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Strong laws at zero for trimmed Lévy processes*

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Abstract

We study the almost sure (a.s.) behaviour of a Lévy process $(X_t)_{t\geq 0}$ on $I\!R$ with extreme values removed, giving necessary and sufficient conditions for the a.s. convergence as $t\downarrow 0$ of normed and centered versions of "trimmed" processes, in which the r largest positive jumps or the r largest jumps in modulus of X up to time t are subtracted from it. Integral criteria in terms of the canonical measure of X are given for the required convergences, under natural conditions on the norming functions. Random walk results of Mori (1976, 1977) and Lévy process results of Shtatland (1965) and Rogozin (1968) are thereby generalised. Another application is to characterise the relative stability at 0 of the trimmed processes, in probability and almost surely.

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1 Introduction

Suppose that $X = \{X_t : t \geq 0\}$, $X_0 = 0$, is a Lévy process with triplet (γ, σ^2, Π). Thus the characteristic function of X is given by the Lévy-Khintchine representation, $E(e^{i\theta X_t}) = e^{t\Psi(\theta)}$, where

$$\Psi(\theta) = \mathrm{i}\theta\gamma - \sigma^2\theta^2/2 + \int_{I\!R\setminus\{0\}} (e^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x \mathbf{1}_{\{|x|<1\}})\Pi(\mathrm{d}x), \text{ for } \theta \in I\!R,\ t \geq 0. \tag{1.1}$$

Here $\gamma \in I\!R$, $\sigma^2 \geq 0$ and Π is a Borel measure on $I\!R_* := I\!R\setminus\{0\}$ such that $\int_{I\!R_*}(x^2\wedge 1)\Pi(\mathrm{d}x)$ is finite. The positive, negative and two-sided tails of Π are

$$\overline{\Pi}^+(x) := \Pi\{(x,\infty)\},\ \overline{\Pi}^-(x) := \Pi\{(-\infty,-x)\},\ \text{and } \overline{\Pi}(x) := \overline{\Pi}^+(x) + \overline{\Pi}^-(x),\ x > 0,$$

assumed right-continuous. We are only interested in small time behaviour of X_t, so we eliminate trivial cases by assuming $\overline{\Pi}(0+) = \infty$ or $\overline{\Pi}^+(0+) = \infty$, as appropriate.

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Denote the jump process of X by $(\Delta X_t)_{t\ge 0}$, where $\Delta X_t = X_t - X_{t-}$, $t>0$, with $\Delta X_0 \equiv 0$. Recall that X is of bounded variation if $\sum_{0<s\le t}|\Delta X_s|<\infty$ a.s. for all $t>0$, equivalently, if $\sigma^2=0$ and $\int_{|x|\le 1}|x|\Pi(\mathrm{d}x)<\infty$. If this is the case (1.1) takes the form

$$\mathrm{i}\theta \mathrm{d}_X + \int_{\mathbb{R}_*}(e^{\mathrm{i}\theta x}-1)\Pi(\mathrm{d}x),$$

where d_X is the drift of X.

For any integer $r=1,2,\ldots$, let $\Delta X_t^{(r)}$ and $\widetilde{\Delta X}_t^{(r)}$ be the r–th largest positive jump and the r-th largest jump in modulus up to time t respectively. Formal definitions of these, allowing for the possibility of tied values (we choose the order uniformly among the ties), are given in Buchmann, Fan and Maller (2014). "One-sided" and "modulus" trimmed versions of X are then defined for $r=1,2,\ldots$ as

$${}^{(r)}X_t := X_t - \sum_{i=1}^{r}\Delta X_t^{(i)} \quad \text{and} \quad {}^{(r)}\widetilde{X}_t := X_t - \sum_{i=1}^{r}\widetilde{\Delta X}_t^{(i)}.$$

When $r=0$ we take ${}^{(0)}X_t = {}^{(0)}\widetilde{X}_t = X_t$. For $x>0$ define truncated moment functions by

$$\nu(x)=\gamma-\int_{x<|y|\le 1} y\Pi(\mathrm{d}y) \quad \text{and} \quad V(x)=\sigma^2+\int_{|y|\le x} y^2\Pi(\mathrm{d}y). \tag{1.2}$$

Our aim is to study the a.s. behaviour of centered and normed versions of ${}^{(r)}X_t$ and ${}^{(r)}\widetilde{X}_t$ when $t\downarrow 0$. We introduce centering and norming functions $a(t)\in\mathbb{R}$ and $b(t)>0$ and characterise the a.s. finiteness or otherwise of $({}^{(r)}X_t - a(t))/b(t)$ and $({}^{(r)}\widetilde{X}_t - a(t))/b(t)$, and some possible a.s. limits of these quantities, when $t\downarrow 0$. In particular, we characterise the relative stability at 0 of the trimmed processes, i.e., convergences of the type ${}^{(r)}X_t/b(t)\to\pm 1$ and ${}^{(r)}\widetilde{X}_t/b(t)\to\pm 1$, for some $b(t)>0$, both in the almost sure and "in probability" senses, as $t\downarrow 0$.

Previous investigations of this sort have been restricted to the case $r=0$. An early result of Khinchin (1939) (Sato 1999, Prop. 47.11, p.358) states that, for any Lévy process X with triplet (γ,σ^2,Π), $\sigma^2\ge 0$,

$$\limsup_{t\downarrow 0}\frac{|X_t|}{\sqrt{2t\log|\log t|}}=\sigma, \text{ a.s.} \tag{1.3}$$

From this we see that $X_t/t^{1/\alpha}\to 0$ a.s. as $t\downarrow 0$ for all $\alpha>2$, and in view of this the norming sequences $b(t)$ we consider will satisfy $b(t)=O(t^{1/\alpha})$ as $t\downarrow 0$, for some $\alpha<2$, $\alpha>0$. Then $b(t)=o(\sqrt{t})$ as $t\downarrow 0$, so in this sense $b(t)$ is not too close to the square root function.

The case $\alpha=2$, of a square root norming, is special, and we do not consider it in detail here (but see Remark (ii) following Theorem 2.1 below). However, as a consequence of Lemma 3.1 below, we observe that $\widetilde{\Delta X}_t^{(1)}=o(\sqrt{t})$ a.s. and $\Delta X_t^{(1)}=o(\sqrt{t})$ a.s. as $t\downarrow 0$ are always true, so we always have ${}^{(r)}\widetilde{X}_t = X_t+o(\sqrt{t})$ a.s. and ${}^{(r)}X_t=X_t+o(\sqrt{t})$ a.s. as $t\downarrow 0$. One implication of this is that (1.3) is also true with X_t replaced by ${}^{(r)}X_t$ or ${}^{(r)}\widetilde{X}_t$, $r=1,2,\ldots$.

The behaviour of X_t relative to powers of t, as $t\downarrow 0$, has been studied in Blumenthal and Getoor (1961), Bertoin, Doney and Maller (2008), and others. The heavily-cited article by Blumenthal and Getoor (1961) has recently received renewed prominence by virtue of its application in time series/financial mathematics areas; cf., e.g., Aït-Sahalia and Jacod (2012). Bertoin, Doney and Maller (2008) extended and completed the Blumenthal and Getoor analysis, in a certain sense, and in particular added in the

$\sqrt{t}$ case. Savov (2009, 2010), among other results, extended the results of Bertoin et al. (2008) to more general norming sequences.

We refer to Bertoin (1996, Sect. III.4) and Sato (1999, Sect. 9.47, p.351) for further background on local behaviour of Lévy processes.

The paper is organised as follows. In Section 2, Theorem 2.1 gives necessary and sufficient conditions for the existence of a centering function $a(t) \in \mathbb{R}$ such that, for a specified norming function $b(t) > 0$, not too close to the square root function, $({}^{(r)}X_t - a(t))/b(t)$ and $({}^{(r)}\widetilde{X}_t - a(t))/b(t)$ are a.s. bounded when $t \downarrow 0$. If this is the case, then these quantities in fact tend to 0 a.s. as $t \downarrow 0$, when $a(t)$ is chosen as $t\nu(b(t))$. Some preliminary results needed for the proof of Theorem 2.1, concerning order statistics of the jumps and a version of Prokhorov's inequality for Lévy processes, are in Section 3. Theorem 2.1 is then proved in Section 4. Relative stability at 0 of the trimmed processes is dealt with in Section 5, using the results in Section 2.

2 Results

Throughout, assume the norming function $b(\cdot)$ is positive and nondecreasing. Keeping in mind (1.3), for our main result we will also impose the condition: there are constants $c > 0$, $\alpha \in (0, 2)$, and $t_0 > 0$ such that

$$\frac{b(s)}{s^{1/\alpha}} \leq \frac{cb(t)}{t^{1/\alpha}} \tag{2.1}$$

whenever $0 < s \leq t \leq t_0$. (2.1) implies in particular that $b(t) = O(t^{1/\alpha})$ as $t \downarrow 0$. The functions $b(t) = t^{1/\alpha}$ with $\alpha \in (0, 2)$, or $b(t)$ strictly nondecreasing and regularly varying at 0 with index greater than $1/2$, satisfy (2.1).

Define a function inverse to $b(t)$ by

$$B(x) := b^{\leftarrow}(x) = \inf\{t > 0 : b(t) > x\},\ x > 0.$$

Then $B(x)$ is nondecreasing and right continuous. It may have intervals of constancy corresponding to jumps in $b(t)$ or jumps corresponding to intervals of constancy in $b(t)$. If $b(t)$ is assumed continuous and strictly increasing then $B(x)$ is continuous and strictly increasing. For integers $r = 1, 2, \ldots$, define the integrals

$$J_r := \int_0^1 \overline{\Pi}^r(x)\mathrm{d}B^r(x), \tag{2.2}$$

and also the "one-sided" versions

$$J_r^{(\pm)} := \int_0^1 \left(\overline{\Pi}^{\pm}(x)\right)^r dB^r(x),\ r = 1, 2, \ldots. \tag{2.3}$$

The main results for both one- and two-sided trimming are stated in Theorem 2.1.

Theorem 2.1. *Assume* $\sigma^2 = 0$ *and* $\overline{\Pi}(0+) = \infty$. *Suppose* $b(t) > 0$ *is continuous, strictly increasing and satisfies* (2.1) *with* $0 < \alpha < 2$, *and fix* $r = 0, 1, 2, \ldots$.

(i) Then for some function $a(t) \in \mathbb{R}$

$$\limsup_{t\downarrow 0} \frac{|{}^{(r)}\widetilde{X}_t - a(t)|}{b(t)} < \infty \text{ a.s.} \tag{2.4}$$

iff $J_{r+1} < \infty$. *If this holds we can take* $a(t) = t\nu(b(t))$, $t > 0$ *(see* (1.2)*), and then in fact*

$$\lim_{t\downarrow 0} \frac{{}^{(r)}\widetilde{X}_t - t\nu(b(t))}{b(t)} = 0 \text{ a.s.} \tag{2.5}$$

(ii) Further: assume $\overline{\Pi}^{+}(0+) = \infty$*. Then* (2.4) *holds for some function* $a(t) \in I\!R$ *with* ${}^{(r)}\widetilde{X}_t$ *replaced by* ${}^{(r)}X_t$ *iff* $J^{(+)}_{r+1} < \infty$ *and* $J^{(-)}_1 < \infty$*. If this is the case we can take* $a(t) = t\nu(b(t))$*,* $t > 0$*, and then* (2.5) *holds with* ${}^{(r)}\widetilde{X}_t$ *replaced by* ${}^{(r)}X_t$*.*

Remarks: (i) In view of (1.3) and the remarks following it, we exclude the case $\sigma^2 > 0$ from Theorem 2.1.

(ii) The case $\alpha = 2$ also is not included in Theorem 2.1. Since, as remarked in Section 1, we always have ${}^{(r)}\widetilde{X}_t = X_t + o(\sqrt{t})$ and ${}^{(r)}X_t = X_t + o(\sqrt{t})$ a.s., as $t \downarrow 0$, the J_r integrals cannot test for boundedness of the type in (2.4) when $b(t) = \sqrt{t}$. For related results in this case see Bertoin, Doney and Maller (2008) and Savov (2009, 2010).

(iii) Theorem 2.1 yields as a corollary an uncentered version of (2.5). Under the assumptions of the theorem, we can deduce that

$$\limsup_{t\downarrow 0} \frac{|{}^{(r)}\widetilde{X}_t|}{b(t)} < \infty \text{ a.s.}$$

iff $J_{r+1} < \infty$ and $t\nu(b(t)) = O(b(t))$, as $t \downarrow 0$. For example, when $b(t) \equiv t$, we deduce that ${}^{(r)}\widetilde{X}_t = O(t)$ a.s as $t \downarrow 0$ iff $J_{r+1} < \infty$ and $\nu(t) = O(1)$. Similarly, we can characterise the boundedness condition ${}^{(r)}X_t = O(t)$ a.s., as $t \downarrow 0$.

(iv) The case $r = 0$ is included in Theorem 2.1. This allows us to recover a result of Shtatland (1965) and Rogozin (1968) to the effect that $X_t = O(t)$ a.s. as $t \downarrow 0$ iff X is of bounded variation. When $r = 0$ and $b(t) = t$, so $B(x) = x$, the convergence of J_1 together with $\sigma^2 = 0$ is equivalent to the bounded variation of X, and the convergence of J_1 also implies $\nu(t) = O(1)$. So we obtain the Rogozin-Shtatland result from the case $r = 0$ of the previous paragraph. (Recall that ${}^{(0)}X_t = {}^{(0)}\widetilde{X}_t = X_t$.) The cases $r > 0$ in the previous paragraph constitute a generalisation of the Rogozin-Shtatland result[1], when $b(t) = t$.

(v) We observe following Lemma 3.1 below that $J_r < \infty$ implies $J_{r+1} < \infty$ for $r = 1, 2, \ldots$. As a simple example, suppose $\overline{\Pi}(x) \sim 1/(x|\log x|)$ as $x \downarrow 0$, and take $b(t) = t$. Then $J_1 = \infty$ so (2.4) does not hold for any $a(t)$ (X is not of bounded variation), but $J_2 < \infty$ so (2.5) holds with $r = 1$, in fact, with $r = 1, 2, \ldots$.

(vi) Theorem 2.1 can be seen as a refinement of Theorem 2.1 of Bertoin et al. (2008) and, particularly, Proposition 2.1 and Corollary 2.1 of Savov (2009), in which the contribution of the large jumps to X near 0 is quantified in an explicit way. See also Maller (2008) for some related results.

(vii) The genesis of Theorem 2.1 is in papers of Mori (1976, 1977), who considered the corresponding strong laws for random walks at large times. (See also Hatori, Maejima and Mori (1979).) As far as possible we adapt his methods for the small time behaviour of the Lévy, adding in variants for one-sided trimming and relative stability. Of course some quite different arguments are needed in places.

3 Preliminary Results

Before proving the theorems we present some preliminary results relating to the order statistics of the jumps (Subsection 3.1) and a version of Prokhorov's inequality for Lévy processes (Subsection 3.2).

3.1 Some Properties of the Jumps

The point measure associated with the jumps of X is a Poisson point process on $[0,\infty) \times I\!R_*$ with intensity measure $\mathrm{d}s \otimes \mathrm{d}\Pi(x)$. So the tail of the distribution of $|\widetilde{\Delta X}_t^{(r+1)}|$

[1] We remark that Rogozin and Shtatland prove a little more; they show that, when X is not of bounded variation, then $-\infty = \liminf_{t\downarrow 0} X_t/t < \limsup_{t\downarrow 0} X_t/t = +\infty$, a.s.

can be calculated as

$$\begin{aligned} P\left(|\widetilde{\Delta X}_t^{(r+1)}| > x\right) &= P\left(\#\{\text{jumps } \Delta X_s \text{ with } 0 < s \le t \text{ and } |\Delta X_s| > x\} \ge r+1\right) \\ &= e^{-t\overline{\Pi}(x)} \sum_{i \ge r+1} \frac{\left(t\overline{\Pi}(x)\right)^i}{i!}, \ x > 0. \end{aligned} \tag{3.1}$$

From this we derive the inequalities

$$\begin{aligned} e^{-t\overline{\Pi}(x)} \frac{\left(t\overline{\Pi}(x)\right)^{r+1}}{(r+1)!} &\le P\left(|\widetilde{\Delta X}_t^{(r+1)}| > x\right) \\ &= e^{-t\overline{\Pi}(x)} \left(t\overline{\Pi}(x)\right)^{r+1} \sum_{i \ge r+1} \frac{\left(t\overline{\Pi}(x)\right)^{i-r-1}}{i!} \\ &\le \frac{\left(t\overline{\Pi}(x)\right)^{r+1}}{(r+1)!}, \ x > 0. \end{aligned} \tag{3.2}$$

Now we can prove:

Lemma 3.1. *Assume $b(t) > 0$ is nondecreasing and fix $r = 0, 1, 2, \ldots$ and $a > 0$.*
(i) The following are equivalent:

$$\int_0^1 \overline{\Pi}^{r+1}(ab(x)) \mathrm{d}x^{r+1} < \infty; \tag{3.3}$$

$$\sum_{n \ge 0} \left(2^{-n}\overline{\Pi}(ab(2^{-n}))\right)^{r+1} < \infty; \tag{3.4}$$

$$P\left(|\widetilde{\Delta X}_t^{(r+1)}| > ab(t) \text{ i.o. as } t \downarrow 0\right) = 0; \tag{3.5}$$

$$\sum_{n \ge 0} P\left(|\widetilde{\Delta X}_{2^{-n}}^{(r+1)}| > ab(2^{-n})\right) < \infty. \tag{3.6}$$

If any of these hold then

$$\lim_{t \downarrow 0} t\overline{\Pi}(ab(t)) = 0. \tag{3.7}$$

(ii) Next assume $b(t) > 0$ is right-continuous, nondecreasing, and satisfies (2.1) *with $\alpha > 0$. Then any of* (3.3)–(3.6) *are equivalent to*

$$J_{r+1} < \infty. \tag{3.8}$$

Further, since (3.8) *does not depend on a, any of conditions* (3.3)–(3.6) *hold for all $a > 0$ if they hold for some $a > 0$, and* (3.7) *then holds for all $a > 0$.*

Remarks: (i) Note that the restriction $\alpha < 2$ is not required in Part (ii) of Lemma 3.1. (2.1) is not assumed at all in Part (i).

(ii) As a consequence of (3.7) we see from (3.3) that the convergence of J_r implies the convergence of J_{r+1}, $r = 1, 2, \ldots$.

(iii) When $r = 0$, $a = 1$ and $b(x) = \sqrt{x}$, the integral in (3.3) becomes $\int_0^1 \overline{\Pi}(\sqrt{x})\mathrm{d}x = 2\int_0^1 x\overline{\Pi}(x)\mathrm{d}x$, which is finite as a consequence of the basic relation $\int_{\mathbb{R}_*}(x^2 \wedge 1)\Pi(\mathrm{d}x) < \infty$. Thus (3.5) always holds for all $a > 0$ when $r = 0$ and $b(x) = \sqrt{x}$, hence $\widetilde{\Delta X}_t^{(1)} = o(\sqrt{t})$ a.s. and $\Delta X_t^{(1)} = o(\sqrt{t})$ a.s. as $t \downarrow 0$ are always true.

Proof of Lemma 3.1: Fix $r=0,1,2,\dots$ and $a>0$. First, (3.3) and (3.4) are equivalent because, by the monotonicity of $b(\cdot)$ and $\overline{\Pi}$,

$$\begin{aligned}(1-2^{-r-1})\sum_{n\ge 0}\left(2^{-n}\overline{\Pi}(ab(2^{-n}))\right)^{r+1} &\le \sum_{n\ge 0}\int_{2^{-n-1}}^{2^{-n}}\overline{\Pi}^{r+1}(ab(x))\mathrm{d}x^{r+1}\\ &= \int_0^1 \overline{\Pi}^{r+1}(ab(x))\mathrm{d}x^{r+1}\\ &\le (2^{r+1}-1)\sum_{n\ge 0}\left(2^{-n-1}\overline{\Pi}(ab(2^{-n-1}))\right)^{r+1}.\end{aligned}$$

Next, assume (3.4). Then

$$\begin{aligned}&P\left(|\widetilde{\Delta X}_t^{(r+1)}|>ab(t)\text{ i.o. as }t\downarrow 0\right)=\lim_{m\to\infty}P\bigcup_{n\ge m}\bigcup_{2^{-n}<t\le 2^{-n+1}}\left\{|\widetilde{\Delta X}_t^{(r+1)}|>ab(t)\right\}\\ \le\;& \lim_{m\to\infty}\sum_{n\ge m}P\left(|\widetilde{\Delta X}_t^{(r+1)}|>ab(t)\text{ for some }t\in(2^{-n},2^{-n+1}]\right)\\ \le\;& \lim_{m\to\infty}\sum_{n\ge m}P\left(|\widetilde{\Delta X}_{2^{-n+1}}^{(r+1)}|>ab(2^{-n})\right)\le\lim_{m\to\infty}\frac{1}{(r+1)!}\sum_{n\ge m}\left(2^{-n+1}\overline{\Pi}(ab(2^{-n}))\right)^{r+1}\\ =\;& 0\text{ (by (3.4))},\end{aligned}$$

where we used the righthand inequality in (3.2) with $x=ab(2^{-n})$ in the last inequality. Thus (3.5) holds.

Conversely, suppose the series in (3.4) diverges. Let

$$A_n:=\left\{|\Delta X_t|>ab(t)\text{ for at least }r+1\text{ values of }t\text{ in }(2^{-n-1},2^{-n}]\right\},\ n=1,2,\dots.$$

The A_n are independent events and we note that

$$\begin{aligned}&P\left(|\widetilde{\Delta X}_{2^{-n-1}}^{(r+1)}|>ab(2^{-n})\right)\\ =\;& P\left(|\Delta X_t|>ab(2^{-n})\text{ for at least }r+1\text{ values of }t\text{ in }(0,2^{-n-1}]\right)\\ =\;& P\left(|\Delta X_t|>ab(2^{-n})\text{ for at least }r+1\text{ values of }t\text{ in }(2^{-n-1},2^{-n}]\right)\\ \le\;& P\left(|\Delta X_t|>ab(t)\text{ for at least }r+1\text{ values of }t\text{ in }(2^{-n-1},2^{-n}]\right)\\ =\;& P(A_n).\end{aligned}$$

Suppose $2^{-n}\overline{\Pi}(ab(2^{-n}))\to 0$. Then by the lefthand inequality in (3.2) with $x=ab(2^{-n})$,

$$\sum_{n\ge 0}P(A_n)\ge\sum_{n\ge 1}P\left(|\widetilde{\Delta X}_{2^{-n-1}}^{(r+1)}|>ab(2^{-n})\right)\ge c_1\sum_{n\ge 1}\left(2^{-n-1}\overline{\Pi}(ab(2^{-n}))\right)^{r+1},\qquad (3.9)$$

for some constant $c_1>0$. The series on the right of (3.9) is infinite since the series in (3.4) diverges. If $2^{-n}\overline{\Pi}(ab(2^{-n}))\nrightarrow 0$ take a subsequence $n_k\to\infty$ such that $x_k:=2^{-n_k}\overline{\Pi}(ab(2^{-n_k}))\to h\in(0,\infty]$ as $k\to\infty$. Then by (3.1)

$$P\left(|\widetilde{\Delta X}_{2^{-n_k}}^{(r+1)}|>ab(2^{-n_k})\right)=1-e^{-x_k}\sum_{i=0}^{r}\frac{x_k^i}{i!}\to 1-e^{-h}\sum_{i=0}^{r}\frac{h^i}{i!}>0,$$

so the middle series in (3.9) is infinite. In either case $\sum_n P(A_n)$ diverges and so by the Borel-Cantelli lemma, $P(A_n\text{ i.o. as }n\to\infty)=1$. But then $P(|\widetilde{\Delta X}_t^{(r+1)}|>ab(t)\text{ i.o. as }t\downarrow 0)=1$, contrary to (3.5). So (3.5) implies (3.4).

It follows from (3.4) that $\lim_{n\to\infty}2^{-n}\overline{\Pi}(ab(2^{-n}))=0$. Given $0<t<1$ choose $n(t)=\lfloor -\log_2 t\rfloor$, so $2^{-n-1}\le t\le 2^{-n}$, and $t\overline{\Pi}(ab(t))\le 2^{-n}\overline{\Pi}(ab(2^{-n-1}))\to 0$ as $t\downarrow 0$, thus we get (3.7).

Assume (3.4), so (3.7) holds. (3.2) with $t = 2^{-n}$ and $x = ab(2^{-n})$ then gives

$$P\left(|\widetilde{\Delta X}_{2^{-n}}^{(r+1)}| > ab(2^{-n})\right) \sim \frac{\left(2^{-n}\overline{\Pi}(ab(2^{-n}))\right)^{r+1}}{(r+1)!}, \text{ as } t \to 0.$$

Similarly we deduce this also if (3.6) holds. The equivalence of (3.6) with (3.4) follows.

Finally, assume $b(t) > 0$ is right-continuous, nondecreasing and satisfies (2.1) with $\alpha > 0$. Fix $r = 0, 1, 2, \ldots$ and $a > 0$. By change of variable[2] we have

$$\int_0^{B(1)} \overline{\Pi}^{r+1}(b(x))\mathrm{d}x^{r+1} = \int_0^1 \overline{\Pi}^{r+1}(x)\mathrm{d}B^{r+1}(x) = J_{r+1}. \tag{3.10}$$

When $0 < \delta \le 1$, (2.1) gives

$$\frac{b(\delta^\alpha x)}{(\delta^\alpha x)^{1/\alpha}} \le \frac{cb(x)}{x^{1/\alpha}}.$$

Assume (3.8), and that $a \le c$, where c is the constant in (2.1). Let $\delta := a/c \le 1$, then $b(\delta^\alpha x) \le c\delta b(x) = ab(x)$, so

$$\begin{aligned}\int_0^{B(1)/\delta^\alpha} \overline{\Pi}^{r+1}(ab(x))\mathrm{d}x^{r+1} &\le \int_0^{B(1)/\delta^\alpha} \overline{\Pi}^{r+1}(b(\delta^\alpha x))\mathrm{d}x^{r+1} \\ &= \delta^{-(r+1)\alpha}\int_0^{B(1)} \overline{\Pi}^{r+1}(b(x))\mathrm{d}x^{r+1} = \delta^{-(r+1)\alpha}J_{r+1}.\end{aligned}$$

Thus (3.3) holds when $a \le c$ and hence when $a = c$, and hence also when $a > c$ by the monotonicity of $\overline{\Pi}$. Thus (3.8) implies (3.3). Conversely, assume (3.3) and take $a \ge 1/c$. Let $\delta = 1/(ac) \le 1$, then $ab(\delta^\alpha x) \le b(x)$, so

$$\begin{aligned}\int_0^{B(1)\delta^\alpha} \overline{\Pi}^{r+1}(ab(x))\mathrm{d}x^{r+1} &= \delta^{(r+1)\alpha}\int_0^{B(1)} \overline{\Pi}^{r+1}(ab(\delta^\alpha x))\mathrm{d}x^{r+1} \\ &\ge \delta^{(r+1)\alpha}\int_0^{B(1)} \overline{\Pi}^{r+1}(b(x))\mathrm{d}x^{r+1} = \delta^{(r+1)\alpha}J_{r+1}.\end{aligned}$$

Thus (3.3) implies (3.8) when $a \ge 1/c$ and hence also when $0 < a < 1/c$, by the monotonicity of $\overline{\Pi}$. □

The next lemma gives formulae for the increments of ${}^{(r)}\widetilde{X}_t$ and ${}^{(r)}X_t$. These are denoted by $(\Delta^{(r)}\widetilde{X}_t)_{t\ge 0}$ and $(\Delta^{(r)}X_t)_{t\ge 0}$.

Lemma 3.2. *(i) Suppose* $\overline{\Pi}(0+) = \infty$. *Then for* $r = 0, 1, 2, \ldots$

$$\sup_{0<s\le t} |\Delta^{(r)}\widetilde{X}_s| = |\widetilde{\Delta X}_t^{(r+1)}|, \; t > 0. \tag{3.11}$$

(ii) Suppose $\overline{\Pi}^+(0+) = \infty$. *Then for* $r = 0, 1, 2, \ldots$

$$\sup_{0<s\le t} \Delta^{(r)}X_s = \Delta X_t^{(r+1)} \quad \text{and} \quad \sup_{0<s\le t} |\Delta^{(r)}X_s| = \max\left((\Delta X^-)_t^{(1)}, \Delta X_t^{(r+1)}\right), \; t > 0. \tag{3.12}$$

Proof of Lemma 3.2: (i) Take $t > 0$ and $0 < \varepsilon < t$ and consider

$${}^{(r)}\widetilde{X}_t - {}^{(r)}\widetilde{X}_{t-\varepsilon} = X_t - \sum_{i=1}^r \widetilde{\Delta X}_t^{(i)} - X_{t-\varepsilon} + \sum_{i=1}^r \widetilde{\Delta X}_{t-\varepsilon}^{(i)}.$$

[2] To get (3.10), set $a(t) := B^{r+1}(t)$ in Theorem T16, p.300, of Bremaud (1981), so that his $c(t) = b(t^{1/(r+1)})$.

Letting $\varepsilon \downarrow 0$ gives

$$\Delta^{(r)}\widetilde{X}_t = \Delta X_t - D_t,$$

where

$$D_t := \sum_{i=1}^{r} \left(\widetilde{\Delta X}_t^{(i)} - \widetilde{\Delta X}_{t-}^{(i)} \right)$$

and $\widetilde{\Delta X}_{t-}^{(i)}$ is the jump with i–th largest modulus among $(\Delta X_s)_{0<s<t}$. Now if $|\Delta X_t| < |\widetilde{\Delta X}_{t-}^{(r)}|$ then the r largest in modulus of the ΔX do not change from $t-$ to t, so

$$\left\{ \widetilde{\Delta X}_t^{(1)}, \ldots, \widetilde{\Delta X}_t^{(r)} \right\} = \left\{ \widetilde{\Delta X}_{t-}^{(1)}, \ldots, \widetilde{\Delta X}_{t-}^{(r)} \right\},$$

$|\Delta X_t| \le |\widetilde{\Delta X}_t^{(r+1)}|$, and $D_t = 0$. Then $\Delta^{(r)}\widetilde{X}_t = \Delta X_t$. Alternatively if $|\Delta X_t| > |\widetilde{\Delta X}_{t-}^{(r)}|$ then ΔX_t displaces $\widetilde{\Delta X}_{t-}^{(r)}$ among the r largest in modulus to time t, so

$$\left\{ \widetilde{\Delta X}_t^{(1)}, \ldots, \widetilde{\Delta X}_t^{(r)} \right\} = \left\{ \widetilde{\Delta X}_{t-}^{(1)}, \ldots, \Delta X_t, \ldots, \widetilde{\Delta X}_{t-}^{(r-1)} \right\},$$

and $D_t = \Delta X_t - \widetilde{\Delta X}_{t-}^{(r)}$. Then $\Delta^{(r)}\widetilde{X}_t = \widetilde{\Delta X}_{t-}^{(r)}$ and $|\Delta^{(r)}\widetilde{X}_t| = |\widetilde{\Delta X}_{t-}^{(r)}| = |\widetilde{\Delta X}_t^{(r+1)}|$. This also holds if $|\Delta X_t| = |\widetilde{\Delta X}_{t-}^{(r)}|$ regardless of the way ties if any may be broken. Thus

$$|\Delta^{(r)}\widetilde{X}_t| = |\Delta X_t| \wedge |\widetilde{\Delta X}_t^{(r+1)}|.$$

Replacing t by s then taking a supremum over $0 < s \le t$ gives (3.11).

(ii) The proof of (3.12) is similar. We obtain

$$\Delta^{(r)}X_t = \Delta X_t \wedge \Delta X_t^{(r+1)} \tag{3.13}$$

and this implies (3.12). □

Remark: (i) Note that we don't have $\sup_{0<s\le t} |\Delta^{(r)}X_s| = \Delta X_t^{(r+1)}$ in Lemma 3.2 because (3.13) does not imply $|\Delta^{(r)}X_t| = |\Delta X_t| \wedge \Delta X_t^{(r+1)}$ (we could have $\Delta X_t < -\Delta X_t^{(r+1)}$).

3.2 Prokhorov's Inequality for $X_t^{(S,h)}$

Prokhorov's inequality (Prokhorov (1960)) for random walks[3] reads as follows: let $S_n = \sum_{i=1}^n \xi_i$, where $(\xi_i)_{i=1,2,\ldots}$ are i.i.d random variables with $|\xi_i| \le h$ for some $h > 0$ and $E\xi_1 = 0$. Then for $x > 0$ and $n = 1, 2, \ldots$

$$P(S_n > x) \le \exp\left(-\frac{x}{2h} \sinh^{-1}\left(\frac{xh}{2\mathrm{Var}S_n} \right) \right), \tag{3.14}$$

where $\sinh^{-1}$ is inverse function to the sinh function, $sinh(x) = (e^x - e^{-x})/2$, $x \in I\!R$. In this section we give a version of Prokhorov's inequality for a Lévy process. Recall the Itô decomposition in the form (e.g., Doney and Maller (2002, Lemma 6.1)): for $h > 0$, $t > 0$,

$$X_t = t\nu(h) + X_t^{(S,h)} + X_t^{(B,h)}, \tag{3.15}$$

where $\nu(\cdot)$ is defined in (1.2), $X_t^{(S,h)}$ is the compensated small jump component of X, i.e., having jumps of magnitude less than or equal to h in modulus, and $X_t^{(B,h)}$ has jumps of magnitude greater than h in modulus.

[3]Prokhorov's inequality holds in fact for independent, not necessarily distributed random variables. For a refinement of Prokhorov's inequality, see Kruglov (2006). The method of Lemma 3.3 can also be used to derive Lévy versions of, e.g., Bernstein's inequality.

Lemma 3.3. *Assume* $\overline{\Pi}(0+) = \infty$*. For* $x > 0$*,* $h > 0$ *and* $t > 0$

$$P\left(X_t^{(S,h)} > x\right) \le \exp\left(-\frac{x}{2h}\sinh^{-1}\left(\frac{xh}{2\mathrm{Var}X_t^{(S,h)}}\right)\right). \tag{3.16}$$

Proof of Lemma 3.3. Take $t > 0$, $h > 0$, and $\varepsilon \in (0, h)$, let

$$N_t(\varepsilon) := \#\{\text{jumps } \Delta X_s \text{ with } 0 < s \le t \text{ and } \varepsilon < |\Delta X_s| \le h\},$$

and let $J_i(\varepsilon)$, $i = 1, 2, \ldots, N_t(\varepsilon)$, be the magnitudes of those jumps. The $J_i(\varepsilon)$ are i.i.d., independent of $N_t(\varepsilon)$, with $|J_i(\varepsilon)| \le h$, and

$$EJ_1(\varepsilon) = \frac{\int_{\varepsilon<|x|\le h} x\Pi(\mathrm{d}x)}{C(\varepsilon)}.$$

Here we abbreviate $C(\varepsilon) := \overline{\Pi}(\varepsilon) - \overline{\Pi}(h)$, which is positive for ε small enough and tends to ∞ as $\varepsilon \downarrow 0$. For any $x > 0$ we can write

$$\begin{aligned} P\left(X_t^{(S,h)} > x\right) &= \lim_{\varepsilon\downarrow 0} P\left(\sum_{i=1}^{N_t(\varepsilon)} J_i(\varepsilon) - t\int_{\varepsilon<|x|\le h} x\Pi(\mathrm{d}x) > x\right) \\ &= \lim_{\varepsilon\downarrow 0} P\left(\sum_{i=1}^{N_t(\varepsilon)} (J_i(\varepsilon) - EJ_1(\varepsilon)) + \left(\frac{N_t(\varepsilon)}{C(\varepsilon)} - t\right)\int_{\varepsilon<|x|\le h} x\Pi(\mathrm{d}x) > x\right). \end{aligned} \tag{3.17}$$

Now $N_t(\varepsilon)$ is Poisson with $EN_t(\varepsilon) = tC(\varepsilon)$, so for $t > 0$

$$\frac{N_t(\varepsilon) - tC(\varepsilon)}{\sqrt{C(\varepsilon)}} \xrightarrow{\mathrm{D}} N(0, t), \text{ as } \varepsilon \downarrow 0.$$

Thus for any $\delta \in (0, h)$,

$$\left(\frac{N_t(\varepsilon)}{C(\varepsilon)} - t\right)\int_{\delta<|x|\le h} x\Pi(\mathrm{d}x) \xrightarrow{\mathrm{P}} 0, \text{ as } \varepsilon \downarrow 0,$$

while for $0 < \varepsilon < \delta$, by the Cauchy-Schwarz inequality,

$$\begin{aligned} \left|\left(\frac{N_t(\varepsilon)}{C(\varepsilon)} - t\right)\int_{\varepsilon<|x|\le\delta} x\Pi(\mathrm{d}x)\right|^2 &= \left(\frac{N_t(\varepsilon) - tC(\varepsilon)}{\sqrt{C(\varepsilon}}\right)^2 \left|\frac{1}{\sqrt{C(\varepsilon)}}\int_{\varepsilon<|x|\le\delta} x\Pi(\mathrm{d}x)\right|^2 \\ &\le O_P(1)\left(\frac{\overline{\Pi}(\varepsilon) - \overline{\Pi}(\delta)}{\overline{\Pi}(\varepsilon) - \overline{\Pi}(h)}\right)\int_{\varepsilon<|x|\le\delta} x^2\Pi(\mathrm{d}x) \\ &\le O_P(1)\int_{\varepsilon<|x|\le\delta} x^2\Pi(\mathrm{d}x). \end{aligned} \tag{3.18}$$

This is arbitrarily small for choice of ε and δ. So we have

$$\left(\frac{N_t(\varepsilon)}{C(\varepsilon)} - t\right)\int_{\varepsilon<|x|\le h} x\Pi(\mathrm{d}x) \xrightarrow{\mathrm{P}} 0, \text{ as } \varepsilon \downarrow 0. \tag{3.19}$$

Now employ Prokhorov's inequality (3.14) for random walks to write

$$P\left(\sum_{i=1}^{N_t(\varepsilon)} (J_i(\varepsilon) - EJ_1(\varepsilon)) > x\right)$$

$$
\begin{aligned}
&= \sum_{n\ge 0} P\left(N_t(\varepsilon)=n\right) P\left(\sum_{i=1}^{n}(J_i(\varepsilon)-EJ_1(\varepsilon))>x\right)\\
&\le \sum_{n\ge 0} P\left(N_t(\varepsilon)=n\right)\exp\left(-\frac{x}{2h}\sinh^{-1}\left(\frac{xh}{2n\mathrm{Var}J_1(\varepsilon)}\right)\right)\\
&= E\exp\left(-\frac{x}{2h}\sinh^{-1}\left(\frac{xh}{2N_t(\varepsilon)\mathrm{Var}J_1(\varepsilon)}\right)\right). \qquad (3.20)
\end{aligned}
$$

(Here we interpret $\sum_{i=1}^{0}=0$, and $e^{-\infty}=0$.) But

$$
\begin{aligned}
N_t(\varepsilon)\mathrm{Var}J_1(\varepsilon) &= N_t(\varepsilon)\left(EJ_1^2(\varepsilon)-E^2(J_1(\varepsilon))\right)\\
&= \frac{N_t(\varepsilon)}{C(\varepsilon)}\left(\int_{\varepsilon<|x|\le h}x^2\Pi(\mathrm{d}x)-\frac{1}{C(\varepsilon)}\left(\int_{\varepsilon<|x|\le h}x\Pi(\mathrm{d}x)\right)^2\right),
\end{aligned}
$$

in which $N_t(\varepsilon)/C(\varepsilon)\xrightarrow{\mathrm{P}} t$ as $\varepsilon\downarrow 0$, and

$$
\frac{1}{\sqrt{C(\varepsilon)}}\int_{\varepsilon<|x|\le h}x\Pi(\mathrm{d}x)\to 0,
$$

which follows as in (3.18). Hence

$$
\begin{aligned}
N_t(\varepsilon)\mathrm{Var}J_1(\varepsilon) &\xrightarrow{\mathrm{P}} t\int_{|x|\le h}x^2\Pi(\mathrm{d}x)\\
&= tV(h)\\
&= \mathrm{Var}X_t^{(S,h)}.
\end{aligned}
$$

Letting $\varepsilon\downarrow 0$ in (3.20) gives

$$
\limsup_{\varepsilon\downarrow 0} P\left(\sum_{i=1}^{N_t(\varepsilon)}(J_i(\varepsilon)-EJ_1(\varepsilon))>x\right)\le \exp\left(-\frac{x}{2h}\sinh^{-1}\left(\frac{xh}{2\mathrm{Var}X_t^{(S,h)}}\right)\right). \qquad (3.21)
$$

Given $0<\delta<x$ and $\eta>0$, take ε small enough so that probability of the term on the left of (3.19) exceeding η in modulus is less than δ. Then from (3.17) and (3.21)

$$
\begin{aligned}
P\left(X_t^{(S,h)}>x\right) &\le \limsup_{\varepsilon\downarrow 0} P\left(\sum_{i=1}^{N_t(\varepsilon)}(J_i(\varepsilon)-EJ_1(\varepsilon))>x-\delta\right)+\eta\\
&\le \exp\left(-\frac{(x-\delta)}{2h}\sinh^{-1}\left(\frac{(x-\delta)h}{2\mathrm{Var}X_t^{(S,h)}}\right)\right)+\eta.
\end{aligned}
$$

Letting $\delta\downarrow 0$ and $\eta\downarrow 0$ proves (3.16). □

4 Proof of Theorem 2.1

Assume $\sigma^2=0$ and $\overline{\Pi}(0+)=\infty$, and $b(t)>0$ is a continuous, strictly increasing function satisfying (2.1) with $0<\alpha<2$, and having continuous, strictly increasing inverse function $B(x)$. Choose $x_0>0$ so that $\overline{\Pi}(x)>0$ for $0<x\le x_0$. We divide the proof into two sections, considering two-sided and one-sided cases separately.

(i) Two-sided Case. Suppose first that $J_{r+1}<\infty$ for an $r\ge 0$, and we will prove (2.5). For $0<x\le x_0$ define

$$
\psi(x)=\sqrt{\frac{B(x)}{\overline{\overline{\Pi}}(x)}}, \qquad (4.1)
$$

with inverse function

$$\phi(x) = \psi^{\leftarrow}(x) = \inf\{y > 0 : \psi(y) > x\}.$$

Then $\psi(x)$ and $\phi(x)$ are positive and nondecreasing in $0 < x \le x_0$ with $\psi(0) = \phi(0) = 0$ and ψ is right-continuous (since $\overline{\Pi}$ is right continuous). These functions have the additional properties:

$$\frac{B(x)}{\psi(x-)} \to 0, \quad \frac{B(\phi(x))}{x} \to 0, \quad \text{and} \quad \frac{\phi(x)}{b(x)} \to 0, \ \text{as } x \downarrow 0. \tag{4.2}$$

The first of these follows from (4.1) because $B(x)\overline{\Pi}(x-) \to 0$ as a consequence of $J_{r+1} < \infty$ (which implies (3.7)). The second follows from the first by replacing x with $\phi(x)$ and noting that $\psi(\phi(x)-) \le x \le \psi(\phi(x)+) = \psi(\phi(x))$, so

$$\frac{B(\phi(x))}{x} \le \frac{B(\phi(x))}{\psi(\phi(x)-)} \to 0.$$

The third property in (4.2) follows from the second by using (2.1) to argue

$$\phi(x) = b(B(\phi(x))) \le b(\delta x) \ (\text{for small } x) \ \le c\delta^{1/\alpha} b(x), \ \text{for any } 0 < \delta < 1.$$

An additional property,

$$x\overline{\Pi}(\phi(x)) = \frac{xB(\phi(x))}{\psi^2(\phi(x))} \le \frac{B(\phi(x))}{x} \to 0, \ \text{as } x \downarrow 0 \tag{4.3}$$

then follows because $\psi(\phi(x)) \ge x$.

Recall the Itô decomposition in (3.15), and from now on write X_t^h for $X_t^{(S,h)}$. From (3.15) we have

$$\begin{aligned} {}^{(r)}\widetilde{X}_t - t\nu(h) &= X_t^h + X_t^{(B,h)} - \sum_{i=1}^{r} \widetilde{\Delta X}_t^{(i)} \\ &= X_t^h + {}^{(*)}\sum \Delta X_s \mathbf{1}_{\{|\Delta X_s| > h\}} - \sum_{i=1}^{r} \widetilde{\Delta X}_t^{(i)} \mathbf{1}_{\{|\widetilde{\Delta X}_t^{(i)}| \le h\}}, \end{aligned} \tag{4.4}$$

where ${}^{(*)}\sum$ denotes summation of jumps ΔX_s, $0 < s \le t$, with $|\Delta X_s| > h$ and terms corresponding to $\widetilde{\Delta X}_t^{(1)}, \dots, \widetilde{\Delta X}_t^{(r)}$ removed. Thus

$$\begin{aligned} & P\left(\left|{}^{(*)}\sum \Delta X_s \mathbf{1}_{\{|\Delta X_s| > h\}}\right| > 0 \text{ i.o. as } t \downarrow 0\right) \\ \le \ & P\left(\exists\, s \le t \text{ such that } |\Delta X_s| \le |\widetilde{\Delta X}_t^{(r+1)}| \text{ and } |\Delta X_s| > h \text{ i.o. as } t \downarrow 0\right) \\ \le \ & P\left(|\widetilde{\Delta X}_t^{(r+1)}| > h \text{ i.o. as } t \downarrow 0\right). \end{aligned}$$

Now choose $h = \delta b(t)$, $\delta > 0$. Then the last term is 0 by Lemma 3.1, and the last term in (4.4) is $\le r\delta b(t)$ in magnitude. So we deduce

$${}^{(r)}\widetilde{X}_t - t\nu(\delta b(t)) = X_t^{\delta b(t)} + O(\delta b(t)), \ \text{a.s.}, \ \text{as } t \downarrow 0. \tag{4.5}$$

Take $x > 0$ and define

$$N_t^{\phi(x)} := \#\{\text{jumps } \Delta X_s \text{ with } 0 < s \le t \text{ and } |\Delta X_s| > \phi(x)\}.$$

Then for $k = 1, 2, \dots$

$$\{N_t^{\phi(x)} \ge k\} \quad = \quad \{|\Delta X_s| > \phi(x) \text{ for at least } k \text{ values of } s \le t\}$$

$$= \quad \{|\widetilde{\Delta X}_t^{(k)}| > \phi(x)\}.$$

Now choose $t = 2^{-n+1}$ and $x = 2^{-n}$. Then from (3.2)

$$\begin{aligned}\sum_n P\left(N_{2^{-n+1}}^{\phi(2^{-n})} \geq k\right) &= \sum_n P\left(|\widetilde{\Delta X}_{2^{-n+1}}^{(k)}| > \phi(2^n)\right) \\ &\leq \frac{1}{(k+1)!}\sum_n \left(2^{-n+1}\overline{\Pi}(\phi(2^n))\right)^k.\end{aligned}$$

Now, with $c_k := k(2^k - 1)$,

$$\begin{aligned}\sum_n \left(2^{-n+1}\overline{\Pi}(\phi(2^n))\right)^k &\leq c_k \sum_n \int_{2^{-n-1}}^{2^{-n}} x^{k-1}\overline{\Pi}^k(\phi(x))\mathrm{d}x \\ &\leq c_k \int_0^1 x^{-k-1} B^k(\phi(x))\mathrm{d}x \quad \text{(by (4.3))} \\ &\leq \frac{c_k}{k}\int_0^1 x^{-k}\mathrm{d}B^k(\phi(x)) \quad \text{(integrate by parts)} \\ &\leq \frac{c_k}{k}\int_0^{\phi(1)} \psi^{-k}(y)\mathrm{d}B^k(y) \quad \text{(change variable)} \\ &= \frac{c_k}{k}\int_0^{\phi(1)} \frac{\overline{\Pi}^{k/2}(y)}{B^{k/2}(y)}\mathrm{d}B^k(y) \quad \text{(by (4.1))}. \end{aligned} \tag{4.6}$$

We assumed $b(\cdot)$ is strictly increasing, so $B(\cdot)$ is continuous. This means that $\mathrm{d}B^k(y) = kB^{k-1}(y)\mathrm{d}B(y)$ and the last expression is of the order of

$$\begin{aligned}c_k \int_0^1 \overline{\Pi}^{k/2}(y)B^{k/2-1}(y)\mathrm{d}B(y) &= \frac{2c_k}{k}\int_0^1 \overline{\Pi}^{k/2}(y)\mathrm{d}B^{k/2}(y) \\ &= \frac{2c_k}{k}J_{k/2} \quad \text{(see (2.2))}.\end{aligned}$$

This is finite when $k \geq 2r + 2$. So

$$\limsup_n N_{2^{-n+1}}^{\phi(2^{-n})} \leq 2r + 2 \text{ a.s.} \tag{4.7}$$

Given $t > 0$ choose $n = n(t)$ so that $2^{-n} < t \leq 2^{-n+1}$. Then

$$\begin{aligned}N_t^{\phi(t)} &= \#\{\Delta X_s \text{ with } 0 < s \leq t \text{ and } |\Delta X_s| > \phi(t)\} \\ &\leq \#\{\Delta X_s \text{ with } 0 < s \leq 2^{-n+1} \text{ and } |\Delta X_s| > \phi(2^{-n})\} \\ &= N_{2^{-n+1}}^{\phi(2^{-n})},\end{aligned}$$

giving

$$\limsup_{t\downarrow 0} N_t^{\phi(t)} \leq 2r + 2 \text{ a.s.} \tag{4.8}$$

Recall that X_t^h is the compensated sum of jumps less than or equal to h in modulus, so when $0 < \phi < h$,

$$\begin{aligned}X_t^h - X_t^\phi &= \lim_{\varepsilon\downarrow 0}\left(\sum_{0<s\leq t}\Delta X_s \mathbf{1}_{\{\varepsilon<|\Delta X_s|\leq h\}} - t\int_{\varepsilon<|x|\leq h} x\Pi(\mathrm{d}x)\right) \\ &\quad - \lim_{\varepsilon\downarrow 0}\left(\sum_{0<s\leq t}\Delta X_s \mathbf{1}_{\{\varepsilon<|\Delta X_s|\leq \phi\}} - t\int_{\varepsilon<|x|\leq \phi} x\Pi(\mathrm{d}x)\right)\end{aligned}$$

$$= \sum_{0<s\le t} \Delta X_s \mathbf{1}_{\{\phi<|\Delta X_s|\le h\}} - t\int_{\phi<|x|\le h} x\Pi(\mathrm{d}x) \qquad (4.9)$$

in which we set $\phi = \phi(t)$ and $h = \delta b(t)$, $\delta > 0$. From (4.5) and (4.9)

$$\begin{aligned}
|{}^{(r)}\widetilde{X}_t - t\nu(\delta b(t)) - X_t^{\phi(t)}| &\le |X_t^{\delta b(t)} - X_t^{\phi(t)}| + O(\delta b(t)) \text{ a.s.} \\
&\le \delta b(t) N_t^{\phi(t)} + t\Big|\int_{\phi(t)<|x|\le\delta b(t)} x\Pi(\mathrm{d}x)\Big| + O(\delta b(t)) \text{ a.s.} \\
&\le O(\delta b(t)) + \delta t b(t)\overline{\Pi}(\phi(t)) \quad (\text{a.s., by } (4.8)) \\
&= O(\delta b(t)) \quad (\text{a.s., using } (4.3)). \qquad (4.10)
\end{aligned}$$

Since δ may be arbitrarily small it remains only to show $X_t^{\phi(t)} = o(b(t))$ a.s.

Given $t > 0$ choose $n = n(t)$ so that $2^{-n} < t \le 2^{-n+1}$. Write

$$\begin{aligned}
|X_t^{\phi(t)} - X_t^{\phi(2^{-n})}| &= \Big|\sum_{0<s\le t} \Delta X_s \mathbf{1}_{\{\phi(2^{-n})<|\Delta X_s|\le\phi(t)\}} - t\int_{\phi(2^{-n})<|x|\le\phi(t)} x\Pi(\mathrm{d}x)\Big| \\
&\le \phi(t) N_{2^{-n+1}}^{\phi(2^{-n})} + 2^{-n+1}\phi(t)\overline{\Pi}(\phi(2^{-n})) \\
&= O(\phi(t)) \quad (\text{by } (4.7) \text{ and } (4.3)) \\
&= o(b(t)), \text{ a.s.} \quad (\text{by } (4.2)). \qquad (4.11)
\end{aligned}$$

So we need only deal with $X_t^{\phi(2^{-n})}$ when $2^{-n} < t \le 2^{-n+1}$.

We need some more calculations. Note that $0 < x \le y$ implies $B(x) \le B(y)$ implies

$$\frac{b(B(x))}{(B(x))^{1/\alpha}} \le \frac{cb(B(y))}{(B(y))^{1/\alpha}}$$

(by (2.1)), and this implies

$$\frac{x}{(B(x))^{1/\alpha}} \le \frac{cy}{(B(y))^{1/\alpha}}$$

(since $b(B(x)) = x$). Thus

$$\frac{B(y)}{B(x)} \le \left(\frac{cy}{x}\right)^{\alpha}, \ 0 < x \le y.$$

Hence (recall $\sigma^2 = 0$, and the definition of $V(x)$ in (1.2))

$$\begin{aligned}
y^{-2}B(y)V(y) &\le 2y^{-2}B(y)\int_0^y x\overline{\Pi}(x)\mathrm{d}x \\
&\le 2c^\alpha y^{-2}\int_0^y x(y/x)^\alpha B(x)\overline{\Pi}(x)\mathrm{d}x \\
&= 2c^\alpha y^{-2+\alpha}\int_0^y x^{1-\alpha}o(1)\mathrm{d}x \quad (\text{by } (3.7)) \\
&= o(1),
\end{aligned}$$

and so

$$V(y) = o\left(\frac{y^2}{B(y)}\right), \qquad (4.12)$$

or, equivalently,

$$\frac{xV(b(x))}{b^2(x)} = o(1), \text{ as } x \downarrow 0. \qquad (4.13)$$

From (4.2), (4.13) and Chebychev's inequality we get for $\eta > 0$ and small t

$$P\left(|X_t^{\phi(t)}| > \eta b(t)\right) \le \frac{tV(\phi(t))}{\eta^2 b^2(t)} \le \frac{tV(b(t))}{\eta^2 b^2(t)} = o(1).$$

Thus $X_t^{\phi(t)} = o_P(b(t))$.

Now we need the following maximal inequality: for $h > 0$, $x > 0$, with m_t^h as a median of $X_t^{(S,h)}$,

$$\begin{aligned} P\left(\sup_{0<s\le t} |X_s^{(S,h)} - m_s^h| > 2x\right) &= \lim_k P\left(\max_{1\le j\le \lceil kt\rceil} |X_{j/k}^{(S,h)} - m_{j/k}^h| > 2x\right) \\ &\le 2\lim_k P\left(|X_{\lceil kt\rceil/k}^{(S,h)}| > x\right) \\ &= 2P\left(|X_t^{(S,h)}| > x\right). \end{aligned}$$

Here we used the strong symmetrisation inequality (Stout (1974, p.116)) applied to the random walk

$$X_{j/k}^{(S,h)} = \sum_{i=1}^{j} \left(X_{i/k}^{(S,h)} - X_{(i-1)/k}^{(S,h)}\right),\ j = 1, 2, \dots,\ k = 1, 2, \dots$$

Since $X_t^{\phi(t)} = X_t^{(S,\phi(t))} = o_P(b(t))$, we have $\sup_{0<s\le t} |m_s^{\phi(t)}| = o(b(t))$. So, with $t = 2^{-n+1}$, $h = \phi(2^{-n})$ and $x = \delta b(2^{-n})$, we get

$$P\left(\sup_{2^{-n}<s\le 2^{-n+1}} |X_s^{\phi(2^{-n})}| > 2\delta b(2^{-n})\right) \le 2P\left(|X_{2^{-n+1}}^{\phi(2^{-n})}| > \delta b(2^{-n})\right),$$

for large n.

Using Prokhorov's inequality (in Lemma 3.3), the last expression does not exceed

$$\exp\left(-\frac{\delta b(2^{-n})}{2\phi(2^{-n})} \sinh^{-1} q_n\right), \tag{4.14}$$

where

$$q_n := \frac{\delta b(2^{-n})\phi(2^{-n})}{2^{-n+2} V(\phi(2^{-n}))}.$$

If $\sinh^{-1} q_n > 2/\delta$ then (4.14) is bounded by

$$\exp\left(-\frac{b(2^{-n})}{\phi(2^{-n})}\right). \tag{4.15}$$

Alternatively, $\sinh^{-1} q_n \le 2/\delta$. Since the function $x \mapsto \sinh x$ is convex, we can find $c_\delta > 0$ so that $\sinh(c_\delta x) \le x$ for $0 < x \le \sinh(2/\delta)$. Then $0 < \sinh^{-1} q_n \le 2/\delta$ implies $\sinh(c_\delta q_n) \le q_n$, so $\sinh^{-1} q_n \ge c_\delta q_n$.

Now $B(\phi(x)) = o(x)$ (see (4.2)) implies, for small x,

$$\frac{b(B(\phi(x)))}{B^{1/\alpha}(\phi(x))} \le \frac{cb(x)}{x^{1/\alpha}} \quad \text{(by (2.1))},$$

hence

$$\frac{\phi(x)}{b(x)} \le \frac{cB^{1/\alpha}(\phi(x))}{x^{1/\alpha}} \text{ or, equivalently, } \frac{x}{B(\phi(x))} \le \left(\frac{cb(x)}{\phi(x)}\right)^{\alpha}. \tag{4.16}$$

Thus, by (4.12),

$$\frac{yV(\phi(y))}{b^2(y)} = o\left(\frac{y\phi^2(y)}{B(\phi(y))b^2(y)}\right)$$

$$\begin{aligned} &\leq o\left(\left(\frac{b(y)}{\phi(y)}\right)^{\alpha}\frac{\phi^2(y)}{b^2(y)}\right) \\ &= o\left(\frac{\phi(y)}{b(y)}\right)^{2-\alpha}, \text{ as } y\downarrow 0. \end{aligned}$$

So

$$\frac{\delta b(2^{-n})}{2\phi(2^{-n})}\sinh^{-1} q_n \geq \frac{c_\delta \delta^2 b^2(2^{-n})}{2^{-n+2}V(\phi(2^{-n}))} \geq \left(\frac{b(2^{-n})}{\phi(2^{-n})}\right)^{2-\alpha},$$

for large n, and (4.14) is bounded in this case by

$$\exp\left(-\left(\frac{b(2^{-n})}{\phi(2^{-n})}\right)^{2-\alpha}\right), \tag{4.17}$$

for large n. Thus (4.15) and (4.17) give

$$P\left(\sup_{0<s\leq t}|X_s^{\phi(2^{-n})}| > 2\delta b(2^{-n})\right) \leq 2\exp\left(-\left(\frac{b(2^{-n})}{\phi(2^{-n})}\right)^{\min(1,2-\alpha)}\right) \leq \left(\frac{\phi(2^{-n})}{b(2^{-n})}\right)^k,$$

for any $k>0$ and all large n.

Now by (4.16)

$$\begin{aligned} \sum_{n\geq 1}\left(\frac{\phi(2^{-n})}{b(2^{-n})}\right)^k &\leq c^k \sum_{n\geq 1}\frac{B^{k/\alpha}(\phi(2^{-n}))}{2^{-nk/\alpha}} \\ &\leq kc^k(2^{k/\alpha}-1)\sum_{n\geq 1}\int_{2^{-n}}^{2^{-n+1}}\frac{B^{k/\alpha}(\phi(x))}{x^{k/\alpha+1}}\mathrm{d}x \\ &= kc^k(2^{k/\alpha}-1)\int_0^1 \frac{B^{k/\alpha}(\phi(x))}{x^{k/\alpha+1}}\mathrm{d}x. \end{aligned}$$

In (4.6) the last integral was shown to be smaller than a constant multiple of $J_{k/2\alpha}$. But $J_{k/2\alpha}$ is finite when $k \geq 2\alpha(r+1)$, so by choosing k large enough we can deduce that

$$\sum_n P\left(\sup_{2^{-n}<s\leq 2^{-n+1}}|X_s^{\phi(2^{-n})}| > 2\delta b(2^{-n})\right) < \infty.$$

Hence, since δ is arbitrary,

$$\sup_{2^{-n}<s\leq 2^{-n+1}} X_s^{\phi(2^{-n})} = o(b(2^{-n})) = o(b(t)) \text{ a.s.}, \tag{4.18}$$

when $2^{-n} < t \leq 2^{-n+1}$. (2.5) now follows from (4.10), (4.11) and (4.18), after letting $t\downarrow 0$ then $\delta\downarrow 0$, and noting that, for $0<\delta<1$,

$$\begin{aligned} \frac{t|\nu(b(t))-\nu(b(\delta t))|}{b(t)} &= \frac{t|\int_{b(\delta t)<|x|\leq b(t)} x\Pi(\mathrm{d}x)|}{b(t)} \\ &\leq t\overline{\Pi}(b(\delta t)) = \delta^{-1}(t\delta)\overline{\Pi}(b(t\delta)) \\ &\to 0, \text{ as } t\downarrow 0. \end{aligned}$$

Conversely, assume (2.4) holds for some function $a(t)\in \mathbb{R}$, so that

$$\limsup_{t\downarrow 0}\frac{|{}^{(r)}\widetilde{X}_t - a(t)|}{b(t)} < M < \infty \text{ a.s.} \tag{4.19}$$

for some constant $M > 0$. Proposition 3.3 of Fan (2015) and (3.1) give

$$\begin{aligned} 4P\left(|{}^{(r)}\widetilde{X}_t - a(t)| > Mb(t)\right) &\geq P\left(|\widetilde{\Delta X}_t^{(r+1)}| > 4Mb(t)\right) \\ &= 1 - e^{-t\overline{\Pi}(4Mb(t))} \sum_{i=0}^{r} \frac{\left(t\overline{\Pi}(4Mb(t))\right)^i}{i!}. \end{aligned} \tag{4.20}$$

If $t_k\overline{\Pi}(4Mb(t_k)) \to \xi \in (0, \infty]$ for a subsequence $t_k \downarrow 0$ then the RHS of (4.20) converges to

$$1 - e^{-\xi} \sum_{i=0}^{r} \frac{\xi^i}{i!} > 0,$$

contradicting the fact that (in view of (4.19)) the LHS of (4.20) converges to 0 as $t \downarrow 0$. Thus $\lim_{t\downarrow 0} t\overline{\Pi}(4Mb(t)) = 0$. Then

$$P\left(|\widetilde{\Delta X}_t^{(1)}| > 4Mb(t)\right) = 1 - e^{-t\overline{\Pi}(4Mb(t))} \to 0, \tag{4.21}$$

so we get

$$\begin{aligned} P\left(|X_t - a(t)| > (4r+1)Mb(t)\right) &\leq P\left(|{}^{(r)}\widetilde{X}_t - a(t)| > Mb(t)\right) \\ &\quad + P\left(|\widetilde{\Delta X}_t^{(1)}| > 4Mb(t)\right) \to 0, \end{aligned} \tag{4.22}$$

for the particular value of M. Taken any sequence $t_k \downarrow 0$ and a further subsequence $t_{k'} \downarrow 0$ so that

$$\frac{X_{t_{k'}} - a(t_{k'})}{b(t_{k'})} \xrightarrow{\mathrm{D}} Z',$$

where Z' is an infinitely divisible rv (by Lemma 4.1 of Maller and Mason (2008)) such that $P(|Z'| > (4r+1)M) = 0$. As a bounded infinitely divisible rv, Z' is degenerate at a constant, $Z' = z'$, say. So

$$\frac{X_{t_{k'}} - a(t_{k'})}{b(t_{k'})} \xrightarrow{\mathrm{P}} z',$$

and then (by Theorem 15.14 in Kallenberg (2002)),

$$t_{k'}\overline{\Pi}(\delta b(t_{k'})) = o(1), \quad a(t_{k'}) = t_{k'}\nu(b(t_{k'})) + o(b(t_{k'})), \quad t_{k'}V(\delta b(t_{k'})) = o(b^2(t_{k'})), \tag{4.23}$$

as $k' \to \infty$, for all $\delta > 0$. Since this holds for all subsequences, we in fact have

$$t\overline{\Pi}(\delta b(t)) = o(1), \quad a(t) = t\nu(b(t)) + o(b(t), \quad \text{and} \quad tV(\delta b(t)) = o(b^2(t)), \text{ as } t \downarrow 0, \tag{4.24}$$

for all $\delta > 0$. Then from (4.19) we deduce that

$$\limsup_{t\downarrow 0} \frac{|{}^{(r)}\widetilde{X}_t - t\nu(b(t))|}{b(t)} < \infty \text{ a.s.} \tag{4.25}$$

Using Δ to denote a difference, we can calculate

$$\begin{aligned} |\Delta\nu(b(t))| &= \lim_{\varepsilon\downarrow 0} |\nu(b(t)) - \nu(b(t \pm \varepsilon))| \\ &= \lim_{\varepsilon\downarrow 0} |\int_{b(t\pm\varepsilon) < |x| \leq b(t)} x\Pi(\mathrm{d}x)| \\ &\leq \limsup_{\varepsilon\downarrow 0} b(t+\varepsilon)\overline{\Pi}(b(t-\varepsilon)) \\ &= b(t)\overline{\Pi}(b(t)-) \end{aligned}$$

(since $b(\cdot)$ is continuous). But $t\overline{\Pi}(\delta b(t)) \to 0$ for all $\delta > 0$ (by (4.24)) implies $t\overline{\Pi}(b(t)-) \to 0$, as $t \downarrow 0$, so $t|\Delta\nu(b(t))| = o(b(t))$ as $t \downarrow 0$. Then we get from (4.25) and the monotonicity of $b(\cdot)$ that

$$\begin{aligned} &\left|\frac{\Delta\left({}^{(r)}\widetilde{X}_t - t\nu(b(t))\right)}{b(t)}\right| \\ &\leq \frac{|{}^{(r)}\widetilde{X}_t - t\nu(b(t))|}{b(t)} + \frac{\lim_{\varepsilon\downarrow 0}|{}^{(r)}\widetilde{X}_{t-\varepsilon} - t\nu(b(t-\varepsilon))|}{b(t)} + \frac{t|\Delta\nu(b(t))|}{b(t)} \\ = \quad & O(1), \text{ a.s., as } t \downarrow 0. \end{aligned}$$

Consequently $\sup_{0<s\leq t}|\Delta^{(r)}\widetilde{X}_s| = O(b(t))$ a.s. as $t \downarrow 0$. It follows then from (3.11) that $\widetilde{\Delta X}_t^{(r+1)} = O(b(t))$ a.s. as $t \downarrow 0$, and we conclude $J_{r+1} < \infty$ from Lemma 3.1.

(ii) One-sided Case. Assume $\overline{\Pi}^+(0+) = \infty$. Then there are infinitely many positive jumps a.s. in any neighbourhood of 0. Hence $\Delta X_t^{(i)} = (\Delta X^+)_t^{(i)}$, $i = 1, 2, \ldots$, where $(\Delta X^+)_t^{(i)}$ is the i–th largest among ΔX_s^+ for $s \leq t$.

Recall the definitions of $J_r^{(\pm)}$ in (2.3) and assume at first that $J_{r+1}^{(+)} < \infty$ and $J_1^{(-)} < \infty$. Rewrite (3.15) in the form

$$ {}^{(r)}X_t - t\nu(h) = X_t^{(S,+,h)} - \sum_{i=1}^{r}(\Delta X^+)_t^{(i)} + X_t^{(B,+,h)} + X_t^{(S,-,h)} + X_t^{(B,-,h)},\ t > 0, h > 0, \tag{4.26}$$

where

$$X_t^{(S,\pm,h)} = \text{a.s.}\lim_{\varepsilon\downarrow 0}\left(\sum_{0<s\leq t}\Delta X_s \mathbf{1}_{\{\varepsilon<\pm\Delta X_s\leq h\}} - t\int_{\varepsilon<\pm x\leq h} x\Pi(\mathrm{d}x)\right)$$

and

$$X_t^{(B,\pm,h)} = \sum_{0<s\leq t}\Delta X_s \mathbf{1}_{\{\pm\Delta X_s>h\}}.$$

In these, we take $h = b(t)$. Then apply Part (i) of Theorem 2.1 to the positive jump process (so, replace ${}^{(r)}\widetilde{X}_t$ by ${}^{(r)}X_t$). Since $J_{r+1}^{(+)} < \infty$, we can infer from (2.5) that

$$X_t^{(S,+,b(t))} - \sum_{i=1}^{r}(\Delta X_t^+)^{(i)} + X_t^{(B,+,b(t))} = o(b(t)) \text{ a.s.},$$

and since $J_1^{(-)} < \infty$, we similarly have $X_t^{(S,-,b(t))} + X_t^{(B,-,b(t))} = o(b(t))$ a.s. (Note that the corresponding centering terms which would be denoted by $\nu^{(\pm)}(\cdot)$ are zero in these applications.) Substituting in (4.26), we get (2.5) with ${}^{(r)}\widetilde{X}_t$ replaced by ${}^{(r)}X_t$.

Conversely assume (2.4) holds with ${}^{(r)}\widetilde{X}_t$ replaced by ${}^{(r)}X_t$. Proposition 3.3 of Fan (2015) (one-sided version) and the one-sided version of the lower bound in (3.2) give, for some $M > 0$,

$$\begin{aligned} 4P\left(|{}^{(r)}X_t - a(t)| > Mb(t)\right) \quad &\geq \quad P\left(\Delta X_t^{(r+1)} > 4Mb(t)\right) \\ &= \quad 1 - e^{-t\overline{\Pi}^+(4Mb(t))}\sum_{i=0}^{r}\frac{\left(t\overline{\Pi}^+(4Mb(t))\right)^i}{i!}. \end{aligned}$$

Following the same argument as in (4.21), we get

$$P\left(\Delta X_t^{(1)} > Mb(t)\right) = 1 - e^{-t\overline{\Pi}^+(Mb(t))} \to 0, \text{ as } t \downarrow 0,$$

and consequently $P\left(|^{(r)}\widetilde{X}_t - a(t)| > (4r+1)Mb(t)\right) \to 0$, just as in (4.22). From this we deduce a one-sided version of (4.25), namely,

$$\limsup_{t\downarrow 0} \frac{|^{(r)}X_t - t\nu(b(t))|}{b(t)} < \infty \text{ a.s.}$$

We again have $t|\Delta\nu(b(t))| = o(b(t))$ as $t \downarrow 0$, and, by (3.12),

$$\Delta X_t^{(r+1)} = \sup_{0<s\le t} \Delta^{(r)}X_s \quad \text{and} \quad (\Delta X^-)_t^{(1)} \le \sup_{0<s\le t} |\Delta^{(r)}X_s|, \ t > 0.$$

So we can conclude $\Delta X_t^{(r+1)} = O(b(t))$ a.s. and $(\Delta X^-)_t^{(1)} = O(b(t))$ a.s. Then by applying Lemma 3.1 to the positive and negative jumps separately we get $J_{r+1}^{(+)} < \infty$ and $J_1^{(-)} < \infty$. □

5 Relative Stability

X_t is said to be *relatively stable* in probability as $t \downarrow 0$ if there is a non-stochastic function $b(t) > 0$ such that $X_t/b(t)$ tends in probability to a nonzero constant, which by rescaling we can take to be ± 1; thus, if for some $b(t) > 0$ we have

$$\frac{X_t}{b(t)} \xrightarrow{\text{P}} \pm 1, \text{ as } t \downarrow 0. \tag{5.1}$$

If either of these holds, $b(t)$ may be chosen to be continuous, strictly increasing on $(0, \infty)$, and regularly varying with index 1 as $t \downarrow 0$. Further, (5.1) is equivalent to $|X_t|/b(t) \xrightarrow{\text{P}} 1$, as $t \downarrow 0$; thus, X does not change sign near 0 with probability approaching 1, when $|X_t|$ is relatively stable in probability at 0. These properties and various other equivalences for (5.1) are in Doney and Maller (2002a) and Griffin and Maller (2013). Among them, we note two in particular to be used in the present paper. For the first, assume $\overline{\Pi}(0+) = \infty$, and define the function

$$A(x) = \gamma + \overline{\Pi}^+(1) - \overline{\Pi}^-(1) - \int_x^1 (\overline{\Pi}^+(y) - \overline{\Pi}^-(y))\mathrm{d}y, \ x > 0.$$

Then (5.1) implies that $A(x)$ is of constant sign near 0, i.e., $A(x) > 0$ for all small x or $A(x) < 0$ for all small x, the sign corresponding to that in (5.1), and[4]

$$\lim_{x\downarrow 0} \frac{\pm A(x)}{x\overline{\Pi}(x)} = \infty. \tag{5.2}$$

Conversely, (5.2) implies (5.1), where $b(t)$ can be taken to satisfy $b(t) = t|A(b(t))|$ for all small t, in the sense that it is asymptotically equivalent to a function satisfying this, for small t. The function $t \mapsto t^{-\beta}b(t)$, where $0 < \beta < 1$ and $t > 0$, is regularly varying with index $1 - \beta$ as $t \downarrow 0$, hence is asymptotically equivalent to a monotone function (Bingham, Goldie and Teugels (1987, p.23)). Thus $b(\cdot)$ can be taken to satisfy (2.1) with $\alpha = 1/\beta > 1$.

For the second property: X is relatively stable in probability at 0 iff there is a nonstochastic function $b^*(t) > 0$ such that every sequence $t_k \to 0$ contains a subsequence $t_{k'} \to 0$ with

$$\frac{X_{t_{k'}}}{b^*(t_{k'})} \xrightarrow{\text{P}} c', \tag{5.3}$$

where c' is a constant with $0 < |c'| < \infty$ which may depend on the choice of subsequence (Griffin and Maller (2013)).

[4] When (5.2) holds, $A(x) \sim \nu(x)$ (see (1.2)) as $x \downarrow 0$, so $A(x)$ can be replaced by $\nu(x)$ in (5.2), but there is some advantage to working with the continuous function $A(x)$.

In this section we extend the idea of relative stability to describe the convergences ${}^{(r)}\widetilde{X}_t/b(t) \to \pm 1$ or ${}^{(r)}X_t/b(t) \to \pm 1$, where the convergence may be in probability or almost sure, as $t \downarrow 0$. Since we also consider the modulus convergences, $|{}^{(r)}\widetilde{X}_t|/b(t) \to 1$ or $|{}^{(r)}X_t|/b(t) \to 1$, we split the almost sure results into two theorems, Theorem 5.1 and Theorem 5.3. Relative stability in probability is characterised in Proposition 5.2.

Theorem 5.1. *Assume $\sigma^2 = 0$ and $\overline{\Pi}(0+) = \infty$ and fix $r = 0, 1, 2, \ldots$. Then*
(a) ${}^{(r)}\widetilde{X}_t$ is a.s. relatively stable as $t \downarrow 0$, i.e., there is a function $b(t) > 0$ on $(0, \infty)$ such that

$$\frac{{}^{(r)}\widetilde{X}_t}{b(t)} \to \pm 1 \text{ a.s.,} \tag{5.4}$$

iff $\pm A(x) > 0$ for all small x, $0 < x \le x_0$, say, and

$$\int_0^{x_0} \left(\frac{x\overline{\Pi}(x)}{\pm A(x)} \right)^{r+1} \frac{\mathrm{d}x}{x} < \infty \tag{5.5}$$

(where the $+$ and $-$ signs are to be taken together);
(b) there is a function $b(t) > 0$ on $(0, \infty)$ such that

$$\frac{|{}^{(r)}\widetilde{X}_t|}{b(t)} \to 1 \text{ a.s.} \tag{5.6}$$

iff $|A(x)| > 0$ for all small x, $0 < x \le x_0$, say, and

$$\int_0^{x_0} \left(\frac{x\overline{\Pi}(x)}{|A(x)|} \right)^{r+1} \frac{\mathrm{d}x}{x} < \infty. \tag{5.7}$$

The sign in (5.6) *is determined by the sign of $A(x)$ for small x, which is constant.*
(c) The conditions in (5.4) *and* (5.6) *are equivalent, as are* (5.5) *and* (5.7). *When $r = 0$ they hold iff $X \in bv$ with drift $\mathrm{d}_X \neq 0$, in which case $\lim_{t\downarrow 0} X_t/(t|\mathrm{d}_X|) = 1$ a.s.*

Remark: The case $r = 0$ in Part (c) of Theorem 5.1 is proved in Doney and Maller (2002a, Thm. 4.2), so the cases $r = 1, 2, \ldots$ constitute a generalisation of this. Similarly for Theorem 5.3 below.

Before beginning the proof of Theorem 5.1, we prove the following proposition characterising relative stability in probability of the trimmed processes.

Proposition 5.2. *Suppose $\overline{\Pi}(0+) = \infty$. Then for $r = 1, 2, \ldots$,*
(a) (i) There is a function $b(t) > 0$ on $(0, \infty)$ such that

$$\frac{|{}^{(r)}\widetilde{X}_t|}{b(t)} \xrightarrow{\mathrm{P}} 1 \quad \text{iff} \quad \frac{|X_t|}{b(t)} \xrightarrow{\mathrm{P}} 1 \text{ as } t \downarrow 0. \tag{5.8}$$

(ii) There is a function $b(t) > 0$ on $(0, \infty)$ such that

$$\frac{{}^{(r)}\widetilde{X}_t}{b(t)} \xrightarrow{\mathrm{P}} \pm 1 \quad \text{iff} \quad \frac{X_t}{b(t)} \xrightarrow{\mathrm{P}} \pm 1 \text{ as } t \downarrow 0. \tag{5.9}$$

All conditions in (5.8) *and* (5.9) *are equivalent, and equivalent to* (5.2) *with the appropriate correspondences in signs.*

(b) Assuming $\overline{\Pi}^{+}(0+) = \infty$, the results remain true if ${}^{(r)}\widetilde{X}_t$ is replaced by ${}^{(r)}X_t$ throughout.

Proof of Proposition 5.2: (a) Suppose $\overline{\Pi}(0+) = \infty$. (i) Assume the first condition in (5.8). Then

$$\lim_{t\downarrow 0} P\left(|{}^{(r)}\widetilde{X}_t| > 2b(t) \right) = 0.$$

The inequality in (4.20) with $a(t) = 0$ then gives

$$\lim_{t\downarrow 0} P\left(|\widetilde{\Delta X}_t^{(r+1)}| > 8b(t)\right) = 0.$$

The same argument as in (4.21) and (4.22) with $a(t) = 0$ gives

$$\lim_{t\downarrow 0} P\left(|X_t| > (8r+2)b(t)\right) = 0.$$

Taken any sequence $t_k \downarrow 0$ and a further subsequence $t_{k'} \downarrow 0$ so that

$$\frac{X_{t_{k'}}}{b(t_{k'})} \xrightarrow{\mathrm{D}} Z',$$

where Z' is a bounded infinitely divisible rv, $|Z'| \le 8r+2$ a.s. Thus Z' is degenerate at a constant, $Z' = z'$, say. If $z' = 0$ then $t_{k'}\overline{\Pi}(\delta b(t_{k'})) \to 0$ for every $\delta > 0$ by (4.23), so $\widetilde{\Delta X}_{t_{k'}}^{(1)}/b(t_{k'}) \xrightarrow{\mathrm{P}} 0$ and consequently ${}^{(r)}\widetilde{X}_{t_{k'}}/b(t_{k'}) \xrightarrow{\mathrm{P}} 0$, which is not possible. Thus $z' \neq 0$ and so every sequence t_k contains a subsequence $t_{k'} \downarrow 0$ for which $X_{t_{k'}}/b(t_{k'})$ converges in probability to a nonzero constant. This is (5.3), and implies relative stability of X which in turn implies the second condition in (5.8). Conversely the second condition in (5.8) is equivalent to the relative stability of X, and it implies $t\overline{\Pi}(\delta b(t)) \to 0$ for every $\delta > 0$, by (4.24), hence $\widetilde{\Delta X}_t^{(1)}/b(t) \xrightarrow{\mathrm{P}} 0$, and hence the first condition in (5.8).

(ii) The first condition in (5.9) implies the first condition in (5.8), hence the relative stability of X, that is, the second condition in (5.9). The converse result follows as in Part (i).

The second conditions in (5.8) and (5.9) are equivalent, as mentioned after (5.1).

(b) The proofs with ${}^{(r)}\widetilde{X}_t$ replaced by ${}^{(r)}X_t$ are similar; we use instead of the inequality in (4.20) the one-sided version

$$4P\left(|{}^{(r)}X_t - a(t)| > Mb(t)\right) \ge P\left(\Delta X_t^{(r+1)} > 4Mb(t)\right),\ t > 0,$$

which is also proved in Fan (2015). □

Proof of Theorem 5.1 Assume $\sigma^2 = 0$ and $\overline{\Pi}(0+) = \infty$.

(a) Suppose (5.4) holds (with a " + " sign, as we shall assume henceforth). Then the conditions in (5.9) hold with a " + " sign, as well as

$$\frac{{}^{(r)}\widetilde{X}_t - b(t)}{b(t)} \to 0 \text{ a.s., as } t \downarrow 0. \tag{5.10}$$

The (positive) relative stability (in probability) of X_t implies $A(x) > 0$ for all small x and we can take $b(t)$ to be continuous, strictly increasing, regularly varying with index 1 as $t \downarrow 0$, such that $b(t) = tA(b(t))$, and such that $b(t)$ satisfies (2.1). From Theorem 2.1 and (5.10) we then deduce that $J_{r+1} < \infty$, in which the inverse function $B(x)$ of $b(t)$ in (2.2) equals $x/A(x)$. Note that

$$\begin{aligned} \frac{\mathrm{d}B(x)}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{A(x)}\right) \\ &= \frac{A(x) - x(\overline{\Pi}^+(x) - \overline{\Pi}^-(x))}{A^2(x)} \\ &\sim \frac{1}{A(x)},\ \text{as } x \downarrow 0 \text{ (by (5.2))}. \end{aligned} \tag{5.11}$$

Then, via (2.2), the convergence of J_{r+1} implies (5.5) with a " + " sign.

Conversely, assume $A(x) > 0$ for all small x and (5.5) holds (in which we take $x_0 = 1$, and the " $+$ " sign). First we want to show that these imply positive relative stability in probability of X_t. Proceed as follows. Use the mean value theorem for integrals and the continuity of $A(x)$ to write

$$\begin{aligned}
\int_0^1 \left(\frac{x\overline{\Pi}(x)}{A(x)}\right)^{r+1} \frac{\mathrm{d}x}{x} &= \sum_{n\ge 1} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \left(\frac{\overline{\Pi}(x)}{A(x)}\right)^{r+1} x^r \mathrm{d}x \\
&= \sum_{n\ge 1} \frac{1}{A^{r+1}(\xi_n)} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \overline{\Pi}^{r+1}(x) x^r \mathrm{d}x \\
&\ge \sum_{n\ge 1} \left(\frac{\overline{\Pi}(1/n)}{A(\xi_n)}\right)^{r+1} \frac{1}{(n+1)^r}\left(\frac{1}{n} - \frac{1}{n+1}\right) \qquad (5.12)
\end{aligned}$$

where $\frac{1}{n+1} \le \xi_n \le \frac{1}{n}$. Now

$$\frac{1}{(n+1)^r}\left(\frac{1}{n} - \frac{1}{n+1}\right) \sim \frac{1}{n^{r+2}} \sim \frac{\xi_n^{r+1}}{n}, \text{ as } n \to \infty,$$

so we conclude that

$$\sum_{n\ge 1} \frac{1}{n}\left(\frac{\xi_n\overline{\Pi}(1/n)}{A(\xi_n)}\right)^{r+1} < \infty.$$

From the convergence of this series we can infer the existence of a sequence $n_i \uparrow \infty$ with $n_{i+1} \sim n_i$ such that

$$\lim_{i\to\infty} \frac{\xi_{n_i}\overline{\Pi}(1/n_i)}{A(\xi_{n_i})} = 0$$

(e.g. Loève (1977, p.277)) and

$$\frac{\xi_{n_{i+1}}}{\xi_{n_i}} \sim \frac{n_i}{n_{i+1}} \to 1.$$

Given $x > 0$ choose i so that $\frac{1}{n_{i+1}} \le x \le \frac{1}{n_i}$. Note also that $\frac{1}{n_{i+1}} \le \xi_{n_i} \le \frac{1}{n_i}$. Thus

$$\frac{1}{n_{i+1}} \le \min(x, \xi_{n_i}) \le \max(x, \xi_{n_i}) \le \frac{1}{n_i}.$$

Then

$$\begin{aligned}
\frac{A(x)}{A(\xi_{n_{i+1}})} &= 1 + \frac{\int_{\xi_{n_{i+1}}}^x (\overline{\Pi}^+(y) - \overline{\Pi}^-(y))\mathrm{d}y}{A(\xi_{n_{i+1}})} \\
&= 1 + \frac{O\left(\max(x, \xi_{n_{i+1}})\overline{\Pi}(\min(x, \xi_{n_{i+1}}))\right)}{A(\xi_{n_{i+1}})} \\
&= 1 + \frac{O\left(\xi_{n_{i+1}}\overline{\Pi}(1/n_{i+1})\right)}{A(\xi_{n_{i+1}})} \\
&= 1 + o(1),
\end{aligned}$$

and

$$\frac{x\overline{\Pi}(x)}{A(x)} \le \left(\frac{\xi_{n_{i+1}}\overline{\Pi}(1/n_{i+1})}{A(\xi_{n_{i+1}})}\right)\left(\frac{1}{n_i\xi_{n_{i+1}}}\right)\left(\frac{A(\xi_{n_{i+1}})}{A(x)}\right) \to 0. \qquad (5.13)$$

This implies (5.2) with a " $+$ " sign, and proves positive relative stability in probability of X_t.

The relative stability allows us to define a norming function $b(t) > 0$ such that $X_t/b(t) \xrightarrow{\mathrm{P}} 1$, with $b(t)$ having the regularity properties listed in the first part of the proof.

From the convergence in (5.5) we then deduce that of J_{r+1} in (2.2) with $B(x) = x/A(x)$ as the inverse function to $b(t)$, satisfying (5.11). We then get (5.4) from (2.5), on noting that $a(t) = t\nu(b(t)) + o(b(t))$ (by (4.24)) implies $a(t) = tA(b(t)) + o(b(t))$, hence $a(t) \sim b(t)$ as $t \downarrow 0$.

(b) Suppose (5.6) holds. Then $|{}^{(r)}\widetilde{X}_t|/b(t) \xrightarrow{\mathrm{P}} 1$, so by Proposition 5.2 we have the (positive, say) relative stability of X. Thus $A(x) > 0$ for all small x and $b(t) \sim b^*(t)$ where $b^*(t)$ is continuous, strictly increasing, regularly varying with index 1 as $t \downarrow 0$, and satisfies $b^*(t) = tA(b^*(t))$. It follows that $|{}^{(r)}\widetilde{X}_t|/b^*(t) \to 1$ a.s. and hence by (3.11), $\limsup_{t\downarrow 0} |\widetilde{\Delta X}_t^{(r+1)}|/b^*(t) \le 2$ a.s. Since $b^*(t)$ satisfies (2.1), Lemma 3.1 then gives $J_{r+1} < \infty$, where in (5.5), the inverse function $B(x)$ of $b^*(\cdot)$ equals $x/A(x)$. This implies (5.7), in which $|A(x)| = A(x)$. An analogous proof with $A(x) < 0$ for small x works if X is negatively relatively stable.

Conversely if $|A(x)| > 0$ for all small x then by continuity $A(x) > 0$ for all small x or $A(x) < 0$ for all small x, and this together with (5.7) implies (5.5), hence, (5.6).

(c) That the conditions in (5.4)–(5.7) are all equivalent is shown in the course of proving Parts (a) and (b) above. When $r = 0$, convergcence of the integral in (5.7) is shown in Doney and Maller (2002a, Thm. 4.2) to be equivalent to $X \in bv$ with drift $\mathrm{d}_X \neq 0$, and then $\lim_{t\downarrow 0} X_t/t = \mathrm{d}_X = 1$ a.s. □

Theorem 5.3. *Assume $\sigma^2 = 0$ and $\overline{\Pi}^+(0+) = \infty$ and fix $r = 0, 1, 2, \ldots$. Then*
(a) ${}^{(r)}X_t$ is a.s. relatively stable as $t \downarrow 0$, i.e., there is a function $b(t) > 0$ on $(0, \infty)$ such that

$$\frac{{}^{(r)}X_t}{b(t)} \to \pm 1 \text{ a.s.,} \tag{5.14}$$

iff $\pm A(x) > 0$ for all small x, $0 < x \le x_0$, say, and

$$\int_0^{x_0} \left(\frac{x\overline{\Pi}^+(x)}{\pm A(x)} \right)^{r+1} \frac{\mathrm{d}x}{x} < \infty \quad \text{and} \quad \int_0^{x_0} \left(\frac{x\overline{\Pi}^-(x)}{\pm A(x)} \right) \frac{\mathrm{d}x}{x} < \infty \tag{5.15}$$

(where the $+$ and $-$ signs are to be taken together);
(b) conditions (5.14) and (5.15) remain equivalent if ${}^{(r)}X_t$, ± 1 and $\pm A(x)$ are replaced by $|{}^{(r)}X_t|$, 1 and $|A(x)|$.

Proof of Theorem 5.3 (a) Assume $\sigma^2 = 0$ and $\overline{\Pi}^+(0+) = \infty$. Suppose (5.14) holds with a " $+$ " sign. Then by Part (b) of Proposition 5.2, X is relatively stable with norming function $b(t)$ which we can take as having the regularity properties listed earlier, and with inverse function $B(x)$ equal to $x/A(x)$ and satisfying (5.11). Also

$$\frac{{}^{(r)}X_t - b(t)}{b(t)} \to 0 \text{ a.s., as } t \downarrow 0,$$

hence by Part (ii) of Theorem 2.1

$$J_{r+1}^{(+)} = \int_0^1 \left(\overline{\Pi}^+(x) \right)^{r+1} dB^{r+1}(x) < \infty \quad \text{and} \quad J_1^{(-)} = \int_0^1 \overline{\Pi}^-(x) dB(x) < \infty. \tag{5.16}$$

Substituting $B(x) = x/A(x)$ and using (5.11) gives (5.15) with " $+$ " signs in both places. Similarly, with " $-$ " signs in place of " $+$ ", throughout.

Conversely, assume (5.15) with " $+$ " signs. The same argument as in (5.12)–(5.13) with $\overline{\Pi}^+$ or $\overline{\Pi}^-$ replacing $\overline{\Pi}$ shows that

$$\lim_{x\downarrow 0} \frac{A(x)}{x\overline{\Pi}^+(x)} = \infty \quad \text{and} \quad \lim_{x\downarrow 0} \frac{A(x)}{x\overline{\Pi}^-(x)} = \infty,$$

hence (5.2) holds with a " $+$ " sign. This is relative stability again, so we can define $b(t)$ and its inverse function $B(x)$ as before to obtain (5.16), and thus (2.5) with ${}^{(r)}X_t$ replacing ${}^{(r)}\widetilde{X}_t$. Since $t\nu(b(t)) \sim tA(b(t)) \sim b(t)$ we get (5.14) (with a " $+$ " sign). □

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