The one-arm exponent
for mean-field long-range percolation

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Abstract

Consider a long-range percolation model on $\mathbb{Z}^d$ where the probability that an edge $\{x, y\} \in \mathbb{Z}^d \times \mathbb{Z}^d$ is open is proportional to $\|x - y\|_2^{-d-\alpha}$ for some $\alpha > 0$ and where $d > 3 \min\{2, \alpha\}$. We prove that in this case the one-arm exponent equals $\frac{1}{2} \min\{4, \alpha\}$. We also prove that the maximal displacement for critical branching random walk scales with the same exponent. This establishes that both models undergo a phase transition in the parameter α when $\alpha = 4$.

1 Introduction and main result

In this paper we study the asymptotic behavior of the one-arm event for two closely related statistical mechanical models: branching random walk (BRW) and long-range percolation (LRP). We study LRP only in the mean-field setting. That is, we require that the dimension of the space is high enough that self-interactions become negligible.

Roughly speaking, we say that a percolation model on $\mathbb{Z}^d$ is long-range with parameter α if the edges $\{x, y\}$ of the graph with vertex set $\mathbb{Z}^d$ and edge set $\mathbb{Z}^d \times \mathbb{Z}^d$ are independently retained with a probability that is asymptotically proportional to $\|x - y\|_2^{-d-\alpha}$ (where $\| \cdot \|_2$ denotes the Euclidean norm on $\mathbb{Z}^d$), and otherwise removed. We say a retained edge is 'open' and a removed edge is 'closed'. We assume throughout that the edge retention probabilities are invariant under the symmetries of $\mathbb{Z}^d$ (translations, rotations, and reflections).

Our main result requires that we rigorize this notion of long-range percolation and that we state some definitions and conditions, so let us first give a simple, somewhat imprecise version. To state the theorem we define Q_r as the intersection of the cube with sides $2r + 1$ and the lattice (i.e., $Q_r := \{x \in \mathbb{Z}^d : \|x\|_\infty \le r\}$) and its complement $Q_r^c := \mathbb{Z}^d \setminus Q_r$ (where $\| \cdot \|_\infty$ denotes the supremum norm on $\mathbb{Z}^d$). We write $\mathbb{P}_{p_c}$ for the critical percolation measure, $\{x \longleftrightarrow y\}$ for the event that x and y belong to the same open cluster, and $\{0 \longleftrightarrow Q_r^c\}$ for the event that the open cluster of the origin intersects Q_r^c. We write $f(n) \asymp g(n)$ if there exist $C \ge c > 0$ such that $cg(n) \le f(n) \le Cg(n)$.

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Theorem. *Critical long-range percolation models on $\mathbb{Z}^d$ that satisfy*

$$\mathbb{P}_{p_c}(x \longleftrightarrow y) \asymp \|x - y\|_2^{\min\{2,\alpha\}-d}$$

with $\alpha \in (0,2) \cup (2,\infty)$ and $d > 3\min\{2,\alpha\}$ also satisfy

$$\mathbb{P}_{p_c}(0 \longleftrightarrow Q_r^c) \asymp r^{-\min\{4,\alpha\}/2}.$$

The condition on $\mathbb{P}_{p_c}(x \longleftrightarrow y)$ is known to hold for a class of LRP models [3]. We discuss this in more detail below.

The theorem implies that the *one-arm exponent* for high-dimensional LRP equals $\frac{1}{2}\min\{4,\alpha\}$, so the model undergoes a *phase transition* in α when $\alpha = 4$. We also show below that for critical BRW with long-range jumps the analogous exponent (for the maximal displacement) also equals $\frac{1}{2}\min\{4,\alpha\}$.

It should moreover be noted that while there is another (well-known) phase transition in α when $\alpha = 2$ that is more readily apparent (for instance in the condition of the theorem above and in (2.8) below), this paper presents the first rigorous proof of the existence of the '$\alpha = 4$' phase transition for mean-field statistical mechanical models with long-range interactions. The phase transition at $\alpha = 4$ was conjectured by Heydenreich, van der Hofstad, and the author [11], based on the observation that the maximal displacement of simple random walk on large clusters of critical LRP appears to undergo a change at $\alpha = 4$. Moreover, the (easier) lower bound in the above theorem has been established by the same authors in [10]. This paper, then, provides an upper bound that confirms the conjecture.

Before we give the explicit version of this theorem, let us formally introduce the models under consideration in this paper, starting with the simpler one: critical BRW. In this paper we view branching random walk as a random embedding of a Galton-Watson tree into $\mathbb{Z}^d$. We will restrict ourselves to *critical* Galton-Watson trees, i.e., trees with a non-trivial i.i.d. offspring distribution $\{p_m\}_{m=0}^\infty$ such that

$$\sum_{m=1}^\infty m p_m = 1 \qquad \text{and} \qquad \sigma_p^2 := \sum_{m=2}^\infty m(m-1)p_m < \infty.$$

Writing ξ_v for the number of children of the vertex v in the rooted tree T, the probability of any finite tree T under the Galton-Watson process is given by

$$\mathbb{P}(T) = \prod_{v \in T} p_{\xi_v}.$$

We say that ϕ is an embedding of T into $\mathbb{Z}^d$ if ϕ maps the root of T to the origin of $\mathbb{Z}^d$ and ϕ maps any other vertex $v \in T$ to a single vertex $\phi(v) \in \mathbb{Z}^d$. We define the branching random walk measure as the measure on configurations (T, ϕ) such that if w is a child of v in T, then the probability that $\phi(w) = y$ if $\phi(v) = x$ is equal to $D(x,y)$, where D is the *random walk one-step distribution*. That is,

$$\mathbb{P}((T,\phi)) = \mathbb{P}(T) \prod_{(v,w) \in T} D(\phi(v), \phi(w)), \tag{1.1}$$

where the product is over pairs (v,w) such that w is a child of v in T.

In this paper we will consider a the following class of one-step distributions:

Definition 1.1. *We always assume that D is invariant under the symmetries of $\mathbb{Z}^d$, i.e., that $D(x,y)$ only depends on $x, y \in \mathbb{Z}^d$ through $x - y$, and that $D(x,y) = D(x',y')$ for any*

$x', y' \in \mathbb{Z}^d$ *such that* $x' - y'$ *is equal to* $x - y$ *up to permutations and sign-changes of the coordinates. We also assume that*

$$D(0, x) = \frac{h(x/\Lambda)}{\sum_{x \in \mathbb{Z}^d} h(x/\Lambda)} \qquad \forall x \in \mathbb{Z}^d, \tag{1.2}$$

where $\Lambda \in (0, \infty)$ *and* h *is a nonnegative bounded function on* $\mathbb{R}^d$ *that is piecewise continuous, has the aforementioned symmetries, satisfies*

$$\int_{\mathbb{R}^d} h(x) \mathrm{d}^d x = 1,$$

and that is either

   a. *supported in* $[-1, 1]^d$,

   b. *exponentially decaying (i.e., that there exist* $C, \kappa > 0$ *such that* $h(x) \leq C\mathrm{e}^{-\kappa \|x\|_\infty}$ *),*

   c. *or else that there exists* $\alpha, c_1, c_2 > 0$ *and* $k < \infty$ *such that*

$$c_1 \|x\|_2^{-d-\alpha} \leq h(x) \leq c_2 \|x\|_2^{-d-\alpha} \qquad \text{for all } \|x\|_2 \geq k. \tag{1.3}$$

We consider α *as a parameter of the model. In such cases where* h *has bounded support or decays exponentially we set* $\alpha \equiv \infty$.

Our focus will be on distributions that satisfy Definition 1.1(c). The canonical example of a distribution that satisfies Definition 1.1(c) is

$$D(x, y) = \mathcal{N} \max \left\{ \frac{\|x - y\|_2}{\Lambda}, 1 \right\}^{-d-\alpha} \tag{1.4}$$

where $\mathcal{N}$ is a normalizing constant.

Our choice to set $\alpha \equiv \infty$ for models with bounded or exponentially decaying h corresponds to the fact that for these models all spatial moments are finite. Indeed, with this definition, the parameter α indicates which spatial moments are finite, i.e.,

$$\sum_{x \in \mathbb{Z}^d} \|x\|_2^q D(0, x) < \infty \qquad \text{iff} \qquad q < \alpha,$$

for any D that satisfies Definition 1.1.

We say that a branching random walk (T, ϕ) *hits* a set $A \subseteq \mathbb{Z}^d$ if there exists at least one vertex $v \in T$ such that $\phi(v) \in A$. For the event that a BRW started at x hits the set A we write $\{x \longrightarrow A\}$, and if $A = \{y\}$ for some $y \in \mathbb{Z}^d$ we simply write $\{x \longrightarrow y\}$.

Now we introduce the long-range percolation model. We start with the complete graph on $\mathbb{Z}^d$, i.e., the graph $(\mathbb{Z}^d, \mathbb{Z}^d \times \mathbb{Z}^d)$. We choose the *percolation parameter* $p \in [0, \|D\|_\infty^{-1}]$ and remove each edge $\{x, y\} \in \mathbb{Z}^d \times \mathbb{Z}^d$ independently with probability $1 - pD(x, y)$ (where D is a one-step distribution), and retain it otherwise. In this paper we only consider LRP models with one-step distributions that satisfy Definition 1.1(c).

We write $\mathbb{P}_p$ for the LRP measure at parameter p, and we write $\mathcal{C}(x)$ for the set of all vertices that can be reached from x through a path of open edges. In this paper we consider LRP at the *critical point* $p_c := \inf\{p : \mathbb{P}_p(|\mathcal{C}(x)| = \infty) > 0\}$. This does not depend on our choice of x, and it is well established that p_c is a non-trivial value, i.e., LRP undergoes a phase transition in p at p_c. Given a vertex x and a vertex set A, we say that x is connected to A, and write $\{x \longleftrightarrow A\}$ for the event $\{\mathcal{C}(x) \cap A \neq \emptyset\}$. If $A = \{y\}$ we simply write $\{x \longleftrightarrow y\}$.

We make one further assumption on D: we assume that our choice of D produces the following *two-point function* asymptotics:

$$\mathbb{P}_{p_c}(x \longleftrightarrow y) \asymp \|x - y\|_2^{(2 \wedge \alpha) - d}, \tag{1.5}$$

where we write $(a \wedge b) := \min\{a, b\}$. Chen and Sakai [3] recently proved that there exists a class of LRP models for which this bound holds when $\alpha \in (0, 2) \cup (2, \infty)$ and $d > 3(2 \wedge \alpha)$ and Λ is sufficiently large. In Remark 1.9 below we briefly discuss this class and the assumptions.

Theorem 1.2 (The one-arm exponent for BRW and LRP). *(a) Consider critical branching random walk on $\mathbb{Z}^d$ with $\alpha \in (0, 2) \cup (2, \infty)$ and $d > (2 \wedge \alpha)$ and with one-step distribution D that satisfies Definition 1.1. Then,*

$$\mathbb{P}(0 \longrightarrow Q_r^c) \asymp r^{-(4 \wedge \alpha)/2}.$$

(b) Consider long-range percolation on $\mathbb{Z}^d$ with $\alpha \in (0, 2) \cup (2, \infty)$ and $d > 3(2 \wedge \alpha)$ and with one-step distribution D that satisfies Definition 1.1(c) and (1.5). Then,

$$\mathbb{P}_{p_c}(0 \longleftrightarrow Q_r^c) \asymp r^{-(4 \wedge \alpha)/2}.$$

Let us make a few remarks about the theorem:

Remark 1.3. The lower bound in Theorem 1.2(b) has already been proved by Heydenreich, van der Hofstad, and the author [10, Section 5]. We repeat it here for completeness, but will not prove it. For part (b) we moreover do not include the cases of Definition 1.1(a) and (b). Our proof does not work for these cases, and for models in case (a) Kozma and Nachmias [15] have already proved that the one-arm exponent equals 2. In fact, our proof is based on their work, and we will use their results throughout this paper. Proving that the one-arm exponent equals 2 for case (b) requires several modifications to the proof of Kozma and Nachmias, but none of these are interesting enough to warrant the lengthy discussion that they require.

Remark 1.4. We do not treat the case '$\alpha = 2$'. The reason is that our proof of Theorem 1.2(a) uses asymptotics of the random walk Green's function $G(x)$ (see (2.6) below) and our proof of Theorem 1.2(b) uses asymptotics of the percolation two-point function (1.5), and these are not known when $\alpha = 2$. When $\alpha = 2$ it is conjectured (see [3]) that

$$G(x) \asymp \frac{\|x\|_2^{2-d}}{\log \|x\|_2} \qquad \text{and} \qquad \mathbb{P}_{p_c}(0 \longleftrightarrow x) \asymp \frac{\|x\|_2^{2-d}}{\log \|x\|_2},$$

when $d \geq 2$ and $d \geq 6$, respectively. Assuming these relations hold, a straightforward modification of the proof of Theorem 1.2 shows that

$$\mathbb{P}(0 \longrightarrow Q_r^c) \asymp \frac{\log r}{r} \qquad \text{and} \qquad \mathbb{P}_{p_c}(0 \longleftrightarrow Q_r^c) \asymp \frac{\log r}{r}, \tag{1.6}$$

when $d \geq 2$ and $d \geq 6$, respectively.

Remark 1.5. The event $\{0 \longrightarrow Q_r^c\}$ is equivalent to the event that the *maximal displacement* of the BRW exceeds r (i.e., the maximal distance of any particle produced by the BRW from the starting point of the BRW).

There are a few related results about the maximal displacement of critical BRW. The first is due to Kesten [14]: Consider one-dimensional critical BRW with a translation invariant one-step distribution $D(0, x)$ that satisfies $\sum_x x D(0, x) = 0$ and $\sum_x |x|^\alpha D(0, x) < \infty$ for some $\alpha > 4$ (no symmetry assumptions are made on D). Let M_n denote the maximal displacement of a particle at level n of the tree T. Condition T on having height

at least An for some large constant A. Then $M_n/\sqrt{n}$ converges in distribution, and there exists a constant $C > 0$ such that

$$\mathbb{P}(M_n \geq z\sqrt{n} \mid \text{height}(T) > An) \leq Cz^{-\alpha}(1 + \log(z)^2) \qquad \forall z > 0.$$

Kesten also proves that $M_n/\sqrt{n}$ is not tight on $\{\text{height}(T) > An\}$ when $\alpha \leq 4$. Note that even though this result is in some sense stronger than Theorem 1.2 when $\alpha > 4$, it does not imply the upper bound when $\alpha > 4$.

Kesten's result can be contrasted with an earlier result by Durrett, Kesten and Waymire [5], where it is proved that when $\sum_x xD(0,x) \in (0,\infty)$ (i.e., the BRW has non-zero drift), then roughly speaking, the maximal displacement of a critical BRW conditioned to have n particles scales to the maximum of a Brownian excursion when $\alpha > 2$, while it scales to an α-dependent function when $\alpha < 2$. In particular, there appears to be no transition at $\alpha = 4$ in the presence of a drift.

In a third related result, Janson and Marckert [13] determine the scaling limit of the *tour of the discrete snake*. Roughly speaking, this process codifies the displacements of critical BRW conditioned on offspring size. This scaling limit is different when $\alpha < 4$ and when $\alpha \geq 4$.

Remark 1.6. The restriction to the lattice $\mathbb{Z}^d$ is easily adapted to any other transitive lattice or to $\mathbb{R}^d$. (But to achieve this one must reevaluate the bounds on moments of the expected volume of the intersection of the BRW or the percolation cluster and the cube Q_r, such as (2.8) below.)

Remark 1.7. It is possible to extend the result of Theorem 1.2(a) to dimensions 1 and 2 for all $\alpha \in (0,2) \cup (2,\infty)$. As it stands, the proof uses the existence of the random walk Green's function, which is not defined for recurrent random walks, whence we require that $d > (2 \wedge \alpha)$. But it is not hard to see that with some minor modifications to the definition (2.4) below it is possible to extend these results to $d = 1$ and $d = 2$ by instead considering a branching random walk that is killed upon exiting a large cube Q_R with $R \gg r$.

Remark 1.8. The proof of the upper bound uses an induction argument that is inspired by [15], where Kozma and Nachmias prove that the one-arm exponent for high-dimensional finite-range percolation equals 2. In [15] the main difficulty of the proof comes from the complicated dependence structure of percolation.

BRW has straightforward dependencies, and as a result parts of the proof presented here are less involved for BRW. (Other parts are more involved because we consider unbounded one-step distributions, whereas [15] only considers bounded steps, which make the induction easier.)

The proof of Theorem 1.2(b) also uses a 'regularity analysis' that is similar to the one in [15]. Indeed, our Claim 5.2 and [15, Theorem 2] are basically the same statement. Our setting is different, but not so different that their proof fails completely. In fact, we only need to modify their proof. Considering that the proof of [15, Theorem 2] takes more than twenty pages, and that it 'almost' applies to our setting, we will only give an outline of the proof of Claim 5.2 (and a detailed description of the modifications to the proof of [15, Theorem 2]) in the appendix. This appendix should be read as a companion to [15]. That is, we do not present a completely self-contained proof: the reader will have to refer to [15] for some parts of the proof of the upper bound in Theorem 1.2(b).

Remark 1.9. Chen and Sakai [3] prove that the two-point function bound (1.5) holds for models that satisfy Definition 1.1(c) if $\alpha \in (0,2) \cup (2,\infty)$ and $d > 3(2 \wedge \alpha)$, if Λ is sufficiently large, and if the following technical bound on the discrete second derivative of the n-fold discrete convolution of D is satisfied:

$$|D^{*(n)}(0,x) - \tfrac{1}{2}(D^{*(n)}(x,y) + D^{*(n)}(x,-y))| \leq CnL^{(2\wedge\alpha)}\frac{(\|y\|_2 \wedge \Lambda)^2}{(\|x\|_2 \wedge \Lambda)^{d+(2\wedge\alpha)+2}}, \qquad (1.7)$$

for some $C > 0$ and for all $x \in \mathbb{Z}^d$ and all $y \in \mathbb{Z}^d$ such that $\|y\|_2 \leq \frac{1}{3}\|x\|_2$. (We write $D^{*(n)}(x,y) := \sum_{x_1,\dots,x_{n-1} \in \mathbb{Z}^d} \prod_{i=0}^{n-1} D(x_i, x_{i+1})$ with $x_0 = x$ and $x_n = y$.) In [3] it is further shown that there exists a class of models that satisfy this bound, although it should be noted that it is not known whether (1.7) holds for the canonical example (1.4). Nevertheless, it is widely conjectured that the two-point function asymptotics (1.5) hold for any model that satisfies Definition 1.1(c). Supporting evidence for this conjecture is that the analogous bound holds without further assumptions for BRW when $d > (2 \wedge \alpha)$, see (2.7) below, and for percolation models with one-step distributions as in Definition 1.1(a) and (b) when $d > 6$ and Λ is sufficiently large.

1.1 Notation and definitions

Parts of the proofs of Theorem 1.2(a) and (b) are presented separately, but they are similar and use several analogous quantities. To make these similarities visible we will denote analogous quantities for both models with the same symbols. That is, the meaning of symbols will sometimes depend on the model under consideration.

General notation

For convenience of notation we will often write

$$\gamma(r) := \mathbb{P}(0 \longrightarrow Q_r^c) \qquad \text{for BRW,}$$

and similarly

$$\gamma(r) := \mathbb{P}_{p_c}(0 \longleftrightarrow Q_r^c) \qquad \text{for LRP.}$$

We will often state bounds in terms of the generic multiplicative constants C and c. We always assume that $0 < c, C < \infty$ but otherwise the value of these constants may change from line to line, even within equations.

When dealing with numbers, we will write a rather than $\lfloor a \rfloor$ or $\lceil a \rceil$ when it is clear that the statement only holds for integers (e.g. in summation limits).

Notation for BRW

We write $|T|$ for the number of vertices of the tree T, and in general, we write $|S|$ or $\#S$ for the cardinality of a set S. We say that $v \in T$ is an *ancestor* of $w \in T$ if the unique path from the root of T to w passes through the vertex v. We say that w is a *child* of v if v is the direct ancestor of w, and write (v, w). We write $|v| = n$ if v is a vertex at graph distance n from the root of T. We will abuse notation a bit and let (T, ϕ) also denote the *random* realization of the BRW. We will also not distinguish the different probability measures typographically, and just write $\mathbb{P}$ for whichever measure is appropriate in the given context.

Notation for LRP

We abbreviate $\mathcal{C}(0)$ by $\mathcal{C}$. Sometimes we abuse notation by letting $\mathcal{C}$ denote the induced subgraph of the vertex set $\mathcal{C}$ in the percolation configuration. We write $\{x, y\} \in \mathcal{C}$ for the event that $\{x, y\}$ is an open edge with both end-points in $\mathcal{C}$. Given a configuration ω and a vertex set A, we define the *restriction of ω to A* as

$$\omega_A(\{x,y\}) := \begin{cases} \omega(\{x,y\}) & \text{if } x, y \in A, \\ 0 & \text{otherwise,} \end{cases}$$

for all $\{x, y\} \in \mathbb{Z}^d \times \mathbb{Z}^d$. That is, ω_A is the configuration ω where all edges that do not have both end-points in A have been closed. Given an event E we then define the event

$\{E \text{ on } A\}$ as the set of all configurations ω such that $\omega_A \in E$. For connection events we write $\xleftrightarrow{A}$ to denote that the connection occurs on A. We similarly define $\{E \text{ off } A\} := \{E \text{ on } A^c\}$ and $\{E \text{ only on } A\} := E \setminus \{E \text{ off } A\}$.

We will frequently use the *van den Berg-Kesten inequality* (BK-inequality) [16], which states that two increasing events that occur on edge-disjoint sets are negatively correlated under the percolation measure. More precisely, we write $A \circ B$ if for each $\omega \in A \cap B$ there exists a set $K(\omega) \subset \mathbb{Z}^d \times \mathbb{Z}^d$ such that $\omega_K \in A$ and $\omega_{K^c} \in B$. If moreover A and B are increasing (i.e., $\mathbb{P}_p(A) \leq \mathbb{P}_q(A)$ if $p < q$), then the BK-inequality is that $\mathbb{P}_p(A \circ B) \leq \mathbb{P}_p(A)\mathbb{P}_p(B)$.

1.2 The structure of this paper

In Section 2 we prove the lower bound in Theorem 1.2 for BRW. Then, in Section 3 we state Proposition 3.1, and use it to prove Theorem 1.2 for both BRW and LRP. In Section 4 we prove Proposition 3.1 for BRW, and in Section 5 we prove Proposition 3.1 for LRP subject to Claim 5.2. Finally, in Appendix A we give an outline of the proof of Claim 5.2, closely following the proof of [15, Theorem 2].

2 Proof of the lower bound in Theorem 1.2(a)

In this section we prove the lower bound on the one-arm probability for critical BRW. We prove the lower bound in two steps: first we show that the bound $\gamma(r) \geq cr^{-\alpha/2}$ using a truncation argument, and then we show that $\gamma(r) \geq cr^{-2}$ using the second-moment method.

We start with the former. If a particle of the BRW at any time takes a step of length greater than $2r + 1$, then the BRW must hit Q_r^c. Moreover, we can restrict ourselves to BRWs that have many children, so for any n we bound

$$\gamma(r) \geq \mathbb{P}(|T| \geq n + 1, \exists (v, w) \in T \text{ such that } \|\phi(w) - \phi(v)\|_\infty > 2r + 1).$$

Let Y be a $(0, \infty)$-valued random variable whose law is the supremum norm of a single step according to the one-step distribution D, i.e., Y is characterized by

$$\mathbb{P}(Y \geq t) = \sum_{x \in Q_t^c} D(0, x). \tag{2.1}$$

It follows by independence of the steps that conditionally on the event $\{|T| \geq n + 1\}$, we have the bound

$$\gamma(r) \geq \mathbb{P}(|T| \geq n + 1)(1 - \mathbb{P}(Y \leq 2r + 1)^n),$$

since the probability of the event that there is a large step is bounded from below by one minus the probability that the first n steps of (T, ϕ) are not large. Since $1 - x \leq e^{-x}$, we can further bound

$$(1 - (1 - \mathbb{P}(Y > 2r + 1))^n) \geq 1 - \exp(-n\,\mathbb{P}(Y > 2r + 1)).$$

It follows from (1.2) and (1.3) that there exists a constant $\kappa > 0$ such that for all r,

$$\mathbb{P}(Y > 2r + 1) = \sum_{x \in Q_{2r+1}^c} D(0, x) \geq \kappa r^{-\alpha}.$$

Now since $1 - e^{-x} \geq \frac{1}{2}x$ when $x < 1$,

$$1 - \exp(-n\,\mathbb{P}(Y > 2r + 1)) \geq \frac{\kappa n}{2r^\alpha}, \qquad \text{for all } n < \frac{2r^\alpha}{\kappa}.$$

We thus have the bound

$$\gamma(r) \geq \frac{\kappa n}{2r^\alpha}\mathbb{P}(|T| \geq n + 1).$$

To bound the right-hand side we use a special instance of the *Hitting Time Theorem* [6]: Let (Y_i) be a sequence of i.i.d. random variables with $\mathbb{P}(Y_i = m - 1) = p_m$ (where $\{p_m\}$ is the offspring distribution of T), and let $S_n = Y_1 + \cdots + Y_n$. Then

$$\mathbb{P}(|T| = n) = \frac{1}{n} \mathbb{P}(S_n = -1).$$

Since S_n is a simple random walk and the Y_i have mean one and variance σ_p, it easily follows from the *Local Central Limit Theorem* (see e.g. [4, Theorem 3.5.2]) that

$$\mathbb{P}(|T| \geq n) \asymp \frac{1}{\sqrt{n}}. \tag{2.2}$$

Setting $n = 2r^\alpha/\kappa - 1$ we thus obtain the bound

$$\gamma(r) \geq \mathbb{P}(|T| \geq 2r^\alpha/\kappa)(1 - o(1)) \geq cr^{-\alpha/2}. \tag{2.3}$$

Now we prove the bound $\gamma(r) \geq cr^{-2}$. We use the second moment method. Let $\mathcal{V}(S)$ denote the set of all particles of the BRW in the set $S \subseteq \mathbb{Z}^d$, i.e.,

$$\mathcal{V}(S) := \{v \in T : \phi(v) \in S\}. \tag{2.4}$$

Let k be some sufficiently large number to be determined later, and define the random variable

$$V := |\mathcal{V}(Q_{kr} \setminus Q_r)| = |\mathcal{V}(Q_{kr})| - |\mathcal{V}(Q_r)|.$$

Clearly, V is supported on $\{0, 1, 2, \ldots\}$ and $\{V > 0\} \subseteq \{0 \longrightarrow Q_r^c\}$, so that by the Paley-Zygmund inequality,

$$\gamma(r) \geq \mathbb{P}(V > 0) \geq \frac{\mathbb{E}[V]^2}{\mathbb{E}[V^2]}. \tag{2.5}$$

Now define the *BRW two-point function*

$$\tau_n(x) := \mathbb{E}\Big[\sum_{v \in T : |v| = n} \mathbb{1}_{\{\phi(v) = x\}} \Big].$$

It is an easy corollary of [17, Proposition 2.3] that $\tau_n(x) = D^{*(n)}(0, x)$. Therefore,

$$\mathbb{E}[|\mathcal{V}(Q_r)|] = \sum_{x \in Q_r} \sum_{n=0}^{\infty} \tau_n(x) = \sum_{x \in Q_r} \sum_{n=0}^{\infty} D^{*(n)}(0, x) =: \sum_{x \in Q_r} G(x), \tag{2.6}$$

where $G(x)$ is the *RW Green's function* at its critical point, with step distribution D. For the one-step distributions D considered in this paper it is a well-established fact (see e.g. [3, 7, 9]) that when $\alpha \in (0, 2) \cup (2, \infty)$ and $d > (2 \wedge \alpha)$,

$$G(x) \asymp \|x\|_2^{(2 \wedge \alpha) - d}, \tag{2.7}$$

so it follows that

$$\mathbb{E}[|\mathcal{V}(Q_r)|] \asymp \sum_{x \in Q_r} \|x\|_2^{(2 \wedge \alpha) - d} \asymp r^{(2 \wedge \alpha)}, \tag{2.8}$$

and therefore, if we choose k sufficiently large,

$$\mathbb{E}[V] = \mathbb{E}[|\mathcal{V}(Q_{kr})|] - \mathbb{E}[|\mathcal{V}(Q_r)|] \geq c(kr)^{(2 \wedge \alpha)} - Cr^{(2 \wedge \alpha)} = cr^{(2 \wedge \alpha)}.$$

To bound $\mathbb{E}[V^2]$ we define the *three-point function*

$$\tau_{n,m}(x, y) := \mathbb{E}\Big[\sum_{v \in T : |v| = n} \sum_{w \in T : |w| = m} \mathbb{1}_{\{\phi(v) = x\}} \mathbb{1}_{\{\phi(w) = y\}} \Big].$$

It similarly follows from [17, Proposition 2.3] that

$$\tau_{n,m}(x,y) = \sigma_p^2 \sum_{k=0}^{(n\wedge m)} \sum_{z\in\mathbb{Z}^d} D^{*(k)}(z)D^{*(n-k)}(x-z)D^{*(m-k)}(y-z).$$

Again using (2.7) to perform this sum we get when $\alpha \in (0,2)\cup(2,\infty)$ and $d > (2\wedge\alpha)$ that

$$\begin{aligned}
\mathbb{E}[|\mathcal{V}(Q_{kr})|^2] &= \sum_{x,y\in Q_{kr}} \sum_{n,m=0}^{\infty} \tau_{n,m}(x,y) \\
&= \sigma_p^2 \sum_{x,y\in Q_{kr}} \sum_{z\in\mathbb{Z}^d} G(z)G(x-z)G(y-z) \\
&\asymp \sum_{x,y\in Q_{kr}} \sum_{z\in\mathbb{Z}^d} \|z\|_2^{(2\wedge\alpha)-d}\|x-z\|_2^{(2\wedge\alpha)-d}\|y-z\|_2^{(2\wedge\alpha)-d}.
\end{aligned}$$

We evaluate the sum in two parts. If $\|z\|_2 \leq 10dr$, then we first sum over x and y and then z to obtain

$$\sum_{\|z\|_2\leq 10dr} \sum_{x,y\in Q_{kr}} \|z\|_2^{(2\wedge\alpha)-d}\|x-z\|_2^{(2\wedge\alpha)-d}\|y-z\|_2^{(2\wedge\alpha)-d} \asymp r^{3(2\wedge\alpha)}.$$

If, on the other hand, $\|z\|_2 > 10dr$, then, since $\|x\|_2 < \sqrt{d}r$ for all $d \geq 1$, it follows that $\frac{1}{2}\|z\|_2 < \|x-z\|_2 < 2\|z\|_2$, and similarly, $\frac{1}{2}\|z\|_2 < \|y-z\|_2 < 2\|z\|_2$. Therefore,

$$\sum_{x,y\in Q_{kr}} \sum_{\|z\|_2>10dr} \|z\|_2^{(2\wedge\alpha)-d}\|x-z\|_2^{(2\wedge\alpha)-d}\|y-z\|_2^{(2\wedge\alpha)-d} \asymp r^{2d} \sum_{\|z\|_2>10dr} \|z\|^{3(2\wedge\alpha)-3d} \asymp r^{3(2\wedge\alpha)}.$$

As a result

$$\mathbb{E}[|\mathcal{V}(Q_{kr})|^2] \asymp r^{3(2\wedge\alpha)}, \tag{2.9}$$

and it follows that $\mathbb{E}[V^2] \leq Cr^{3(2\wedge\alpha)}$ as well.

Substituting the bounds on $\mathbb{E}[V]$ and $\mathbb{E}[V^2]$ into (2.5) we obtain

$$\gamma(r) \geq \frac{cr^{2(2\wedge\alpha)}}{Cr^{3(2\wedge\alpha)}} \geq \frac{c}{r^2}. \tag{2.10}$$

Taking the maximum among the bounds (2.3) and (2.10) completes the proof of the lower bound of Theorem 1.2(a). □

3 Proof of the upper bounds in Theorem 1.2

From here on we write ρ for the long-range one-arm exponent, i.e., $\rho := \frac{1}{2}(4\wedge\alpha)$.

For both BRW and LRP we will prove the upper bound with an induction argument. The following proposition provides the crucial bound for this induction:

Proposition 3.1. Let $\alpha \in (0,2)\cup(2,\infty)$ and choose β such that

$$\frac{11}{10(4\wedge\alpha)} < \beta < \frac{(4\wedge\alpha)+1}{(4\wedge\alpha)^2}. \tag{3.1}$$

For BRW with $d > (2\wedge\alpha)$ that satisfies Definition 1.1, and for LRP with $d > 3(2\wedge\alpha)$ that satisfies Definition 1.1(c) and (1.5), there exists $C_1, c_2 > 0$ such that for all $\lambda \in (0,1]$ there exists $\varepsilon_0(\lambda)$ such that for all $\varepsilon \in (0,\varepsilon_0)$,

$$\gamma(r(1+\lambda)) \leq \frac{C_1}{\sqrt{\varepsilon}r^\rho} + \varepsilon^{\beta\rho}r^\rho\gamma(r)\gamma\left(\tfrac{1}{2}\lambda r\right) + (1-c_2)\gamma(r). \tag{3.2}$$

Remark 3.2. (a) The constraints on the exponent β are of a technical nature. They summarize a number of constraints that are needed in the proofs of Theorem 1.2 and Proposition 3.1. In particular, these constraints guarantee that β satisfies $-(\beta \wedge 1/(2\rho)) < 1 - 2\beta\rho < 0$ and $2\beta\rho > \frac{11}{10}$, which are needed in the proofs of both BRW and LRP, and that β satisfies the constraint $-1 < \beta\left(\frac{\alpha}{2\alpha+2} - \rho\right) < 0$, which is needed in the proof of LRP.

(b) The choice of $\varepsilon_0(\lambda)$ above should be so that $\varepsilon_0 < (\frac{\lambda}{4})^{2\rho}$ and small enough so that (4.15) below holds for all $\varepsilon < \varepsilon_0$ for BRW, and small enough so that (5.7) and (5.15) hold for all $\varepsilon < \varepsilon_0$ for LRP.

We will prove this proposition for BRW in Section 4, and for LRP in Section 5. Before we do this, let us give the proof of the upper bounds in Theorem 1.2.

Proof of the upper bounds in Theorem 1.2 subject to Proposition 3.1. The proof is by induction. For ease of notation, we will prove the claim for $\gamma(r(1+\lambda))$. We assume that there exists a constant M (to be determined below) such that

$$\gamma(s) \leq \frac{M}{s^\rho} \qquad \forall s \leq r. \tag{3.3}$$

Choose λ sufficiently small so that

$$(1+\lambda)^\rho \leq 2 \qquad \text{and} \qquad (1-c_2)(1+\lambda)^\rho \leq (1 - \tfrac{1}{2}c_2). \tag{3.4}$$

Now choose $\varepsilon_0(\lambda)$ as in Proposition 3.1. Fix M so large, and $\eta \in (0,1)$ so that

$$M^{-2\eta} \leq \varepsilon_0(\lambda) \tag{3.5}$$

and

$$2C_1 M^{\eta-1} + 2\left(\frac{2}{\lambda}\right)^\rho M^{1-2\eta\beta\rho} \leq \tfrac{1}{2}c_2. \tag{3.6}$$

Note that such an M only exists when β satisfies $2\eta\beta\rho > 1$. Recall the constraints on β in (3.1) and observe that $2\beta\rho \geq \frac{11}{10}$ for all $\alpha > 0$, so there always exists an $\eta \in (0,1)$ such that this constraint is true.

For $s = (r(1+\lambda))^\rho \leq M$ the assumption (3.3) is true vacuously. We advance the induction by applying Proposition 3.1 and (3.3)—(3.6) with $\varepsilon = M^{-2\eta}$:

$$\gamma(r(1+\lambda)) \leq \frac{C_1}{\sqrt{\varepsilon}r^\rho} + \varepsilon^{\beta\rho}r^\rho\gamma(r)\gamma\left(\tfrac{1}{2}\lambda r\right) + (1-c_2)\gamma(r)$$

$$\leq C_1 M^\eta \frac{1}{r^\rho} + \left(\frac{2}{\lambda}\right)^\rho M^{2-2\eta\beta\rho}\frac{1}{r^\rho} + (1-c_2)M\frac{1}{r^\rho}$$

$$\leq \frac{M}{(r(1+\lambda))^\rho}\left(2C_1 M^{\eta-1} + 2\left(\frac{2}{\lambda}\right)^\rho M^{1-2\eta\beta\rho} + 1 - \tfrac{1}{2}c_2\right)$$

$$\leq \frac{M}{(r(1+\lambda))^\rho}.$$

This completes the proof of the upper bounds of Theorem 1.2. $\qquad\square$

4   The proof of Proposition 3.1 for BRW

Since the tree is critical it is unlikely that $|T|$ is very large, but if $|T|$ is very large, then the BRW is likely to hit Q_r^c, so we choose $\varepsilon > 0$ sufficiently small (to be determined later) and bound

$$\gamma(r(1+\lambda)) \leq \mathbb{P}(0 \longrightarrow Q_{r(1+\lambda)}^c, |T| \leq \varepsilon r^{2\rho}) + \mathbb{P}(|T| > \varepsilon r^{2\rho}). \tag{4.1}$$

For the second term in (4.1) we use (2.2), from which it follows that

$$\mathbb{P}(|T| \geq \varepsilon r^{2\rho}) \leq \frac{C}{\sqrt{\varepsilon} r^{\rho}}, \tag{4.2}$$

so this term contributes to the first term in (3.2).

To analyze the first term of (4.1) we split the event according to whether (T, ϕ) contains a step longer than δr, with $\delta := \varepsilon^{1/(2\rho)}$. That is, we define the event

$$\mathcal{E}_{\delta r} := \{\exists (i, j) \in T \text{ such that } \|\phi(j) - \phi(i)\|_{\infty} \geq \delta r\}.$$

If $|T| \leq \varepsilon r^{2\rho}$, then it is not very likely that the BRW ever makes a jump longer than δr, but if it does, then it has a good chance of hitting Q_r^c, so we bound

$$\mathbb{P}(0 \longrightarrow Q_{r(1+\lambda)}^c, |T| \leq \varepsilon r^{2\rho}) \leq \mathbb{P}(|T| \leq \varepsilon r^{2\rho}, \mathcal{E}_{\delta r}) + \mathbb{P}(0 \longrightarrow Q_{r(1+\lambda)}^c, |T| \leq \varepsilon r^{2\rho}, \mathcal{E}_{\delta r}^c). \tag{4.3}$$

We will first bound the first term. Let Y again denote a random variable whose law is characterized by (2.1). There exists a constant $\zeta > 0$ such that for all r,

$$\mathbb{P}(Y > r) = \sum_{x \in Q_r^c} D(x) \leq \zeta r^{-\alpha}. \tag{4.4}$$

Conditioning on $|T| = n$, the event $\mathcal{E}_{\delta r}$ is just the event that at least one of n i.i.d. random variables with the law of Y exceeds δr, so that by the union bound and (2.2),

$$
\begin{aligned}
\mathbb{P}(|T| \leq \varepsilon r^{2\rho}, \mathcal{E}_{\delta r}) &= \sum_{n=1}^{\varepsilon r^{2\rho}} \mathbb{P}(\mathcal{E}_{\delta r} \mid |T| = n) \mathbb{P}(|T| = n) \leq \mathbb{P}(Y > \delta r) \sum_{n=1}^{\varepsilon r^{2\rho}} n \mathbb{P}(|T| = n) \\
&\leq \frac{\zeta}{(\delta r)^{\alpha}} \sum_{n=1}^{\varepsilon r^{2\rho}} \mathbb{P}(|T| \geq n) \leq \frac{C}{(\delta r)^{2\rho}} \sum_{n=1}^{\varepsilon r^{2\rho}} \frac{1}{\sqrt{n}} \leq \frac{C\sqrt{\varepsilon}}{\delta^{2\rho} r^{\rho}} = \frac{C}{\sqrt{\varepsilon} r^{\rho}}.
\end{aligned} \tag{4.5}
$$

(Here we used for the first inequality that

$$\sum_{n=1}^{N} \mathbb{P}(|T| \geq n) = \sum_{n=1}^{N} \sum_{i=n}^{\infty} \mathbb{P}(|T| = i) \geq \sum_{n=1}^{N} n \mathbb{P}(|T| = n),$$

and for the last equality we used that $\delta = \varepsilon^{1/(2\rho)}$.) The term (4.5) contributes to the first term in (3.2).

Now we bound the second term in (4.3). Fix $L = \varepsilon^{\beta} r$. For all integer j we define the shell of outer radius j and thickness δr as

$$\partial Q_{j, \delta r} := \{x \in \mathbb{Z}^d : j - \delta r < \|x\|_{\infty} \leq j\} = Q_j \setminus Q_{j - \delta r},$$

and we define the set

$$\mathcal{X}_j := \{v \in T : \phi(v) \in \partial Q_{j, \delta r} \text{ and all ancestors } w \text{ of } v \text{ satisfy } \phi(w) \in Q_{j - \delta r}\} \tag{4.6}$$

and the random variable $X_j := |\mathcal{X}_j|$. A moment's reflection will convince the reader that if the event $\{0 \longrightarrow Q_{r(1+\lambda)}^c\} \cap \{|T| \leq \varepsilon r^{2\rho}\} \cap \{\mathcal{E}_{\delta r}^c\}$ occurs, and $\delta < \frac{1}{4}\lambda$, then either

$$\mathcal{B}_1 := \{\exists j \in [r(1 + \tfrac{1}{4}\lambda), r(1 + \tfrac{1}{2}\lambda)] \text{ s.t. } 0 < X_j < L^{\rho}\} \cap \{0 \longrightarrow Q_{r(1+\lambda)}^c\} \cap \mathcal{E}_{\delta r}^c$$

occurs, or

$$\mathcal{B}_2 := \{X_j \geq L^{\rho} \, \forall j \in [r(1 + \tfrac{1}{4}\lambda), r(1 + \tfrac{1}{2}\lambda)]\} \cap \{|T| \leq \varepsilon r^{2\rho}\} \cap \mathcal{E}_{\delta r}^c$$

occurs. Note in particular that on the event $\{0 \longrightarrow Q^c_{r(1+\lambda)}\} \cap \mathcal{E}^c_{\delta r}$ it cannot happen that $X_j = 0$ for some (or all) $j \in [r(1 + \frac{1}{4}\lambda), r(1 + \frac{1}{2}\lambda)]$ because $\{0 \longrightarrow Q^c_{r(1+\lambda)}\}$ ensures that there exists a connection to $Q^c_{r(1+\lambda)}$, while $\mathcal{E}^c_{\delta r}$ ensures that this connection cannot skip more than δr levels at a time, and if there is a particle at level j, then $X_k \neq 0$ for all $k \in \{j + 1, \dots, j + \delta r\}$.

We bound $\mathbb{P}(\mathcal{B}_1)$ with a regeneration argument: let j_0 be the first j in $[r(1 + \frac{1}{4}\lambda), r(1 + \frac{1}{2}\lambda)]$ such that $0 < X_j \leq L^\rho$. Given (T, ϕ), define the pair $(\mathcal{T}_{j_0}, \phi)$ as the BRW that is indexed by the tree $\mathcal{T}_{j_0}$ that is induced by the set of vertices

$$\{v \in T : 0 \longrightarrow \phi(v) \text{ and all ancestors } w \text{ of } v \text{ satisfy } \phi(w) \in Q_{j_0 - \delta r}\}.$$

Observe that on $\mathcal{E}^c_{\delta r}$ this vertex set is equivalent to the set

$$\mathcal{X}_{j_0} \cup \{w \in T : w \text{ is an ancestor of some } v \in \mathcal{X}_{j_0}\},$$

and that $\mathcal{X}_{j_0}$ is the set of leaves of $\mathcal{T}_{j_0}$ in $\partial Q_{j_0, \delta r}$.

We condition on the realizations of $(\mathcal{T}_{j_0}, \phi)$:

$$\mathbb{P}(\mathcal{B}_1) = \sum_{(S, \psi) \text{ admissible}} \mathbb{P}\left(\{0 \longrightarrow Q^c_{r(1+\lambda)}\} \cap \mathcal{E}^c_{\delta r} \mid (\mathcal{T}_{j_0}, \phi) = (S, \psi)\right) \mathbb{P}((\mathcal{T}_{j_0}, \phi) = (S, \psi)),$$

where '(S, ψ) admissible' means that $\mathbb{P}((\mathcal{T}_{j_0}, \phi) = (S, \psi)) > 0$. Note that '$(S, \psi)$ admissible' implies that $\{0 \longrightarrow Q^c_r\}$ occurs for (S, ψ), since otherwise j_0 would not be well-defined, and thus (S, ψ) would not be admissible.

If $\{0 \longrightarrow Q^c_{r(1+\lambda)}\} \cap \mathcal{E}^c_{\delta r}$ occurs, then there must exist some $x \in \mathcal{X}_{j_0}$ such that $\{x \longrightarrow Q^c_{r(1+\lambda)}\}$ occurs. The BRW coming out of any $x \in \mathcal{X}_{j_0}$ is an independent BRW started at x, since the condition $\{\mathcal{T}_{j_0} = S\}$ tells us nothing about the BRWs of the descendants of S besides their initial position. Moreover, since (S, ψ) is admissible, it must be the case that ∂S, the set of leaves of S corresponding to $\mathcal{X}_{j_0}$, satisfies $|\partial S| \leq L^\rho$. Moreover, $\|\psi(v)\|_\infty \leq r(1 + \frac{1}{2}\lambda)$ for all $v \in \partial S$ by our choice of j_0. Therefore it follows that

$$\mathbb{P}\left(\{0 \longrightarrow Q^c_{r(1+\lambda)}\} \cap \mathcal{E}^c_{\delta r} \mid (\mathcal{T}_{j_0}, \phi) = (S, \psi)\right) \leq \sum_{v \in \partial S} \mathbb{P}(\psi(v) \longrightarrow Q^c_{r(1+\lambda)})$$
$$\leq L^\rho \gamma\left(\tfrac{1}{2}\lambda r\right),$$

where the second inequality also uses translation invariance and monotonicity of the one-arm event, i.e., $\mathbb{P}(x \longrightarrow Q^c_{a+b}) \leq \mathbb{P}(0 \longrightarrow Q^c_b)$ for all $x \in Q_a$. As a result,

$$\mathbb{P}(\mathcal{B}_1) \leq L^\rho \gamma\left(\tfrac{1}{2}\lambda r\right) \sum_{S \text{ admissible}} \mathbb{P}(\mathcal{T}_{j_0} = S) \leq L^\rho \gamma\left(\tfrac{1}{2}\lambda r\right) \gamma(r), \tag{4.7}$$

where the second inequality follows since 'S admissible' implies that $\{0 \longrightarrow Q^c_r\}$. Because $L = \varepsilon^\beta r$, this gives the second term in (3.2).

Finally we bound $\mathbb{P}(\mathcal{B}_2)$. Let

$$N := \tfrac{1}{4}\lambda(\varepsilon^\beta + \delta)^{-1} = \tfrac{1}{4}\lambda(\varepsilon^\beta + \varepsilon^{1/(2\rho)})^{-1}.$$

For every integer $1 \leq i \leq N$ let

$$j_i = r + \tfrac{1}{4}\lambda r + i(L + \delta r) \in [r(1 + \tfrac{1}{4}\lambda), r(1 + \tfrac{1}{2}\lambda)].$$

We use these numbers below to (roughly) partition $Q_{r(1+\lambda/2)} \setminus Q_{r(1+\lambda/4)}$ into N non-overlapping annuli of width L.
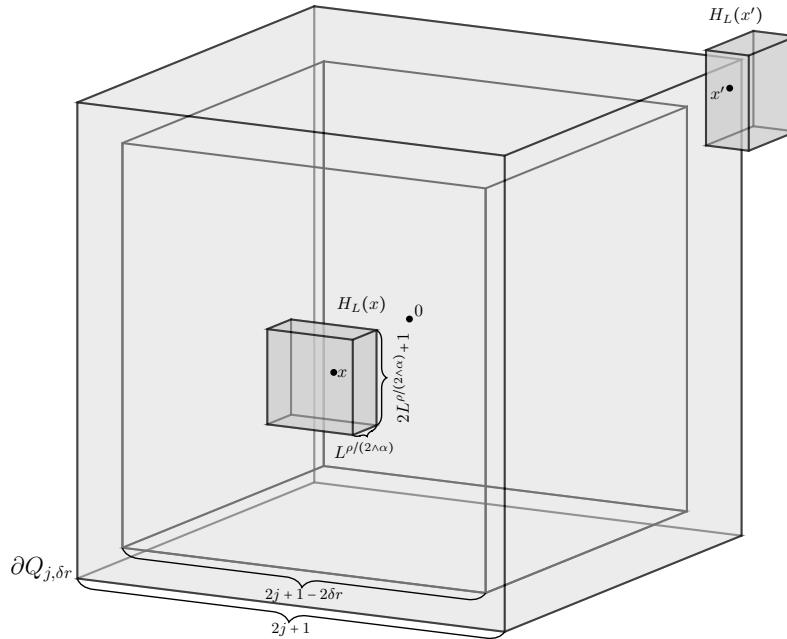
Figure 1: The annulus $\partial Q_{j,\delta r}$ with half-cubes $H_L(x)$ and $H_L(x')$ for $x, x' \in \partial Q_{j,\delta r}$.

Let $\iota_x \in \{1, \ldots d\}$ be the index of the first component of $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$ that satisfies $\|x\|_\infty = |x_{\iota_x}|$. Define the *outward-facing half-cube*

$$H_L(x) := \{y \in \mathbb{Z}^d : \|x - y\|_\infty \leq L^{\rho/(2 \wedge \alpha)} \text{ and } |y_{\iota_x}| \geq |x_{\iota_x}| \text{ and } \operatorname{sign}(y_{\iota_x}) = \operatorname{sign}(x_{\iota_x})\}. \tag{4.8}$$

(See Figure 1.)

Recall the definition of $\mathcal{X}_j$, (4.6), and define the random variables

$$A_j := \#\{w \in T : \exists v \in \mathcal{X}_j \text{ s.t. } v \text{ is an ancestor of } w \text{ and } \phi(w) \in H_L(\phi(v))\}$$

and

$$I := \#\{i : X_{j_i} \geq L^\rho \text{ and } A_{j_i} < aL^{2\rho}\}, \tag{4.9}$$

where a is a constant that will be determined later. Observe that on $\mathcal{E}_{\delta r}^c$, by the definition of j_i, A_j, and H_L, the random variables A_{j_i} and A_{j_k} count disjoint sets of particles when $i \neq k$. So if $|T| \leq \varepsilon r^{2\rho}$, then it follows deterministically that

$$J := \#\{i : A_{j_i} \geq aL^{2\rho}\} < \frac{\varepsilon r^{2\rho}}{aL^{2\rho}} = \frac{\varepsilon r^{2\rho}}{a(\varepsilon^\beta r)^{2\rho}} = a^{-1}\varepsilon^{1-2\beta\rho}. \tag{4.10}$$

The event $\mathcal{B}_2$ implies that all $X_{j_i} \geq L^\rho$ and that $|T| \leq \varepsilon r^{2\rho}$. Since the total number of j_i's is N, it follows that

$$\mathbb{P}(\mathcal{B}_2) \leq \mathbb{P}(I \geq N - J). \tag{4.11}$$

For (4.11) to be non-trivial (when ε is sufficiently small) we need that $N \gg J \gg 1$, which implies that

$$-(\beta \wedge 1/(2\rho)) < 1 - 2\beta\rho < 0. \tag{4.12}$$

The constraints on β in (3.1) ensure that this is the case.

We use Markov's inequality to bound the right-hand side of (4.11). We write

$$\mathbb{E}[I] = \sum_{i=1}^N \mathbb{P}(X_{j_i} \geq L^\rho \text{ and } A_{j_i} < aL^{2\rho}). \tag{4.13}$$

We use the following claim to complete the proof:

Claim 4.1. *For BRW with $\alpha \in (0,2) \cup (2,\infty)$ and $d > (2 \wedge \alpha)$ that satisfies Definition 1.1, there exists $0 < a < \frac{1}{2}$ such that*

$$\mathbb{P}(X_j \geq L^\rho \text{ and } A_j < aL^{2\rho}) \leq (1-a)\gamma(r) \tag{4.14}$$

for all $\delta > 0$ and all integer $j > r + \delta r$.

Before we prove this claim, let us see how it completes the proof. By Markov's inequality, (4.13) and (4.14),

$$
\begin{aligned}
\mathbb{P}(\mathcal{B}_2) \leq \mathbb{P}(I \geq N - J) &\leq \frac{(1-a)N}{N-J}\gamma(r) \\
&\leq \frac{1-a}{1 - \frac{4}{a\lambda}\varepsilon^{1-2\beta\rho}(\varepsilon^\beta + \varepsilon^{1/(2\rho)})}\gamma(r) \leq (1-c_3)\gamma(r)
\end{aligned}
\tag{4.15}
$$

where the last inequality holds if we can choose ε small enough so that the denominator in the third inequality is arbitrarily close to one. That this is possible follows from (4.12).

Gathering together the bounds (4.1), (4.2), (4.3), (4.5), (4.7), and (4.15) gives the bound in the proposition's hypothesis. This completes the proof of Proposition 3.1 for BRW. □

Proof of Claim 4.1. We write

$$\mathbb{P}(X_j \geq L^\rho \text{ and } A_j \leq aL^{2\rho}) = \sum_{n=L^\rho}^{\infty} \mathbb{P}(A_j \leq aL^{2\rho} \mid X_j = n)\mathbb{P}(X_j = n). \tag{4.16}$$

We again use the Paley-Zygmund inequality: if $\mathbb{E}[A_j \mid X_j = n] > aL^{2\rho}$, then

$$\mathbb{P}(A_j > aL^{2\rho} \mid X_j = n) \geq \frac{(\mathbb{E}[A_j \mid X_j = n] - aL^{2\rho})^2}{\mathbb{E}[A_j^2 \mid X_j = n]}. \tag{4.17}$$

Recall the definition of the half-cube H_L in (4.8). By independence and translation invariance of the BRW, the symmetries of D, and (2.8), there exists a constant $0 < b \leq 1$ such that

$$\mathbb{E}[A_j \mid X_j = n] = n\,\mathbb{E}[|\mathcal{V}(H_L(0))|] \geq \tfrac{1}{2}n\mathbb{E}[|\mathcal{V}(Q_{L^{\rho/(2\wedge\alpha)}})|] \geq bnL^\rho. \tag{4.18}$$

Again using translation invariance and independence, the bounds (2.8) and (2.9), and using in addition the fact that two vertices counted in A_j can either have the same parent or different parents in $\mathcal{X}_j$, we bound

$$
\begin{aligned}
\mathbb{E}[A_j^2 \mid X_j = n] &= n(n-1)\mathbb{E}[|\mathcal{V}(H_L(0))|]^2 + n\mathbb{E}[|\mathcal{V}(H_L(0))|^2] \\
&\leq n^2 \mathbb{E}[|\mathcal{V}(Q_{L^{\rho/(2\wedge\alpha)}})|]^2 + n\mathbb{E}[|\mathcal{V}(Q_{L^{\rho/(2\wedge\alpha)}})|^2] \\
&\leq Cn^2 L^{2\rho} + C'nL^{3\rho} \leq Kn^2 L^{2\rho},
\end{aligned}
\tag{4.19}
$$

where the last bound only holds when $n \geq L^\rho$. We can assume that $K \geq 1$ and choose $a = b/(K+1)$ so that the condition $\mathbb{E}[A_j \mid X_j = n] > aL^{2\rho}$ is satisfied. Substituting (4.18) and (4.19) into (4.17) we obtain that for all $n \geq L^\rho$,

$$\mathbb{P}(A_j \leq aL^{2\rho} \mid X_j = n) \leq 1 - \frac{(b-a)n^2 L^{2\rho}}{Kn^2 L^{2\rho}} = 1 - a.$$

Substituting this bound into (4.16) we get

$$\mathbb{P}(X_j \geq L^\rho \text{ and } A_j \leq aL^{2\rho}) \leq (1-a)\mathbb{P}(X_j \geq L^\rho) \leq (1-a)\gamma(r),$$

since $\{X_j \geq L^\rho\} \subseteq \{0 \longrightarrow Q_r^c\}$ whenever $j > r + \delta r$. This completes the proof of Claim 4.1. □

5 The proof of Proposition 3.1 for LRP

The proof of Proposition 3.1 for LRP that we give in this section is similar to the proof for BRW in the previous section, so we will supply fewer details in some of the calculations. It should however be noted that LRP is a more complicated model than BRW, because it has complicated dependencies. These dependencies will mostly manifest themselves in Claim 5.2 below.

To prove the one-arm exponent for high-dimensional finite-range percolation, [15] uses the Barsky-Aizenman bound [2, 8]

$$\mathbb{P}_{p_c}(|\mathcal{C}| \geq n) \asymp \frac{1}{\sqrt{n}} \tag{5.1}$$

to establish an ersatz upper bound $\gamma(r) \leq \mathbb{P}_{p_c}(|\mathcal{C}| \geq r/\Lambda) \leq C/\sqrt{r}$ (where Λ is the maximal edge length). Although (5.1) also holds for LRP [12, Theorem 1.3], the bound $\gamma(r) \leq C/\sqrt{r}$ does not, because there is no maximal edge length (and because it contradicts the lower bound in Theorem 1.2(b) when $\alpha < 1$).

In the proof of Proposition 3.1 we will need an ersatz upper bound on $\gamma(r)$ in several places. The following lemma establishes such a bound:

Lemma 5.1. *For LRP with $d > 3(2 \wedge \alpha)$ that satisfies Definition 1.1(c), let $\xi = \frac{\alpha}{2\alpha+2}$. There exists $C > 0$ such that*

$$\gamma(r) \leq \frac{C}{r^\xi}. \tag{5.2}$$

Proof. We use (5.1) to bound

$$\begin{aligned}
\gamma(r) &\leq \mathbb{P}_{p_c}(|\mathcal{C}| > r^{2\xi}) + \mathbb{P}_{p_c}(|\mathcal{C}| \leq r^{2\xi}, 0 \longleftrightarrow Q_r^c) \\
&\leq \frac{C}{r^\xi} + \mathbb{P}_{p_c}(|\mathcal{C}| \leq r^{2\xi}, \exists\{v,w\} \in \mathcal{C} \text{ such that } \|v - w\|_\infty > r^{1-2\xi}).
\end{aligned} \tag{5.3}$$

To bound the second term we define the *ℓ-truncated cluster* $\widetilde{\mathcal{C}}_\ell$ as the subgraph of $\mathcal{C}$ with vertex set

$$\{x \in \mathcal{C} \,:\, 0 \longleftrightarrow x \text{ with edges of } \| \cdot \|_\infty\text{-length } < \ell\},$$

and with the set of edges that this vertex set induces in $\mathcal{C}$ and that are shorter than ℓ as its edge set. That is, $\widetilde{\mathcal{C}}_\ell$ is the part of the cluster that can be reached from 0 by only using open edges of length less than ℓ.

We set $\ell = r^{1-2\xi}$ and condition on the size of $|\widetilde{\mathcal{C}}_{r^{1-2\xi}}|$. Under this conditioning, independent percolation models have a certain independence for short and long edges: knowing $\widetilde{\mathcal{C}}_{r^{1-2\xi}}$ tells us which vertices can be reached from the origin via 'short' edges (i.e., edges of length less than $r^{1-2\xi}$) but it does not provide us with any information about the 'long' edges attached to these vertices. Since moreover the status of edges is independent under $\mathbb{P}_{p_c}$, and $\mathbb{P}_{p_c}$ is translation invariant, we bound

$$\mathbb{P}_{p_c}(\exists\{v,w\} \in \mathcal{C} \text{ s.t. } \|v - w\|_\infty > r^{1-2\xi}, |\widetilde{\mathcal{C}}_{r^{1-2\xi}}| \leq r^{2\xi})$$

$$= \sum_{s=1}^{r^{2\xi}} \mathbb{P}_{p_c}(\exists\{v,w\} \in \mathcal{C} \text{ s.t. } \|v - w\|_\infty > r^{1-2\xi} \mid |\widetilde{\mathcal{C}}_{r^{1-2\xi}}| = s)\mathbb{P}_{p_c}(|\widetilde{\mathcal{C}}_{r^{1-2\xi}}| = s)$$

$$\leq \mathbb{P}_{p_c}(\exists x \in Q_{r^{1-2\xi}}^c \text{ s.t. } \{0,x\} \text{ is open}) \sum_{s=1}^{r^{2\xi}} s\,\mathbb{P}_{p_c}(|\widetilde{\mathcal{C}}_{r^{1-2\xi}}| = s).$$

$$\tag{5.4}$$

Using (4.4) we bound

$$\mathbb{P}_{p_c}(\exists x \in Q_{r^{1-2\xi}}^c \text{ s.t. } \{0,x\} \text{ is open}) \leq \sum_{x \in Q_{r^{1-2\xi}}^c} p_c D(0,x) \leq Cr^{-(1-2\xi)\alpha}. \tag{5.5}$$

Using (5.1) we bound

$$\sum_{s=1}^{r^{2\xi}} s\, \mathbb{P}_{p_c}(|\widetilde{\mathcal{C}}_{r^{1-2\xi}}| = s) \leq \sum_{s=1}^{r^{2\xi}} \mathbb{P}_{p_c}(|\widetilde{\mathcal{C}}_{r^{1-2\xi}}| \geq s) \leq \sum_{s=1}^{r^{2\xi}} \mathbb{P}_{p_c}(|\mathcal{C}| \geq s) \leq \sum_{s=1}^{r^{2\xi}} \frac{C}{\sqrt{s}} \leq Cr^{\xi}. \quad (5.6)$$

Combining (5.3), (5.4), (5.5), and (5.6), and choosing $\xi = \frac{\alpha}{2\alpha+2}$, we obtain

$$\gamma(r) \leq C\left(r^{-\xi} + r^{\xi - (1-2\xi)\alpha}\right) \leq Cr^{-\xi}. \quad \square$$

Proof of Proposition 3.1 for LRP subject to Claim 5.2 below. Recall that $\rho = \frac{1}{2}(4 \wedge \alpha)$. It follows from Lemma 5.1 that it suffices to show that Proposition 3.1 holds for $\varepsilon \leq r^{\xi-\rho} \ll 1$, since

$$\gamma(r(1+\lambda)) \leq \frac{C}{r^{\xi}} \leq \frac{C}{\sqrt{\varepsilon}r^{\rho}}. \quad (5.7)$$

From here on we will thus assume that $\varepsilon > r^{\xi-\rho}$ (this will be important for the validity of Claim 5.2 below). Using (5.1) again we bound

$$\gamma(r(1+\lambda)) \leq \frac{C}{\sqrt{\varepsilon}r^{\rho}} + \mathbb{P}_{p_c}(0 \longleftrightarrow Q^c_{r(1+\lambda)}, |\mathcal{C}| \leq \varepsilon r^{2\rho}), \quad (5.8)$$

so the first term here contributes to the first term in (3.2).

To analyze the second term of (5.8) we split the event according to whether $\mathcal{C}$ contains an open edge longer than δr, with $\delta := \varepsilon^{1/(2\rho)}$. That is, we define the event

$$\mathcal{E}_{\delta r} := \{\exists \{x,y\} \in \mathcal{C} \text{ such that } \|x-y\|_\infty \geq \delta r\}.$$

We bound

$$\mathbb{P}_{p_c}(0 \longleftrightarrow Q^c_{r(1+\lambda)}, |\mathcal{C}| \leq \varepsilon r^{2\rho}) \leq \mathbb{P}_{p_c}(|\mathcal{C}| \leq \varepsilon r^{2\rho}, \mathcal{E}_{\delta r}) + \mathbb{P}_{p_c}(0 \longleftrightarrow Q^c_{r(1+\lambda)}, |\mathcal{C}| \leq \varepsilon r^{2\rho}, \mathcal{E}^c_{\delta r}). \quad (5.9)$$

We start with the first term. Consider the δr-truncated cluster $\widetilde{\mathcal{C}}_{\delta r}$. As explained in the proof of Lemma 5.1 above, the long and short edges are conditionally independent on $\widetilde{\mathcal{C}}_{\delta r}$, i.e.,

$$\begin{aligned}
\mathbb{P}_{p_c}(|\mathcal{C}| \leq \varepsilon r^{2\rho}, \mathcal{E}_{\delta r}) &\leq \mathbb{P}_{p_c}(|\widetilde{\mathcal{C}}_{\delta r}| \leq \varepsilon r^{2\rho}, \mathcal{E}_{\delta r}) \\
&= \sum_{n=1}^{\varepsilon r^{2\rho}} \mathbb{P}_{p_c}(\mathcal{E}_{\delta r} \mid |\widetilde{\mathcal{C}}_{\delta r}| = n)\mathbb{P}_{p_c}(|\widetilde{\mathcal{C}}_{\delta r}| = n) \\
&\leq \mathbb{P}_{p_c}(\exists x \in Q^c_{\delta r} \text{ s.t. } \{0,x\} \text{ is open})\sum_{n=1}^{\varepsilon r^{2\rho}} n\, \mathbb{P}_{p_c}(|\widetilde{\mathcal{C}}_{\delta r}| = n) \\
&\leq \frac{C}{(\delta r)^{\alpha}}\sum_{n=1}^{\varepsilon r^{2\rho}} n\, \mathbb{P}_{p_c}(|\widetilde{\mathcal{C}}_{\delta r}| = n),
\end{aligned} \quad (5.10)$$

where the second inequality follows from translation invariance of the model and the union bound, and the third from (5.5). Applying the same argument as used in (5.6) to the right-hand side of (5.10) yields

$$\mathbb{P}_{p_c}(|\mathcal{C}| \leq \varepsilon r^{2\rho}, \mathcal{E}_{\delta r}) \leq \frac{C}{(\delta r)^{2\rho}}\sum_{n=1}^{\varepsilon r^{2\rho}} \frac{1}{\sqrt{n}} \leq \frac{C\sqrt{\varepsilon}}{\delta^{2\rho}r^{\rho}} = \frac{C}{\sqrt{\varepsilon}r^{\rho}}, \quad (5.11)$$

where for the equality we used that $\delta = \varepsilon^{1/(2\rho)}$. The term (5.11) contributes to the first term in (3.2).

Now we bound the second term in (5.9). Fix $L = \varepsilon^\beta r$. Observe that the constraints (3.1) on β ensure that $1 \ll L \ll r$ when $\varepsilon \geq r^{\xi - \rho}$ (that $L \ll r$ simply follows from $\beta > 0$). We define the set

$$\mathcal{X}_j := \left\{ x \in \partial Q_{j, \delta r} : 0 \xleftrightarrow{Q_j} x \right\} \tag{5.12}$$

and the random variable $X_j := |\mathcal{X}_j|$. If the event $\{0 \longleftrightarrow Q^c_{r(1+\lambda)}\} \cap \{|\mathcal{C}| \leq \varepsilon r^{2\rho}\} \cap \mathcal{E}^c_{\delta r}$ occurs and $\delta < \frac{1}{4}\lambda$, then either

$$\mathcal{B}_1 := \{\exists j \in [r(1 + \tfrac{1}{4}\lambda), r(1 + \tfrac{1}{2}\lambda)] \text{ s.t. } 0 < X_j < L^\rho\} \cap \{0 \longleftrightarrow Q^c_{r(1+\lambda)}\} \cap \mathcal{E}^c_{\delta r}$$

occurs, or

$$\mathcal{B}_2 := \{X_j \geq L^\rho \, \forall j \in [r(1 + \tfrac{1}{4}\lambda), r(1 + \tfrac{1}{2}\lambda)]\} \cap \{|\mathcal{C}| \leq \varepsilon r^{2\rho}\} \cap \mathcal{E}^c_{\delta r}$$

occurs.

We start with the bound on $\mathbb{P}_{p_c}(\mathcal{B}_1)$: let j_0 be the first j in $[r(1 + \tfrac{1}{4}\lambda), r(1 + \tfrac{1}{2}\lambda)]$ such that $0 < X_j \leq L^\rho$. As in [15], we define the *cluster until level j* as $\mathcal{C}_j := \mathcal{C}(0; Q_j)$ as the connected cluster of the origin in the modified configuration where all edges that do not have both vertices in Q_j are closed. That is, an edge $\{x, y\}$ is in $\mathcal{C}_j$ if $\{x, y\}$ is in the cluster of the origin in ω_{Q_j}. Observe that $\mathcal{X}_{j_0}$ is the vertex set of $\mathcal{C}_{j_0}$ in $\partial Q_{j_0, \delta r}$.

We condition on the realizations of $\mathcal{C}_{j_0}$ (i.e., we condition on all the information needed to determine $\mathcal{C}_{j_0}$: the cluster until level j_0, and all the closed edges touching this cluster), to obtain

$$\mathbb{P}_{p_c}(\mathcal{B}_1) = \sum_{S \text{ admissible}} \mathbb{P}_{p_c}\left(\{0 \longleftrightarrow Q^c_{r(1+\lambda)}\} \cap \mathcal{E}^c_{\delta r} \mid \mathcal{C}_{j_0} = S\right) \mathbb{P}_{p_c}(\mathcal{C}_{j_0} = S),$$

where 'S admissible' means that $\mathbb{P}_{p_c}(\mathcal{C}_{j_0} = S) > 0$. Observe that by the definition of $\mathcal{C}_{j_0}$, any admissible S must satisfy $S \cap Q^c_r \neq \emptyset$. If $\{0 \longleftrightarrow Q^c_{r(1+\lambda)}\} \cap \mathcal{E}^c_{\delta r}$ occurs, then there must exist a connection from x to $Q^c_{r(1+\lambda)}$ 'off $\mathcal{C}_{j_0}$' for some $x \in \mathcal{X}_{j_0}$, so if we write $\partial S = S \cap \partial Q_{j_0, \delta r}$, then

$$\mathbb{P}_{p_c}\left(\{0 \longleftrightarrow Q^c_{r(1+\lambda)}\} \cap \mathcal{E}^c_{\delta r} \mid \mathcal{C}_{j_0} = S\right) \leq \sum_{x \in \partial S} \mathbb{P}_{p_c}\left(x \longleftrightarrow Q^c_{r(1+\lambda)} \text{ off } S \mid \mathcal{C}_{j_0} = S\right).$$

We can tell whether $j = j_0$ by inspecting $\mathcal{C}_j$ (since j_0 is the first j such that $0 < X_j < L^\rho$), so conditioning on $\{\mathcal{C}_{j_0} = S\}$ reveals no information about the configuration 'off S'. Therefore,

$$\mathbb{P}_{p_c}\left(x \longleftrightarrow Q^c_{r(1+\lambda)} \text{ off } S \mid \mathcal{C}_{j_0} = S\right) = \mathbb{P}_{p_c}\left(x \longleftrightarrow Q^c_{r(1+\lambda)} \text{ off } S\right) \leq \mathbb{P}_{p_c}\left(x \longleftrightarrow Q^c_{r(1+\lambda)}\right),$$

where the inequality follows by monotonicity of the one-arm event. It follows that

$$\mathbb{P}_{p_c}\left(0 \longleftrightarrow Q^c_{r(1+\lambda)} \mid \mathcal{C}_{j_0} = S\right) \leq \sum_{x \in \partial S} \mathbb{P}_{p_c}\left(x \longleftrightarrow Q^c_{r(1+\lambda)}\right) \leq L^\rho \gamma\left(\tfrac{1}{2}\lambda r\right),$$

where the second inequality also uses translation invariance and monotonicity of $\gamma(r)$. As a result,

$$\mathbb{P}_{p_c}(\mathcal{B}_1) \leq L^\rho \gamma\left(\tfrac{1}{2}\lambda r\right) \sum_{S \text{ admissible}} \mathbb{P}_{p_c}(\mathcal{C}_{j_0} = S) \leq L^\rho \gamma\left(\tfrac{1}{2}\lambda r\right) \gamma(r), \tag{5.13}$$

since 'S admissible' implies that $\{0 \longleftrightarrow Q^c_r\}$. This gives the second term in (3.2).

Finally we bound $\mathbb{P}_{p_c}(\mathcal{B}_2)$. The same computations as in Section 4 will complete the proof, but we do need a percolation variant of Claim 4.1. Let

$$N := \tfrac{1}{4}\lambda(\varepsilon^\beta + \delta)^{-1} = \tfrac{1}{4}\lambda(\varepsilon^\beta + \varepsilon^{1/(2\rho)})^{-1}.$$

Recall that $L = \varepsilon^\beta r$ and for every integer $1 \le i \le N$ let

$$j_i = r + \tfrac{1}{4}\lambda r + i(L + \delta r) \ \in [r(1 + \tfrac{1}{4}\lambda), r(1 + \tfrac{1}{2}\lambda)].$$

Recall the definition of $\mathcal{X}_j$, (5.12), and define the random variable

$$A_j := \#\{y \in Q_{j+L} \setminus Q_j : 0 \longleftrightarrow y\}.$$

Claim 5.2. *For LRP with* $\alpha \in (0,2) \cup (2,\infty)$ *and* $d > 3(2 \wedge \alpha)$ *that satisfies Definition 1.1(c) and (1.5) there exist constants* $0 < a < \tfrac{1}{2}$ *such that for any* j *sufficiently large, any* $\theta \in (0,1)$*, and any* $L \ge j^\theta$*,*

$$\mathbb{P}_{p_c}(X_j \ge L^\rho \text{ and } A_j < aL^{2\rho}) \le (1-a)\gamma(r). \tag{5.14}$$

With this claim we can complete the proof of Proposition 3.1 for LRP. Earlier, we chose $\varepsilon > r^{\xi-\rho}$, $L = \varepsilon^\beta r$, and $j \in [r(1 + \tfrac{1}{4}\lambda), r(1 + \tfrac{1}{2}\lambda)]$, so by our constraints on β in (3.1) we can find a $\theta > 0$ such that $L \ge j^\theta$. Define I and J as in the BRW case (see (4.9) and (4.10)). Then, by Markov's inequality and Claim 5.2,

$$\begin{aligned}
\mathbb{P}_{p_c}(\mathcal{B}_2) \le \mathbb{P}_{p_c}(I \ge N - J) &\le \frac{(1-a)N}{N-J}\gamma(r) \\
&\le \frac{1-a}{1 - \frac{4}{a\lambda}\varepsilon^{1-2\beta\rho}(\varepsilon^\beta + \varepsilon^{1/(2\rho)})}\gamma(r) \le (1-c_3)\gamma(r)
\end{aligned} \tag{5.15}$$

where the last inequality holds if we can choose ε small enough so that the denominator in the third inequality is arbitrarily close to one. The constraint (3.1) on β guarantees that we can do this.

Gathering together the bounds (5.8), (5.9), (5.11), (5.13), and (5.15) gives the bound in the proposition's hypothesis for $\varepsilon > r^{\xi-\rho}$. $\qquad\square$

All that remains is to prove Claim 5.2. We only give an outline of the proof of Claim 5.2 here, since, as mentioned before, the proof of the claim is very similar to the proof of [15, Theorem 2]. In the appendix we give all the necessary modifications to that proof so that it proves Claim 5.2, but we omit many details.

A An outline of the proof of Claim 5.2

This appendix contains an abridged version of the proof of [15, Theorem 2]. It is *not* self-contained: it is intended to be read alongside [15]. Indeed, the difference between the proof of [15, Theorem 2] and what we need is only superficial. The proof makes hardly any direct use of properties of percolation that do not also hold for LRP, and in the few places where it does, these properties are not crucial.

Let us start with some general remarks about the modifications. The main idea of the proof of [15, Theorem 2] is the same as the proof of Claim 4.1 above. Namely, we use the second moment method to bound the probability that there are many vertices on the boundary $\mathcal{X}_j$ but that they have a combined cluster in $Q_{j+L} \setminus Q_j$ that is not too big. For BRW we could use the independence of the offspring of different particles to bound the first and second moment. For percolation this independence does not exist, because two vertices may for instance connect to the same cluster. To deal with these dependencies, Kozma and Nachmias perform a 'regularity analysis' on the boundary: they show that the clusters emanating from most vertices on the boundary are close to independent (i.e., 'regular') by showing that there are only a few vertices that are not regular. (Roughly speaking, a vertex in $\mathcal{C}$ is regular if it has close to the expected number of other vertices in $\mathcal{C}$ in it's vicinity, and if these vertices are 'nicely spread out'.) For the proof of Claim 5.2 we need a similar analysis.

One of the first things one might notice when comparing this paper with [15] is that [15] frequently states events, summations, etc. in terms of ∂Q_r, the surface of the cube Q_r. In our setting ∂Q_r does not mean very much, since LRP can produce very sparse clusters that may leave many surfaces ∂Q_r empty while still reaching a great distance. For this reason we consider Q_r^c instead when defining the one-arm event. In [15] also uses the surface ∂Q_j in regeneration arguments. Clearly that does not work for LRP either, since X_j and A_j are defined in terms of $\partial Q_{j,\delta r}$. The problem is easily solved by replacing ∂Q_j with $\partial Q_{j,\delta r}$.

Before we proceed with a detailed treatment, let us remark that these are the key modifications to the proof of [15, Theorem 2]:

- Replace ∂Q_r with Q_r^c in the definition of the one-arm event.
- Replace ∂Q_j with $\partial Q_{j,\delta r}$ when using ∂Q_j as a regeneration surface.
- Replace the ersatz bound $\gamma(r) \leq C/\sqrt{r}$ (which does not hold for LRP) with the bound from Lemma 5.1, $\gamma(r) \leq C/r^\xi$ with $\xi = \frac{\alpha}{2\alpha+2}$ (which does hold). To accommodate this replacement we need to 'stretch' some length scales by a factor $1/\xi$.
- Replace the two-point function bound $\mathbb{P}_{p_c}(x \longleftrightarrow y) \asymp \|x - y\|_2^{2-d}$ with the bound $\mathbb{P}_{p_c}(x \longleftrightarrow y) \asymp \|x - y\|_2^{(2\wedge\alpha)-d}$ (i.e., (1.5)).
- Replace exponents that are integer multiples of 2 with exponents that are multiples of $(2 \wedge \alpha)$ if the objects in question are related to the volume of a cluster, and replace them with multiples of ρ if the exponents are related to the one-arm event. (Powers of logarithms will remain unchanged.)
- Replace $x + Q_L$ by $H_{L,j}(x)$ (as defined in (A.1) below) where needed.

In what follows, we give the required modifications to the proof of [15, Theorem 2] per section (the proof spans [15, Sections 3, 4, and 5]). For ease of reference we will maintain the numbering scheme of [15] in this appendix, but affix the letter 'M' to the label to indicate that it is a modification.

Modifications for [15, Section 3]

This section proves [15, Lemma 1.1], which states a lower bound on a connection probability. This bound is not used directly. What is used is the lemma's corollary, [15, Corollary 3.2]. This corollary is used only in the proof of [15, Lemma 4.5]. It turns out that for LRP the bound that is needed in the proof of [15, Lemma 4.5] is trivial, since connection probabilities in LRP can always be bounded from below by the probability of a connection by a single edge, which is large enough. Hence we may skip [15, Section 3].

Modifications for [15, Section 4.1]

This section states the regularity result [15, Theorem 4], modified below. Roughly speaking, this theorem states that it is unlikely that the cluster has many 'dense patches' on the boundary of a cube Q_j.

We need to make several modifications in this section. We start by redefining 'typical clusters':

$$\mathcal{T}_s(x) := \{|\mathcal{C}(x) \cap (x + Q_s)| < s^{2(2\wedge\alpha)} \log^7 s\}.$$

Recall that $\mathcal{C}(x; Q_j)$ is defined as the connected cluster of the vertex x in the modified configuration where all edges that do not have both vertices in Q_j are closed. With $\partial Q_{j,\delta r}$ replacing ∂Q_j and the new definition of $\mathcal{T}_s(x)$, the definitions of s-bad and K-irregular are essentially the same:

Definition M4.1. *For $x \in \partial Q_{j,\delta r}$ and positive integers s and K:*

(i) *We say that x is s-bad if $\mathcal{C}(x; Q_j)$ satisfies*

$$\mathbb{P}_{p_c}(\mathcal{T}_s(x) \mid \mathcal{C}(x; Q_j)) \leq 1 - \mathrm{e}^{-\log^2 s}.$$

(ii) *We say that $x \in \partial Q_{j,\delta r}$ is K-irregular if there exists $s \geq K$ such that x is s-bad.*

We redefine the number of K-irregular vertices at level j,

$$X_j^{K-\mathrm{irr}} := \#\{x \in \partial Q_{j,\delta r} : 0 \overset{Q_j}{\longleftrightarrow} x \text{ and } x \text{ is } K\text{-irregular}\}.$$

With these redefinitions we can leave [15, Theorem 4] otherwise unchanged:

Theorem M4. *There exist constants $C, c > 0$ such that for any K sufficiently large and any j and M we have*

$$\mathbb{P}_{p_c}\left(X_j \geq M \text{ and } X_j^{K-\mathrm{irr}} \geq \tfrac{1}{2} X_j\right) \leq C j^d \mathrm{e}^{-c \log^2 M}.$$

Modifications for [15, Section 4.2]

To verify whether a vertex x is K-regular we need to inspect the entire percolation configuration. In this section, the notion of K-local-regularity is introduced, and it is shown that K-local-regularity implies K-regularity. The advantage of K-local-regularity is of course that we only need to inspect a finite region around x to verify whether x is K-locally-regular. We again need to make several modifications.

Recall from Lemma 5.1 that $\gamma(r) \leq C/r^\xi$ with $\xi = \frac{\alpha}{2\alpha+2}$. We redefine what it means to be locally regular:

Definition M4.2. *For $x \in \partial Q_{j,\delta r}$ and a positive integer s we say that the event $\mathcal{T}_s^{\mathrm{loc}}(x)$ occurs if the following two events occur:*

a. *For all $y \in x + Q_s$,*

$$|\mathcal{C}(y; x + Q_{s^{2d/\xi}}) \cap (x + Q_s)| < s^{2(2\wedge\alpha)} \log^4 s.$$

b. *There exist at most $\log^3 s$ disjoint open paths starting in $x + Q_s$ and ending at $x + Q_{s^{2d/\xi}}^c$.*

Claim M4.1. *For any $x \in \mathbb{Z}^d$ and positive integer s,*

$$\mathcal{T}_s^{\mathrm{loc}}(x) \Rightarrow \mathcal{T}_s(x).$$

The proof of [15, Claim 4.1] holds, mutatis mutandis.
We continue.

Definition M4.3. *For $x \in \partial Q_{j,\delta r}$ and positive integers s and K we define:*

(i) *We say that a cluster $\mathcal{C}$ in $B := (x + Q_{s^{4d^2/\xi}}) \cap Q_j$ is a 'spanning cluster' if $x \in \mathcal{C}$ or if $\mathcal{C} \cap Q_j$ intersects both $x + Q_{s^{4d^2/\xi}}^c$ and $x + Q_{s^{2d/\xi}}$.*

(ii) *We say that x is s-locally-bad if there exist spanning clusters $\mathcal{C}_1, \ldots, \mathcal{C}_m$ in B such that*

$$\mathbb{P}_{p_c}(\mathcal{T}_s^{\mathrm{loc}}(x) \mid \mathcal{C}_1, \ldots, \mathcal{C}_m) \leq 1 - \mathrm{e}^{-\log^2 s}.$$

(iii) *We say that $x \in \partial Q_{j,\delta r}$ is K-locally-irregular if there exists $s \geq K$ such that x is s-locally-bad.*

We modify [15, Claim 4.2]:

Claim M4.2. *For any $x \in \partial Q_{j,\delta r}$ and a positive s we have that if x is s-locally-good, then x is s-good.*

Claim M4.2 has the same proof as [15, Claim 4.2], mutatis mutandis.

We redefine

$$X_j^{K-\mathrm{loc-irr}} := \#\{x \in \partial Q_{j,\delta r} : 0 \xleftrightarrow{Q_j} x \text{ and } x \text{ is } K\text{-locally-irregular}\}.$$

With these new definitions, we can leave [15, Theorem 5] otherwise unchanged:

Theorem M5. *There exist constants $C > c > 0$ such that for any K sufficiently large and any j and M we have*

$$\mathbb{P}_{p_c}\left(X_j \geq M \text{ and } X_j^{K-\mathrm{loc-irr}} \geq \tfrac{1}{2}X_j\right) \leq Cj^d \mathrm{e}^{-c\log^2 M}.$$

Theorem M4 follows directly from Theorem M5 and Claim M4.2.

Modifications for [15, Section 4.3]

This section states two lemmas that give a large deviations type bound on the probability that a vertex is s-locally bad. We state two modified lemmas:

Lemma M4.3. *For $x \in \partial Q_{j,\delta r}$ and positive integer s we have*

$$\mathbb{P}_{p_c}(x \text{ is } s\text{-locally-bad}) \leq C\mathrm{e}^{-c\log^4 s}.$$

Lemma M4.4. *There exists some constant $c > 0$ such that for all $s > 0$ and $\lambda > 0$ we have*

$$\mathbb{P}_{p_c}\left(\max_{y \in Q_s} |\mathcal{C}(y) \cap Q_s| > \lambda s^{2(2\wedge\alpha)}\right) \leq s^{d-3(2\wedge\alpha)}\mathrm{e}^{-c\lambda}.$$

Lemma M4.4 is used to prove Lemma M4.3. Note that the exponents 4 and 6 in [15, Lemma 4.4] have been replaced here with $2(2\wedge\alpha)$ and $3(2\wedge\alpha)$, corresponding to the fact that this is a volume estimate. Write $\mathcal{C}_{\max}$ for the maximum in Lemma M4.4. The proof of Lemma M4.4 is the same as [15, Lemma 4.4], except that we now use the bound

$$\mathbb{E}_{p_c}[\mathcal{C}_{\max}^k] \leq k!C_1^k s^{d-3(2\wedge\alpha)+2(2\wedge\alpha)k},$$

which follows from [1, §4.3, Lemma 2] (Aizenman's η is $(2\wedge\alpha)-2$ in our case, corresponding to the assumption (1.5)).

The proof of [15, Lemma 4.3] does not work for Lemma M4.3 because it uses the bound $\gamma(r) \leq \mathbb{P}_{p_c}(|\mathcal{C}| \geq r/\Lambda) \leq C/\sqrt{r}$, which does not hold for LRP. Therefore we need to redo two of the bounds using Lemma 5.1, but the rest of the proof holds, mutatis mutandis.

The first of these bounds is used to bound the probability of the complement of the event in Definition M4.2(b): We use (5.2), translation invariance, monotonicity of the one-arm event, and the union bound to bound

$$\mathbb{P}_{p_c}(x + Q_s \longleftrightarrow x + Q_{s^{2d/\xi}}^c) \leq Cs^d \mathbb{P}_p(0 \longleftrightarrow Q_{s^{2d/\xi}-s}^c) \leq \frac{Cs^d}{(s^{2d/\xi}-s)^\xi} \leq \frac{C}{s}.$$

The second bound goes along exactly the same lines: the probability that there exist at least $\log^3 s$ spanning clusters is at most

$$\left(Cs^{2d/\xi}\right)^{\log^3 s}\left(C(s^{4d^2/\xi}-s^{2d/\xi})^{-\xi}\right)^{\log^3 s} \leq C\mathrm{e}^{-c\log^4 s}.$$

All other parts of the proof are the same.

Modifications for [15, Section 4.4]

This section presents an exploration process that is used to define two martingales. These martingales are used to control the number of vertices and s-locally-bad vertices on the boundary of Q_j. To define the exploration for LRP we need a different notion of a connection event 'through' a set B:

For a vertex x and vertex sets A and B, we write $\{x \overset{B}{\leadsto} A\}$ for the event that there exists a path of open edges from x to A such that all vertices along that path, except possibly the terminal vertex of the path in A, belong to B. Another way of looking at this event for the case $A \cap B = \emptyset$, is that there exists a path from x to some $y \in B$ in the modified percolation configuration where all edges that do not have both end-points in B are are made closed, and there exists an open edge from y to a vertex $z \in A$ in the unmodified configuration.

We redefine the set of boxes

$$G(w) := \{(Q_{2s^{4d^2/\xi}} + v) \cap Q_j \,:\, v \in (4s^{4d^2/\xi} + 1)\mathbb{Z}^d + w\} \setminus \emptyset,$$

and the sets of explored and active boxes: at time 1,

$$E_1 := \{q \in G(w) : q \cap Q_{j-\delta r}^c = \emptyset\},$$
$$A_1 := \{q \in G(w) : \exists x \in q \text{ s.t. } 0 \overset{\cup E_1}{\leadsto} x\} \setminus E_1,$$

and at time i:

$$E_i := E_{i-1} \cup \{q_i\},$$
$$A_i := \left(A_{i-1} \cup \{q \in G(w) : \exists x \in q \text{ s.t. } 0 \overset{\cup E_i}{\leadsto} x\}\right) \setminus E_i.$$

We say that $q \in G(w)$ is s-bad if there exists an $x \in \partial Q_{j,\delta r}$ that is s-locally bad and such that $(x + Q_{s^{4d^2/\xi}}) \cap Q_j \subset q$. The definition of the filtration $\{\mathcal{F}_i\}$ and the martingale (β_i) remain unchanged: The filtration $\{\mathcal{F}_i\}$ is the configuration restricted to $\bigcup_{j \leq i} E_j$, and

$$\beta_i = \beta_{i-1} + \mathbb{1}_{\{q_i \text{ is } s\text{-bad}\}} - \mathbb{P}_{p_c}(q_i \text{ is } s\text{-bad} \mid \mathcal{F}_{i-1}),$$

with $\beta_1 = 0$. We redefine the martingale (γ_i): start with $\gamma_1 = 0$ and let

$$\gamma_i = \gamma_{i-1} + \mathbb{1}_{\left\{\exists x \in q_i \cap \partial Q_{j,\delta r} \,:\, 0 \overset{\cup E_i}{\leadsto} x\right\}} - \mathbb{P}_{p_c}(\exists x \in q_i \cap \partial Q_{j,\delta r} \,:\, 0 \overset{\cup E_i}{\leadsto} x \mid \mathcal{F}_{i-1}).$$

We restate [15, Lemma 4.5] and its proof here with the needed modifications:

Lemma M4.5. *There exist constants $C_1 > 0$ and $c_1 > 0$ such that for any j, s, and M we have*

$$\mathbb{P}_{p_c}(c_1 e^{-C_1 \log^2 s} \tau \geq X_j \geq M) \leq C e^{-cM + C \log^2 s},$$

where τ is the stopping time of the exploration defined above.

Proof of Lemma M4.5. For every i,

$$X_j \geq \gamma_i + \sum_{k=1}^{i} \mathbb{P}_{p_c}(\exists x \in q_k \cap \partial Q_{j,\delta r} \,:\, 0 \overset{\cup E_k}{\leadsto} x \mid \mathcal{F}_{k-1}).$$

Since we explored q_k, there exists some $z \in q_k^c$ such that $\{0 \overset{\cup E_{k-1}}{\longleftrightarrow} z\}$ occurs and $\{z \overset{q_k}{\leadsto} x\}$ occurs, so given $\mathcal{F}_{k-1}$, the probability that there exists $x \in q_k \cap \partial Q_{j,\delta r}$ such that $\{0 \overset{\cup E_k}{\leadsto} x\}$ occurs is at least the probability of $\{z \overset{q_k}{\leadsto} x\}$. The probability of this event is bounded from below by the probability that there exists an open edge between z and x, which, by Definition 1.1(c), is at most $c(s^{4d^2/\xi})^{-d-\alpha} \geq c_2 e^{-C_2 \log^2 s}$. From here the proof continues

as in [15]. Observe that this short argument completely circumvents the calculations in [15, Section 3] (but it only works for LRP). □

Redefining
$$X_j^{s\text{-loc-bad}} := \#\{x \in \partial Q_{j,\delta r} \, : \, x \text{ is } s\text{-locally-bad}\}$$

and

$$X_j^{s\text{-loc-bad}}(w) := \#\{x \in \partial Q_{j,\delta r} \, : \, x \text{ is } s\text{-locally-bad, and } \exists q \in G(w) \text{ s.t. } x \in q\},$$

the statement and proof of [15, Lemma 4.6] becomes

Lemma M4.6. *There exist constants $C_3 > 0$ and $c_3 > 0$ such that for any j, s, w, and M, and for any real number $\mu \geq C_3 \mathrm{e}^{-c_3 \log^4 s}$ we have*

$$\mathbb{P}_{p_c}(\mu^{-1} X_j^{s\text{-loc-bad}}(w) \geq \tau \geq M) \leq C s^{8d^2/\xi} \mu^{-2} \exp\left(-c s^{-8d^2 \xi} \mu^2 M\right).$$

The proof is basically the same as given in [15], but the proof given there contains a small mistake, so we give a short version of the proof to indicate how the different exponents come about.

Proof. By Lemma M4.3 it follows that

$$\mathbb{P}_{p_c}(q_k \text{ is bad} \mid \mathcal{F}_{k-1}) \leq \sum_{x \,:\, x + Q_{s^{4d^2/\xi}} \subset q_k} C \mathrm{e}^{-c \log^4 s} \leq \tfrac{1}{2} C_3 \mathrm{e}^{-c_3 \log^4 s}.$$

It thus holds deterministically that

$$X_j^{s\text{-loc-bad}}(w) \leq (2s^{4d^2/\xi} + 1)^d |\{k \leq \tau \,:\, q_k \text{ is bad}\}| \leq (2s^{4d^2/\xi} + 1)^d \left(\beta_\tau + \tfrac{1}{2} C_3 \mathrm{e}^{-c_3 \log^4 s} \tau\right)$$

(In [15] this bound is stated with a factor s^{d-1} multiplying only the first term, but this cannot hold deterministically even in the nearest neighbor case. It does not matter: the polynomial factor is negligible compared to the exponential factor.) Now $X_j^{s\text{-loc-bad}}(w) \geq \mu\tau$ implies

$$\beta_\tau \geq c s^{-4d^3/\xi} \tau \left(\mu - \tfrac{1}{2} C_3 \mathrm{e}^{-c_3 \log^4 s}\right) \geq c_4 s^{-4d^3/\xi} \mu\tau.$$

Continuing the proof with this bound we find

$$\mathbb{P}_{p_c}(\mu^{-1} X_j^{s\text{-loc-bad}}(w) \geq \tau \geq M) \leq \mathbb{P}_{p_c}\left(c_4^{-1} s^{4d^3/\xi} \mu^{-1} \beta_\tau \geq \tau \geq M\right)$$

$$\leq \sum_{i=M}^{\infty} \mathbb{P}_{p_c}(\beta_i \geq c_4 s^{-4d^3/\xi} \mu i)$$

$$\leq \sum_{i=M}^{\infty} \exp\left(-c s^{-8d^3/\xi} \mu^2 i\right)$$

$$\leq C s^{8d^2/\xi} \mu^{-2} \exp\left(-c s^{-8d^2 \xi} \mu^2 M\right). \quad \square$$

Theorem M5 is proved using Lemmas M4.5 and M4.6, and goes as in [15], mutatis mutandis.

Modifications for [15, Section 5]

This section uses a second moment method to bound the probability that the percolation cluster contains many vertices on the boundary of Q_j, given that there are precisely M K-regular vertices. This is combined with Theorem M4 (which bounds the probability

that there are too many K-irregular vertices) to give the desired bound. The only changes needed in this section pertain to the volume and one-arm exponents (i.e., replacing 2's with $(2 \wedge \alpha)$'s or ρ's).

We start by defining the *shifted outward-facing half-cube*

$$H_{L,j}(x) := \{y \in \mathbb{Z}^d : \|x + \text{sign}(x_{\iota_x})(j - \|x\|_\infty)e_{\iota_x} - y\|_\infty \leq L^{\rho/(2\wedge\alpha)}$$
$$\text{and } |y_{\iota_x}| > j \text{ and sign}(y_{\iota_x}) = \text{sign}(x_{\iota_x})\}, \quad \text{(A.1)}$$

where e_{ι_x} is the unit vector in direction ι_x (similar to (4.8)). Observe that $H_{L,j}(x) \subset Q_{j+L} \setminus Q_j$ for all $x \in \partial Q_{j,\delta r}$ (this is the result of the extra outward shift with respect to (4.8)).

We redefine the notion of (j, L, K)-admissible to be a pair of vertices (x, y) that satisfies:

- $x \in \partial Q_{j,\delta r}$ and $y \in H_{L,j}(x)$.
- $0 \xleftarrow{Q_j} x$ and $x \longleftrightarrow y$.
- x is K-regular.
- There exists a vertex $\tilde{x} \in Q_j^c$ such that $\{x, \tilde{x}\}$ is an open and pivotal edge for the event $0 \longleftrightarrow y$. (If there is more than one such edge, choose the lexicographically first one).

With $Y(j, L, K)$ denoting the number of pairs (x, y) that are (j, L, K)-admissible, [15, Lemmas 5.1 and 5.2] can now be restated as follows:

Lemma M5.1. *Let K be sufficiently large, and let j, M and L be integers such that $M \geq L^\rho/2$. Then there exists a constant $c = c(K) > 0$ such that*

$$\mathbb{E}_{p_c}[Y(j, L, K)\mathbb{1}_{\{X_j^{K\text{-reg}}=M\}}] \geq cML^\rho \mathbb{P}_{p_c}(X_j^{K\text{-reg}} = M).$$

Lemma M5.2. *Let j, K, M and L be integers. Then*

$$\mathbb{E}_{p_c}[Y(j, L, K)^2\mathbb{1}_{\{X_j^{K\text{-reg}}=M\}}] \leq CM^2L^{2\rho}\mathbb{P}_{p_c}(X_j^{K\text{-reg}} = M).$$

From here on the proof of Claim 5.2 easily follows: Using Theorem M4 we bound

$$\mathbb{P}_{p_c}(X_j \geq L^\rho \text{ and } A_j \leq aL^{2\rho}) \leq \mathbb{P}_{p_c}(X_j \geq L^\rho \text{ and } X_j^{K\text{-irr}} \geq \tfrac{1}{2}L^\rho)$$
$$\leq Cj^d e^{-c\log^2 j} + \sum_{M \geq L^\rho/2} \mathbb{P}_{p_c}(X_j^{K\text{-reg}} = M \text{ and } A_j \leq cL^{2\rho})$$
$$\text{(A.2)}$$

(here we also used that $L \geq j^\theta$ for some $\theta > 0$.)

Now we use that $A_j \geq Y(j, L, K)$ by definition,[1] and we use the Paley-Zygmund inequality and Lemmas M5.1 and M5.2 to bound

$$\mathbb{P}_{p_c}(X_j^{K\text{-reg}} = M \text{ and } A_j \leq cL^{2\rho}) \leq (1 - c')\mathbb{P}_{p_c}(X_j^{K\text{-reg}} = M).$$

Substituting this bound in (A.2), using that the exponential term is negligible and using that $\{X_j^{K\text{-reg}} \geq \tfrac{1}{2}L^\rho\}$ implies $\{0 \longleftrightarrow Q_r^c\}$, we complete the proof of Claim 5.2.

It remains to prove Lemmas M5.1 and M5.2. These proofs can be given exactly as in [15] if we consistently adhere to the following substitutions:

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[1] Note that there is a small mistake in [15]: in [15, (5.2)] it is claimed that $A \geq Y$. Using their definitions this inequality does not hold since the random variable $Y(j, L, K)$ may count vertices outside of $Q_{j+L} \setminus Q_j$, while A_j does not. Our use of $H_{L,j}(x)$ rather than $x + Q_L$ in the definition of (j, L, K)-admissible takes care of this.

- As above, we should always use $x \in \partial Q_{j,\delta r}$ instead of $x \in \partial Q_j$ and $y \in H_{L,j}(x)$ instead of $y \in x + Q_L$.
- Two-point function estimates of the form $\|x - y\|_2^{2-d}$ should be replaced with $\|x - y\|_2^{(2\wedge\alpha)-d}$.
- Corresponding to the above two substitutions, all exponents pertaining to volume estimates should be replaced with their corresponding multiples of ρ if they arise due to summing over sets of the form $H_{L,j}(x)$, and with $(2 \wedge \alpha)$ if they are due to bounds that use $\mathcal{T}_s$.

For instance, the estimate [15, (5.5)] becomes

$$\sum_{y\in H_{L,j}(x)} \mathbb{P}_{p_c}(x' \longleftrightarrow y \text{ only on } A) \leq CL^\rho \sum_{z\in A} \|z - x'\|_2^{(2\wedge\alpha)-d},$$

and the claim [15, (5.7)] becomes

$$|A_t| < 2^{2(2\wedge\alpha)(t+1)}(t+1)^7 \qquad \forall t \text{ such that } 2^t \geq K/2.$$

Indeed, in these final steps we estimate the first and second moment of the number of (j, L, K)-admissible pairs by using the properties of K-regularity (i.e., that it is unlikely that there are many vertices on the boundary near a K-regular vertex).

As a last remark: at the end of the proof of Lemma M5.1 we should note that the local modification involves not $(2K)^d$ as in the proof of [15, Lemma 5.1], but $(2K)^{2d}$ edges. This is of course of no importance.

This concludes the description of all the ways that the proof of [15, Theorem 2] should be modified to prove Claim 5.2 above.

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