit of $\mathbb{M}_I^{\text{fct}}$ -valued processes \mathcal{X}^n .

Unsurprisingly, if the set of \mathbb{P} -measure smaller or equal to ε in Theorem 3.7 is independent of t , the result is true for all t simultaneously, almost surely. The modulus of continuity may also depend on t in a continuous way; or be arbitrary if the limiting process has continuous paths:

Theorem 3.9. *Let $J \subseteq \mathbb{R}_+$ be an interval, and $\mathcal{X}^n = (\mathcal{X}_t^n)_{t \in J}$, $n \in \mathbb{N}$, a sequence of \mathbb{M}_I -valued càdlàg processes converging in distribution to an \mathbb{M}_I -valued càdlàg process $\mathcal{X} = (\mathcal{X}_t)_{t \in J}$. Assume that for every $t \in J$, $\varepsilon > 0$, there exists $h_{t,\varepsilon} \in \mathcal{H}$ such that*

$$\limsup_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}(\{\mathcal{X}_t^n \in \mathbb{M}_I^{\delta, h_{t,\varepsilon}(\delta)} \forall t \in J\}) \geq 1 - \varepsilon. \quad (3.17)$$

Then \mathcal{X} is an $\mathbb{M}_I^{\text{fct}}$ -valued càdlàg process, that is (3.16) is satisfied, if at least one of the following two conditions holds:

- (i) \mathcal{X} has continuous paths a.s.
- (ii) $t \mapsto h_{t,\varepsilon}(\delta)$ is continuous for every $\varepsilon, \delta > 0$.

If additionally \mathcal{X}^n is M_I^{fct} -valued almost surely for all $n \in \mathbb{N}$, (3.17) can be replaced by existence of random measurable sets $Y_{t,\varepsilon,\delta}^n \subseteq X_t^n$, in addition to the $h_{t,\varepsilon} \in \mathcal{H}$, satisfying the following two conditions for every $\varepsilon > 0$:

$$\limsup_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}(\{ \nu_t^n(X_t^n \setminus Y_{t,\varepsilon,\delta}^n) \leq h_{t,\varepsilon}(\delta) \ \forall t \in J \}) \geq 1 - \varepsilon, \tag{3.18}$$

$$\forall n \in \mathbb{N}, t \in J, \delta > 0, x, y \in Y_{t,\varepsilon,\delta}^n : r_n(x, y) < \delta \implies d(\kappa_n(x), \kappa_n(y)) \leq h_{t,\varepsilon}(\delta). \tag{3.19}$$

Proof. Due to the Skorohod representation theorem, we may assume that $\mathcal{X}^n \rightarrow \mathcal{X}$ almost surely in the Skorohod topology. For condition (i) respectively (ii) we obtain

- (i) If \mathcal{X} has continuous paths a.s., the convergence in Skorohod topology implies uniform convergence of $\mathcal{X}_t^n(\omega)$ on J a.s. with respect to d_{mGP} . Hence we have $\mathcal{X}_t^n \xrightarrow[n \rightarrow \infty]{\text{mGw}} \mathcal{X}_t$ for all $t \in J$, almost surely, and we can proceed as in the proof of Theorem 3.7.
- (ii) There are (random) continuous $w^n : J \rightarrow J$, converging to the identity uniformly on compacta, such that $\mathcal{X}_{w^n(t)}^n \rightarrow \mathcal{X}_t$ for all $t \in J$, almost surely. We can use the moduli of continuity $\hat{h}_{t,\varepsilon}(\delta) := h_{t,\varepsilon}(\delta) + \delta$ and proceed as in the proof of Theorem 3.7. Note here that, due to continuity of $h_{t,\varepsilon}(\delta)$ in t , there is for every compact subinterval \mathcal{J} of J an $N_{\mathcal{J},\varepsilon,\delta} \in \mathbb{N}$ such that $\hat{h}_{t,\varepsilon}(\delta) \geq h_{w^n(t),\varepsilon}(\delta)$ for all $n \geq N_{\mathcal{J},\varepsilon,\delta}$ and $t \in \mathcal{J}$.

The same arguments apply for left limits with w_-^n such that $\mathcal{X}_{w_-^n(t)}^n \rightarrow \mathcal{X}_{t-}$. □

To use Theorem 3.9, we have to check in (3.17) or (3.18) a condition for uncountably many t simultaneously, which is often much more difficult than for every t individually. One situation, where it is easy to pass from individual t to all t simultaneously is the case where the moduli of continuity $h_{t,\varepsilon}$ actually do not depend on t and ε (see Corollary 3.13). The independence of ε , however, is a strong requirement. Therefore, we relax it to not blowing up too fast as $\varepsilon \downarrow 0$, where the “too fast” is determined by the following modulus of càdlàgness of the limiting process.

Definition 3.10 (modulus of càdlàgness). *Let J be an interval, (E, r) a metric space, and $e = (e_t)_{t \in J} \in \mathcal{D}_E(J)$ a càdlàg path on J with values in E . Following [2, (14.44)], set*

$$w''(e, \delta) := \sup_{t, t_1, t_2 \in J : t_1 \leq t \leq t_2, t_2 - t_1 \leq \delta} \min\{r(e(t), e(t_1)), r(e(t_2), e(t))\}. \tag{3.20}$$

We say that e admits $w \in \mathcal{H}$ as modulus of càdlàgness if $w''(e, \delta) \leq w(\delta)$ for all $\delta > 0$.

Theorem 3.11. *Fix an interval $J \subseteq \mathbb{R}_+$. Let $\mathcal{X} = (\mathcal{X}_t)_{t \in J}$ and $\mathcal{X}^n = (\mathcal{X}_t^n)_{t \in J}$, $n \in \mathbb{N}$, be M_I -valued càdlàg processes such that \mathcal{X}^n converges in distribution to \mathcal{X} . Furthermore, assume that there is a dense set $Q \subseteq J$ and $w_\varepsilon, h_\varepsilon \in \mathcal{H}$, such that for all $\varepsilon > 0$*

$$\limsup_{n \rightarrow \infty} \mathbb{P}(\{\mathcal{X}_t^n \in M_I^{\delta, h_\varepsilon(\delta)}\}) \geq 1 - \varepsilon \quad \forall \delta > 0, t \in Q, \tag{3.21}$$

$$\mathbb{P}(\{t \mapsto \mathcal{X}_t \text{ admits } w_\varepsilon \text{ as modulus of càdlàgness w.r.t. } d_{\text{mGP}}\}) \geq 1 - \varepsilon, \text{ and} \tag{3.22}$$

$$\liminf_{\delta \downarrow 0} h_{\varepsilon \cdot \delta}(2w_\varepsilon(\delta)) = 0. \tag{3.23}$$

Then \mathcal{X} is an M_I^{fct} -valued càdlàg process, that is (3.16) holds.

Recall the decomposition $M_I \setminus M_I^{\text{fct}} = \bigcup_{m \in \mathbb{N}} F_m$ with F_m defined in (2.4). The basic idea of the proof is to use the following lemma about càdlàg paths to show that, almost surely, the path of \mathcal{X} avoids F_m . The assertion of the lemma follows easily using the triangle-inequality.

Lemma 3.12. *Let J be an interval, (E, r) a metric space, and $e = (e_t)_{t \in J} \in \mathcal{D}_E(J)$ a càdlàg path admitting modulus of càdlàgness $w \in \mathcal{H}$. Let $F \subseteq E$ be any set, $\delta > 0$, and $Q \subseteq J$ such that for all $t \in J$ there is $t_1, t_2 \in Q$ with $t_1 \leq t \leq t_2 \leq t_1 + \delta$. Then*

$$r(e_t, F) > w(\delta) \quad \forall t \in Q \quad \implies \quad e_t \notin F \text{ and } e_{t-} \notin F \quad \forall t \in J. \quad (3.24)$$

Proof of Theorem 3.11. Because $\mathbb{M}_I^{h_\varepsilon} = \bigcap_{\delta > 0} \mathbb{M}_I^{\delta, h_\varepsilon(\delta)}$ is closed by Lemma 2.9, the Portmanteau theorem and (3.21) imply

$$\mathbb{P}(\{\mathcal{X}_t \notin \mathbb{M}_I^{h_\varepsilon}\}) < \varepsilon \quad \forall t \in Q, \varepsilon > 0. \quad (3.25)$$

Due to the Skorohod representation theorem, we may assume that $\mathcal{X}^n \rightarrow \mathcal{X}$ almost surely in Skorohod topology. In order to simplify notation, we assume $J = [0, 1]$ and $Q = \bigcup_{k \in \mathbb{N}} Q_k$ with $Q_k = \{i2^{-k} : i = 0, \dots, 2^k\}$. It is enough to show for every $\varepsilon > 0$, $m \in \mathbb{N}$ and F_m as defined in (2.4) that

$$p_m := \mathbb{P}(\{\exists t \in [0, 1] : \mathcal{X}_t \text{ or } \mathcal{X}_{t-} \in F_m\}) \leq 3\varepsilon. \quad (3.26)$$

To show (3.26), fix $\varepsilon > 0$ and $m \in \mathbb{N}$, and let $\mathcal{X}_t = (X_t, r_t, \mu_t)$. Because \mathcal{X} has càdlàg paths, we find $K = K(\varepsilon) < \infty$ such that

$$\mathbb{P}(\{\sup_{t \in [0, 1]} \|\mu_t\| \geq K - 3\}) < \varepsilon. \quad (3.27)$$

According to (3.23) and (3.25), we can choose $k \in \mathbb{N}$ big enough such that for $h := h_{\varepsilon 2^{-k}}$ we have

$$h(2w_\varepsilon(2^{-k})) < (Km)^{-1} - 2w_\varepsilon(2^{-k}) \quad \text{and} \quad \mathbb{P}(\{\mathcal{X}_t \notin \mathbb{M}_I^h\}) < \varepsilon 2^{-k}. \quad (3.28)$$

Assume without loss of generality that $w_\varepsilon(2^{-k}) \leq 1$. Now Proposition 2.7(iii) implies that, whenever $\mathcal{X}_t \in \mathbb{M}_I^h$ and $\|\mu_t\| < K - 3$, we have

$$d_{\text{mGP}}(\mathcal{X}_t, F_m) > w_\varepsilon(2^{-k}). \quad (3.29)$$

Combining (3.22) and Lemma 3.12, we obtain

$$p_m \leq \varepsilon + \mathbb{P}(\{\exists t \in Q_k : d_{\text{mGP}}(\mathcal{X}_t, F_m) \leq w_\varepsilon(2^{-k})\}). \quad (3.30)$$

Using (3.27), (3.29), and (in the last step) (3.28), we conclude

$$p_m \leq 2\varepsilon + 2^k \sup_{t \in Q_k} \mathbb{P}(\{\|\mu_t\| < K - 3, \mathcal{X}_t \notin \mathbb{M}_I^h\}) \leq 3\varepsilon. \quad (3.31)$$

Thus (3.26) holds for all $\varepsilon > 0$, and $\mathbb{P}(\{\exists t \in [0, 1] : X_t \notin \mathbb{M}_I^{\text{fct}}\}) = \sup_{m \in \mathbb{N}} p_m = 0$ follows. \square

If, in Theorem 3.11, we can choose the modulus of continuity $h_\varepsilon = h \in \mathcal{H}$, independent of ε , such that (3.21) holds, we do not need to check (3.22) and (3.23).

Corollary 3.13 (ε -independent modulus of continuity). *Assume that $\mathcal{X}^n = (\mathcal{X}_t^n)_{t \in J}$ converges in distribution to an \mathbb{M}_I -valued càdlàg process \mathcal{X} , and $Q \subseteq J$ is dense. Then \mathcal{X} is an $\mathbb{M}_I^{\text{fct}}$ -valued càdlàg process if, for some $h \in \mathcal{H}$,*

$$\limsup_{n \rightarrow \infty} \mathbb{P}(\{\mathcal{X}_t^n \in \mathbb{M}_I^h\}) = 1 \quad \forall t \in Q. \quad (3.32)$$

Proof. Let $h \in \mathcal{H}$ be such that (3.32) is satisfied and set $h_\varepsilon := h$. Then (3.23) is satisfied for every choice of $w_\varepsilon \in \mathcal{H}$, $\varepsilon > 0$. For every càdlàg process, in particular for \mathcal{X} , there exist moduli of càdlàgness w_ε such that (3.22) holds (cf. [2, (14.6), (14.8) and (14.46)]). Thus, Theorem 3.11 yields the claim. \square

4 Examples

The (neutral) tree-valued Fleming-Viot dynamics is constructed in [10] using the formalism of metric measure spaces. In [5], (allelic) types – encoded as marks of marked metric measure spaces – are included, in order to be able to model mutation and selection.

In [5, Remark 3.11] and [6, Theorem 6] it is stated that the resulting tree-valued Fleming-Viot dynamics with mutation and selection (TFVMS) admits a mark function at all times, almost surely. The given proof, however, contains a gap, because it relies on the criterion claimed in [6, Lemma 7.1], which is wrong in general (see Example 4.1). The reason why the criterion may fail is a lack of homogeneity of the measure ν , in the sense that there are parts with high and parts with low mass density. Consequently, if we condition two samples to have distance less than ε , the probability that they are from the high-density part tends to one as $\varepsilon \downarrow 0$, and we do not “see” the low-density part. This phenomenon occurs if ν has an atom but is not purely atomic. We also give two non-atomic examples, one a subset of Euclidean space, and the other one ultrametric.

Example 4.1 (counterexamples). In both examples, it is straight-forward to see that (X, r, μ) , with $\mu = \nu \otimes K$, satisfies the assumptions of [6, Lemma 7.1], but does not admit a mark function. The mark space is $I = \{0, 1\}$.

- (i) Let λ_A be Lebesgue measure of appropriate dimension on a set A . Define $X := [0, 1]^2 \cup [2, 3]$, where $[2, 3]$ is identified with $[2, 3] \times \{0\} \subseteq \mathbb{R}^2$,

$$\nu := \frac{1}{2}(\lambda_{[0,1]^2} + \lambda_{[2,3]}) \quad \text{and} \quad K_x := \begin{cases} \frac{1}{2}(\delta_0 + \delta_1), & x \in [0, 1]^2, \\ \delta_0, & x \in [2, 3]. \end{cases} \quad (4.1)$$

- (ii) In this example think of a tree consisting of a left part with tertiary branching points and a right part with binary branching points. The leaves correspond to $X := A \cup B$ with $A = \{0, 1, 2\}^{\mathbb{N}}$ and $B = \{3, 4\}^{\mathbb{N}}$, and we choose as a metric

$$r((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}) := \max_{n \in \mathbb{N}} e^{-n} \cdot \mathbb{1}_{x_n \neq y_n}. \quad (4.2)$$

Note that (X, r) is a compact, ultrametric space. The measure ν is constructed as follows: choose the left respectively right part of the tree with probability $\frac{1}{2}$ each. Going deeper in the tree, at each branching point a branch is chosen uniformly. That is, let ν_A and ν_B be the Bernoulli measures on A and B with uniform marginals on $\{0, 1, 2\}$ and $\{3, 4\}$, respectively. Define

$$\nu := \frac{1}{2}(\nu_A + \nu_B) \quad \text{and} \quad K_x := \begin{cases} \frac{1}{2}(\delta_0 + \delta_1), & x \in A, \\ \delta_0, & x \in B. \end{cases} \quad (4.3)$$

4.1 The tree-valued Fleming-Viot dynamics with mutation and selection

In the following, we prove the existence of a mark function for the TFVMS by verifying the assumptions of Theorem 3.9 for a sequence of approximating tree-valued Moran models. Due to the Girsanov transform given in [5, Theorem 2], it is enough to consider the neutral case, that is without selection.

We briefly recall the construction of the tree-valued Moran model with mutation (TMMM) with finite population $U_N = \{1, \dots, N\}$, $N \in \mathbb{N}$, and types from the mark space I . For details and more formal definitions, see [5, Subsections 2.1–2.3]. In the underlying Moran model with mutation (MMM), every pair of individuals “resamples” independently at rate $\gamma > 0$. Here, resampling means that one of the individuals (chosen uniformly at random among the two) is replaced by an offspring of the other one, and the offspring

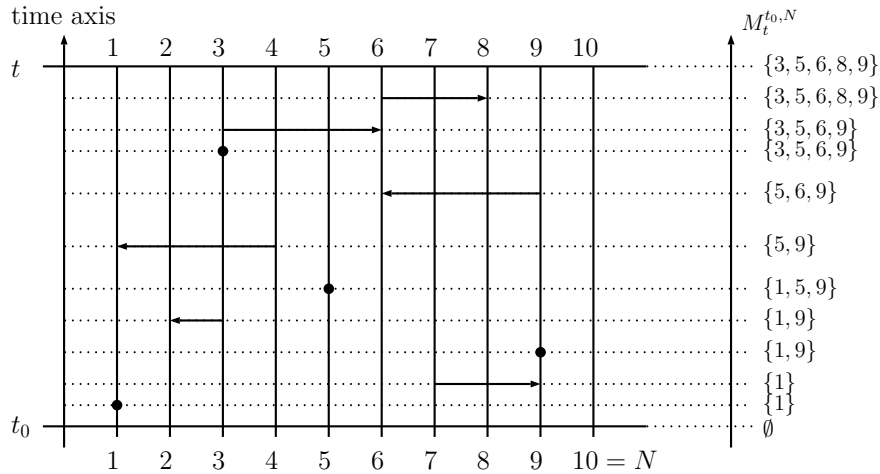


Figure 1: Graphical construction of the MMM for $N = 10$ for the time-period $[t_0, t]$, and the resulting process $(M_s^{t_0, N})_{s \in [t_0, t]}$. Resampling arrows are drawn at points of $\eta_{\text{res}}^{k, \ell}$, and mutation dots at points of η_{mut}^k .

gets the same type as the parent. Furthermore, every individual mutates independently at rate $\vartheta \geq 0$, which means that it changes its type according to a fixed stochastic kernel $\beta(\cdot, \cdot)$ on I . Denote the resulting type of individual $x \in U_N$ at time $t \geq 0$ by $\kappa_t^N(x)$. To obtain the tree-valued dynamics, define the distance $r_t^N(x, y)$ between two individuals $x, y \in U_N$ at time $t \geq 0$ as twice the time to the most recent common ancestor (MRCA) (cf. [5, (2.7)]), provided that a common ancestor exists, and as $2t + r_0^N(x, y)$ otherwise. The TMMM is the resulting process $\mathcal{X}_t^N = (U_N, r_t^N, \nu_N, \kappa_t^N)$, with sampling measure $\nu_N = \frac{1}{N} \sum_{k=1}^N \delta_k$. It is easy to check that, by definition, (U_N, r_t^N) is an ultrametric space, provided that the initial metric space (U_N, r_0^N) is ultrametric. This explains the name *tree-valued* (cf. [5, Remark 2.7]).

Next recall the graphical construction of the MMM from [5, Definition 2.2]. A resampling event is modeled by means of a family of independent Poisson point processes $\{\eta_{\text{res}}^{k, \ell} : k, \ell \in U_N\}$ on \mathbb{R}_+ , where each $\eta_{\text{res}}^{k, \ell}$ has rate $\gamma/2$. If $t \in \eta_{\text{res}}^{k, \ell}$, draw an arrow from (k, t) to (ℓ, t) to represent a resampling event at time t , where ℓ is an offspring of k . Similarly, model mutation times by a family of independent Poisson point processes $\{\eta_{\text{mut}}^k : k \in U_N\}$, where each η_{mut}^k has rate ϑ . If $t \in \eta_{\text{mut}}^k$, draw a dot at (k, t) to represent a mutation event changing the type of individual k (see Figure 1).

Let $(M_t^{t_0, N})_{t \geq t_0}$, $M_t^{t_0, N} \subseteq U_N$ with $M_{t_0}^{t_0, N} = \emptyset$ be the process that records the individuals of the population at time t with an ancestor at a time $t_0 < s \leq t$ involved in a mutation event. By a coupling argument, this process can be constructed by means of the Poisson point processes $(\eta_{\text{res}}^{k, \ell}, \eta_{\text{mut}}^k, k, \ell \in U_N)$ as follows (compare Figures 1–2):

$$M_t^{t_0, N} = \begin{cases} M_{t-}^{t_0, N} \cup \{\ell\} & \text{if there is an arrow from } k \in M_{t-}^{t_0, N} \text{ to } \ell \in U_N \text{ at time } t, \\ M_{t-}^{t_0, N} \cup \{k\} & \text{if there is a mutation event at } k \in U_N \text{ at time } t, \\ M_{t-}^{t_0, N} \setminus \{k\} & \text{if there is an arrow from } k \notin M_{t-}^{t_0, N} \text{ to } \ell \in U_N \text{ at time } t. \end{cases} \tag{4.4}$$

Let $\xi_t^N := \frac{1}{N} \# M_{t_0+t}^{t_0, N}$ be the proportion of individuals at time t_0+t , $t \geq 0$ whose ancestors have mutated after (the for the moment fixed) time t_0 .

Lemma 4.2. *Let $C := \frac{1}{2}\vartheta(2\vartheta + \gamma)$. Then for all $a, \delta > 0$*

$$\limsup_{N \rightarrow \infty} \mathbb{P} \left(\sup_{t \in [0, \delta]} \xi_t^N \geq a \right) \leq Ca^{-2}\delta^2. \tag{4.5}$$

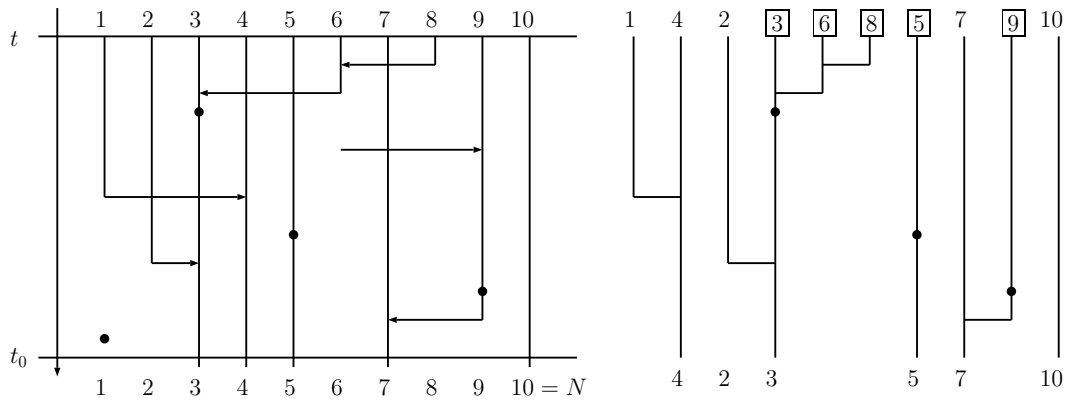


Figure 2: Tracing the ancestor backwards in time in Figure 1: This dual construction is also known as the coalescent backwards in time. Reverse the arrows to see for instance that 3 at time t_0 is an ancestor of 8 at time t . The elements of $M_t^{t_0, N} \subseteq U_N$ are highlighted by boxes in the right part of the picture.

Proof. By definition, $(\xi_t^N)_{t \geq 0}$ is a (continuous time) Markov jump process on $[0, 1]$ with $\xi_0^N = 0$ and transitions

$$\begin{cases} x \mapsto x - 1/N & \text{at rate } \frac{\gamma}{2} N^2 x(1-x), \\ x \mapsto x + 1/N & \text{at rate } \frac{\gamma}{2} N^2 x(1-x) + \vartheta N(1-x). \end{cases} \quad (4.6)$$

This process converges weakly with respect to the Skorohod topology to the solution $(Z_t)_{t \geq 0}$ of the stochastic differential equation (SDE)

$$dZ_t = \vartheta(1 - Z_t)dt + \sqrt{\gamma Z_t(1 - Z_t)} dB_t, \quad Z_0 = 0. \quad (4.7)$$

Indeed, to establish tightness use [7, Theorem III.9.4]. Note that, as $[0, 1]$ is compact, it suffices to show the convergence of the generators applied to a set of appropriate test-functions. For existence and uniqueness of solutions to (4.7) reason as for the Bessel SDE in [19, (48.1) and below]. Moreover, $Z_t \in [0, 1]$ is a bounded non-negative right-continuous submartingale. Hence, with Doob's submartingale inequality (see for instance [7, Proposition II.2.16(a)]), we obtain

$$\mathbb{P}\left(\sup_{t \in [0, \delta]} Z_t \geq a\right) = \mathbb{P}\left(\sup_{t \in [0, \delta]} Z_t^2 \geq a^2\right) \leq a^{-2} \mathbb{E}[Z_\delta^2]. \quad (4.8)$$

As $Z_t \in [0, 1]$, we further deduce using Itô's formula that for all $t \geq 0$,

$$\mathbb{E}[Z_t] \leq \vartheta t \quad \text{and} \quad (4.9)$$

$$\mathbb{E}[Z_t^2] = \mathbb{E}\left[\int_0^t 2Z_s \vartheta(1 - Z_s) + \gamma Z_s(1 - Z_s) ds\right] \leq Ct^2. \quad (4.10)$$

Then

$$\limsup_{N \rightarrow \infty} \mathbb{P}\left(\sup_{t \in [0, \delta]} \xi_t^N \geq a\right) \leq \mathbb{P}\left(\sup_{t \in [0, \delta]} Z_t \geq a\right) \leq Ca^{-2} \delta^2 \quad (4.11)$$

follows. □

As the construction of the TFVMS in [5] is only given for a compact type-space I , we make the same assumption. Note, however, that our proof itself does not use compactness and is therefore valid for non-compact I , provided that the TFVMS is the limit of the corresponding Moran models, and there exists a Girsanov transform allowing us to reduce to the neutral case.

Theorem 4.3 (the TFVMS admits a mark-function). *Let I be compact and $\mathcal{X} = (\mathcal{X}_t)_{t \geq 0}$ be the tree-valued Fleming-Viot dynamics with mutation and selection as defined in [5]. Then*

$$\mathbb{P}(\mathcal{X}_t \in \mathbb{M}_I^{\text{fct}} \text{ for all } t > 0) = 1. \tag{4.12}$$

In particular, $(\mathcal{X}_t)_{t > 0}$ is an $\mathbb{M}_I^{\text{fct}}$ -valued càdlàg process.

Proof. By [5, Theorem 2], there exists a Girsanov transform that enables us to assume without loss of generality that selection is not present. In this case, according to [5, Theorem 3], \mathcal{X} is the limit in distribution of TMMMs $\mathcal{X}^N = (\mathcal{X}_t^N)_{t \geq 0}$, as discussed above. Let $\mathcal{X}_t^N = (U_N, r_t^N, \nu_N, \kappa_t^N)$ with $U_N = \{1, \dots, N\}$ and ν_N the uniform distribution on U_N . Let $\delta > 0$ be fixed for the moment, and recall that the distance $r_t^N(x, y)$ between two individuals $x, y \in U_N$ at time $t \geq \delta/2$ is twice the time to the MRCA. Hence, if $r_t^N(x, y) < \delta$, then x and y at time t have a common ancestor at time $t - \delta/2$. Further recall that $(M_t^{t_0, N})_{t \geq t_0}$, with $M_t^{t_0, N} \subseteq U_N$ and $M_{t_0}^{t_0, N} = \emptyset$, records the individuals of the population at time t with an ancestor at a time $s \in (t_0, t]$ involved in a mutation event (cf. (4.4)).

Fix an arbitrary time horizon $T > 0$ and $i \in \mathbb{N}$, $i \leq 2T/\delta$. Using the notation of Theorem 3.9, for $t \in [i\delta/2, (i+1)\delta/2)$, let $Y_{t, \varepsilon, \delta}^N := U_N \setminus M_t^{(i-1)\delta/2, N}$, independent of $\varepsilon > 0$. Set $Y_{t, \varepsilon, \delta}^N := \emptyset$ for $t < \delta/2$. We claim that (3.19) is satisfied for any choice of $h_{t, \varepsilon} \in \mathcal{H}$. Indeed, if $x, y \in Y_{t, \varepsilon, \delta}^N$ satisfy $r_t^N(x, y) < \delta$, then they have a common ancestor at time $t_0 := (i-1)\delta/2 \leq t - \delta/2$, and after this point in time no mutation occurred along their ancestral lineages. In particular, $d(\kappa_t^N(x), \kappa_t^N(y)) = 0$, and (3.19) is obvious. Moreover, \mathcal{X}^N is $\mathbb{M}_I^{\text{fct}}$ -valued by construction, and \mathcal{X} has continuous paths by [5, Theorem 1]. According to Theorem 3.9, it is therefore enough to find moduli of continuity $h_{t, \varepsilon} \in \mathcal{H}$ such that (3.18) holds for every $\varepsilon > 0$.

By Lemma 4.2, we obtain a constant $C > 0$ such that for every $a > 0$,

$$\limsup_{N \rightarrow \infty} \mathbb{P}\left(\sup_{t \in [i\delta/2, (i+1)\delta/2)} \nu_N(U_N \setminus Y_{t, \varepsilon, \delta}^N) \geq a\right) \leq Ca^{-2}\delta^2. \tag{4.13}$$

After summation over $i \in \{1, \dots, \lfloor 2T/\delta \rfloor\}$, we obtain

$$\limsup_{N \rightarrow \infty} \mathbb{P}\left(\sup_{t \in [\delta/2, T]} \nu_N(U_N \setminus Y_{t, \varepsilon, \delta}^N) \geq a\right) \leq 2TC\delta a^{-2}. \tag{4.14}$$

For $\varepsilon > 0$ arbitrary, we use this inequality with $a := \sqrt{\varepsilon^{-1}2TC\delta}$, together with $\|\nu_N\| \leq 1$ for $t < \delta/2$, to see that (3.18) is satisfied for $h_{t, \varepsilon} \in \mathcal{H}$ with

$$h_{t, \varepsilon}(\delta) \geq \sqrt{\varepsilon^{-1}2TC\delta} + \mathbb{1}_{[2t, \infty)}(\delta). \tag{4.15}$$

□

4.2 The tree-valued Λ -Fleming-Viot process

Let Λ be a finite measure on $[0, 1]$, and recall the Λ -coalescent, introduced in [18]. It is a coalescent process, where each k -tuple out of N blocks merges independently at rate

$$\lambda_{N, k} := \int_0^1 y^{k-2}(1-y)^{N-k} \Lambda(dy). \tag{4.16}$$

For fixed N , it is elementary to construct a finite, random (ultra-)metric measure space encoding the random genealogy of the Λ -coalescent, where the distance is defined as the time to the MRCA (recall the construction of Figures 1–2 and see Figure 3). In [9, Theorem 4], existence and uniqueness of a Gromov-weak limit in distribution, as $N \rightarrow \infty$, is proven to be equivalent to the so-called “dust-free”-property, namely $\int_0^1 y^{-1} \Lambda(dy) = \infty$. The resulting limit is called Λ -coalescent measure tree.

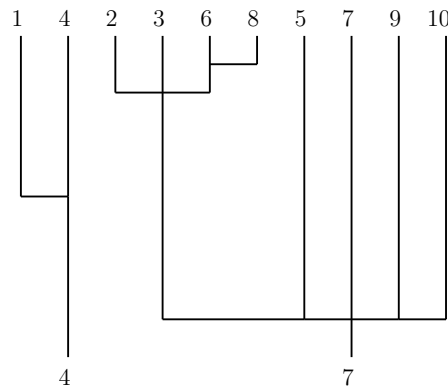


Figure 3: Tracing the ancestor backwards in time: The Λ -coalescent allows for one parent to have more than one child.

Now, replace the tree-valued Moran models considered in Subsection 4.1 and [5] by so-called tree-valued Λ -Cannings models with Λ satisfying the dust-free-property. That is, leave the mutation- and selection-part as it is and change the resampling-part of the Moran models as follows: For $k = 2, \dots, N$, at rate $\binom{N}{k} \lambda_{N,k}$ a block of k individuals is chosen uniformly at random among the N individuals of the population. Upon such a resampling event, all individuals in this block are replaced by an offspring of a single individual which is chosen uniformly from this block. Note that the genealogy (disregarding types) of the resulting Λ -Cannings model with N individuals is dual to the Λ -coalescent starting with N blocks. We call any limit point (in path space) of the tree-valued Λ -Cannings processes, as N tends to infinity and Λ is fixed, *tree-valued Λ -Fleming-Viot process (TLFV)*. In the neutral case, existence and uniqueness of such a limit point follows as a special case of the forthcoming work [8]. Here, we show that, whenever limit points exist, all of them admit mark functions.

Theorem 4.4 (the TLFV admits a mark-function). *Suppose there is no selection, that is $\alpha = 0$, and $\mathcal{X} = (\mathcal{X}_t)_{t \geq 0}$ is a tree-valued Λ -Fleming-Viot process with mutation. Then*

$$\mathbb{P}(\mathcal{X}_t \in \mathbb{M}_I^{\text{fct}} \text{ for all } t > 0) = 1. \tag{4.17}$$

Proof. By passing to a subsequence if necessary, we may assume that the Λ -Cannings models converge in distribution to \mathcal{X} . We proceed as in Subsection 4.1. Again, let $(M_t^{t_0, N})_{t \geq t_0}$, $M_t^{t_0, N} \subseteq U_N$ with $M_{t_0}^{t_0, N} = \emptyset$ be the process that records the individuals of the population at time t with an ancestor at a time $t_0 < s \leq t$ involved in a mutation event and $\xi_t^N := \frac{1}{N} \# M_{t_0+t}^{t_0, N}$ be the proportion of individuals at time $t_0 + t, t \geq 0$ whose ancestors have mutated after (the for the moment fixed) time t_0 . By definition, $(\xi_t^N)_{t \geq 0}$ is a (continuous time) Markov jump process on $[0, 1]$ with $\xi_0^N = 0$ and generator

$$\begin{aligned} (\Omega^N f)(x) &= \vartheta N(1-x)(f(x+1/N) - f(x)) \\ &+ \sum_{k=2}^N \lambda_{N,k} \sum_{m=0}^{\binom{Nx}{k}} \binom{Nx}{m} \binom{N(1-x)}{k-m} \\ &\times \left(\frac{m}{k} (f(x+(k-m)/N) - f(x)) + \frac{k-m}{k} (f(x-m/N) - f(x)) \right), \end{aligned} \tag{4.18}$$

where $x \in [0, 1], N \cdot x \in \mathbb{N} \cup \{0\}, f \in \mathcal{C}_b^2([0, 1])$. Due to Taylor's formula, there is $x_{m,k,N}^+ \in$

$[x, x + (k - m)/N]$, $x_{m,k,N}^- \in [x - m/N, x]$ with

$$\begin{aligned}
 (\Omega^N f)(x) &= \vartheta N(1 - x)(f(x + 1/N) - f(x)) \\
 &+ \sum_{k=2}^N \lambda_{N,k} \sum_{m=0}^{(Nx) \wedge k} \binom{Nx}{m} \binom{N(1-x)}{k-m} \\
 &\times \left(\frac{f''(x_{m,k,N}^+)}{2} \frac{m(k-m)^2}{kN^2} + \frac{f''(x_{m,k,N}^-)}{2} \frac{(k-m)m^2}{kN^2} \right) \\
 &= \vartheta(1-x)f'(x) + O(N^{-1}) + x(1-x) \sum_{k=2}^N \lambda_{N,k} \Delta_{N,k}(x),
 \end{aligned}
 \tag{4.19}$$

where, using $\binom{n}{i} = \frac{n}{i} \binom{n-1}{i-1}$ for $i \geq 1$,

$$\Delta_{N,k}(x) = \sum_{m=1}^{(Nx) \wedge (k-1)} \binom{Nx-1}{m-1} \binom{N(1-x)-1}{k-m-1} \left(f''(x_{m,k,N}^+) \frac{k-m}{2k} + f''(x_{m,k,N}^-) \frac{m}{2k} \right).
 \tag{4.20}$$

Recall that $\sum_{m=0}^k \binom{\ell}{m} \binom{N-\ell}{k-m} = \binom{N}{k}$ and $\lambda_{N,k} = \int_0^1 y^{k-2}(1-y)^{N-k} \Lambda(dy)$ with a finite measure Λ on $[0, 1]$ to see that

$$\begin{aligned}
 \sum_{k=2}^N |\Delta_{N,k}(x)| &\leq \|f''\|_\infty \sum_{k=2}^N \lambda_{N,k} \sum_{m=0}^{k-2} \binom{Nx-1}{m} \binom{N(1-x)-1}{k-2-m} \\
 &= \|f''\|_\infty \int_0^1 \sum_{k=2}^N \binom{N-2}{k-2} y^{k-2}(1-y)^{N-k} \Lambda(dy) \\
 &= \|f''\|_\infty \Lambda([0, 1]).
 \end{aligned}
 \tag{4.21}$$

Therefore,

$$(\Omega^N f)(x) = \vartheta(1-x)f'(x) + O(N^{-1}) + x(1-x)O(1).
 \tag{4.22}$$

Use $f(x) = x$, $x \in [0, 1]$ in (4.19) to see that $(\xi_t^N)_{t \geq 0}$ is a non-negative right-continuous submartingale with $\xi_0^N = 0$ and $\mathbb{E}[\xi_t^N] \leq \vartheta t$. Use $f(x) = x^2$ to deduce from (4.22) that

$$\mathbb{E}[(\xi_t^N)^2] \leq Ct^2 + O(N^{-1})t.
 \tag{4.23}$$

Now reason as for the TFMMS in the proofs of Lemma 4.2 and Theorem 4.3 to complete the claim. □

4.3 Future application: Evolving phylogenies of trait-dependent branching

In [14] the results of the present paper will be applied in a context of evolving genealogies to establish the existence of a mark function with the help of Theorem 3.9. These genealogies are random marked metric measure spaces, constructed as the limit of approximating particle systems. The individual birth- respectively death-rates in the N^{th} -approximating population depend on the present trait of the individuals alive and are of order $O(N)$. At each birth-event, mutation happens with a fixed probability. Each individual is assigned mass $1/N$. The metric under consideration is genetic distance: in the N^{th} -approximating population genetic distance is increased by $1/N$ at each birth with mutation. Hence, genetic distance of two individuals is counted in terms of births with mutation backwards in time to the MRCA rather than in terms of the time to the MRCA.

Because of the use of exponential times in the modeling of birth- and death-events in this therefore non-ultrametric setup the analysis of the modulus of continuity of the

trait-history of a particle in combination with the evolution of its genetic age plays a major role in establishing tightness of the approximating systems and existence of a mark function. In [13, Lemma 3.9], control on the modulus of continuity is obtained by transferring the model to the context of historical particle systems. In a first step, time is related to genetic distance by means of the modulus of continuity. The extend of the change of trait of an individual in a small amount of time (recall (3.9) and (3.3)) can then be controlled by means of the modulus of continuity of its trait-path in combination with a control on the height of the largest jump during this period of time. This can in turn be ensured by appropriate assumptions on the mutation transition kernels of the approximating systems.

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