On the rate of convergence in the Kesten renewal theorem[*]

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Abstract

We consider the stochastic recursion $X_{n+1} = M_{n+1}X_n + Q_{n+1}$ on $\mathbb{R}^d$, where (M_n, Q_n) are i.i.d. random variables such that Q_n are translations, M_n are similarities of the Euclidean space $\mathbb{R}^d$. Under some standard assumptions the sequence X_n converges to a random variable R and the law ν of R is the unique stationary measure of the process. Moreover, the weak limit of properly dilated measure ν exists, defining thus a homogeneous tail measure Λ. In this paper we study the rate of convergence of dilations of ν to Λ

In particular in the one dimensional setting, when $(M_n, Q_n) \in \mathbb{R}^+ \times \mathbb{R}$ and $X_n \in \mathbb{R}$, the Kesten renewal theorem says that $t^\alpha \mathbb{P}[|R| > t]$ converges to some strictly positive constant C_+. Our main result says that

$$\left| t^\alpha \mathbb{P}[|R| > t] - C_+ \right| \leq C(\log t)^{-\sigma},$$

for some $\sigma > 0$ and large t. It generalizes the previous one by Goldie.

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1 Introduction

We consider the stochastic difference equation on $\mathbb{R}^d$

$$X_n = M_n X_{n-1} + Q_n, \quad n \geq 1 \tag{1.1}$$

where (M_n, Q_n) is a sequence of i.i.d. (independent identically distributed) random variables with values in $\mathrm{GL}(\mathbb{R}^d) \times \mathbb{R}^d$ and $X_0 \in \mathbb{R}^d$ is the initial distribution. The generic

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element of the sequence (M_n, Q_n) will be denoted by (M, Q). Under mild contractivity hypotheses the sequence X_n converges in law to a random variable R, which is the unique solution of the random difference equation

$$R =_d MR + Q, \qquad R \text{ independent of } (M, Q). \tag{1.2}$$

Moreover, the solution R can be explicitly written as

$$R = \sum_{n=0}^{\infty} M_1 .. M_n Q_{n+1}. \tag{1.3}$$

Existence of R is related to the contraction properties of M expressed in terms of the so called top Liapunov exponent of M. Following work by Furstenberg, Kesten [32], Vervaat [55] and Brandt [17] definitive sufficient and necessary conditions for convergence of the above series were achieved by Bougerol and Picard [14] and by Goldie and Goldie, Maller in the one dimensional case [33, 34]

There is a considerable interest in study various aspects of the iteration (1.1) and, in particular, the tail behavior of R. The story started with the seminal paper of Kesten [42] who formulated reasonable conditions for R to have a heavy tail in the case when the matrices M_n have positive entries. For one dimensional situation this means that

$$\lim_{t \to \infty} t^{\alpha} \mathbb{P}[|R| > t] = C_+, \tag{1.4}$$

exists, where α is the Cramér coefficient, i.e. the unique positive number such that $\mathbb{E}|M|^{\alpha} = 1$ (see Lemma 2.1). Later on the proof was essentially simplified and improved by Goldie [33] who also studied the rate of convergence in (1.4). (1.4) found an enormous number of applications both in pure and applied mathematics, see [12, 24, 49] and the comprehensive bibliography there. For this reason it is important to study this result further and to describe both the limiting constant (this is a subject of recent work, see [22, 29]) and the rate of convergence in (1.4). Here we consider the second problem and we prove that

$$\left| t^{\alpha} \mathbb{P}[|R| > t] - C_+ \right| \le C(\log t)^{-\sigma} \tag{1.5}$$

for some $\sigma > 0$ and large t (see Theorem 2.2 for the one dimensional situation). (1.5) should be compared with the result of Goldie [33] that says

$$\left| t^{\alpha} \mathbb{P}[|R| > t] - C_+ \right| \le Ct^{-\sigma},$$

but under assumption that the law of (M, Q) is spread out. We have a slower rate of convergence but the law of (M, Q) may be much less regular which is of interest nowadays.

Goldie studied not only recursion (1.1) but also more general iterative one dimensional systems i.e. recursions of the type

$$X_n = f_n(X_{n-1}), \quad n = 1, 2, ..., \quad X_0 = x$$

for certain affine like functions (in particular Lipschitz) and he adopted to them the approach working for (1.1). Beginning from the early nineties the general Lipschitz iterative systems have attracted a lot of attention: Alsmeyer, Arnold and Crauel, Diaconis and Freedman, Duflo, Elton, Henion and Hervé, Mirek [7, 8, 11, 24, 27, 28, 39, 47] and they still do. In particular, it seems that modeling them after (1.1) has been very fruitful Alsmeyer [8], Mirek [47]. Therefore, studying asymptotic properties of multidimensional R may be of a broader impact.

In parallel the matrix recursion (1.1) was studied in various contexts (not always fitting into the Diaconis-Freedman Lipschitz scheme) and appropriate sharp estimates

have been proved: de Saporta, Guivarch, Le Page [23, 36, 37, 35, 44], Klüppelberg, Pergamenchtchikov [43] Alsmeyer, Mentemeier [10], Buraczewski, Damek, Mirek, Urban [19, 20].

However nobody has approached the corresponding rate of convergence i.e. a multidimensional analog of (1.5). So we are trying to take the first step.

We consider here matrices M being similarities, when the precise asymptotics of the tail of R is known due to our paper with Y. Guivarc'h [19]. More precisely, we consider the d-dimensional Euclidean space $\mathbb{R}^d$, endowed with the scalar product $\langle x, y \rangle = \sum_1^d x_i y_i$ and the corresponding norm: $|x|^2 = \sum_1^d |x_i|^2$. The norm of a linear transformation g of $\mathbb{R}^d$ is denoted $|g|$. An element $g \in \text{GL}(\mathbb{R}^d)$ is a similarity in the sense of Euclidean geometry, if

$$|gx| = |g||x|, \quad x \in \mathbb{R}^d. \tag{1.6}$$

If g is a similarity, then $\frac{1}{|g|}g$ preserves the norm on $\mathbb{R}^d$. Hence the subgroup $\text{G} \subseteq \text{GL}(\mathbb{R}^d)$ of all the similarities is isomorphic to the direct product of the multiplicative group of real positive numbers $\mathbb{R}^+$ and the orthogonal group $\text{O}(\mathbb{R}^d)$. The isomorphism map is given by $g \mapsto (|g|, g/|g|)$. We shall identify $\text{G} = \mathbb{R}^+ \times \text{O}(\mathbb{R}^d)$. We consider the group $\text{H} = \mathbb{R}^d \rtimes \text{G}$ of transformations

$$\mathbb{R}^d \ni x \to hx = gx + q \in \mathbb{R}^d,$$

where $h = (q, g)$ with $g \in \text{G}$, $q \in \mathbb{R}^d$ and study the stochastic recursion defined in (1.1). Then (Q_n, M_n) is an H valued i.i.d. sequence with distribution $\bar{\mu}$. (Here $Q_n \in \mathbb{R}^d$ and $M_n \in \text{G}$.)

If $\mathbb{E} \log |M| < 0$ and $\mathbb{E} \log^+ |Q| < \infty$, then the sequence X_n converges in law to a random variable R, which is the unique solution of the random difference equation (1.2). The main result of [19] shows that under appropriate assumptions the random variable R is regularly varying, i.e. if ν denotes the law of R, then for some class of functions (containing e.g. bounded continuous functions supported outside 0),

$$\lim_{|g| \to 0, g \in \text{G}_\mu} |g|^{-\alpha}(g\nu)(f) = \lim_{|g| \to 0, g \in \text{G}_\mu} |g|^{-\alpha} \int_{\mathbb{R}^d} f(gx)\nu(dx) = \Lambda(f) \tag{1.7}$$

for a Radon measure Λ on $\mathbb{R}^d \setminus \{0\}$, see Theorem 4.1. (Here $\text{G}_\mu = \mathbb{R}^+ \times K$ is the closed subgroup of G generated by the support of the law μ of M i.e. the image of $\bar{\mu}$ under the map $(q, g) \mapsto g$.) (1.7) is the right analogue of (1.4) here.

The goal of this paper is to study the rate of convergence of $|g|^{-\alpha}(g\nu)$ to Λ on natural function spaces like the Hölder space or the Zolotarev space, Theorem 4.5 being our main result (see also Theorems 4.2, 4.3). We obtain

$$||g|^{-\alpha} g\nu(f) - \Lambda(f)| \leq C|\log |g||^{-\sigma} \tag{1.8}$$

for small $|g|$, $g \in \text{G}_\mu$ and C independent of a function f. Again, as in (1.5), regularity of μ is very mild.

In (1.8) the spectral gap of the measure $|g|^\alpha \mu(dg) = \mu_\alpha(dg)$ as the convolution operator on $L_0^2(\text{G}_\mu) = \{\phi \in L^2(\text{G}_\mu) : \int_K \phi(gk) \, dk = 0\}$ is crucial. Let $U = \sum_{n=0}^\infty \mu_\alpha^n$. We write

$$|g|^{-\alpha}(g\nu)(f) = \psi_1 * U(g) + \psi_2 * U(g)$$

as the sum of two potentials of very good functions ψ_1, ψ_2 on G with additional properties:

$$\psi_1 \in L_0^2(\text{G}_\mu), \quad \psi_2(gk) = \psi_2(g), \quad (g \in \text{G}, \ k \in \text{K}). \tag{1.9}$$

To $U * \psi_2$, which reduces to the one dimensional situation, we apply the Implicit Renewal Theory as developed by Goldie in [33]. To handle $U * \psi_1$ we need a spectral gap i.e. that

$$\|\psi_1 * \mu_\alpha\|_{L^2} \leq \lambda \|\psi_1\|_{L^2} \tag{1.10}$$

for a $\lambda < 1$ and all $\psi_1 \in L_0^2(G_\mu)$. Inequality (1.10) is clearly satisfied if μ is spread out (see paragraph 5.2), but due to the action of K much less is needed. If we disintegrate μ as

$$\mu(dg) = \mu_a(dk)\mu_{\mathbb{R}^+}(da).$$

then the existence of a spectral gap for μ_a on $L_0^2(K) = \{\phi \in L^2(K) : \int_K \phi(k)\, dk = 0\}$ for a in a set A with $\mu_{\mathbb{R}^+}(A) > 0$ is sufficient (Proposition 5.4). In particular, if

$$\mu(dg) = \mu_1(dk) \times \mu_{\mathbb{R}^+}(da)$$

is a product measure and μ_1 has a spectral gap then (1.10) holds.

For a measure on K to have a spectral gap it is sufficient to be spread out but much less will do. This phenomenon has been intensively studied by a number of authors. The most important are the measures

$$\eta = \frac{1}{2k} \sum_{j=1}^{k} (\delta_{g_j} + \delta_{g_j^{-1}})$$

on $SU(d)$ Bourgain, Gamburd ([15, 16]) or $SO(d)$ HeeOh, Lubotzky, Phillips, Sarnak ([48, 45, 46]), related to so called Hecke operators ($Tf = f * \nu$). Also there are other examples: measures supported by special rotations Diaconis, Saloff-Coste, Janvresse, Rosenthal ([25, 41, 50]) or η being a measure uniformly distributed on the set of reflections (elements of $O(d), SU(d), Sp(d)$) leaving one hyperplane fixed, Porod[53], [54]). So our result applies e.g. to the situation when, say "K part" of the measure is of this kind.

The structure of this paper is as follows. In Section 2 we state the one dimensional result and explain the scheme of the proof. In Section 3 we introduce notation and definitions needed to consider the multidimensional situation. Section 4 contains statements of our main results preceded by some discussions concerning distance between measures on appropriate function spaces. The rest of the work is devoted to proofs. In Section 5 we collect some auxiliary results concerning the spectral cut off and discuss conditions implying existence of the spectral gap. (A part of this discussion is postponed to Appendix A.) Main steps of the proofs are contained in Sections 6 and 7. In Appendix B we describe relations between two different metrics on compact groups.

Finally let us mention that we separate the one dimensional arguments from the technically more involved multidimensional case. Thus, if the reader is interested only in the one dimensional situation, the complete proof is contained in Section 2, Lemmas 6.1, 7.1 and Proposition 7.2 and this part can be read independently of the rest of the paper.

2 The main result in the one dimensional case

The asymptotic behavior of R in the one dimensional case was described by Goldie, who proved the following result.

Theorem 2.1 (Kesten [42], Goldie [33]). *Assume that*

- *there exists $\alpha > 0$ such that $\mathbb{E}M^\alpha = 1$;*
- *$\mathbb{E}[M^\alpha|\log M|]$ and $\mathbb{E}|Q|^\alpha$ are both finite;*
- *the law of $\log M$ is nonarithemetic;*
- *for every $x \in \mathbb{R}$, $\mathbb{P}[Mx + Q = x] < 1$.*

Then there exist nonnegative constants C_+ and C_- such that

$$\lim_{x \to \infty} x^\alpha \mathbb{P}[R > x] = C_+, \quad \lim_{x \to -\infty} |x|^\alpha \mathbb{P}[R < x] = C_-. \tag{2.1}$$

Moreover $C_+ + C_- > 0$.

In his paper Goldie considered also rate of convergence in (2.1). He proved that if $\mathbb{E}M^{\alpha+\varepsilon} < \infty$, $\mathbb{E}|Q|^{\alpha+\varepsilon} < \infty$ and the law of $\log M$ is spread-out (i.e. some of its convolution powers have a positive absolutely continuous component), then the rate of approach is of the order $o(|x|^{-\delta})$, for some $\delta > 0$.

Theorem 2.1 is a consequence of the classical two-sided renewal theorem for the probability measure μ_α viewed as a measure on $\mathbb{R}$, via the logarithm. Under additional hypotheses stated above the measure μ_α has finite exponential moments and it is spread out. Thus Stone's decomposition theorem [52] provides also control of the rate of convergence of the renewal measure $U = \sum_{n=0}^{\infty} \mu_\alpha^n$, where μ_α^n denotes the nth convolution power of μ_α, to the Lebesgue measure.

In this paper we study the rate of approach of $t^\alpha \mathbb{P}[R > t]$ to C_+, however under weaker assumptions. We develop here Goldie's remarks that the control of the rate of convergence in (2.1) is a consequence of a description of the distance between the renewal measure and the Lebesgue measure. More precisely, according to the classical Renewal Theorem, [30, Theorem 1, Chapter XI, S1], for any directly Riemann integrable function (dRi) F

$$\frac{1}{m} \int_{\mathbb{R}} F(s)\, ds = \lim_{t \to \infty} \sum_{n=0}^{\infty} \mu_\alpha^n * F(t) = \lim_{t \to \infty} \int_{\mathbb{R}} F(t - s)\, U(ds), \qquad (2.2)$$

where $m = \int_{\mathbb{R}} x \mu_\alpha(dx)$. Let $H(x) = U(-\infty, x]$ be the "distribution function" of U. It is well known that if the measure μ_α has finite second moment m_2 then

$$H(x) = Ax + B + o(1)$$

as x goes to $+\infty$. (Here $A = \frac{1}{m}$ and $B = \frac{m_2}{2m^2}$.)

In short, our main result, Theorem 2.2 below, says that if the term $o(1)$ is of the order $O(x^{-\delta})$, then the speed of convergence in (2.1) is of logarithmic order.

Theorem 2.2. *Assume the conditions of Theorem 2.1 are satisfied,*

$$H(s) = \begin{cases} As + B + O(s^{-\delta}) & \text{as } s \to +\infty, \\ O(|s|^{-\delta}) & \text{as } s \to -\infty. \end{cases} \qquad (2.3)$$

for some $\delta > 0$ and

$$\mathbb{E}\big[M^\alpha |\log M|^\gamma\big] < \infty, \qquad (2.4)$$

$$\mathbb{E}\big[(M^\alpha + |Q|^\alpha)(|\log M|^\chi + |\log Q|^\chi)\big] < \infty \qquad (2.5)$$

for some $\gamma > 2$ and $\chi > 1$. Then

$$
\begin{aligned}
x^\alpha \mathbb{P}[R > x] &= C_+ + o(|\log x|^{-\eta}) & \text{as } x \to \infty, \\
|x|^\alpha \mathbb{P}[R < x] &= C_- + o(|\log |x||^{-\eta}) & \text{as } x \to -\infty,
\end{aligned}
$$

where

$$\eta = \frac{1}{2} \min\big\{\chi - 1, \gamma - 2, \delta\big\}.$$

Hypothesis (2.3) is quite natural. It is satisfied e.g. when the measure μ_α is absolutely continuous, but it is valid also in more general settings. Carlsson [21] proved (2.3) for nonlattice measure of p-type, i.e for measures satisfying

$$\liminf_{|t| \to \infty} \big|t^p(1 - \widehat{\mu}_\alpha(t))\big| > 0,$$

where $\widehat{\mu}_\alpha(t)$ is the Fourier transform of μ_α: $\widehat{\mu}_\alpha(t) = \int_{\mathbb{R}} e^{-itx} \mu_\alpha(dx)$. Then, assuming also (2.4), expansion (2.3) holds for $\delta = \min\big\{\gamma - 2, \frac{\lfloor \gamma \rfloor}{(1+p(\lfloor \gamma \rfloor + 1))}\big\}$.

Assuming some technical lemmas, that are presented in the latter sections, the proof of Theorem 2.2 is immediate. We include it here in order to clarify our presentation.

Proof of Theorem 2.2. We proceed as in [33]. Define

$$r(t) = e^{\alpha t} \mathbb{P}[R > e^t].$$

We intend to consider r as a solution of a renewal equation. (We would like to apologize in advance that later we shall also use the same letter r to denote a positive number - the radius of a sphere.) Clearly, r does not have to be dRi. Hence we need to study its smoothed version, i.e. its convolution with $K(t) = \mathbf{1}_{[0,\infty)}(t)e^{-t}$:

$$\breve{r}(t) = r * K(t) = \int_{-\infty}^{\infty} K(t-u)r(u)\,du. \tag{2.6}$$

First let us sketch the idea of the proof of Theorem 2.1. After a number of calculations one proves that

$$\breve{r}(t) = \breve{g} * U(t), \tag{2.7}$$

where U is the potential of μ_α and

$$g(t) = e^{\alpha t}\big(\mathbb{P}(R > e^t) - \mathbb{P}(MR > e^t)\big), \tag{2.8}$$

see [33, (9.8)]. Next, since g belongs to $L^1(\mathbb{R})$, $\breve{g}$ is dRi, see [33, Lemma 9.2]. Then by the renewal theorem

$$\lim_{t\to\infty} \breve{r}(t) = \frac{1}{m}\int_{\mathbb{R}} \breve{g}(u)du = \frac{1}{m}\int_{\mathbb{R}} g(u)du$$

and finally unsmoothing the function, [33, Lemma 9.3], one deduces

$$\lim_{t\to\infty} r(t) = \frac{1}{m}\int_{\mathbb{R}} g(u)du =: C_+.$$

In order to prove Theorem 2.2 one has to proceed more carefully.

Our goal is to control the rate of approach in the last limit. For this purpose some further properties of functions g and r are needed. First, applying (2.5), one has to prove that $\int_{\mathbb{R}} |g(t)||t|^\chi dt < \infty$ and this is done in Lemma 6.1. Secondly, in view of (2.1), g is bounded. Moreover g is continuous because R has no atoms. This follows immediately from the result of Alsmeyer, Iksanov and Rösler [9] and was also proved in [19] (Proposition 2.4) in a much more general setting. Next we need to prove that r satisfies some Tauberian condition. Namely, for $t > s$, by (2.1),

$$\begin{aligned}
r(t) - r(s) &= \big(e^{\alpha t} - e^{\alpha s}\big)\mathbb{P}[R > e^t] + e^{\alpha s}\big(\mathbb{P}[R > e^t] - \mathbb{P}[R > e^s]\big) \\
&\le \big(e^{\alpha(t-s)} - 1\big)e^{\alpha s}\mathbb{P}[R > e^t] \le C(t-s)
\end{aligned}$$

for $s \le t \le s + \eta_0$ provided s_0 sufficiently big and η_0 sufficiently small.

The conclusion follows now from Proposition 7.2, being the main part of Section 7. In this Proposition one uses the asymptotic expansion of the potential U given by (2.3) to estimate the difference

$$\left| \breve{r}(t) - \frac{1}{m}\int_{\mathbb{R}} \breve{g}(u)du \right|$$

and here all the properties of the function g stated above are essential. Finally thanks to the Tauberian property of r we can use a result due to Frennemo [31] to unsmooth the estimates. $\qquad\square$

3 Some preliminaries

Before we state our main results in the multidimensional case we recall some standard notation concerning function spaces on groups.

Let K be a compact group and let $G = \mathbb{R}^+ \times K$ be the direct product of the multiplicative group of the real numbers $\mathbb{R}^+$ and K. We shall write the elements of G as $g = ak$, where $a \in \mathbb{R}^+$ and $k \in K$ and set $|g| = a$. Let da denote the Haar measure on $\mathbb{R}^+$ normalized so that $\int_1^e da = 1$ and let dk be the Haar measure on K normalized so that the volume of K is 1. The resulting direct product $dg = da\,dk$ is a Haar measure on G.

The group G acts on the space $C(G)$ of the continuous complex valued functions via the left regular representation

$$\lambda(h)\phi(g) = \phi(h^{-1}g) \qquad (g, h \in G;\ \phi \in C(G))$$

and by the right regular representation

$$\rho(h)\phi(g) = \phi(gh) \qquad (g, h \in G;\ \phi \in C(G)).$$

If μ is a bounded measure on G, define the convolutions of ϕ and μ on the right and on the left by

$$\phi * \mu(g) = \rho(\mu)\phi(g) = \int_G \phi(gh)\,\mu(dh),$$
$$\mu * \phi(g) = \lambda(h)(\mu)\phi(g) = \int_G \phi(h^{-1}g)\,\mu(dh) \qquad (g \in G). \tag{3.1}$$

(If μ is absolutely continuous with respect to the Haar measure then we use the first or the second formula, whichever is more convenient.) In particular, if $\breve{\phi}$ denotes the reflection of ϕ with respect to the , $\breve{\phi}(g) = \phi(g^{-1})$, then

$$\phi * \mu(g^{-1}) = \mu * \breve{\phi}(g) \qquad (g \in G). \tag{3.2}$$

The reflected measure $\breve{\mu}$ is defined by $\breve{\mu}(\psi) = \mu(\breve{\psi})$. For an integer $n = 1, 2, 3, \ldots$ the n-fold convolution μ^n of μ is defined by

$$\int_G \psi(g)\,d\mu^n(g) = \int_G \int_G \ldots \int_G \psi(g_1 g_2 \ldots g_n)\,\mu(dg_n) \ldots \mu(dg_2)\,\mu(dg_1). \tag{3.3}$$

Then $\rho(\mu^n) = \rho(\mu)^n$ and $\lambda(\mu^n) = \lambda(\mu)^n$. Also, we shall adopt the convention that μ^0 is the Dirac delta at the identity of the group.

The convolution with the Haar measure on K defines the projection onto the space of the K-invariants:

$$C(G) \ni \phi \to \phi^K \in C(G)^K, \quad \phi^K(g) = \int_K \phi(gk)\,dk \qquad (g \in G). \tag{3.4}$$

Lemma 3.1. *If μ is a bounded measure on G then*

$$(\phi * \mu)^K = \phi^K * \mu \qquad (\phi \in C(G)).$$

Proof. This is straightforward:

$$\begin{aligned}
(\phi * \mu)^K(g) &= \int_K \int_G \phi(gkh)\,\mu(dh)\,dk = \int_G \int_K \phi(gkh)\,dk\,\mu(dh) \\
&= \int_G \int_K \phi(gh(h^{-1}kh))\,dk\,\mu(dh) = \int_G \int_K \phi(ghk)\,dk\,\mu(dh) \\
&= \int_G \int_K \phi^K(gh)\,\mu(dh) = \phi^K * \mu(g),
\end{aligned}$$

where the forth equality follows from the fact that G normalizes K and that the Haar measure on K is invariant under the conjugation action of G. □

If $F(G)$ is a space of complex valued functions on G, which are integrable over K, we shall denote by $F(G)_0 \subseteq F(G)$ the subspace of all the functions whose average over K is zero. Thus

$$F(G)_0 = \{\phi \in F(G); \ \phi^K = 0\}. \tag{3.5}$$

(Later we shall use this definition for very particular function spaces L^∞, L^2, C.)

For a bounded measure μ on G, let $\mu_{\mathbb{R}^+}$ denote the projection of μ onto $\mathbb{R}^+$:

$$\mu_{\mathbb{R}^+}(B) = \mu(B \times K) \qquad (B \text{ a Borel subset of } \mathbb{R}^+).$$

Equivalently, for $\phi \in C(G)^K$,

$$\phi * \mu(ak) = \phi|_{\mathbb{R}^+} * \mu_{\mathbb{R}^+}(a) \qquad (a \in \mathbb{R}^+, \ k \in K),$$

where $\phi|_{\mathbb{R}^+}$ is the restriction of ϕ to $\mathbb{R}^+$, that is $\phi|_{\mathbb{R}^+}(a) = \phi(a\mathbf{1})$, where $\mathbf{1}$ is the identity of K. (In fact $\phi(a\mathbf{1}) = \phi(ak)$, for every $k \in K$.) Indeed, the left hand side is equal to

$$\int_G \phi((ak)(bl)) \, d\mu(bl) = \int_G \phi(abkl) \, d\mu(bl) = \int_G \phi(ab) \, d\mu(bl) = \int_G \phi(ab) \, d\mu_{\mathbb{R}^+}(b),$$

which coincides with the right hand side. Hence, Lemma 3.1 together with the induction on n show that

$$\phi * \mu^n(ak) = \phi|_{\mathbb{R}^+} * \mu_{\mathbb{R}^+}^n(a) \qquad (\phi \in C(G)^K, \ a \in \mathbb{R}^+, \ k \in K, \ n = 1, 2, 3, \dots). \tag{3.6}$$

4 Main results in the multidimensional case

4.1 Behavior of the tail of ν

In the multidimensional case we have the following description of the tail of ν. Let μ be the marginal law of M i.e the image of $\bar{\mu}$ under the map $(q, g) \mapsto g$. Let G_μ be the closed group generated by the support of μ and let $K_\mu = G_\mu \cap O(\mathbb{R}^d)$. For $g \in G_\mu$, set

$$\nu_g(f) = |g|^{-\alpha}(g\nu)(f) = |g|^{-\alpha} \int_{\mathbb{R}^d} f(gx)\nu(dx) = |g|^{-\alpha} \mathbb{E}f(gR), \tag{4.1}$$

where R is the solution to (1.2). Denote by $\overline{R}$ the projection of R onto the unit sphere, i.e. $\overline{R} = R/|R|$. From now on we shall denote by R, R_1, M_1, Q_1 the random variables such that

$$R = M_1 R_1 + Q_1, \qquad \text{a.s.}$$

and M_1, Q_1 are independent of R_1.

The following theorem describes the tail of ν, [19].

Theorem 4.1. *Assume that the action of* $\mathrm{supp}\mu$ *on* $\mathbb{R}^d$ *has no fixed point,*

- $\mathbb{E}(\log |M|) < 0$
- *there is* $\alpha > 0$ *such that* $\mathbb{E}|M|^\alpha = 1$
- $m_\alpha = \mathbb{E}(|M|^\alpha \log |M|)$ *and* $\mathbb{E}|Q|^\alpha$ *are both finite*
- μ *is not arithmetic.*

Then there is a Radon measure Λ *on* $\mathbb{R}^d \setminus \{0\}$ *such that for every bounded continuous function* f *that vanishes in a neighborhood of zero*

$$\lim_{|g| \to 0, g \in G_\mu} \langle f, \nu_g \rangle = \langle f, \Lambda \rangle, \tag{4.2}$$

where, given a measure π, $\langle f, \pi \rangle$ *denotes the integral of a function* f *with respect to the measure* π.

Moreover, there is a finite K_μ - invariant measure σ_μ on S^{d-1} such that, in radial coordinates,

$$\langle f, \Lambda \rangle = \int_{\mathbb{R}^+ \times S^{d-1}} f(r\omega) \frac{\alpha}{r^\alpha} \, dr \, \sigma_\mu(d\omega).$$

The family of measures σ_t on S^{d-1} defined by

$$\sigma_t(W) = t^\alpha \mathbb{P}\{|R| > t, \bar{R} \in W\},$$

for $W \subset S^{d-1}$, converges weakly to σ_μ as $t \to +\infty$.

Finally, (4.2) holds for every function f such that $0 \notin \operatorname{supp} f$, the measure Λ of the set of discontinuities of f is 0 and for some $\varepsilon > 0$

$$\sup_{x \neq 0} \left(|x|^{-\alpha} |\log |x||^{1+\varepsilon} |f(x)| \right) < \infty. \tag{4.3}$$

4.2 Function spaces

We are interested in the rate of convergence of $\langle f, \nu_g \rangle$ to $\langle f, \Lambda \rangle$ in terms of a distance between measures ν_g and Λ. Traditionally, in order to define a distance between bounded measures of the same mass one takes some family $\mathcal{F}$ of functions and computes

$$\rho_{\mathcal{F}}(\nu_1, \nu_2) = \sup\{|\langle f, \nu_1 \rangle - \langle f, \nu_2 \rangle| : f \in \mathcal{F}\}. \tag{4.4}$$

Typically, more than just the continuity of functions $f \in \mathcal{F}$ is required. We are going to use here the so called Zolotarev distance [56, 57] between two probability measures ν_1 and ν_2. It is defined, as in (4.4), by fixing $\varepsilon > 0$ and taking

$$\mathcal{F} = \mathcal{F}_\varepsilon = \{f \in H^\varepsilon(\mathbb{R}^d), \ \|f\|_\varepsilon \leq 1\},$$

where $H^\varepsilon(\mathbb{R}^d)$ is the Hölder space consisting of all the functions $f : \mathbb{R}^d \to \mathbb{C}$ for which the seminorm

$$\|f\|_\varepsilon = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\varepsilon}$$

is finite. Clearly, locally $\rho_{\mathcal{F}_{\varepsilon_2}} \leq C \rho_{\mathcal{F}_{\varepsilon_1}}$ if $\varepsilon_1 \leq \varepsilon_2$, so ideally we would like to be able to estimate $\rho_{\mathcal{F}_\varepsilon}$ for any $\varepsilon > 0$. It is essential here for the measures to have the same mass. Otherwise the supremum is infinite.

In our situation we have two specific issues to address. First, Λ is unbounded. Second, $\nu_g(\mathbb{R}^d)$ varies with g, although for any $r > 0$ the function $\nu_g(\mathbb{R}^d \setminus B_r(0))$ approaches $\Lambda(\mathbb{R}^d \setminus B_r(0))$ as $|g| \to 0$. Therefore we fix $r > 0$, restrict measures to $\mathbb{R}^d \setminus B_r(0)$ and take a class of functions supported away from 0. This eliminates the main obstacle, the constants, so that the definition makes sense.

Specifically, we fix $\varepsilon > 0$, denote by H_r^ε the space of Hölder functions on $\mathbb{R}^d$ supported in $\mathbb{R}^d \setminus B_r(0)$ and consider (4.4), with

$$\mathcal{F} = \mathcal{F}_{r,\varepsilon} = \{f \in H_r^\varepsilon; \ \|f\|_\varepsilon \leq 1\}.$$

Then $\rho_{\mathcal{F}_{r,\varepsilon}}(\nu_g, \Lambda)$ is indeed a distance between ν_g and Λ restricted to $\mathbb{R}^d \setminus B_r(0)$.

We are able to estimate $\rho_{\mathcal{F}_{r,1}}(\nu_g, \Lambda)$ only (the space of Lipschitz functions) and under some more assumptions [1] we obtain the following inequality

$$\rho_{\mathcal{F}_{r,1}}(\nu_g, \Lambda) \leq C |\log |g||^{-\sigma} \tag{4.5}$$

---

[1] In particular $\alpha \geq 1$. For $\alpha \leq 1$ we need a further restriction: supports of functions f must be contained in a compact set.

for $|g| < 1$. See Theorem 4.5. This will be more carefully discussed at the end of this section. It is interesting that with this choice of functions vanishing around zero we can make the quantity $\sup\{|\langle f, \nu_g \rangle - \langle f, \Lambda \rangle|\}$ finite even without the requirement of equality of total masses. Finally, together with Theorem 4.2 we have (4.5) also for ν_g and Λ restricted to $\mathbb{R}^d \setminus B_r(0)$ and normalized.

Notice that ν_g, unlike Λ, are not K_μ invariant so (4.5) has an additional value of showing how fast ν_g become K_μ invariant. The existence of the spectral gap on K_μ or G_μ is vital here. See Proposition 4.7.

For $\varepsilon < 1$ the distance $\rho_{\mathcal{F}_{r,\varepsilon}}(\nu_g, \Lambda)$ is difficult to estimate. We impose further restriction on the class $\mathcal{F}$. More precisely, let

$$\mathcal{F}_{r,\varepsilon,+} = \{f \in H_r^\varepsilon, \|f\|_\varepsilon \leq 1, f(kx) = f(x), f(a_1 x) \leq f(a_2 x), \text{for } 0 < a_1 \leq a_2\}.$$

Then

$$\rho_{\mathcal{F}_{r,\varepsilon,+}}(\nu_g, \Lambda) \leq C|\log|g||^{-\sigma} \qquad (|g| < 1).$$

This is formulated in Theorem 4.3 and leads directly to an approximation of the measure σ_μ, which is interesting as a straightforward generalization of Theorem 2.2. In the one dimensional case the measure σ_μ is determined by the constants C_+, C_-.

For $\alpha > 1$ we may also think of other Zolotarev distances, namely those defined in terms of functions growing faster at infinity. Let $m \in \mathbb{N}$, $0 < \varepsilon \leq 1$ and $m + \varepsilon < \alpha$. The point of this assumption is that under the conditions of Theorem 4.1

$$\mathbb{E}|M|^{m+\varepsilon} < 1. \tag{4.6}$$

Let $H^{m,\varepsilon}$ be the space of functions such that for every multiindex I of length m, $\partial^I f$ is a Hölder function, $\partial^I f \in H^\varepsilon$. Define

$$\|f\|_{m,\varepsilon} = \sup_{|I|=m} \sup_{x \neq y} \frac{|\partial^I f(x) - \partial^I f(y)|}{|x-y|^\varepsilon} \tag{4.7}$$

and let

$$\mathcal{F}_{m;\varepsilon} = \{f \in H^{m,\varepsilon}, \|f\|_{m,\varepsilon} \leq 1\}.$$

Then $\rho_{\mathcal{F}_{m;\varepsilon}}(\nu_1, \nu_2)$ is well defined provided all the moments of ν_1, ν_2 of order at most m are equal. This latter condition is not satisfied in our situation, but we take functions supported outside a ball of radius r, $H_r^{m,\varepsilon} = \{f \in H^{m,\varepsilon} : \mathrm{supp} f \cap B_r(0) = \emptyset\}$ and set

$$\mathcal{F}_{m;r,\varepsilon} = \{f \in H_r^{m,\varepsilon}, \|f\|_{m,\varepsilon} \leq 1\}.$$

This kills polynomials, $\rho_{\mathcal{F}_{m;r,\varepsilon}}(\nu_1, \nu_2)$ is well defined and again we have,

$$\rho_{\mathcal{F}_{m;r,\varepsilon}}(\nu_1, \nu_2) \leq C|\log|g||^{-\sigma}, \tag{4.8}$$

for small $|g|$. See Theorem 4.5.

4.3 Main results for K-invariant functions

Now we are going to formulate our results more precisely and show how do they follow from more technical theorems and lemmas contained in later chapters. All the theorems stated below require slightly different assumptions and provide estimates in different metrics. However the schemes of the proofs are similar.

We start with estimating the difference

$$\nu_g(\mathbb{R}^d \setminus B_r(0)) - \Lambda(\mathbb{R}^d \setminus B_r(0)),$$

as $|g| \to 0$, i.e.

$$t^\alpha \mathbb{P}[|R| > t] - \sigma_\mu(S^{d-1}), \tag{4.9}$$

as $t \to +\infty$. This is in full analogy with Theorem 2.2. Indeed, the proof is as in the one dimensional case. Let μ_α be the measure on G_μ defined by $d\mu_\alpha(g) = |g|^\alpha d\mu(g)$ and let $U = \sum_{n=0}^\infty \mu_\alpha^n$ be its potential on G. In order to estimate (4.9) we need only the image $U_\mathbb{R}$ of U on $\mathbb{R}$ via $g \mapsto \log |g|$.

Theorem 4.2. *Assume that the conditions of Theorem 4.1 are satisfied and that*

$$\mathbb{E}\big(|Q|^\alpha + |M|^\alpha\big)\big((\log(1+|Q|))^\chi + (\log(1+|M|))^\chi\big) < \infty \tag{4.10}$$

holds with $\chi > 1$ as well as (2.3) holds for $U_\mathbb{R}$. Then there are constants C, σ such that

$$\Big|\nu_g\big(\mathbb{R}^d \setminus B_r(0)\big) - \Lambda\big(\mathbb{R}^d \setminus B_r(0)\big)\Big| \leq C|\log|g||^{-\sigma}, \tag{4.11}$$

for $|g| < 1$, i.e.

$$\big|t^\alpha \mathbb{P}[|R| > t] - \sigma_\mu(S^{d-1})\big| \leq C(\log t)^{-\sigma}, t > r. \tag{4.12}$$

Proof. We are going to apply Proposition 7.2 to the function

$$F(t) = e^{\alpha t}\Big(\mathbb{P}[|R| > e^t] - \mathbb{P}[|MR| > e^t]\Big).$$

Proceeding as in the one dimensional case, following Goldie, we prove that F is bounded, $r(t) = U * F$ and $\breve{r}(t) = U * \breve{F}$ are well defined and

$$U * F(t) = e^{\alpha t}\mathbb{P}[|R| > e^t].$$

Moreover F is continuous because, as shown in [19, Proposition 2.4], $\nu(S) = 0$ for every sphere S. The condition (7.4) is satisfied by Lemma 6.1 and (7.6) can be checked as in the proof of Theorem 2.2. $\qquad\square$

We can approximate the measure σ_μ, not just its total mass, as follows. Take a continuous function f of the form

$$f(r\omega) = f_1(r)f_2(\omega), 0 \notin \text{supp} f_1. \tag{4.13}$$

Then

$$\langle f, \Lambda \rangle = \int_{\mathbb{R}^+} f_1(s) \frac{\alpha}{s^\alpha} \, ds \int_{S^{d-1}} f_2(\omega)\sigma_\mu(d\omega) = C(f_1)\langle f_2, \sigma_\mu \rangle.$$

Suppose we fix f_1 such that $C(f_1) \neq 0$ and for a class of functions f_2 we have

$$\big|\langle f, \nu_g \rangle - \langle f, \Lambda \rangle\big| \leq C|\log|g||^{-\sigma}. \tag{4.14}$$

Then

$$\Big|\langle f_2, \sigma_\mu \rangle - C(f_1)^{-1}\langle f, \nu_g \rangle\Big| \leq C|\log|g||^{-\sigma}. \tag{4.15}$$

Clearly taking f_2 in (4.14) invariant under K_μ does not change anything because σ_μ is K_μ invariant. In order to get (4.14) we need to impose some more regularity on f_2. Being Hölder is fine.

Theorem 4.3. *Assume that the conditions of Theorem 4.2 are satisfied. Let $f \in H_r^\varepsilon$, $\|f\|_\varepsilon \leq 1$, $f(kx) = f(x)$, $k \in K_\mu$ and $f(a_1 x) \leq f(a_2 x)$, if $a_1 \leq a_2$. Then there are constants C, σ such that*

$$\Big|\langle f, \nu_g \rangle - \langle f, \Lambda \rangle\Big| \leq C|\log|g||^{-\sigma}, \tag{4.16}$$

for every f as above and $|g| < 1$.

Hence for product functions we have the following Corollary.

Corollary 4.4. *Assume that the conditions of Theorem 4.2 are satisfied. Let $f \in H_r^\varepsilon$ be as in (4.13) with f_1 is non-decreasing, $f_2(k\omega) = f_2(\omega)$, $k \in \mathrm{K}$ and $\|f\|_\varepsilon \le 1$. Then there are constants $C, C(f_1), \sigma$ such that*

$$\left| \langle f_2, \sigma_\nu \rangle - C(f_1)^{-1} \langle f, \nu_g \rangle \right| \le C |\log |g||^{-\sigma}, \tag{4.17}$$

holds for every f_2 as above and $|g| < 1$.

The measure σ_μ represents the weight of the tail of ν in various directions. Indeed, if $\sigma_\mu(\partial W) = 0$ then

$$\sigma_\mu(W) = \lim_{t \to \infty} t^\alpha \mathbb{P}\{|R| > t, \bar{R} \in W\}.$$

Suppose we could take in (4.15) $f_1 = \mathbf{1}_{[0,\infty)}$, $f_2 = \mathbf{1}_W$, then we could estimate

$$\sigma_\mu(W) - t^\alpha \mathbb{P}\{|R| > t, \bar{R} \in W\}.$$

We cannot do that, but taking Hölder functions f_1, f_2 close to functions $\mathbf{1}_{[0,\infty)}, \mathbf{1}_W$, we obtain something analogous.

Proof of Theorem 4.3. Let

$$\psi(g) = |g|^{-\alpha} \mathbb{E}\big[f(gR) - f(gM_1R_1) \big]. \tag{4.18}$$

Then, by [19],

$$\nu_g(f) = \psi * U(g). \tag{4.19}$$

Let $\bar{f}(g) = \nu_g(f)$. Notice that with our assumptions $\bar{f}$ and ψ are K-invariant, i.e. abusing slightly the notation, for $g = e^t k$ we have

$$\bar{f}(g) = \bar{f}(e^t), \text{ and } \psi(g) = \psi(e^t).$$

Moreover, for $F(t) = \psi(e^{-t})$ we have

$$\bar{f}(e^{-t}) = \int_{\mathbb{R}} \psi(e^{-t+s}) \, U_{\mathbb{R}}(ds) = \int_{\mathbb{R}} F(t-s) \, U_{\mathbb{R}}(ds) = U_{\mathbb{R}} * F(t).$$

So $|g| \to 0$ corresponds to $t \to \infty$. Since

$$\langle f, \Lambda \rangle = \frac{1}{m_\alpha} \int_{\mathrm{G}} \psi(g) \, dg = \frac{1}{m_\alpha} \int_{\mathbb{R}} F(t) \, dt,$$

where m_α is the mean of μ_α (see Theorem 4.1), we have

$$\langle f, \nu_g \rangle - \langle f, \Lambda \rangle = U_{\mathbb{R}} * F(t) - \frac{1}{m_\alpha} \int_{\mathbb{R}} F(t) \, dt.$$

We are going to apply Proposition 7.2 to F. Clearly F is continuous. In view of Lemma 6.3, $|\psi|$ is dRi and so is $|F|$. Therefore, $U_{\mathbb{R}} * \check{F} = \check{r}$ is well defined, where $\check{F}$ is the smoothing operator defined in (2.6). In particular F is bounded. The condition (7.4) is satisfied by (6.5). Since $f \in H_r^\varepsilon$ and since f is radially non-decreasing, it is real valued and non-negative. Hence, for $s < t$,

$$
\begin{aligned}
F * U_{\mathbb{R}}(t) - F * U_{\mathbb{R}}(s) &= \bar{f}(e^{-t}) - \bar{f}(e^{-s}) \\
&= e^{\alpha t}\big(\mathbb{E}f(e^{-t}R) - \mathbb{E}f(e^{-s}R) \big) + (e^{\alpha t} - e^{\alpha s})\mathbb{E}f(e^{-s}R), \\
&\le (e^{\alpha(t-s)} - 1)e^{\alpha s}\mathbb{E}f(e^{-s}R)
\end{aligned}
$$

because
$$e^{\alpha t}\big(\mathbb{E}f(e^{-t}R) - \mathbb{E}f(e^{-s}R)\big) \leq 0.$$

Hence
$$F * U_{\mathbb{R}}(t) - F * U_{\mathbb{R}}(s) \leq e^{\alpha \eta_0} \alpha(t-s) e^{\alpha s} \mathbb{E}f(e^{-s}R)$$

as soon as $s \leq t \leq s + \eta_0$. Now

$$
\begin{aligned}
0 &\leq e^{\alpha s} \mathbb{E}f(e^{-s}R) = e^{\alpha s}\mathbb{E}|f(e^{-s}R) - f(0)| \\
&\leq e^{(\alpha-\varepsilon)s}\|f\|_\varepsilon \mathbb{E}\big[|R|^\varepsilon \mathbf{1}_{\{|R| \geq re^s\}}\big] \leq Cr^{-\alpha+\varepsilon}\|f\|_\varepsilon.
\end{aligned}
$$

To get the last inequality we apply Theorem 4.1 to $s > 0$ and to function $|x|^\varepsilon \mathbf{1}_{\{|x| \geq 1\}}$. Finally,
$$F * U_{\mathbb{R}}(t) - F * U_{\mathbb{R}}(s) \leq Cr^{-\alpha+\varepsilon}(t-s)\|f\|_\varepsilon$$

and so (7.6) is satisfied. Hence the conclusion follows. $\qquad\square$

4.4 Main results for general functions

Our results for the general functions (not necessary K-invariant) require further assumptions on the measure μ. Recall the definitions of the function spaces $L^\infty(\mathrm{G})_0$, $L^2(\mathrm{G})_0$ given in (3.5). We are going to assume that

$$\|\phi * \mu\|_{L^\infty} \leq \lambda\|\phi\|_{L^\infty}, \ \phi \in L^\infty(\mathrm{G})_0 \tag{4.20}$$

for some $\lambda < 1$.

Alternatively, we are going to assume that

$$\|\phi * \mu\|_{L^2} \leq \lambda\|\phi\|_{L^2}, \ \phi \in L^2(\mathrm{G})_0, \tag{4.21}$$

for some $\lambda < 1$.

In Section 5 we discuss these assumptions in more detail and present conditions ensuring that (4.20) and (4.21) are satisfied. Moreover we have to restrict our attention to compactly supported functions. Our next main result is the following theorem.

Theorem 4.5. *Assume that the hypotheses of Theorem 4.1 are fulfilled, μ satisfies (4.20) or (4.21) and (4.10) with $\chi > 1$. Let (2.3) hold for $U_{\mathbb{R}}$. Fix $\tilde{r} > r > 0$. If $\alpha \leq 1$ let*

$$\mathcal{F} = \big\{f \in H_r^{0,1} : \mathrm{supp}f \subset \bar{B}_{\tilde{r}}(0) : \|f\|_1 \leq 1\big\}.$$

If $\alpha > 1$ and $m + \varepsilon < 1$ or $\varepsilon = 1$ and $m = 0$, let

$$\mathcal{F} = \{f \in H_r^{m,\varepsilon} : \|f\|_{m,\varepsilon} \leq 1\}.$$

Then there are $C, \sigma > 0$ such that

$$\rho_{\mathcal{F}}(\nu_g, \Lambda) \leq C|\log|g||^{-\sigma} \tag{4.22}$$

for small $|g|$.

The proof of this Theorem is similar to the proof of Theorem 4.2, but is technically more involved.

Notice first that definitions (4.18) and (4.19) can be extended to any function $f \in H_r^{m,\varepsilon}$. Indeed we have the following lemma.

Lemma 4.6. *Let $f \in H_r^{m,\varepsilon}$. Then the functions ψ and $\overline{f}$ given by*

$$\psi(g) = |g|^{-\alpha}\mathbb{E}\big[f(gR) - f(gM_1R_1)\big] \tag{4.23}$$

and

$$\bar{f}(g) = \psi * U(g) \tag{4.24}$$

are well defined.

Proof. The function $\psi * U$ is well defined in view of Lemma 6.3 and

$$\sum_{k=0}^{n} \psi * \mu_\alpha^k(g) = \bar{f}(g) - |g|^{-\alpha}(\bar{f} * \mu_\alpha^{n+1})(g).$$

It is enough to prove that for every g

$$\lim_{n \to \infty} \bar{f} * \mu_\alpha^n(g) = 0.$$

But this follows from (4.6) via a straightforward computation. $\qquad \square$

Next, we need to extend the concept of dRi functions to the group G_μ. We say that a bounded Borel function ϕ is dRi (directly Riemann integrable) on G_μ if

- the set of discontinuities of ϕ is negligible with respect to the Haar measure of G_μ;
- $\sum_n \sup_{\{g \in G_\mu : \, n < \log |g| \leq n+1\}} |\phi(g)| < \infty$.

A continuous function satisfying the second condition on G_μ is dRi on any subgroup of G_μ.

Proposition 4.7. *Assume that the potential $U_{\mathbb{R}}$ satisfies (2.3) and (4.10) is satisfied for some $\chi > 1$. Let $\mathcal{F}$ be as in Theorem 4.1. Suppose ψ is a dRi function such that for some constant D*

$$|\psi * \mu^n(g)| \leq C_1 n^D \|\psi\|_{L^\infty}(1 + |\log |g||)^{-\beta_1}, \ n \geq 0, \tag{4.25}$$

and

$$\left|(\psi - \psi^{\mathrm{K}}) * \mu^n(g)\right| \leq C_2(\psi)\lambda^n, \ n \geq 1 \tag{4.26}$$

for some $\lambda < 1$. Then

$$\left|\psi * U(g) - \frac{1}{m}\int_G \psi(h) \, dh\right|$$
$$\leq \Big(2C_1\|\psi\|_{L^\infty} + C_2(\psi)(1-\lambda)^{-1} + C_3(\psi)\Big)(\log |\log |g||)^D |\log |g||^{-\beta},$$

for $|g| < e^{-e}$ and some β.

The Proposition follows from Proposition 7.2 and its proof will be given in Section 7.

Proof of Theorem 4.5. We are going to apply Proposition 4.7 to μ_α and ψ. By Lemma 6.3, ψ is dRi.

Our next step is to justify that hypothesis (4.26) is satisfied. Assume first that (4.20) holds. Then (4.26) follows from Lemma 5.3 and Corollary 6.4.

Under the condition (4.21) this follows from Proposition 5.1 if we know that the norms $\|\psi\|_{L^2(G)}$ and $\|\psi\|_{H^\varepsilon(G)}$ (see (5.3) are finite. The first norm is bounded by Corollary 6.4. For the Hölder norm we apply the property

$$|k - k_1| \leq C\tau(k, k_1),$$

where $|\ |$ is the operator norm as in (1.6) and τ is the Riemannian distance on K (see Appendix B). Therefore, by Lemma 6.5, there is C such that for all $f \in \mathcal{F}$

$$|\psi(e^t k) - \psi(e^s k)| \leq C(|t - s| + \tau(k, k_1))^\varepsilon$$

i.e.

$$\|\psi\|_{H_r^\varepsilon(G)} \leq C.$$

Finally we have to check (4.25). By Lemma 6.3 there is C_4 such that

$$|\psi(g)| \leq C_4(1 + |\log|g||)^{-\chi}.$$

On the other hand (4.10) implies

$$\int_{\mathrm{G}} (1 + |\log|g||)^{\chi} \, \mu_\alpha(dg) < \infty.$$

Therefore, by Lemma 6.6 there is D such that for every $f \in \mathcal{F}$

$$|\psi * \mu_\alpha^n(g)| \leq C_1 n^D (1 + |\log|g||)^{-\chi} \tag{4.27}$$

and so (4.25) is satisfied. Thus, by Proposition 4.7 we conclude the Theorem. $\qquad\square$

5 Spectral cut off

5.1 Some more preliminaries and auxiliary results

Suppose that K is a compact Lie group with the Lie algebra $\mathfrak{k}$. Then both are Riemannian manifolds. There is $r_0 > 0$ small enough such that the Jacobian of the exponential map (see [38, Theorem 7.4, page 139] for the precise formula) is bounded away from zero on the ball of radius r_0 centered at zero in $\mathfrak{k}$ and every ball of radius $r < r_0$ centered at zero in $\mathfrak{k}$ is mapped bijectively onto the ball of radius r centered at the identity in the connected identity component of K, see [38, Proposition 9.4 and Ch.IV]. Let us denote by $v(r)$ the volume of the last ball. Then there is a constant c such that

$$v(r) \geq (cr)^{\dim(\mathrm{K})} \qquad (0 < r < r_0). \tag{5.1}$$

By taking direct products, the group $\mathrm{G} = \mathbb{R}^+ \times \mathrm{K}$ is also a Riemannian manifold with the bi-invariant distance $\mathsf{d}(\,,\,)$ and that there is $r_0 > 0$ and a constant C such that the volume $V(r)$ of the ball of radius r in G satisfies

$$V(r) \geq (Cr)^{\dim(\mathrm{G})} \qquad (0 < r < r_0). \tag{5.2}$$

(Recall [40, (8.6)] that there is an invariant distance on every compact group whose identity is equal to the intersection of countably many open sets, but we shall need an explicit estimate (5.2). Therefore we restrict our attention to compact Lie groups.)

We shall say that a function $\psi \in C(\mathrm{G})$ satisfies Hölder condition of order $\epsilon > 0$ if there is a finite constant $\|\psi\|_{H^\epsilon(\mathrm{G})}$ such that

$$|\psi(g) - \psi(g')| \leq \|\psi\|_{H^\epsilon(\mathrm{G})} \, \mathsf{d}(g, g')^\epsilon \qquad (g, g' \in \mathrm{G}). \tag{5.3}$$

The space of all such functions is denoted by $H^\epsilon(\mathrm{G})$.

Notice that if we equip $\mathbb{R}^+$ with the usual Riemannian structure then

$$\mathsf{d}(a, b) = |\log(ab^{-1})| = |\log(a) - \log(b)| \qquad (a, b \in \mathbb{R}^+).$$

Also, if $\|\,\|$ is the norm defined by the K invariant scalar product on $\mathfrak{k}$, which determines the Riemannian structure on K, then

$$\mathsf{d}(k, l) = \mathsf{d}(kl^{-1}, e) = \|\log(kl^{-1})\|$$

if $k, l \in \mathrm{K}$ are close enough.

Proposition 5.1. *Suppose μ is a probability measure on G and $0 < \lambda < 1$ is a constant such that*

$$\|\phi * \mu\|_{\mathrm{L}^2(\mathrm{G})} \leq \lambda \|\phi\|_{\mathrm{L}^2(\mathrm{G})}. \qquad (\phi \in \mathrm{L}^2(\mathrm{G})_0). \tag{5.4}$$

Then there is a constant C such that for $\epsilon > 0$

$$\|\psi * \mu^n\|_{\mathrm{L}^\infty(\mathrm{G})} \leq C \max\{\|\psi\|_{H^\epsilon(\mathrm{G})}, \|\psi\|_{\mathrm{L}^2(\mathrm{G})}\} \left(\lambda^{\frac{2\epsilon}{2\epsilon+\dim(\mathrm{G})}}\right)^n$$

$$(\psi \in H^\epsilon(\mathrm{G})_o, \ n = 0, 1, 2, \dots). \tag{5.5}$$

Remark 5.2. The idea of the proof comes from [25].

Proof. Let μ_r denote the Haar measure on G multiplied by the indicator function of the ball of radius r centered at the identity of G and divided by the volume of that ball. Then μ_r is a probability measure supported on the ball. Clearly

$$\begin{aligned}
|\psi(g) - \mu_r * \psi(g)| &\leq \int_{\mathrm{G}} |\psi(g) - \psi(h^{-1}g)| \, \mu_r(dh) \leq \int_{\mathrm{G}} \|\psi\|_{H^\epsilon(\mathrm{G})} \mathsf{d}(g, h^{-1}g)^\epsilon \mu_r(dh) \\
&= \int_{\mathrm{G}} \|\psi\|_{H^\epsilon(\mathrm{G})} \mathsf{d}(e, h^{-1})^\epsilon \mu_r(dh) = \int_{\mathrm{G}} \|\psi\|_{H^\epsilon(\mathrm{G})} \mathsf{d}(h, e)^\epsilon \mu_r(dh) \\
&\leq r^\epsilon \|\psi\|_{H^\epsilon(\mathrm{G})}.
\end{aligned}$$

Hence,

$$\|\psi - \mu_r * \psi\|_{\mathrm{L}^\infty(\mathrm{G})} \leq r^\epsilon \|\psi\|_{H^\epsilon(\mathrm{G})}. \tag{5.6}$$

On the other hand, Cauchy's inequality, the assumption (5.4) and Lemma 3.1 show that

$$\|\mu_r * \psi * \mu^n\|_{\mathrm{L}^\infty(\mathrm{G})} \leq V(r)^{-\frac{1}{2}} \|\psi * \mu^n\|_{\mathrm{L}^2(\mathrm{G})} \leq V(r)^{-\frac{1}{2}} \lambda^n \|\psi\|_{\mathrm{L}^2(\mathrm{G})}. \tag{5.7}$$

Hence,

$$\begin{aligned}
\|\psi * \mu^n\|_{\mathrm{L}^\infty(\mathrm{G})} &\leq \|\psi * \mu^n - \mu_r * \psi * \mu^n\|_{\mathrm{L}^\infty(\mathrm{G})} + \|\mu_r * \psi * \mu^n\|_{\mathrm{L}^\infty(\mathrm{G})} \\
&\leq \|\psi - \mu_r * \psi\|_{\mathrm{L}^\infty(\mathrm{G})} + \|\mu_r * \psi * \mu^n\|_{\mathrm{L}^\infty(\mathrm{G})} \\
&\leq r^\epsilon \|\psi\|_{H^\epsilon(\mathrm{G})} + V(r)^{-\frac{1}{2}} \lambda^n \|\psi\|_{\mathrm{L}^2(\mathrm{G})} \\
&\leq \max\{\|\psi\|_{H^\epsilon(\mathrm{G})}, \|\psi\|_{\mathrm{L}^2(\mathrm{G})}\}(r^\epsilon + V(r)^{-\frac{1}{2}} \lambda^n).
\end{aligned}$$

Let $r = C^{-\frac{\dim(\mathrm{G})}{2\epsilon+\dim(\mathrm{G})}} \left(\lambda^{\frac{2}{2\epsilon+\dim(\mathrm{G})}}\right)^n$. Then $r^\epsilon = (Cr)^{-\frac{\dim(\mathrm{G})}{2}} \lambda^n$.

If n is large enough then $r < r_0$ and we have the estimate (5.2). Hence,

$$r^\epsilon + V(r)^{-\frac{1}{2}} \lambda^n \leq r^\epsilon + (Cr)^{-\frac{\dim(\mathrm{G})}{2}} \lambda^n = 2r^\epsilon = 2C^{-\frac{\dim(\mathrm{G})}{2}} \left(\lambda^{\frac{2\epsilon}{2\epsilon+\dim(\mathrm{G})}}\right)^n.$$

Therefore

$$\|\psi * \mu^n\|_{\mathrm{L}^\infty(\mathrm{G})} \leq max\{\|\psi\|_{H^\epsilon(\mathrm{G})}, \|\psi\|_{\mathrm{L}^2(\mathrm{G})}\}2C^{-\frac{\dim(\mathrm{G})}{2\epsilon+\dim(\mathrm{G})}} \left(\lambda^{\frac{2\epsilon}{2\epsilon+\dim(\mathrm{G})}}\right)^n. \tag{5.8}$$

The estimate holds with the constant "C" equal to the supremum over $\epsilon > 0$ of $2C^{-\frac{\dim(\mathrm{G})}{2\epsilon+\dim(\mathrm{G})}}$. $\qquad \square$

Lemma 5.3. *Let μ be a bounded measure on G. Suppose $\lambda \geq 0$ is such that*

$$\|\psi * \mu\|_\infty \leq \lambda \|\psi\|_\infty \qquad (\psi \in C(\mathrm{G})_o). \tag{5.9}$$

Then

$$\|\psi * \mu^n\|_\infty \leq \lambda^n \|\psi\|_\infty \qquad (\psi \in C(\mathrm{G})_o, \ n = 1, 2, 3, \dots). \tag{5.10}$$

Proof. We see from Lemma 3.1 that

$$(\psi * \mu^n)^{\mathrm{K}} = \psi^{\mathrm{K}} * \mu^n.$$

Thus if $\psi^{\mathrm{K}} = 0$ then $(\psi * \mu^n)^{\mathrm{K}} = 0$ for any $n = 1, 2, 3, \dots$. Therefore (5.9) implies (5.10). $\qquad \square$

5.2 Examples

Let us discuss some conditions that imply (4.21). By Lemma A.2 it is enough for μ to have density. This assumption can be relaxed because instead of μ we may consider μ^n. Indeed, ν is the stationary measure also for the recursion (1.1) with (Q, M) having the law μ^n. Also, if the measure μ is of the form

$$\mu = s\mu_1 + (1-s)\mu_2$$

for an s such that $0 < s \le 1$ and μ_1 satisfies (4.21) then so does μ. This means that the assumption for μ to be spread out is sufficient.

We can formulate still weaker hypotheses. Let μ be a probability measure on G and let $\tilde{\mu}$ be the push-forward of μ via the projection $G \to \mathbb{R}^+$. We deduce from the theory of the conditional probability, as in [30], that there are probability measures μ_a, $a \in \mathbb{R}^+$, such that

$$\int_G \phi(g)\,\mu(dg) = \int_{\mathbb{R}^+} \int_K \phi(ak)\,\mu_a(dk)\,\tilde{\mu}(da) \qquad (\phi \in C(G)). \tag{5.11}$$

In our settings, if μ is the law of M, then $\tilde{\mu}$ is the law of $|M|$ and μ_a is a version of the conditional law of $\frac{1}{|M|}M$, given $|M| = a$. We want the operators

$$T_a\phi = \phi * \mu_a$$

to have good properties on $L^2(K)_0$.

Proposition 5.4. *Let* $\|T_a\|_{L^2(K)_0}$ *be the norm of* T_a *on* $L^2(K)_0$ *and let*

$$\beta = \int_{\mathbb{R}^+} \|T_a\|_{L^2(K)_0}\,\tilde{\mu}(da).$$

Then for every $\phi \in L^2(G)_0$

$$\|\phi * \mu\|_{L^2} < \beta\|\phi\|_{L^2}.$$

Remark 5.5. In particular, if $\beta < 1$ then (4.21) holds. This assumption is clearly satisfied when there is a set $A \subset \mathbb{R}^+$ of positive measure $\tilde{\mu}(a)$ such that $\|T_a\| < 1$. For instance μ_a being spread out for $a \in A$ is sufficient. The same can be applied to $(\mu^n)_a$ that is to the measure obtained from disintegration of μ^n.

Proof. Let $\phi, \tilde{\phi} \in L^2(G)_0$ and let $\phi_a(k) = \phi(ak)$ and similarly for $\tilde{\phi}$. Then the absolute value of the $L^2(G)$ - scalar product of $\phi * \mu$ and $\tilde{\phi}$ is equal to

$$|\langle \phi * \mu, \tilde{\phi}\rangle| = \left| \int_{\mathbb{R}^+ \times \mathbb{R}^+} \int_K \phi_{aa_1} * \mu_{a_1}(k)\overline{\tilde{\phi}_a(k)}\,dk\tilde{\mu}(da_1)da \right|$$

$$\le \int_{\mathbb{R}^+ \times \mathbb{R}^+} \|T_{a_1}\|_{L^2(K)_0}\|\phi_{aa_1}\|_{L^2(K)_0}\|\tilde{\phi}_a\|_{L^2(K)_0}\,\tilde{\mu}(da_1)da.$$

Integrating first with respect to a and then with respect to a_1, we get the conclusion. $\square$

Remark 5.6. The same proof holds for the L^∞ norm i.e. we may replace the norm $\|T_a\|_{L^2(K)_0}$ wih the norm $\|T_a\|_{L^\infty(K)_0}$ in Proposition 5.4 and Remark 5.5.

6 Properties of ψ

In this section we collect all the lemmas describing the behavior of the function ψ defined in 4.18. Recall that the function ψ depends on some function f. Thus the behavior of ψ is determined by the properties of f. We consider here pointwise and integral estimates as well as finiteness of norms discussed in Section 4.

We begin with a very particular case considered in Theorems 2.2 and 4.2, where the function f is of the form $f(x) = \mathbf{1}_{(1,\infty)}(x)$ or $f(x) = \mathbf{1}_{\mathbb{R}^d \setminus B_1(0)}(x)$.

Lemma 6.1. *Let*
$$g(t) = e^{\alpha t}\big(\mathbb{P}[R > e^t] - \mathbb{P}[MR > e^t]\big)$$

in the one dimensional case and
$$g(t) = e^{\alpha t}\big(\mathbb{P}[|R| > e^t] - \mathbb{P}[|MR| > e^t]\big)$$

in general. If (4.10) *holds, then*
$$\int_{\mathbb{R}} |g(t)||t|^\chi dt < \infty.$$

Here $g(t) = \psi(e^t k)$, if we take $f(x) = \mathbf{1}_{\mathbb{R}^d \setminus B_1(0)}(x)$ in (4.18).

Proof. It is sufficient to study the integral for large positive values of t, since for negative t:
$$|g(t)| \le 2e^{-\alpha|t|}.$$

Let $X = MR + Q$, $Y = MR$ or $X = |MR + Q|$, $Y = |MR|$, for R independent of (M, Q). Given a real number z we denote $z_e = \max\{z, e\}$. Then

$$
\begin{aligned}
\int_1^\infty |g(t)| t^\chi dt &= \int_1^\infty e^{\alpha t} t^\chi \big|\mathbb{P}[X > e^t] - \mathbb{P}[Y > e^t]\big| dt \\
&\le \int_e^\infty s^{\alpha-1}(\log s)^\chi \mathbb{E}\big[\mathbf{1}_{\{X \ge s \ge Y\}}\big] ds + \int_e^\infty s^{\alpha-1}(\log s)^\chi \mathbb{E}\big[\mathbf{1}_{\{Y > s > X\}}\big] ds \\
&= \mathbb{E}\Big[\mathbf{1}_{\{X \ge Y\}} \int_{Y_e}^{X_e} s^{\alpha-1}(\log s)^\chi ds\Big] + \mathbb{E}\Big[\mathbf{1}_{\{X < Y\}} \int_{Y_e}^{X_e} s^{\alpha-1}(\log s)^\chi ds\Big] \\
&\le \frac{1}{\alpha}\mathbb{E}\big[\mathbf{1}_{\{X \ge Y\}}(\log X_e)^\chi (X_e^\alpha - Y_e^\alpha)\big] + \frac{1}{\alpha}\mathbb{E}\big[\mathbf{1}_{\{X < Y\}}(\log Y_e)^\chi (Y_e^\alpha - X_e^\alpha)\big] \\
&\le \frac{1}{\alpha}\mathbb{E}\Big[\big((\log X_e)^\chi + (\log Y_e)^\chi\big)\big|X_e^\alpha - Y_e^\alpha\big|\Big] = I.
\end{aligned}
$$

Assume first that $\alpha \le 1$. Then using inequalities
$$
\begin{aligned}
|a^\alpha - b^\alpha| &\le |a - b|^\alpha, \qquad a, b > 0, \\
\log(x + y)_e &\le \log x_e + \log y_e, \\
\log(xy)_e &\le \log(1 + |x|) + \log(1 + |y|) + 1,
\end{aligned}
$$

we obtain that I can be estimated by
$$\mathbb{E}\Big[(1 + \log(1 + |M|) + \log(1 + |Q|) + \log(1 + |R|))^\chi |Q|^\alpha\Big],$$

which is finite by (4.10) and the fact that $\mathbb{E}|R|^\beta < \infty$ for any $\beta < \alpha$, see [19, Lemma D.8].

For $\alpha > 1$ we apply the inequality
$$|a^\alpha - b^\alpha| \le \alpha|a - b|\max\{a, b\}^{\alpha-1}$$

and we dominate I by
$$\mathbb{E}\Big[(1 + \log(1 + |M|) + \log(1 + |Q|) + \log(1 + |R|))^\chi |Q|(|M|^{\alpha-1}|R|^{\alpha-1} + |Q|^{\alpha-1})\Big].$$

To prove that the latter expression is finite we use independence of R and (Q, M), (4.10) as well as the Hölder inequality. For instance, by the Hölder inequality with $p = \alpha$, $q = \frac{\alpha}{\alpha-1}$, we estimate

$$\mathbb{E}\big[(1 + \log(1 + |M|))^\chi |Q||M|^{\alpha-1}\big]$$
$$\le \Big(\mathbb{E}\big[(1 + \log(1 + |M|))^\chi |Q|^\alpha\big]\Big)^{1/\alpha} \Big(\mathbb{E}\big[(1 + \log(1 + |M|))^\chi |M|^\alpha\big]\Big)^{(\alpha-1)/\alpha},$$

which is finite, by (4.10). $\square$

The following lemma concerns functions $f \in H_r^{m,\varepsilon}$ and provides the control of their derivatives. It will be used in Lemma 6.3.

Lemma 6.2. *Let I be a multiindex, $|I| \le m$. Then for every $f \in H_r^{m,\varepsilon}$ and $x \in \mathbb{R}^d$*

$$|\partial^I f(x)| \le C_{|I|} \|f\|_{m,\varepsilon} |x|^{m+\varepsilon-|I|}, \tag{6.1}$$

where $C_{|I|} = d^{m-|I|} \left(\Pi_{j=1}^{m-|I|} (j+\varepsilon) \right)^{-1}$.

Moreover, if $m \ge 1$, then there is $C = C(m,\varepsilon,d)$ such that for every $f \in H_r^{m,\varepsilon}$ and every $x, y \in \mathbb{R}^d$

$$|f(x) - f(y)| \le C\|f\|_{m,\varepsilon} (|x| + |y|)^{m+\varepsilon-1} |x-y|. \tag{6.2}$$

Proof. Notice that $\partial^I f(0) = 0$ so for $|I| = m$ (6.1) follows from the definition of the seminorm (4.7). Let $|I| < m$. Then

$$\partial^I f(x) = \partial^I f(x) - \partial^I f(0) = \int_0^1 \frac{\partial}{\partial t} \partial^I f(tx) \, dt = \int_0^1 \sum_{j=1}^d (\partial_{x_j} \partial^I f)(tx) x_j \, dt.$$

Hence, by induction,

$$|\partial^I f(x)| \le C_{|I|+1} d |x|^{m+\varepsilon-|I|} \|f\|_{m,\varepsilon} \int_0^1 t^{m+\varepsilon-|I|-1} \, dt$$

and (6.1) follows for an arbitrary I.

Next, since

$$f(x) - f(y) = \int_0^1 \frac{d}{dt} f(x + t(y-x)) \, dt$$

$$= \int_0^1 \sum_{j=1}^d (\partial_{x_j} f)(x + t(y-x))(y_j - x_j) \, dt$$

we have

$$|f(x) - f(y)| \le C_1 \|f\|_{m,\varepsilon} d \int_0^1 |x + t(y-x)|^{m+\varepsilon-1} |y-x| \, dt$$

$$\le C_1 \|f\|_{m,\varepsilon} d \max\{1, 2^{m+\varepsilon-2}\} (|x| + |y|)^{m+\varepsilon-1} |y-x|,$$

which implies (6.2). $\qquad\square$

The following lemma is crucial. It implies that if $f \in H_r^{m,\varepsilon}$, then the corresponding function ψ is dRi and provides also its estimates.

Lemma 6.3. *Assume (4.10). Let $0 < \varepsilon \le 1$, m be a nonnegative integer and let $\alpha > m+\varepsilon$. Then there is a constant $C = C(\alpha, m, \varepsilon, d, \bar{\mu}, \nu)$ such that for any $f \in H_r^{m,\varepsilon}$,*

$$\sum_{n \in \mathbb{Z}} \sup_{n \le \log|g| \le n+1} |\psi(g)| \le Cr^{-(\alpha-m-\varepsilon)} \|f\|_{m,\varepsilon}. \tag{6.3}$$

Also,

$$\sup_{g \in \mathrm{G}} |\psi(g)| (1 + |\log|g||)^\chi \le Cr^{-(\alpha-m-\varepsilon)} \max \left(r^{\varepsilon/2}, r^{-\varepsilon/2} \right) \|f\|_{m,\varepsilon} \tag{6.4}$$

and

$$\int_{\mathrm{G}} |\psi(g)| (1 + |\log|g||)^\chi \, dg \le Cr^{-(\alpha-m-\varepsilon)} \max \left(r^{\varepsilon/2}, r^{-\varepsilon/2} \right) \|f\|_{m,\varepsilon}. \tag{6.5}$$

Corollary 6.4. *Suppose that the assumptions of Lemma 6.3 are satisfied. Then there is a constant $C = C(\alpha, m, \varepsilon, d, \bar{\mu}, \nu)$ such that for every $f \in H_r^{m,\varepsilon}$*

$$\|\psi\|_{L^\infty}, \|\psi\|_{L^1}, \|\psi\|_{L^2} \le Cr^{-(\alpha-m-\varepsilon)} \max\left(r^{\varepsilon/2}, r^{-\varepsilon/2}\right)\|f\|_{m,\varepsilon}.$$

Proof of Lemma 6.3. In the proof we will denote by C a constant, that may vary from line to line, but depends only on $\alpha, m, \varepsilon, \chi, \bar{\mu}, \nu$ and d.

Step 1. Proof of (6.3). Suppose first that $m \ge 1$. Notice that since $R = Q_1 + M_1 R_1$, $f(gR) - f(gM_1R_1) \ne 0$ implies $|gQ_1| + |gM_1R_1| \ge r$. Let $\mathbf{1}_s$ be the indicator function of the event $\{|Q_1| + |M_1R_1| \ge s\}$. Then for $n \le \log|g| \le n+1$ by Lemma 6.2 we have

$$|\psi(g)| \le C\|f\|_{m,\varepsilon}|g|^{-\alpha}\mathbb{E}\Big[(|gR| + |gM_1R_1|)^{m+\varepsilon-1}|gQ_1|\mathbf{1}_{r|g|^{-1}}\Big]$$
$$\le C\|f\|_{m,\varepsilon}e^{-(\alpha-m-\varepsilon)n}\mathbb{E}\Big[(|Q_1| + 2|M_1R_1|)^{m+\varepsilon-1}|Q_1|\mathbf{1}_{re^{-n-1}}\Big].$$

Therefore, taking $n_0 = \lfloor -\log(|R| + |M_1R_1|) + \log r - 1\rfloor$, we have

$$\sum_{n\in\mathbb{Z}} \sup_{n\le\log|g|\le n+1} |\psi(g)| \le C\|f\|_{m,\varepsilon}\mathbb{E}\Big[(|Q_1| + 2|M_1R_1|)^{m+\varepsilon-1}|Q_1| \sum_{n>n_0} e^{-(\alpha-m-\varepsilon)n}\Big]$$
$$\le Cr^{-(\alpha-m-\varepsilon)}\|f\|_{m,\varepsilon}\mathbb{E}\Big[(|Q_1| + |M_1R_1|)^{\alpha-1}|Q_1|\Big].$$

Now a standard argument using (4.10) and based on the Hölder inequality proves finiteness of the last expectation. Thus (6.3) follows.

If $m = 0$ then in the first step of the proof we write

$$|\psi(g)| \le \|f\|_{m,\varepsilon}|g|^{-\alpha}\mathbb{E}\Big[|gQ_1|^\varepsilon\mathbf{1}_{r|g|^{-1}}\Big]$$

and the rest of the argument caries over.

Step 2. Proof of (6.4). We proceed as in the first step. Let $m \ge 1$. Since for $|g| \ge 1$ we can dominate $(1 + |\log|g||)^\chi$ by $C|g|^\delta$ for any positive δ, we have

$$|\psi(g)|(1 + |\log|g||)^\chi \le C\|f\|_{m,\varepsilon}|g|^{-\alpha+m+\varepsilon+\delta}\mathbb{E}\Big[(|Q_1| + |M_1||R_1|)^{m+\varepsilon-1}|Q_1|\mathbf{1}_{r|g|^{-1}}\Big]$$
$$\le Cr^{-\alpha+m+\varepsilon+\delta}\|f\|_{m,\varepsilon}\mathbb{E}\Big[(|Q_1| + |M_1||R_1|)^{\alpha-1-\delta}|Q_1|\Big].$$

For the last inequality we have used $r|g|^{-1} \le |Q_1| + |M_1||R_1|$. For $m = 0$, we write

$$|\psi(g)|(1 + |\log|g||)^\chi \le C\|f\|_{m,\varepsilon}|g|^{-\alpha+\varepsilon+\delta}\mathbb{E}\big[|Q_1|^\varepsilon\mathbf{1}_{r|g|^{-1}}\big]$$
$$\le Cr^{-\alpha+\varepsilon+\delta}\|f\|_{m,\varepsilon}\mathbb{E}\big[(|Q_1| + |M_1||R_1|)^{\alpha-\varepsilon-\delta}|Q_1|^\varepsilon\big].$$

Again, Hölder inequality implies that both expectations in the last and in the previous formulae are finite. Hence we get the bound (6.4) for $|g| \ge 1$.

For $|g| \le 1$ and $m \ge 1$, we write

$$|\psi(g)|(1 + |\log|g||)^\chi$$
$$\le C\|f\|_{m,\varepsilon}|g|^{-\alpha+m+\varepsilon}(1 + |\log|g||)^\chi\mathbb{E}\Big[(|Q_1| + |M_1||R_1|)^{m+\varepsilon-1}|Q_1|\mathbf{1}_{r|g|^{-1}}\Big].$$

Since for $1 \le |g|^{-1} \le r^{-1}(|Q_1| + |M_1||R_1|)$ we have

$$|\log|g|| \le \log\big(1 + r^{-1}(|Q_1| + |M_1||R_1|)\big)$$
$$\le \log(1 + r^{-1}) + \log(1 + |Q|) + \log(1 + |M_1|) + \log(1 + |R_1|),$$

and we estimate

$$|\psi(g)|(1 + |\log|g||)^\chi \le C\|f\|_{m,\varepsilon} r^{-\alpha+m+\varepsilon} \cdot \mathbb{E}\Big[(|Q_1| + |M_1||R_1|)^{\alpha-1}|Q_1|$$
$$\big(\log(1+r^{-1}) + \log(1+|Q_1|) + \log(1+|M_1|) + \log(1+|R_1|)\big)^\chi\Big].$$

Now using (4.10), independence R_1 of (Q_1, M_1) and the fact that $\mathbb{E}|R_1|^{\alpha-1}\log(1+|R_1|)^\chi < \infty$ we prove that the last expectation is finite. To indicate how to do it let us estimate for instance

$$\mathbb{E}\Big[|M_1|^{\alpha-1}|R_1|)^{\alpha-1}|Q_1|\big(\log(1+|Q_1|)\big)^\chi\Big] = \mathbb{E}[|R_1|^{\alpha-1}]\mathbb{E}\Big[|M_1|^{\alpha-1}|Q_1|\big(\log(1+|Q_1|)\big)^\chi\Big]$$
$$\le \mathbb{E}[|R_1|^{\alpha-1}]\Big(\mathbb{E}\Big[|Q_1|^\alpha\big(\log(1+|Q_1|)\big)^\chi\Big]\Big)^{1/\alpha}\Big(\mathbb{E}\Big[|M_1|^\alpha\big(\log(1+|Q_1|)\big)^\chi\Big]\Big)^{(\alpha-1)/\alpha}.$$

The factor $\log(1 + r^{-1})$ gives an extra factor $\max\{r^{-\varepsilon/2}, 1\}$ and (6.4) follows for $|g| \le 1$ too. If $m = 0$, we proceed similarly starting with

$$|\psi(g)|(1 + |\log|g||)^\chi \le C\|f\|_{m,\varepsilon}|g|^{-\alpha+\varepsilon}(1 + |\log|g||)^\chi \mathbb{E}\big[|Q|^\varepsilon \mathbf{1}_{r|g|^{-1}}\big]. \qquad (6.6)$$

Step 3. Proof of (6.5). For $m \ge 1$ we write

$$\int_{|g|\ge 1} |\psi(g)|(1 + |\log|g||)^\chi dg \le C \int_{|g|\ge 1} |\psi(g)||g|^\delta \, dg$$
$$\le C\|f\|_{m,\varepsilon} \int_{|g|\ge 1} |g|^{-\alpha+m+\varepsilon+\delta} \mathbb{E}\Big[(|Q_1| + |M_1||R_1|)^{m+\varepsilon-1}|Q_1|\mathbf{1}_{r|g|^{-1}}\Big] dg$$
$$\le C\|f\|_{m,\varepsilon} \mathbb{E}\Big[|Q_1|(|Q_1| + |M_1||R_1|)^{m+\varepsilon-1} \int_{r^{-1}(|Q_1|+|M_1||R_1|)}^\infty s^{-\alpha+m+\varepsilon+\delta} \frac{ds}{s}\Big]$$
$$\le C\|f\|_{m,\varepsilon} r^{-\alpha+m+\varepsilon+\delta} \mathbb{E}\Big[|Q_1|(|Q_1| + |M_1||R_1|)^{\alpha-1-\delta}\Big]$$

and

$$\int_{|g|\le 1} |\psi(g)|(1 + |\log|g||)^\chi dg$$
$$\le C\|f\|_{m,\varepsilon} \int_{|g|\le 1} |g|^{-\alpha+m+\varepsilon}(1 + |\log|g||)^\chi \mathbb{E}\Big[(|Q_1| + |M_1||R_1|)^{m+\varepsilon-1}|Q_1|\mathbf{1}_{r|g|^{-1}}\Big] dg$$
$$\le C\|f\|_{m,\varepsilon} \mathbb{E}\Big[|Q_1|(|Q| + |M_1||R_1|)^{m+\varepsilon-1} \int_1^{r^{-1}(|Q_1|+|M_1||R_1|)} s^{\alpha-m-\varepsilon}(1 + |\log s|)^\chi \frac{ds}{s}\Big].$$

Integrating by parts we notice that

$$\int_1^{r^{-1}(|Q_1|+|M_1||R_1|)} s^{\alpha-m-\varepsilon}(1 + |\log s|)^\chi \frac{ds}{s}$$
$$\le Cr^{-(\alpha-m-\varepsilon)}(|Q_1| + |M_1||R_1|)^{\alpha-m-\varepsilon}\big(\log(1+r^{-1}(|Q_1| + |M_1||R_1|))\big)^\chi.$$

Hence

$$\int_{|g|\le 1} |\psi(g)|(1 + |\log|g||)^\chi dg$$
$$\le C\|f\|_{m,\varepsilon} r^{-(\alpha-m-\varepsilon)} \mathbb{E}\Big[(|Q_1| + |M_1||R_1|)^{\alpha-1}\big(\log(1+r^{-1}(|Q_1| + |M_1||R_1|))\big)^\chi\Big].$$

And the rest follows as in the previous step. $\qquad \square$

The following lemma will be used to justify the Tauberian condition (7.6) in Proposition 7.2.

Lemma 6.5. *Let $0 < \varepsilon \leq 1, \alpha > \varepsilon + m$ and $r > 0$. There is a constant $C = C(\alpha, m, \varepsilon, d, \bar{\mu}, \nu)$ such that for every $f \in H_r^{m,\varepsilon}$*

$$|\psi(e^t k) - \psi(e^s k_1)| \leq C \max\{1, r^{-(\alpha-m-\varepsilon)}\}\|f\|_{m,\varepsilon}\big(|t-s|^\varepsilon + |k-k_1|^\varepsilon\big), \qquad (6.7)$$

where $|k - k_1|$ is norm of $k - k_1$, as in (1.6).

Proof. In view of Corollary 6.4 we may restrict our attention to $|t - s| \leq 1$. We have

$$\psi(e^t k) - \psi(e^s k_1) = (e^{-\alpha t} - e^{-\alpha s})\mathbb{E}\big[f(e^t kR) - f(e^t k M_1 R_1)\big]$$
$$+ e^{-\alpha s}\mathbb{E}\big[f(e^t kR) - f(e^s k_1 R)\big] - e^{-\alpha s}\mathbb{E}\big[f(e^t k M_1 R_1) - f(e^s k_1 M_1 R_1)\big]$$
$$= I_1 + I_2 + I_3$$

To estimate I_1, by Corollary 6.4, we write

$$|I_1| \leq \alpha e^\alpha |t - s||\psi(e^t k)| \leq Cr^{-(\alpha-m-\varepsilon)} \max\big(r^{\varepsilon/2}, r^{-\varepsilon/2}\big)\|f\|_{m,\varepsilon}.|t-s|^\varepsilon$$

For I_2 assume $s > t$. Then, if $m = 0$

$$|I_2| \leq e^{-\alpha s}\|f\|_{0,\varepsilon}\mathbb{E}\Big[|e^t kR - e^s k_1 R|^\varepsilon \mathbf{1}_{\{|R|>re^{-s}\}}\Big]$$
$$\leq e^{-\alpha s}\|f\|_{0,\varepsilon}\mathbb{E}\Big[\big(|(e^t - e^s)R|^\varepsilon + |e^s (k - k_1)R|^\varepsilon\big)\mathbf{1}_{\{|R|>re^{-s}\}}\Big]$$
$$= e^{-\alpha s + \varepsilon s}\|f\|_{0,\varepsilon}\mathbb{E}\Big[\big(|e^{t-s} - 1|^\varepsilon |R|^\varepsilon + |k - k_1|^\varepsilon |R|^\varepsilon\big)\mathbf{1}_{\{|R|>re^{-s}\}}\Big]$$
$$\leq e^{-\alpha s + \varepsilon s}\|f\|_{0,\varepsilon}(|t-s|^\varepsilon + |k-k_1|^\varepsilon)\mathbb{E}\Big[|R|^\varepsilon \mathbf{1}_{\{|R|>re^{-s}\}}\Big]$$
$$\leq \max\{1, r^{-\alpha+\varepsilon}\}\|f\|_{0,\varepsilon}(|t-s|^\varepsilon + |k-k_1|^\varepsilon),$$

where for the last inequality we used Theorem 4.1.

If $m \geq 1$ then by Lemma 6.2

$$\mathbb{E}\Big[|f(e^t kR) - f(e^s k_1 R)|\Big] \leq C\|f\|_{m,\varepsilon}(e^t + e^s)^{m+\varepsilon-1}\mathbb{E}\Big[|R|^{m+\varepsilon-1}|e^t kR - e^s k_1 R|\Big]$$
$$\leq C\|f\|_{m,\varepsilon}e^{s(m+\varepsilon)}(e+1)^{m+\varepsilon-1}\mathbb{E}\Big[|R|^{m+\varepsilon-1}(|t-s| + |k-k_1|)|R|\Big].$$

But again we may assume that $|t - s| \leq 1$, also $|k - k_1| \leq 2$. Therefore,

$$\begin{aligned}|I_2| &\leq& C\|f\|_{m,\varepsilon}e^{-(\alpha-m+\varepsilon)s}(|t-s|^\varepsilon + |k-k_1|^\varepsilon)\mathbb{E}\Big[|R|^{m+\varepsilon}\mathbf{1}_{\{|R|>re^{-1-s}\}}\Big]\\ &\leq& \max\{1, r^{-\alpha+m+\varepsilon}\}\|f\|_{0,\varepsilon}(|t-s|^\varepsilon + |k-k_1|^\varepsilon).\end{aligned}$$

$\square$

Lemma 6.6. *Suppose μ is a probability measure on G and $\chi > 1$ is a constant. Let $\psi \in C(G)$ be a function such that, for some constant $C_{\psi,\chi}$,*

$$|\psi(g)| \leq C_{\psi,\chi}(1 + |\log|g||)^{-\chi} \qquad (g \in G).$$

Then

$$|\psi * \mu^n(g)| \leq C_{n,\psi,\chi}(1 + |\log|g||)^{-\chi} \qquad (g \in G, \ n = 1, 2, 3, \dots),$$

where

$$C_{n,\psi,\beta} = 2^{\chi-1}\left(C_{\psi,\chi} + \|\psi\|_\infty n^\chi \int_G |\log|h||^\beta \ \mu(dh)\right).$$

Proof. Fix any number $0 < \eta < 1$. Suppose $|g| < 1$, so that $\log|g| < 0$. Since

$$
\begin{aligned}
\int_G |\log|h||^\chi \, d\mu^n(h) &= \int_G \cdots \int_G \big|\log|h_1 \ldots h_n|\big|^\chi \mu(dh_1) \ldots \mu(dh_n) \\
&= \int_G \cdots \int_G \Big| \sum_{j=1}^n \log|h_j| \Big|^\chi \mu(dh_1) \ldots \mu(dh_n) \\
&\leq \int_G \cdots \int_G n^{\chi-1} \sum_{j=1}^n \big|\log|h_j|\big|^\chi \mu(dh_1) \ldots \mu(dh_n) \\
&= n^\chi \int_G \big|\log|h|\big|^\chi \mu(dh).
\end{aligned}
$$

we have

$$
\begin{aligned}
(1 + |\log|g||)^\chi |\psi * \mu^n(g)| &= \int_G |(1 + |\log|ghh^{-1}||)^\chi \psi(gh)| \, d\mu^n(h) \\
&= \int_G |(1 + |\log|gh|| + |\log|h||)^\chi \psi(gh)| \, d\mu^n(h) \\
&\leq 2^{\chi-1} \Big(\int_G \big(1 + \log|gh|\big)^\chi |\psi(gh)| \, d\mu^n(h) + \int_G \big(1 + \log|h|\big)^\chi |\psi(gh)| \, d\mu^n(h) \Big) \\
&\leq 2^{\chi-1} \Big(C_{\psi,\chi} + \|\psi\|_\infty n^\chi \int_G |\log|h||^\chi \, d\mu(h) \Big)
\end{aligned}
$$

$\square$

The next two lemmas will be used in the proof of Proposition 4.7, which is the main step in the proof of Theorem 4.5.

Lemma 6.7. *Suppose $\alpha > 1$ and $f \in H_r^{0,1}$. Consider the function*

$$
\eta(t) = e^{-\alpha t} \int \mathbb{E} f(e^t k R) dk. \tag{6.8}
$$

Then there is $C = C(\alpha, \nu)$ such that for every $s, t \in \mathbb{R}$,

$$
|\eta(t) - \eta(s)| \leq C \max\{1, r^{-\alpha+1}\} \|f\|_{0,1} |t - s|. \tag{6.9}
$$

If $\alpha \leq 1$, $f \in H_r^{0,1}$ and additionally $\operatorname{supp} f \subset B_{\tilde{r}}(0)$, $r < \tilde{r}$. Then there is $C = C(\alpha, \nu)$ such that for every $s, t \in \mathbb{R}$,

$$
|\eta(t) - \eta(s)| \leq C r^{-\alpha} \tilde{r} \|f\|_{0,1} |t - s|. \tag{6.10}
$$

Proof. Assume first $\alpha > 1$. Notice that for every $t \in \mathbb{R}$,

$$
|\eta(t)| \leq C \max\{1, r^{-\alpha+1}\} \|f\|_{0,1}. \tag{6.11}
$$

Indeed, we have

$$
\begin{aligned}
|\eta(t)| &\leq e^{-\alpha t} \int \mathbb{E}|f(e^t k R) - f(0)| \, dk \leq e^{-\alpha t + t} \|f\|_{0,1} \mathbb{E}\Big[|R| \mathbf{1}_{\{|R| \geq re^{-t}\}}\Big] \\
&\leq e^{-(\alpha-1)t} \|f\|_{0,1} \mathbb{E}|R|.
\end{aligned}
$$

The last expectation being finite. For $t < 0$, applying Theorem 4.1 with $f(x) = |x| \cdot \mathbf{1}_{\{|x| > 1\}}$ we obtain

$$
\mathbb{E}\Big[|R| \mathbf{1}_{\{|R| \geq re^{-t}\}}\Big] \leq C r^{-\alpha+1} e^{(\alpha-1)t},
$$

completing thus proof of (6.11). Hence we may restrict our attention to $|t - s| \leq 1$. Assume $s > t$. Then we have

$$
\begin{aligned}
|\eta(t) - \eta(s)| &= \left| e^{-\alpha t} \int \mathbb{E} f(e^t k R) dk - e^{-\alpha s} \int \mathbb{E} f(e^s k R) dk \right| \\
&\leq |e^{-\alpha t} - e^{-\alpha s}| \left| \int \mathbb{E} f(e^t k R) dk \right| + e^{-\alpha s} \left| \int \Big[\mathbb{E} f(e^t k R) - \mathbb{E} f(e^s k R) \Big] dk \right| \\
&\leq |1 - e^{-\alpha(s-t)}||\eta(t)| + e^{-\alpha s} \|f\|_{0,1} \mathbb{E} \Big[|e^t R - e^s R| \mathbf{1}_{\{|R| \geq re^{-s}\}} \Big] \\
&\leq C \max\{1, r^{-\alpha+1}\} |t - s| \|f\|_{0,1} + e^{(-\alpha+1)s} |t - s| \|f\|_{0,1} \mathbb{E} \Big[|R| \mathbf{1}_{\{|R| \geq re^{-s}\}} \Big] \\
&\leq C \max\{1, r^{-\alpha+1}\} |t - s| \|f\|_{0,1} + C \max\{1, r^{-\alpha+1}\} |t - s| \|f\|_{0,1} \\
&\leq C \max\{1, r^{-\alpha+1}\} |t - s| \|f\|_{0,1}.
\end{aligned}
$$

Hence (6.9) follows.

In the second case, when $\alpha \leq 1$, we have

$$
|\eta(t)| \ \leq \ e^{-\alpha t} \mathbb{E} \Big[|e^t R| \mathbf{1}_{\{re^{-t} \leq |R| \leq \tilde{r}e^{-t}\}} \Big] \ \leq \ e^{-\alpha t} \tilde{r} \mathbb{P}[|R| \ \geq \ re^{-t}] \ \leq \ C\tilde{r} r^{-\alpha} \|f\|_{0,1}.
$$

Again we may restrict our attention to $|t - s| \leq 1$. Proceeding as above, we obtain

$$
\begin{aligned}
|\eta(t) - \eta(s)| &\leq |1 - e^{-\alpha(s-t)}||\eta(t)| + e^{-\alpha s} \|f\|_{0,1} \mathbb{E} \Big[|e^t R - e^s R| \mathbf{1}_{\{re^{-s-1} \leq |R| \leq \tilde{r}e^{-s+1}\}} \Big] \\
&\leq C \tilde{r} r^{-\alpha} |t - s| \|f\|_{0,1} + e^{-\alpha s} \|f\|_{0,1} |t - s| \mathbb{E} \Big[|e^s R| \mathbf{1}_{\{re^{-s-1} \leq |R| \leq \tilde{r}e^{-s+1}\}} \Big].
\end{aligned}
$$

Finally,

$$
\mathbb{E} \Big[|e^s R| \mathbf{1}_{\{re^{-s-1} \leq |R| \leq \tilde{r}e^{-s+1}\}} \Big] \leq e\tilde{r} \mathbb{P}\big[|R| > re^{-s-1} \big] \leq Cr^{-\alpha} \tilde{r} e^{\alpha s}
$$

and (6.10) follows. □

Lemma 6.8. *Let* $m \geq 1, \varepsilon \leq 1, m + \varepsilon < \alpha$, $f \in H_r^{m,\varepsilon}$ *and let*

$$
\phi(t) = \int_{\mathrm{K}} \psi(e^t k) \, dk.
$$

Suppose that (4.10) *is satisfied. Then there is* $C = C(\alpha, \varepsilon, m, \chi, d, r, \mu, \nu)$ *such that*

$$
\int_{\mathbb{R}} |\phi'(t)|(1 + |t|)^\chi \, dt \leq C \|f\|_{m,\varepsilon}.
$$

Proof. We have

$$
\phi'(t) = -\alpha \phi(t) + e^{-\alpha t} \sum_{j=1}^{d} \int \mathbb{E} \big[g_j(e^t k R) - g_j(e^t k M_1 R_1) \big] \, dk,
$$

where

$$
g_j(x) = x_j \cdot \partial_{x_j} f(x).
$$

Consider first the case $m = 1$. Then

$$
\begin{aligned}
|g_j(x) - g_j(y)| &\leq |\partial_{x_j} f(x) - \partial_{x_j} f(y)||x_j| + |\partial_{x_j} f(y) - \partial_{x_j} f(0)||x_j - y_j| \\
&\leq \|f\|_{m,\varepsilon} |x - y|^\varepsilon |x| + \|f\|_{m,\varepsilon} |y|^\varepsilon |x - y|.
\end{aligned}
$$

Hence, for $x = e^t kR$ and $y = e^t kM_1R_1$, we have

$$|\phi'(t)| \leq \alpha|\phi(t)|$$
$$+ de^{-\alpha t}\|f\|_{m,\varepsilon}\mathbb{E}\Big[\big(e^{\varepsilon t+t}|Q_1|^\varepsilon|R| + e^{\varepsilon t+t}|M_1R_1|^\varepsilon|Q_1|\big)\mathbf{1}_{\{|Q_1|+|M_1R_1|\geq re^{-t}\}}\Big].$$

Thus, we need to prove that

$$W = \int_{\mathbb{R}} (1+|t|)^\chi e^{-(\alpha-\varepsilon-1)t}\mathbb{E}\Big[|Q_1|^\varepsilon|R| + |M_1R_1|^\varepsilon|Q_1|\big)\mathbf{1}_{\{|Q_1|+|M_1R_1|\geq re^{-t}\}}\Big]\,dt < \infty.$$

Let $t_0 = -\log(r^{-1}(|Q_1| + |M_1R_1|))$. Since $R = Q_1 + M_1R_1$, we see that

$$W \leq \mathbb{E}\Big[\big(|Q_1|^{1+\varepsilon} + |Q_1|^\varepsilon|M_1||R_1| + |M_1R_1|^\varepsilon|Q_1|\big)\cdot\int_{t_0}^\infty (1+|t|)^\chi e^{-(\alpha-\varepsilon-1)t}\,dt\Big].$$

Let $t_1 = -\log(1 + r^{-1}(|Q_1| + |M_1R_1|))$. Notice that there is a constant $C(\chi)$ such that

$$\int_{t_0}^\infty (1+|t|)^\chi e^{-(\alpha-\varepsilon-1)t}\,dt \leq C(\chi) + \int_{t_1}^0 (1+|t|)^\chi e^{-(\alpha-\varepsilon-1)t}\,dt.$$

Integrating by parts the second integral we conclude that

$$\int_{t_0}^\infty (1+|t|)^\chi e^{-(\alpha-\varepsilon-1)t}\,dt \leq C(\alpha,r,\chi)\big(\log(1 + |Q_1| + |M_1R_1|)\big)^\chi\big(|Q_1| + |M_1R_1|\big)^{\alpha-\varepsilon-1}.$$

Therefore,

$$W \leq C(r,\chi)\mathbb{E}\Big[\big(|Q_1|^{1+\varepsilon} + |Q_1|^\varepsilon|M_1||R_1| + |M_1R_1|^\varepsilon|Q_1|\big)$$
$$\big(\log(1 + |Q_1| + |M_1R_1|)\big)^\chi\big(|Q_1| + |M_1R_1|\big)^{\alpha-\varepsilon-1}\Big],$$

which is finite by the Hölder inequality and (4.10).

Notice that $\|\partial_{x_j} f\|_{m-1,\varepsilon} \leq \|f\|_{m,\varepsilon}$. If $m \geq 2$ then by Lemma 6.2 applied to $\partial_{x_j} f \in H_r^{m-1,\varepsilon}$ we have

$$\big|\partial_{x_j} f(x) - \partial_{x_j} f(y)\big| \leq C\|f\|_{m,\varepsilon}(|x| + |y|)^{m+\varepsilon-2}|x-y|$$

and so

$$|g_j(x) - g_j(y)| \leq C\|f\|_{m,\varepsilon}(|x| + |y|)^{m+\varepsilon-2}|x-y||x| + C\|f\|_{m,\varepsilon}|y|^{m+\varepsilon-2}|x-y|.$$

Then for $x = e^t kR$, $y = e^t kM_1R_1$

$$|\phi'(t)| \leq \alpha|\phi(t)| + de^{-\alpha t}C\|f\|_{m,\varepsilon}e^{t(m+\varepsilon)}$$
$$\cdot\mathbb{E}\Big[\big((|Q_1| + 2|M_1R_1|)^{m+\varepsilon-1}|Q_1| + |M_1R_1|^{m+\varepsilon-1}|Q_1|\big)\mathbf{1}_{\{|Q_1|+|M_1R_1|\geq re^{-t}\}}\Big].$$

As before, we estimate

$$\int_{t_0}^\infty (1+|t|)^\chi e^{-(\alpha-\varepsilon-m)t}\,dt \leq C(r,\chi)\big(\log(1 + |Q_1| + |M_1R_1|)\big)^\chi\big(|Q_1| + |M_1R_1|\big)^{\alpha-\varepsilon-m}.$$

Finally

$$\int_{\mathbb{R}} (1+|t|)^\chi e^{-(\alpha-\varepsilon-1)t}$$
$$\mathbb{E}\Big[\big((|Q_1| + 2|M_1R_1|)^{m+\varepsilon-1}|Q_1| + |M_1R_1|^{m+\varepsilon-1}|Q_1|\big)\mathbf{1}_{\{|Q_1|+|M_1R_1|\geq re^{-t}\}}\Big]dt$$
$$\leq \mathbb{E}\Big[\big((|Q_1| + 2|M_1R_1|)^{\alpha-1}|Q_1| + (|Q_1| + 2|M_1R_1|)^{\alpha-1}|Q_1|\big)$$
$$\cdot\big(\log(1 + |Q_1| + |M_1R_1|)\big)^\chi\Big]$$

is finite as, before. $\qquad\square$

7 Rate of convergence in the renewal theorem.

The main goal of this section is to prove Proposition 7.2, which says that under the assumption (2.3) for some class of functions F one can control the rate of convergence in the classical renewal theorem (2.2). After that we provide a proof of Proposition 4.7. We show first an auxiliary Lemma giving the same result but for differentiable functions.

Lemma 7.1. *Let U be a potential on $\mathbb{R}$ satisfying* (2.3) *Let F be a bounded differentiable function on $\mathbb{R}$ satisfying for some $\beta > 1$ the following conditions*

$$\int_{\mathbb{R}} |F(s)|(1+|s|)\, ds < \infty, \qquad \int_{\mathbb{R}} |F'(s)|(1+|s|)^{\beta}\, ds < \infty, \qquad \lim_{s \to -\infty} sF(s) = 0.$$

Then, as $t \to -\infty$,

$$\left| \int_{\mathbb{R}} F(t-s)\, U(ds) - A \int_{\mathbb{R}} F(s)\, ds \right| \leq C|t|^{-\min\{\beta-1,\delta\}}.$$

Proof. Notice that for $M > 0$ we can write

$$\int_{-M}^{M} F(t-s)\, U(ds) = \int_{-M}^{M} F(t-s)\, dH(s),$$

where the right hand side is the Riemann-Stieltjes integral. Integrating by parts, [51, Theorem 6.30], we have

$$\int_{-M}^{M} F(t-s)\, dH(s) = H(M)F(t-M) - H(-M)F(t+M) + \int_{-M}^{M} F'(t-s)H(s)\, ds. \quad (7.1)$$

Since,

$$\int_{-M}^{M} F'(t-s)s\, ds = -MF(t-M) - MF(t+M) + \int_{-M}^{M} F(t-s)\, ds$$

and

$$\int_{-M}^{M} F'(t-s)\, ds = F(t-M) - F(t+M),$$

we have

$$\int_{-M}^{M} F(t-s)\, dH(s) - A \int_{-M}^{M} F(s)\, ds = \int_{-M}^{M} F'(t-s)(H(s) - As - B)\, ds + W(M), \quad (7.2)$$

where

$$W(M) = (H(M) - AM - B)F(t-M) - (H(-M) + AM - B)F(t+M)$$

and by our assumptions

$$\lim_{M \to \infty} W(M) = 0.$$

Therefore, taking the limit, we obtain

$$\int_{\mathbb{R}} F(t-s)\, dU(s) - A \int_{\mathbb{R}} F(s)\, ds = \int_{\mathbb{R}} F'(t-s)(H(s) - As - B)\, ds. \quad (7.3)$$

We would like to estimate the last integral as $t \to \infty$. Thus from now on let $t > 0$. We split the integral on the right hand side in (7.3) into the following sum

$$\int_{-\infty}^{t/2} F'(t-s)(H(s) - As - B)\, ds + \int_{t/2}^{\infty} F'(t-s)(H(s) - As - B)\, ds.$$

Let us consider the first term. Since by (2.3), $|H(s) - As - B| \leq C(1 + |s|)$ for all $s \in \mathbb{R}$,

$$\left| \int_{-\infty}^{t/2} F'(t - s)(H(s) - As - B) \, ds \right| \leq \int_{-\infty}^{t/2} |F'(t - s)|(1 + |s|) \, ds$$

$$= \int_{-\infty}^{-t/2} |F'(s)|(1 + |s - t|) \, ds \leq 3 \int_{-\infty}^{-t/2} |F'(s)|(1 + |s|) \, ds$$

$$\leq \frac{3 \cdot 2^{\beta-1}}{|t|^{\beta-1}} \int_{-\infty}^{-t/2} |F'(s)|(1 + |s|)^{\beta} \, ds.$$

Also, by (2.3) applied to $s > t/2$ we have

$$\left| \int_{t/2}^{\infty} F'(t - s)(H(s) - As - B) \, ds \right| \leq \int_{t/2}^{\infty} |F'(s + t)||s|^{-\delta} \, ds \leq 2^{\delta} |t|^{-\delta} \int_{\mathbb{R}} |F'(s)| \, ds.$$

$\square$

Proposition 7.2. *Fix $C_0, \eta_0 > 0$. Let F be a bounded continuous function on $\mathbb{R}$ satisfying the following conditions*

$$\int_{\mathbb{R}} |F(s)|(1 + |s|)^{\beta} \, ds < C_0, \tag{7.4}$$

$$r(t) = U * F(t) \text{ is well defined and } \breve{r}(t) = U * \breve{F}(t) \tag{7.5}$$

and there is s_0 such that

$$U * F(t) - U * F(s) < C_0(t - s), \text{ for } s_0 \leq s < t \leq s + \eta_0. \tag{7.6}$$

Let U be a potential satisfying (2.3). Then, as $t \to \infty$,

$$\left| U * F(t) - A \int_{\mathbb{R}} F(s) \, ds \right| \leq O(|t|^{-\min\{\beta-1, \delta\}/2}), \tag{7.7}$$

where the last estimate depends only on C_0 and the bounds in (2.3).

Proof. We are going to prove that $\breve{F}$ satisfies assumptions of Lemma 7.1. Since

$$\breve{F}' = F - \breve{F},$$

it is enough to prove that

$$\int_{\mathbb{R}} |\breve{F}(s)|(1 + |s|)^{\beta} \, ds < C \int_{\mathbb{R}} |F(s)|(1 + |s|)^{\beta} \, ds \tag{7.8}$$

with C independent of F and

$$\lim_{s \to -\infty} s\breve{F}(s) = 0. \tag{7.9}$$

The left hand side of (7.8) is equal to

$$\int_{-\infty}^{\infty} e^{-s} \left(\int_{-\infty}^{s} e^{u} |F(u)| \right)(1 + |s|)^{\beta} \, ds \, du = \int_{-\infty}^{\infty} e^{u} |F(u)| du \int_{u}^{\infty} e^{-s}(1 + |s|)^{\beta} \, ds.$$

Consider first

$$\int_{-\infty}^{0} e^{u} |F(u)| \int_{u}^{\infty} e^{-s}(1 + |s|)^{\beta} \, ds \, du = \int_{-\infty}^{0} e^{u} |F(u)| du \int_{0}^{\infty} e^{-s}(1 + |s|)^{\beta} \, ds$$

$$+ \int_{-\infty}^{0} e^{u} |F(u)| du \int_{u}^{0} e^{-s}(1 + |s|)^{\beta} \, ds.$$

The first integral is clearly finite. The second one is dominated by

$$\int_{-\infty}^{0} e^{u}|F(u)|(e^{-u}-1)(1+|u|)^{\beta}\,du \leq \int_{-\infty}^{0} |F(u)|(1+|u|)^{\beta}\,du < \infty.$$

For the integral on the positive half-line notice that for every $\gamma > 0$ there is $C(\gamma)$ such that for every $u \geq 0$,

$$\int_{u}^{\infty} e^{-s}(1+s)^{\gamma}\,ds \leq C(\gamma)e^{-u}(1+u)^{\gamma}. \tag{7.10}$$

Indeed, integrating by parts we have

$$\int_{u}^{\infty} e^{-s}(1+s)^{\gamma}\,ds = e^{-u}(1+u)^{\gamma} + \gamma\int_{u}^{\infty} e^{-s}(1+s)^{\gamma-1}\,ds \tag{7.11}$$

and so (7.10) holds for $\gamma \leq 1$. If $\gamma > 1$ and $\gamma - 1 \leq 1$, then

$$\int_{u}^{\infty} e^{-s}(1+s)^{\gamma-1}\,ds \leq e^{-u}(1+u)^{\gamma} + \gamma C(\gamma-1)e^{-u}(1+u)^{\gamma-1}$$

and (7.10) follows inductively from (7.11). Therefore,

$$\int_{0}^{\infty} e^{u}|F(u)|\int_{u}^{\infty} e^{-s}(1+|s|)^{\beta}\,ds\,du \leq \int_{0}^{\infty} e^{u}|F(u)|C(\beta)e^{-u}(1+u)^{\beta}\,du < \infty \tag{7.12}$$

and (7.8) follows.

For (7.9) we consider first $t < 0$. Then

$$|t\breve{F}(t)| \leq |t|e^{-t}\int_{-\infty}^{t} e^{u}|F(u)|\,du$$

$$= |t|e^{-t}\int_{-\infty}^{t} e^{u}(1+|u|)^{-\beta}(1+|u|)^{\beta}|F(u)|\,du$$

$$\leq |t|(1+|t|)^{-\beta}\int_{\mathbb{R}} (1+|u|)^{\beta}|F(u)|\,du \leq C_0|t|(1+|t|)^{-\beta}.$$

Hence $\lim_{t\to-\infty} t\breve{F}(t) = 0$. If $t > 0$ then we write

$$e^{-t}\int_{-\infty}^{t/2} e^{u}|F(u)|\,du \leq \|F\|_{\infty}e^{-t}\int_{-\infty}^{t/2} e^{u}\,du \leq Ce^{-t/2}$$

and

$$e^{-t}\int_{t/2}^{t} e^{u}|F(u)|\,du = e^{-t}\int_{t/2}^{t} e^{u}(1+|u|)^{-\beta}(1+|u|)^{\beta}|F(u)|\,du$$

$$\leq C2^{\beta}(2+t)^{-\beta}\int_{\mathbb{R}} (1+|u|)^{\beta}|F(u)|\,du \leq CC_0(2+t)^{-\beta}.$$

Hence again $\lim_{t\to\infty} t\breve{F}(t) = 0$. Notice that

$$\int_{\mathbb{R}} \breve{F}(s)\,ds = \int_{\mathbb{R}} F(s)\,ds.$$

Applying now Lemma 7.1 to $\breve{F}$ we have

$$\left|U * \breve{F}(t) - A\int_{\mathbb{R}} F(u)du\right| \leq C|t|^{-\min\{\beta-1,\delta\}}, \text{ as } t \to \infty \tag{7.13}$$

where C depends only on the bounds in (2.3).

Finally we have to unsmooth inequality (7.13). For this purpose we need to apply a Tauberian remainder theory. However we are not able to apply the Beurling - Ganelius Theorem as in [33] (Theorem 9.6). Instead, we refer to the following Tauberian remainder theorem due to Frennemo (Theorem 2 in [31]). For that we need to introduce the notion of a weight function. p is a weight function on $\mathbb{R}^+$ if $p(x) \geq p(0)$, $p(x + y) \leq p(x)p(y)$, $p(sx) \geq p(x)$ for $s \geq 1$. The following result was proved by Frennemo [31]:

Theorem 7.3. *Let* K *be an integrable function on* $\mathbb{R}$ *such that* $\widehat{\mathrm{K}}(\theta)$ *does not vanish for real arguments and that the function* $g(\theta) = \frac{1}{\widehat{\mathrm{K}}(\theta)}$ *can be holomorphically extended to a strip* $-\alpha < \operatorname{Im}\theta < \beta$ *and*

$$|g(\theta)| \leq C_1 P_1(|\theta|), \quad |g'(\theta)| \leq C_1 P_2(|\theta|), \quad -\alpha < \operatorname{Im}\theta < \beta \tag{7.14}$$

for some weight functions P_1 *and* P_2

Let p *be a weight function such that* $\limsup_{x\to\infty} \frac{\log p(x)}{x} < \beta$ *and let* S *be the inverse of* $x^{\frac{3}{2}}(P_1(x)P_2(x))^{\frac{1}{2}}$. *Fix positive constants* d_0, d_1 *and suppose that* ϕ *be a measurable function satisfying*

$$\|\phi\|_{L^\infty} \leq d_0, \quad \text{and} \quad |K * \phi(x)| \leq d_1 p(x)^{-1}.$$

Assume moreover, there are C, x_0 *such that*

$$\phi(t) - \phi(x) \geq -CS(p(x))^{-1}, \text{for } x_0 \leq x \leq t \leq x + S(p(x))^{-1}.$$

Then there is a constant d_2 *independent of* ϕ *such that*

$$|\phi(x)| \leq d_2 S(p(x))^{-1}, \quad x \to \infty.$$

In our situation $K(t) = \mathbf{1}_{(0,\infty)}(t)e^{-t}$ and $\widehat{K}(\theta) = \frac{1}{1-i\theta}$ is nonzero for all real θ. Next $g(\theta) = 1 - i\theta$ is holomorphic on the whole complex plane and (7.14) holds with $P_1(x) = x + 1$ and $P_2(x) = 1$. Finally, $\phi = -r + A\int_{\mathbb{R}} F$, which by (7.6) satisfies the Tauberian condition. Hence $p(x) = C_0(1 + x)^{\min(\beta-1,\delta)}$ and $S(x)$ behaves like $x^{\frac{1}{2}}$ at ∞ i.e. $S(x)x^{-\frac{1}{2}}$ is between two positive constants.

Therefore,

$$\left| U * F(t) - A\int_{\mathbb{R}} F(u)du \right| \leq C|t|^{-\min(\beta-1,\delta)/2}$$

uniformly for all F satisfying assumptions the Lemma. $\qquad\square$

Proof of Proposition 4.7. **Step 1.** First we prove that the required estimates hold for the function ψ^K:

$$\left| \psi^{\mathrm{K}} * U_{\mathbb{R}^+}(a) - \frac{1}{m}\int_{\mathbb{R}^+} \psi^{\mathrm{K}}(b)\,db \right| \leq C_3(\psi)|\log|a||^{-\beta_2}, \text{ for } |a| < 1. \tag{7.15}$$

Step 1a. Assume $\alpha \leq 1$. Let $F(t) = \psi^K(e^t)$. We are going to apply Proposition 7.2. Its assumptions are satisfied by Lemmas 6.3 and 6.7. Namely, notice that

$$\eta(t) = U_{\mathbb{R}^+} * \psi^{\mathrm{K}}(e^t),$$

for η as in (6.8). Hypothesis (7.6) is satisfied by (6.10) with a constant C_0 independent of F as far as $f \in \mathcal{F}$. Choose $\varepsilon < \alpha$. Then $\|f\|_{0,\varepsilon} \leq \tilde{r}^{1-\varepsilon}$. Now (6.5) implies (7.4) with with C_0 independent of F and (6.4) implies (7.5). Hence (7.15) follows.

Step 1a. Assume $\alpha > 1$.

For $m = 0$ we proceed as in the previous case applying Proposition 7.2 to F defined as above. (7.6) is satisfied by (6.9) with a constant C_0 independent of F, (6.5) implies (7.4) with with C_0 independent of ϕ and (6.4) implies (7.5). Hence (7.15) follows.

For $m \geq 1$ we apply Lemma 7.1 to function F. To justify that its hypotheses are fulfilled in our case we use (6.5), (6.4) and Lemma 6.8. Hence

$$\int_{\mathbb{R}} |\phi'(s)|(1 + |s|)^{\chi} \, ds$$

is bounded uniformly for $f \in \mathcal{F}$. So we obtain (7.15)

Step 2. We estimate for $|g| < e^{-e}$.

$$\left| \psi * U(g) - \frac{1}{m} \int_{\mathrm{G}} \psi(h) \, dh \right| \leq \left| (\psi - \psi^{\mathrm{K}}) * U(g) \right| + \left| \psi^{\mathrm{K}} * U(g) - \frac{1}{m} \int_{\mathrm{G}} \psi(h) \, dh \right|$$
$$= I + II$$

Next, by (4.25) and (4.26), we estimate

$$
\begin{aligned}
I &\leq \sum_{n < \log |\log |g||} |(\psi - \psi^{\mathrm{K}}) * \mu^n(g)| + \sum_{n \geq \log |\log |g||} \|(\psi - \psi^{\mathrm{K}}) * \mu^n\|_{L^\infty} \\
&\leq C_1 (\log |\log |g||)^D \|\psi - \psi^{\mathrm{K}}\|_{L^\infty} (1 + |\log |g||)^{-\beta_1} + \sum_{n \geq \log |\log |g||} C_2(\psi) \lambda^n \\
&\leq C_1 (\log |\log |g||)^D 2\|\psi\|_{L^\infty} (1 + |\log |g||)^{-\beta_1} + C_2(\psi)|\log |g||^{\log \lambda}(1 - \lambda)^{-1} \\
&\leq (2C_1 \|\psi\|_{L^\infty} + C_2(\psi)(1 - \lambda)^{-1})(\log |\log |g||)^D |\log |g||^{-\beta}
\end{aligned}
$$

and finally by (7.15)

$$
\begin{aligned}
II &\leq \left| \psi^{\mathrm{K}} * U_{\mathbb{R}^+}(|g|) - \frac{1}{m} \int_{\mathbb{R}^+} \psi^{\mathrm{K}}(b) \, db \right| \leq C_3(\psi)|\log |g||^{-\beta_2} \\
&\leq C_3(\psi)(\log |\log |g||)^D |\log |g||^{-\beta}.
\end{aligned}
$$

$\square$

A Spectral gap

Let G be a locally compact group and let $\widehat{\mathrm{G}}$ denote the unitary dual of G. For a bounded measure μ on G and for an irreducible unitary representation ρ of G realized on a Hilbert space $\mathbb{V}_\rho$, let

$$\widehat{\mu}(\rho) = \int_{\mathrm{G}} \rho(g) \, \mu(dg) \in \mathrm{End}(\mathbb{V}_\rho)$$

denote the Fourier transform of μ at ρ, as in [26, 18.2]. In particular if μ is of the form $f(g) \, dg$, where dg stands for a Haar measure on G and $f \in \mathrm{L}^1(\mathrm{G})$ then

$$\widehat{f}(\rho) = \int_{\mathrm{G}} f(g) \, dg.$$

The dual measure $d\rho$ on $\widehat{\mathrm{G}}$ is such that the following Plancherel formula holds

$$\int_{\mathrm{G}} |f(g)|^2 \, dg = \int_{\widehat{\mathrm{G}}} \|\widehat{f}(\rho)\|_2^2 \, d\rho$$

for $f \in \mathrm{L}^1(\mathrm{G}) \cap \mathrm{L}^2(\mathrm{G})$, see [26, 18.8.2]. Here $\|\cdot\|_2$ stands for the Hilbert-Schmidt norm on $\mathrm{End}(\mathbb{V}_\rho)$. Denote by $\|\cdot\|_\infty$ the operator norm on $\mathrm{End}(\mathbb{V}_\rho)$.

Lemma A.1. *Assume that every irreducible unitary representation of* G *is finite dimensional. Let μ be a probability measure on* G *such that* suppμ *generates* G. *Then the only representation ρ such that*

$$\|\widehat{\mu}(\rho)\|_\infty = 1 \tag{A.1}$$

is the trivial representation.

Proof. Let ρ be as in (A.1). Since $\dim \mathbb{V}_\rho < \infty$, there are unit vectors $u, v \in \mathbb{V}_\rho$ such that

$$\|\widehat{\mu}(\rho)\|_\infty = (\widehat{\mu}(\rho)u, v).$$

Also, there is be a unitary transformation T of $\mathbb{V}_\rho$ such that $Tu = v$. Hence,

$$1 = (\widehat{\mu}(\rho)u, v) = (\widehat{\mu}(\rho)u, T^{-1}u) = (T\widehat{\mu}(\rho)u, u) = \int_G (T\rho(g)u, u)\, \mu(dg),$$

which implies that $(T\rho(g)u, u) = 1$ for $g \in$ suppμ. Thus we have the equality in Cauchy's inequality $|(T\rho(g)u, u)| \leq \|T\rho(g)u\| \cdot \|u\| = 1$, which shows that there are complex numbers t_g such that $T\rho(g)u = t_g u$. Since the representation ρ is irreducible, the span of the vectors $R\rho(g)u$, $g \in$ G, equals $\mathbb{V}_\rho$. Therefore $\dim \mathbb{V}_\rho = 1$.

Now we see that the transformation T is equal to the multiplication by a complex number t of absolute value 1 and that there is a unitary character $\chi_\rho : G \to \mathbb{C}^\times$ such that $\rho(g)$ coincides with the multiplication by $\chi_\rho(g)$. Furthermore,

$$1 = \int_G (t\chi_\rho(g)u, u)\, \mu(dg) = \int_G t\chi_\rho(g)\, \mu(dg),$$

which implies that $t\chi_\rho(g) = 1$ for $g \in$ suppμ. But suppμ generates G. Hence, $t\chi_\rho(g) = 1$ for all $g \in$ G. In particular $1 = t\chi_\rho(e) = t$. Hence ρ is the trivial representation. $\square$

Recall the right regular representation, ρ, (3.1).

Lemma A.2. *Let* K *be a compact group and let* $G = \mathbb{R}^+ \times K$. *Suppose μ is a probability measure on* G *which is absolutely continuous with respect to the Haar measure and such that* suppμ *generates* G. *Then*

$$\sup\{\|\psi * \mu\|_{L^2(G)};\ \psi \in L^2(G),\ \int_K \lambda(k)\psi\, dk = 0,\ \|\psi\|_{L^2(G)} \leq 1\} < 1. \tag{A.2}$$

Proof. Since $\widehat{G}$ consists of the tensor products $\rho = \chi \otimes \eta$, where $\chi \in \widehat{\mathbb{R}^+}$ and $\eta \in \widehat{K}$, we have an identification of the topological spaces $\widehat{G} = \widehat{\mathbb{R}^+} \times \widehat{K}$ and (A.2) may be rewritten as

$$\sup\{\|\psi * \mu\|_{L^2(G)};\ \psi \in L^2(G),\ \text{supp}\widehat{\psi} \subseteq \Xi,\ \|\psi\|_{L^2(G)} \leq 1\} < 1, \tag{A.3}$$

where $\Xi = \widehat{G} \setminus (\widehat{\mathbb{R}^+} \times \{triv\})$. Since,

$$(\psi * \mu)\widehat{\ }(\rho) = \widehat{\psi}(\rho)\widehat{\mu}(\breve{\rho}),$$

where $\breve{\rho}(g) = \rho(g^{-1})$, Plancherel formula shows that the square of the quantity on the left hand side of (A.3) is equal to

$$\sup\{\int_\Xi \|\widehat{\psi}(\rho)\widehat{\mu}(\breve{\rho})\|_2^2\, d\rho;\ \int_\Xi \|\widehat{\psi}(\rho)\|_2^2\, d\rho \leq 1,\ \psi \in L^2(G)\}. \tag{A.4}$$

However a moment of thought shows that the quantity (A.4) is equal to

$$\sup\{\|\widehat{\mu}(\breve{\rho})\|_\infty^2;\ \rho \in \Xi\}. \tag{A.5}$$

Suppose (A.5) is equal to 1. Then

$$\sup\{\|\widehat{\mu}(\breve{\rho})\|_\infty;\ \rho \in \Xi\} = \sup\{\|\widehat{\mu}(\breve{\rho})\|_\infty;\ \rho \in \Xi \text{ and } \|\widehat{\mu}(\breve{\rho})\|_\infty \geq \tfrac{1}{2}\}$$

Since μ is absolutely continuous with respect to Haar measure, Riemann-Lebesgue Lemma [40, (28.40)] shows that the set

$$\Xi_0 = \left\{\rho \in \widehat{G};\ \|\widehat{\mu}(\breve{\rho})\|_\infty \geq \tfrac{1}{2}\right\}$$

is compact. Then $\Xi \cap \Xi_0$ is compact and

$$\sup\left\{\|\widehat{\mu}(\breve{\rho})\|_\infty;\ \rho \in \Xi \cap \Xi_0\right\} = 1.$$

Hence, there is $\rho \in \Xi \cap \Xi_0$ such that

$$\|\widehat{\mu}(\breve{\rho})\|_\infty = 1.$$

Lemma A.1 implies that $\breve{\rho}$ is trivial, which contradicts the fact that the trivial representation does not belong to Ξ. $\qquad\square$

B Two metrics on a compact group

Let K be a compact connected Lie group with the Lie algebra κ. Choose a Killing form $\langle \cdot, \cdot \rangle$ on κ and let $d(\cdot, \cdot)$ denote the corresponding Riemannian distance on K. There is a neighborhood of identity $U \subseteq$ K on which the logarithm $\log : U \to \kappa$ is well defined and such that

$$\mathsf{d}(k,l) = \mathsf{d}(kl^{-1}, e) = \langle \log(kl^{-1}), \log(kl^{-1}) \rangle^{\frac{1}{2}} \qquad (kl^{-1} \in U). \tag{B.1}$$

Lemma B.1. *Let $(\rho, \mathbb{V})$ be faithful representation of* K *on a finite dimensional Hilbert space $\mathbb{V}$ over $\mathbb{C}$ or $\mathbb{R}$. Denote by $\|\ \|$ the operator norm on* $\mathrm{End}(\mathbb{V})$. *Then there is a constant C such that*

$$\|\rho(k) - \rho(l)\| \leq Cd(k,l) \qquad (k,l \in \text{K}).$$

Proof. Notice that for $A \in \mathrm{End}(\mathbb{V})$,

$$\|\exp(A) - I\| = \|\sum_{n=1}^{\infty} \frac{1}{n!} A^n\| = \|\sum_{n=1}^{\infty} \frac{1}{n!} A^{n-1}\|\|A\|$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n!} \|A\|^{n-1} \|A\| = \frac{\exp(\|A\|) - 1}{\|A\|} \|A\|.$$

Since any two norms on a finite dimensional vector space are equivalent, there is a constant C_0 such that

$$\|\rho(X)\| \leq C_0 \langle X, X \rangle^{\frac{1}{2}} \qquad (X \in \kappa).$$

By combining the above two facts with (B.1) we see that there is a constant C such that

$$\|\rho(k) - \rho(e)\| \leq Cd(k,e) \qquad (k \in U).$$

For two arbitrary elements $k, l \in$ K choose points $l = k_0, k_1, ..., k_n = k$ on the geodesic from l to k such that $k_j k_{j-1}^{-1} \in U$. Then

$$\|\rho(k) - \rho(l)\| \leq \sum_{j=1}^{n} \|\rho(k_j) - \rho(k_{j-1})\| = \sum_{j=1}^{n} \|\rho(k_j k_{j-1}^{-1}) - \rho(e)\|$$

$$\leq C \sum_{j=1}^{n} d(k_j k_{j-1}^{-1}, e) = C \sum_{j=1}^{n} d(k_j, k_{j-1}) = Cd(k,l).$$

$\qquad\square$

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