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On $\mathcal{H}^1$ and entropic convergence for contractive PDMP

Pierre Monmarché*

Abstract

Explicit rate of convergence in variance (or more general entropies) is obtained for a class of Piecewise Deterministic Markov Processes such as the TCP process, relying on functional inequalities. A method to establish Poincaré (and more generally Beckner) inequalities with respect to a diffusion-type energy for the invariant law of such hybrid processes is developed.

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1 Introduction

This work is devoted to the study of convergence to equilibrium for a class of Piecewise Deterministic Markov Process (PDMP). These hybrid processes, satisfying a deterministic differential equation between random jumps, have received much attention recently: we refer to [5] and the references therein for an overview of the topic. Ergodicity and, then, speed of convergence to the steady state are particularly studied. As far as this last point is concerned, coupling methods have recently proved efficient in order to get explicit rate of convergence in Wasserstein distances for PDMP (see [18, 7, 10, 27, 13] for instance, among many others). On the other hand, another classical approach to quantify ergodicity, based on functional inequalities, is hardly used, since the usual methods do not directly apply. Our aim is to adapt them (see also [34] in this direction).

Let Ω be an open set of $\mathbb{R}^d$. The dynamics is defined thanks to a vector field $b : \Omega \to \mathbb{R}^d$, a jump rate $\lambda : \Omega \to \mathbb{R}_+$, and a transition kernel Q which will be seen either as a function from Ω to $\mathcal{P}(\Omega)$ the set of probability measures on Ω, or as an operator on some functional space. For $x \in \Omega$ let $(\varphi_x(t))_{t\geq 0}$ be the flow associated to b, namely the solution of

$$\partial_t \varphi_x(t) = b(\varphi_x(t)), \qquad \varphi_x(0) = x.$$

Starting at point x, the process $(X_t)_{t\geq 0}$ deterministically follows this flow up to its first jump time T_x with law

$$\mathbb{P}(T_x < s) = \int_0^s \lambda(\varphi_x(u))\, e^{-\int_0^u \lambda(\varphi_x(w))dw} du = 1 - e^{-\int_0^s \lambda(\varphi_x(w))dw}.$$

*Université de Neuchâtel. E-mail: pierre.monmarche@unine.ch

At time T_x, the process jumps according to the law $Q(\varphi_x(T_x))$, and starts anew from its new position. The infinitesimal generator of the process is

$$Lf(x) \quad = \quad b(x).\nabla f(x) + \lambda(x)\left(Qf(x) - f(x)\right), \tag{1.1}$$

defined at least for bounded $f \in \mathcal{C}^1(\Omega)$. We note

$$P_t f(x) = \mathbb{E}\left(f(X_t)|X_0 = x\right)$$

the associated semi-group. The following assumptions hold throughout this work:

- The flow is well-defined and it fixes Ω: if $x \in \Omega$ then $\varphi_x(t) \in \Omega$ for all $t > 0$.
- The process is non-explosive: there can't be infinitely many jumps in a finite time interval, so that the process (and therefore the semi-group) is defined for all time. We suppose $\lambda > 0$.
- The functions λ and b are smooth; we write $J_b(x) = [\partial_i b_k(x)]_{1\leq i,k\leq d}$ the Jacobian matrix of $b = (b_k)_{1\leq k\leq d}$.
- The process admits a unique invariant law μ, and P_t is ergodic in the sense $P_t f(x) \underset{t\to\infty}{\longrightarrow} \int f \mathrm{d}\mu$ for all $f \in L^2(\mu)$ and all $x \in \Omega$. Moreover all polynomial moments of μ are finite and, denoting by $\mathcal{A}$ the set of function in $\mathcal{C}^\infty(\Omega)$ whose derivatives grow at most polynomially at infinity, Q, L and $(P_t)_{t\geq 0}$ are well-defined on $\mathcal{A}$ and they fix $\mathcal{A}$.

These strong assumptions allow us to focus only on the quantification of ergodicity. Note that the uniqueness of the invariant measure, the finiteness of its moments and the ergodicity of the process may often be proved by checking it is irreducible and admits a Lyapunov function (cf. [25]). Throughout this work the test functions will always belong to the set $\mathcal{A}$, in order to keep the study at a formal level, all the forthcoming elementary definitions and calculations being licit in this framework.

We recall here some classical arguments (see [4, Chapter 5] for a general introduction to functional inequalities and for the detailed proofs of the assertions in this paragraph). For $f \in \mathcal{A}$, we write $\Gamma(f) = \frac{1}{2}L(f^2) - fLf$ the *carré du champ* operator of L, $\Gamma(f,g)$ the corresponding symmetric bilinear operator obtained by polarization, and

$$\Gamma_2(f) = \frac{1}{2}L\left(\Gamma f\right) - \Gamma\left(f, Lf\right).$$

Writing $\psi(s) = P_s\Gamma\left(P_{t-s}f\right)$, from $\partial_t P_t f = LP_t f = P_t Lf$ one gets

$$\psi'(s) = 2P_s\Gamma_2\left(P_{t-s}f\right).$$

Hence, if the Bakry-Emery (or Γ_2) criterion $\Gamma_2 > \rho\Gamma$ holds for some $\rho > 0$, the Gronwall Lemma yields $\psi(0) \leq e^{-2\rho t}\psi(t)$, namely

$$\Gamma\left(P_t f\right) \quad \leq \quad e^{-2\rho t} P_t \Gamma f. \tag{1.2}$$

For instance for the Ornstein-Uhlenbeck process with generator

$$L_{OU} f(x) = \Delta f(x) - \rho x \cdot \nabla f(x),$$

this reads

$$|\nabla P_t f|^2 \quad \leq \quad e^{-2\rho t} P_t |\nabla f|^2, \tag{1.3}$$

where $|.|$ is the Euclidian norm of $\mathbb{R}^d$. In fact, the sub-commutation (1.2) is equivalent to the Bakry-Emery criterion. Nevertheless the latter does not usually hold in our settings. That said, a simple adaptation of the Γ_2 argument will give, at least in the constant jump rate case, a gradient estimate similar to (1.3). In the following we denote by A^* the usual transpose of a matrix A and thus by u^*v the scalar product of two vectors.

Theorem 1.1. *Assume λ is constant and $|\nabla Qf(x)|^2 \leq M(x)Q|\nabla f|^2(x)$ with M such that*

$$\forall (x,u) \in \Omega \times \mathbb{R}^d, \qquad 2u^* J_b(x)u + \lambda\left(M(x)-1\right)|u|^2 \leq -\eta |u|^2 \tag{1.4}$$

for some $\eta \in \mathbb{R}$. Then for all $t>0$, $f \in \mathcal{A}$ and $x \in \Omega$,

$$|\nabla P_t f|^2(x) \leq e^{-\eta t} P_t |\nabla f|^2(x). \tag{1.5}$$

Inequality (1.4) is a balance condition on the drift and the jumps, reminiscent of the condition on the curvature in [20, Theorem 1.2]. More precisely, suppose $|\nabla Qf(x)|^2 \leq M(x)Q|\nabla f|^2(x)$ for some function M on Ω. If $M<1$, Q is a contraction of the Wasserstein distance (this will be detailed in Section 2); it means two particles that simultaneously jump can be coupled so that they get closer. More generally M measures how two such particles can be coupled in order for them not to get too far away one from the other. On the other hand, J_b measures how two trajectories of the deterministic flow tend to get closer or to drift apart. Indeed,

$$\begin{array}{rcl} \varphi_x(t) - \varphi_y(t) & = & x - y + tJ_b(x)(x-y) + t\underset{y\to x}{o}(x-y) + \underset{t\to 0}{o}(t) \\ \Rightarrow \quad |\varphi_x(t)-\varphi_y(t)|^2 & = & |x-y|^2 + 2t(x-y)^* J_b(x)(x-y) + t\underset{y\to x}{o}\left(|x-y|^2\right) + |x-y|\underset{t\to 0}{o}(t), \end{array}$$

We see that the condition $u^* J_b(x) u < 0$ for all $(x,u) \in \Omega \times \mathbb{R}^d$ implies the flow contracts the space in the neighbourhood of all points of Ω.

Note that by integrating Inequality (1.5) with respect to μ and writing

$$W_t = \int |\nabla P_t f|^2 d\mu,$$

Theorem 1.1 implies $W_t \leq e^{-\eta t} W_0$ for all $t>0,\ f \in \mathcal{A}$, which is equivalent to $\partial_t W_t \leq -\eta W_t$ for all $t>0,\ f\in\mathcal{A}$, or to $\left(\partial_t W_t\right)_{t=0} \leq -\eta W_0$ for all $f \in \mathcal{A}$.

In the non-constant jump rate case, under a condition similar to (1.4), we will prove there exist constants $\beta > 0$ and $\eta \in \mathbb{R}$ such that

$$\partial_t W_t \leq -\eta W_t + 2\beta \mathcal{E}_t \tag{1.6}$$

where $\mathcal{E}_t$ is defined as

$$\mathcal{E}_t \quad = \quad \int \Gamma\left(P_t f\right) d\mu.$$

Both W_t and $\mathcal{E}_t$ are usually called energy ; we may say W_t is the classical (or diffusion-like) energy, while $\mathcal{E}_t$ is the Markovian one. They coincide in the case of the Ornstein-Uhlenbeck process. The Markovian energy usually appears in particular when one is concerned with the variance of $P_t f$ with respect to μ,

$$V_t \quad = \quad \int (P_t f)^2 d\mu - \left(\int P_t f d\mu\right)^2.$$

We say μ satisfies a Poincaré (or spectral gap) inequality with respect to Γ if there exists a constant $c>0$ such that $V_0 \leq c\mathcal{E}_0$ for all $f \in \mathcal{A}$. Since $\partial_t V_t = -2\mathcal{E}_t$, such an inequality is equivalent to $V_t \leq e^{-\frac{2t}{c}} V_0$, namely to an exponential decay in $L^2(\mu)$. The same goes for entropy and Gross log Sobolev inequality, or general Φ-entropies (see [17] and Section 3 for some definitions), at least for diffusion processes.

For reversible processes (*i.e.* when L is symmetric in $L^2(\mu)$) there is a strong link between, on the one hand, Wasserstein distances and coupling and, on the other hand, variance (or entropy) and functional inequalities (see [6, 16, 30]); nevertheless PDMP are not reversible. Furthermore their invariant measures usually do not satisfy a Poincaré inequality for Γ, which is non-local, not easy to handle, satisfying no chain rule (nevertheless, see [15] for a case in which such an inequality does indeed hold).

However, they may satisfy a diffusion-like Poincaré inequality of the form

$$\forall f \in \mathcal{A} \qquad \int f^2 d\mu - \left(\int f d\mu\right)^2 \quad \leq \quad c \int |\nabla f|^2 d\mu, \tag{1.7}$$

in other words $V_t \leq cW_t$. Such an inequality, which involves the classical energy rather than the Markovian one, implies concentration properties for the measure μ (see [4]), but is *a priori* not directly linked to the convergence to equilibrium in general.

Suppose such an inequality holds. Then, from inequality (1.6), if $\eta > 0$,

$$\begin{aligned}\partial_t \left(W_t + \beta V_t\right) \quad &\leq \quad -\eta W_t \\ &\leq \quad -\frac{\eta}{1+\beta c}\left(W_t + \beta V_t\right).\end{aligned}$$

This yields:

Theorem 1.2. *Assume the Poincaré inequality* (1.7) *holds, and* $|\nabla Qf(x)|^2 \leq M(x)Q|\nabla f|^2(x)$ *with* M *such that for* μ*-almost all* $x \in \Omega$ *and for all* $u \in \mathbb{R}^d$*,*

$$u^* \left(2J_b(x) + \frac{\nabla\lambda(x)(\nabla\lambda(x))^*}{\beta\lambda(x)}\right) u + \lambda(x)\left(M(x) - 1\right)|u|^2 \quad \leq \quad -\eta|u|^2 \tag{1.8}$$

for some constants $\eta, \beta > 0$*. Then*

$$W_t + \beta V_t \leq \left(W_0 + \beta V_0\right) e^{-\frac{\eta t}{\beta c+1}}.$$

Note that

$$W_t + \beta V_t = \|\nabla P_t f\|^2_{L^2(\mu)} + \beta\|P_t f - \mu f\|^2_{L^2(\mu)}$$

is equivalent to the square of the usual Sobolev $\mathcal{H}^1$-norm of $P_t f - \mu f$. Thus Theorem 1.2 provides a decay in $\mathcal{H}^1(\mu)$ rather than in $L^2(\mu)$. In this sense, our method can be seen as an hypocoercive method of modified Lyapunov functional (see [38, 24, 9], etc.), although it is quite simple. In these settings, it is usual to assume a Poincaré inequality (1.7) holds. There are classical criteria on a function F on $\mathbb{R}^d$ to decide whether the law $e^{-F(x)}dx$ satisfies such an inequality, and several ways to estimate the constant c. However, for PDMP, the invariant law is usually quite unknown. The second part of this work will thus be dedicated to the obtention of such inequalities, which are interesting by themselves as they provide concentration bounds for the measure μ.

The original motivation of the present work was the study of the so-called TCP process on $\Omega = \mathbb{R}_+$, whose generator is

$$\forall x > 0,\ f \in \mathcal{A}, \qquad Lf(x) \quad = \quad f'(x) + x\left(f(\delta x) - f(x)\right), \tag{1.9}$$

for some $\delta \in (0,1)$. It has been studied in [18], which inspired the main ideas of this work. In addition to the previous difficulties (no Poincaré inequality for Γ, non-constant rate of jump), there is another one which is particular to this process : the jump vanishes at the origin. Nevertheless, as an illustration of the efficiency of our method, we will prove the following:

Proposition 1.3. *For* $f \in \mathcal{A}$*, define*

$$Ent f = \mu\left(f^2 \log f^2\right) - \left(\mu f^2\right)\log\left(\mu f^2\right).$$

Then if $(P_t)_{t>0}$ is the semi-group associated to the generator (1.9)*, there exist $c, r > 0$ such that for all $f \in \mathcal{A}$,*

$$\mathit{Ent} P_t f \quad \leq \quad c e^{-rt} \mu(f')^2.$$

Moreover it is possible to get explicit values for c and r such that this holds.

The paper is organized as follows. Slightly generalized versions of Theorems 1.1 and 1.2 are stated and proved in Section 2. A general strategy to obtain some functional inequalities (including the Poincaré inequality) for PDMP by the study of their embedded chain is exposed in Section 3 and applied in several illustrative models in Section 4, where in particular Proposition 1.3 is proved. A perturbative results for Poincaré and log-Sobolev inequalities is stated and proved in an Appendix.

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2 Exponential decay

We keep the notations and assumptions of the Introduction. In particular we study the semi-group $(P_t)_{t\geq 0}$ with generator L defined by (1.1).

When A is a linear operator on $\mathcal{A}$ and ϕ is a bilinear symmetric one, for $f, g \in \mathcal{A}$ we define

$$\Gamma_{A,\phi}(f,g) = \frac{1}{2}\left(A\phi(f,g) - \phi(f,Ag) - \phi(Af,g)\right).$$

With respect to f, $\Gamma_{A,\phi}(f,f)$ is quadratic, and linear with respect to A and ϕ. We will always note $f \mapsto \phi(f)$ the quadratic form associated to a bilinear form $f, g \mapsto \phi(f,g)$ and similarly we will always note $f, g \mapsto q(f,g)$ the symmetric bilinear form associated by polarization to a quadratic form $f \mapsto q(f)$ on $\mathcal{A}$. Let

$$\psi(s) = P_s \phi\left(P_{t-s} f\right), \qquad s \in [0,t]$$

which interpolates between $\phi\left(P_t f\right)$ and $P_t\left(\phi f\right)$. Then

$$\psi'(s) = 2 P_s \Gamma_{L,\phi}\left(P_{t-s} f\right).$$

To prove Theorems 1.1 and 1.2 we should consider $\phi(f) = |\nabla f|^2$. In fact it will be convenient for the applications to work with a weighted gradient $\phi_a(f) = a|\nabla f|^2$ with $a > 0$ a scalar field on Ω in $\mathcal{A}$ (so that $f \in \mathcal{A} \Rightarrow \phi_a(f) \in \mathcal{A}$).

Lemma 2.1. *1. For all $f \in \mathcal{A}$*

$$\Gamma_{b^*\nabla,\phi_a}(f) = \frac{b^*\nabla a}{2a}\phi_a(f) - a(\nabla f)^* J_b \nabla f.$$

2. Suppose there exists a function M on Ω such that, for all $f \in \mathcal{A}$, $\phi_a(Qf) \leq M Q\left(\phi_a(f)\right)$, and let I be the identity operator on $\mathcal{A}$. Then for all $f \in \mathcal{A}$

$$\Gamma_{\lambda(Q-I),\phi_a}(f) \geq -a(\nabla f)^*(\nabla\lambda)(Qf - f) + \frac{\lambda}{2}(1-M)\phi_a(f).$$

Proof. First we note that

$$\nabla\left(b^*\nabla f\right) \quad = \quad J_b \nabla f + H_f b$$

with $H_f(x) = [\partial_i\partial_k f(x)]_{1\leq i,k\leq d}$ the Hessian of f, and

$$b^*\nabla\left(a|\nabla f|^2\right) \quad = \quad (b^*\nabla a)|\nabla f|^2 + 2ab^*H_f\nabla f$$

Thus

$$\begin{aligned}\Gamma_{b^*\nabla,\phi_a}(f) &= \frac{1}{2}b^*\nabla\left(a|\nabla f|^2\right) - a(\nabla f)^*\nabla\left(b^*\nabla f\right)\\ &= \frac{1}{2}(b^*\nabla a)|\nabla f|^2 - a(\nabla f)^*J_b\nabla f.\end{aligned}$$

As far as the second point is concerned,

$$\begin{aligned}\Gamma_{\lambda(Q-I),\phi_a}(f) &= \frac{1}{2}\lambda\left(Q\left(\phi_a(f)\right) - \phi_a(f)\right) - a(\nabla f)^*(\nabla\lambda)(Qf-f) - \lambda a(\nabla f)^*(\nabla Qf - \nabla f)\\ &\geq \frac{\lambda}{2}\left(Q\left(\phi_a(f)\right) + \phi_a(f) - 2\sqrt{\phi_a(f)\phi_a(Qf)}\right) - a(\nabla f)^*(\nabla\lambda)(Qf-f).\end{aligned}$$

We conclude by

$$2\sqrt{\phi_a(f)\phi_a(Qf)} \leq 2\sqrt{M\phi_a(f)Q\phi_a(f)} \leq M\phi_a(f) + Q\left(\phi_a(f)\right).$$

□

We can now state the following :

Theorem 2.2. *Assume λ is constant and there exist a function M on Ω and a constant $\eta\in\mathbb{R}$ such that, for all $f\in\mathcal{A}$, $\phi_a\left(Qf\right)\leq MQ\left(\phi_a(f)\right)$ and*

$$\forall (x,u)\in\Omega\times\mathbb{R}^d,\qquad 2u^*J_b(x)u + \left(\lambda\left(M(x)-1\right) - \frac{b^*\nabla a(x)}{a(x)} + \eta\right)|u|^2 \leq 0.$$

Then

$$\phi_a(P_tf) \leq e^{-\eta t}P_t\left(\phi_a(f)\right).$$

In particular with $a=1$ we retrieve Theorem 1.1.

Proof. From Lemma 2.1, since in the constant rate case $\nabla\lambda = 0$,

$$\begin{aligned}\Gamma_{L,\phi_a}(f) &\geq -a(\nabla f)^*J_b\nabla f + a\left(\frac{b^*\nabla a}{2a} + \frac{\lambda}{2}(1-M)\right)|\nabla f|^2\\ &\geq \frac{\eta}{2}\phi_a(f).\end{aligned}$$

Hence if $\psi(s) = P_s\phi_a(P_{t-s}f)$,

$$\psi'(s) = 2P_s\Gamma_{L,\phi_a}(P_{t-s}f) \geq \eta\psi(s)$$

and $\psi(t)\geq e^{\eta t}\psi(0)$, which concludes. □

Remark that we did note use the ergodicity of the process here, and that η can be negative.

This commutation between the semigroup and the gradient leads to a contraction in Wasserstein distance. More precisely, define on Ω the distance associated to the weighted gradient $D=\sqrt{a}\nabla$ by

$$d(x,y) = \inf\left\{\int_0^1 \frac{|\gamma'(s)|}{\sqrt{a\left(\gamma(s)\right)}}ds,\ \gamma:[0,1]\to\Omega,\ smooth,\ \gamma(0)=x,\gamma(t)=y\right\}$$

and the associated Wasserstein distance between two probability laws ν_1, ν_2 having a finite p^{th} moment (*i.e.* for which there exists a $x_0 \in \Omega$ with $\nu_i\left[d^p(., x_0)\right] < \infty$) by

$$\mathcal{W}_{d,p}(\nu_1,\nu_2) = \inf_{X\sim\nu_1,\ Y\sim\nu_2} \left(\mathbb{E}\left[d^p(X,Y)\right]\right)^{\frac{1}{p}}.$$

A function f will be called κ-Lipschitz with respect to D if $\forall x, y \in \Omega$,

$$f(x) - f(y) \leq \kappa d(x,y).$$

This is equivalent for a smooth function to $\|Df\|_\infty \leq \kappa$, and we have the Kantorovich-Rubinstein dual representation (see [39])

$$\mathcal{W}_{d,1}(\nu_1,\nu_2) = \sup\left\{\nu_1 f - \nu_2 f,\ \|Df\|_\infty \leq 1\right\},$$

where we use the operator notation $\nu f = \int f d\nu$.

Recall that by duality a Markov semi-group acts on the right on probability laws by

$$\left(\nu_1 P_t\right) f := \nu_1\left(P_t f\right).$$

If P_t were absolutely continuous with respect to the Lebesgue measure for $t > 0$ - which is not the case for a PDMP since for all time t there is a non-zero probability that the process hasn't jumped yet - the gradient estimate of Theorem 2.2 would yield, from [31, Theorem 2.2], a contraction of the $\mathcal{W}_{d,2}$ distance :

$$\mathcal{W}_{d,2}\left(\nu_1 P_t,\ \nu_2 P_t\right) \leq e^{-\frac{\eta}{2}t}\mathcal{W}_{d,2}\left(\nu_1,\ \nu_2\right).$$

Instead of trying to adapt Kuwada's result, since our work is more concerned about variance than Wasserstein distance, we will only state the weaker result :

Corollary 2.3. *In the setting of Theorem 2.2, for all laws ν_1, ν_2 with finite first moment, if $\nu_1 P_t$ and $\nu_2 P_t$ still have finite first moment,*

$$\mathcal{W}_{d,1}\left(\nu_1 P_t,\ \nu_2 P_t\right) \leq e^{-\frac{\eta}{2}t}\mathcal{W}_{d,1}\left(\nu_1,\ \nu_2\right).$$

Proof. Theorem 2.2 yields the weaker gradient estimate

$$\|DP_t f\|_\infty \leq e^{-\frac{\eta}{2}t} P_t \|Df\|_\infty = e^{-\frac{\eta}{2}t}\|Df\|_\infty.$$

This implies the $\mathcal{W}_{d,1}$ decay, thanks to the Kantorovich-Rubinstein dual representation

□

Note that the invariant measure does not intervene neither in Theorem 2.2 nor in Corollary 2.3, so that its existence and uniqueness are not necessary. Besides, on a complete space, a contraction of the Wasserstein distance implies ergodicity, from [19, Theorem 5.23].

We won't push the analysis further concerning the Wasserstein distance, but refer to the study in [7] of the TCP process where an exponential decay is first obtained for a distance equivalent to $d(x,y) = \sqrt{|x-y|}$ and then is transposed to $d(x,y) = |x-y|^p$ via moments estimates and Hölder inequality. For further considerations on gradient-semigroup commutation, one shall consult [12, 3, 31].

We now turn to the non-constant jump rate case. Let $a \in \mathcal{A}$ be a non-negative scalar field on Ω. Throughout all the text we will say a probability measure ν satisfies a weighted Poincaré inequality with constant c and weight a if for all $f \in \mathcal{A}$

$$\nu f^2 - \left(\nu f\right)^2 \quad \leq \quad c\nu\left(a|\nabla f|^2\right). \tag{2.1}$$

Let $V_t = \mu\left(P_t f\right)^2 - \left(\mu f\right)^2$ and $W_t = \mu\phi_a(P_t f)$. Note that in the Introduction W_t was defined with the constant weight $a = 1$, so that the following is slightly more general than Theorem 1.2:

Theorem 2.4. *Assume that μ satisfies the weighted Poincaré inequality* (2.1) *with constant c and weight a and that there exist a function M and constants $\eta, \beta > 0$ such that for μ-almost all $x \in \Omega$, for all $f \in \mathcal{A}$ and for all $u \in \mathbb{R}^d$, $\phi_a(Qf) \leq MQ(\phi_a(f))$ and*

$$u^* \left(2J_b(x) + \frac{a}{\beta\lambda(x)} \nabla\lambda(x)(\nabla\lambda(x))^* + \lambda(x)\left(M(x) - 1\right) - \frac{b^*\nabla a(x)}{a(x)} + \eta \right) u \quad \leq \quad 0. \tag{2.2}$$

Then

$$W_t + \beta V_t \leq e^{-\frac{\eta t}{\beta c + 1}} \left(W_0 + \beta V_0 \right),$$

and

$$W_t \leq (1 + \beta c) e^{-\frac{\eta t}{\beta c+1}} W_0.$$

Proof. Since μ is the invariant measure of the process, $\mu L g = 0$ for all $g \in \mathcal{A}$. In particular if ϕ is a quadratic form on $\mathcal{A}$, $\mu\left(L\phi(f)\right) = 0$ and

$$\begin{aligned} \partial_t \left(\mu \left(\phi(P_t f) \right) \right) &= 2\mu \left(\phi(P_t f, L P_t f) \right) \\ &= -2\mu \Gamma_{L,\phi}(P_t f). \end{aligned}$$

In particular

$$\partial_t W_t = -2\mu\Gamma_{L,\phi_a}(P_t f).$$

From Lemma 2.1,

$$\begin{aligned} \Gamma_{\lambda(Q-I),\phi_a}(f) &\geq -a(\nabla f)^*(\nabla\lambda)(Qf - f) + \frac{\lambda}{2}(1 - M)\phi_a(f) \\ &\geq -\frac{a^2}{2\beta\lambda} |(\nabla f)^* \nabla\lambda|^2 - \frac{\beta\lambda}{2}(Qf - f)^2 + \frac{\lambda}{2}(1-M)\phi_a(f). \end{aligned}$$

Again from Lemma 2.1 and from Inequality (2.2),

$$\Gamma_{L,\phi_a}(f) \geq \frac{\eta}{2}\phi_a(f) - \frac{\beta\lambda}{2}(Qf - f)^2.$$

On the other hand, if $\phi_2(f) = f^2$ then Γ_{L,ϕ_2} is the usual *carré du champ* operator. From the Leibniz rule $\Gamma_{b^*\nabla,\phi_2} f = 0$, so that

$$\begin{aligned} \partial_t V_t &= -2\mu\Gamma_{\lambda(Q-I),\phi_2}(P_t f) \\ &= -\mu\lambda \left(Q(P_t f)^2 + (P_t f)^2 - 2(P_t f)(Q P_t f) \right) \\ &\leq -\mu\lambda \left(Q P_t f - P_t f \right)^2 \end{aligned}$$

the last inequality being a consequence of the Cauchy-Schwartz inequality for Q. At the end of the day, we get

$$\partial_t \left(W_t + \beta V_t \right) \leq -\eta W_t$$

and, thanks to the weighted Poincaré inequality (2.1),

$$\partial_t \left(W_t + \beta V_t \right) \leq -\frac{\eta}{1 + \beta c}(W_t + \beta V_t),$$

which yields the first assertion. Then

$$W_t \leq W_t + \beta V_t \leq e^{-\frac{\eta t}{\beta c+1}} (W_0 + \beta V_0) \leq (1 + \beta c) e^{-\frac{\eta t}{\beta c+1}} W_0.$$

□

Note that η could depend on x, so that the weight that intervenes in the Poincaré inequality may be different from a. For instance for the TCP with linear rate on $\mathbb{R}_+$ (Example 4.4), one could consider $a(x) = x$ and $\eta(x) = -\kappa - \alpha x$ for some $\kappa, \alpha > 0$. Then it would be sufficient to prove an inequality with weight $\tilde{a}(x) = 1 + x$, which is weaker than both the classical inequality with constant weight and the inequality with weight a.

3 Functional inequalities for PDMP

This section is devoted to the obtention of the Poincaré inequality (2.1) and of slightly more general functional inequalities for μ the invariant measure of the process $(X_t)_{t\geq 0}$ with generator (1.1).

3.1 Confining operators

The variance is a way among others to quantify the distance to equilibrium. In this section we suppose that for all $f \in \mathcal{A}$ the so-called p-entropies

$$\begin{aligned} \mathrm{Ent}_p f &= \frac{\mu f^2 - \left(\mu f^{\frac{2}{p}}\right)^p}{p-1} \qquad \text{for } p \in (1,2], \\ \mathrm{Ent}_1 f &= \mu\left(f^2 \log f^2\right) - \left(\mu f^2\right)\log\left(\mu f^2\right) \end{aligned}$$

are well-defined. We say that μ satisfies a Beckner's inequality $\mathcal{B}(p,c)$ if

$$\forall f \in \mathcal{A}, \qquad \mathrm{Ent}_p f \quad \leq \quad c\mu|\nabla f|^2. \tag{3.1}$$

For $p=2$ this is the Poincaré inequality, for $p=1$ this is the Gross log Sobolev one. Since $\mathrm{Ent}_p f$ is non increasing with $p \in (1,2]$ (see [33]; note that we took the definitions of [11]), $\mathcal{B}(p,c)$ implies $\mathcal{B}\left(q,c\right)$ whenever $q \geq p$. On the other hand by Jensen inequality $(p-1)\mathrm{Ent}_p f$ is non decreasing with $p \in [1,2]$. In particular all Beckner's inequalities for $p \in (1,2]$ are equivalent up to some factor. For the global study of these inequalities and of more general Φ-entropies, we refer to [17] and [11].

For $\alpha \in [0,1]$ we say μ satisfies a generalized Poincaré inequality $\mathcal{I}(\alpha,c)$ if

$$\forall f \in \mathcal{A},\ \forall p \in (1,2], \qquad (p-1)^{1-\alpha}\mathrm{Ent}_p f \quad \leq \quad c\mu|\nabla f|^2. \tag{3.2}$$

For $\alpha = 0$ this is still the Poincaré inequality, for $\alpha = 1$ this is the log Sobolev one, and for $\alpha \in (0,1)$ this is an interpolation between these two cases which implies the following concentration property: there exists a constant $L > 0$ such that for any Borel set A with $\mu(A) \geq \frac{1}{2}$, if A_t is the set of points at distance at most t from A, then $\mu(A_t) \geq 1 - e^{Lt^{\frac{2}{2-\alpha}}}$ (see [33]). To prove $\mathcal{I}(\alpha,c)$ is equivalent to prove $\mathcal{B}\left(p, c(p-1)^{\alpha-1}\right)$ for all $p \in (1,2]$.

In this section, for the sake of simplicity, we won't consider weighted inequalities such as the weighted Poincaré inequality (2.1) with $a \neq 1$. Everything would work the same, and, at least in dimension one, a weighted inequality can be seen as a non-weighted one through a change of variable (see an application in Section 4.4).

Remark that if μ satisfies $\mathcal{B}(p,c)$ for $p \in [1,2]$, then it satisfies a Poincaré inequality. In this case, providing the inequality (2.2) of Theorem 2.4 holds, W_t decays exponentially fast, and

$$\mathrm{Ent}_p P_t f \quad \leq \quad cW_t \quad \leq \quad c(1+\beta c)e^{-\frac{\eta t}{1+\beta c}}W_0.$$

Let $\psi : \Omega \to \Omega$ be a smooth function with Jacobian matrix J_ψ, and let $|J_\psi|$ be the Euclidian operator norm of J_ψ, namely

$$|J_\psi| = \sup\left\{|J_\psi u|,\ u \in \mathbb{R}^d,\ |u| = 1\right\}.$$

We say ψ is γ-Lipschitz (where $\gamma \in \mathbb{R}_+$) if for all $x \in \Omega$, $|J_\psi(x)| \leq \gamma$. It is clear that in this case when the law of a random variable Z satisfies $\mathcal{B}(p,c)$ then the law of $\psi(Z)$ satisfies $\mathcal{B}(p,\gamma^2 c)$. In order to get Beckner's inequalities for the invariant law of a PDMP we will prove a generalization of this fact, based on an initial idea of Malrieu and Talay [37].

Let H be a Markov kernel on Ω that fixes $\mathcal{A}$.

Definition 3.1. *Let $c, \gamma > 0$, $p \in [1,2]$. We say that H is (c,γ,p)-confining if both the following conditions are satisfied :*

- Sub-commutation*: $\forall f \in \mathcal{A},\ \forall x \in \Omega$,*

$$\left|\nabla \left(Hf^{\frac{2}{p}}\right)^{\frac{p}{2}}\right|^2(x) \quad \leq \quad \gamma H|\nabla f|^2(x). \tag{3.3}$$

- Local Beckner's inequality: $\forall f \in \mathcal{A},\ \forall x \in \Omega$,,

$$\frac{Hf^2(x) - \left(Hf^{\frac{2}{p}}\right)^p(x)}{p-1} \quad \leq \quad cH|\nabla f|^2(x). \tag{3.4}$$

if $p > 1$ and

$$H\left(f^2 \ln f^2\right)(x) - Hf^2(x) \ln Hf^2(x) \quad \leq \quad cH|\nabla f|^2(x). \tag{3.5}$$

if $p = 1$.

If $\gamma < 1$ we say H is (c,γ,p)-contractive. When there is no ambiguity for p, H will simply be called confining (or contractive) if there exist $c, \gamma > 0$ satisfying both conditions.

Note that (3.5) holds iff (3.4) holds for all $p > 1$

Examples:

- Let ψ be a γ-Lipschitz function and $Hf(x) = f(\psi(x))$. The sub-commutation (3.3) is clear, and the local inequality (3.4) holds with $c = 0$, since $H(x)$ is a Dirac mass.
- The sub-commutation is always satisfied with $\gamma = 0$ if $H(x) = \nu$ is a constant kernel, namely is a probability on Ω, so that ν is confining iff it satisfies a Beckner's inequality.
- If N is a standard Gaussian vector on $\mathbb{R}^d$ and $(B_t)_{t\geq 0}$ a Brownian motion on $\mathbb{R}^d$ then

$$K_t f(x) = \mathbb{E}\left(f(x + B_t)\right) = \mathbb{E}\left(f(x + \sqrt{t}N)\right)$$

  is $(t,1)$-confined for the usual gradient and $p = 1$ (see [4, Chapter 1]). If the Brownian motion is replaced by an elliptic diffusion, a sub-commutation is given by its Bakry-Emery curvature (see [4, Chapter 5]).
- Remark this definition could be extended to a Markov kernel $H : \Omega_1 \to \mathcal{P}(\Omega_2)$ with $\Omega_1 \subset \mathbb{R}^d$ and $\Omega_2 \subset \mathbb{R}^n$. For instance if φ is the flow associated to a vector field b on Ω_1 then $Hf(t) = f(\varphi_x(t))$ is a Markov kernel from $\mathbb{R}_+$ to $\mathcal{P}(\Omega_1)$, and $\partial_t Hf = H(b^* \nabla f)$.

Here is maybe our most important, although very simple result:

Lemma 3.2. *For $i = 1, 2$, let H_i be a (c_i, γ_i, p)-confining Markov kernel on Ω.*

1. *Then $H_1 H_2$ is a $(c_2 + \gamma_2 c_1, \gamma_1\gamma_2, p)$-confining Markov kernel.*
2. *If $\nu \in \mathcal{P}(\Omega)$ satisfies $\mathcal{B}(p,c)$ then νH_2 satisfies $\mathcal{B}(p, c_2 + \gamma_2 c)$.*
3. *Suppose H is (c,γ,p)-contractive and the Markov chain generated by H is ergodic in the sense there exists $\nu \in \mathcal{P}(\Omega)$ such that for all $x \in \Omega$ and $f \in \mathcal{A}$, $H^n f(x)$ goes to νf as n goes to infinity. Then the invariant law ν satisfies $\mathcal{B}\left(p, c(1-\gamma)^{-1}\right)$.*

Proof. Let $p \in (1,2]$ (the case $p=1$ is similar and already treated in [18]). First,

$$\left|\nabla\left(H_1H_2f^{\frac{2}{p}}\right)^{\frac{p}{2}}\right|^2 \leq \gamma_1 H_1\left(\left|\nabla\left(H_2f^{\frac{2}{p}}\right)^{\frac{p}{2}}\right|^2\right) \leq \gamma_1\gamma_2 H_1H_2|\nabla f|^2$$

and

$$\begin{aligned}
\frac{H_1H_2f^2-\left(H_1H_2f^{\frac{2}{p}}\right)^p}{p-1} &= \frac{1}{p-1}\left(H_1\left[H_2f^2-\left(H_2f^{\frac{2}{p}}\right)^p\right]+H_1\left(H_2f^{\frac{2}{p}}\right)^p-\left(H_1H_2f^{\frac{2}{p}}\right)^p\right)\\
&\leq c_2H_1H_2|\nabla f|^2+\frac{1}{p-1}\left(H_1g^2-\left(H_1g^{\frac{2}{p}}\right)^p\right) \qquad \text{with } g=\left(H_2f^{\frac{2}{p}}\right)^{\frac{p}{2}}\\
&\leq c_2H_1H_2|\nabla f|^2+c_1H_1|\nabla g|^2\\
&\leq (c_2+\gamma_2c_1)H_1H_2|\nabla f|^2.
\end{aligned}$$

The second point is obtained from the first one by considering $H_1=\nu$. Concerning the third assertion, by induction from the first one we get for all $n\in\mathbb{N}$

$$\frac{H^nf^2-\left(H^nf^{\frac{2}{p}}\right)^p}{p-1} \leq c\left(\sum_{k=0}^{n}\gamma^k\right)H^n|\nabla f|^2.$$

The weak convergence of H^n to ν concludes. □

Example: Let $(E_k)_{k\geq0}$ be an i.i.d. sequence of standard exponential variables, and $(X_k)_{k\geq0}$ be the Markov chain on $\mathbb{R}_+$ defined by $X_{k+1}=\frac{X_k+E_k}{2}$. Its transition operator is

$$Pf(x) = \mathbb{E}\left(f\left(\frac{x+E_0}{2}\right)\right).$$

Clearly $(Pf)'(x)=\frac{1}{2}P(f')(x)$, so that $|(Pf)'|^2\leq\frac{1}{4}P|f'|^2$. On the other hand $P(x)$, the law of $\frac{x+E}{2}$, is the image by a $\frac{1}{2}$-Lipschitz transformation of the exponential law $\mathcal{E}(1)$, which satisfies a Poincaré inequality $\mathcal{B}(2,4)$ (cf. Theorem [4, Theorem 6.2.2] for instance). Thus P is $(2,\frac{1}{4},2)$-contractive. On the other hand it is clear the chain is irreducible, it admits $C=[0,3]$ as a small set and $V(x)=x+1$ as a Lyapunov function (since $PV(x)\leq\frac{3}{4}V(x)+\mathbb{1}_{x<3}$) so that it is ergodic (see [25] for definitions and proof). According to Lemma 3.2, the invariant measure satisfies a Poincaré inequality $\mathcal{B}\left(2,\frac{8}{3}\right)$.

This chain can be obtained from the TCP process with constant jump rate (Section 4.1 below) if the process is only observed when it jumps. This is the so-called embedded chain associated to the continuous process, which we now introduce in a general framework.

3.2 The embedded chain

Recall $X=(X_t)_{t\geq0}$ is a process on Ω with generator given by (1.1). Let $(S_k)_{k\geq0}$ be the jump times of X and let $Z_k=X_{S_k}$. The Markov chain $(Z_k)_{k\geq0}$ is called the embedded chain associated to X.

For $s\in[S_k,S_{k+1})$, $X_s=\varphi_{Z_k}(s-S_k)$ where we recall φ_x is the flow associated to the vector field b. Since

$$\frac{d}{dt}\big(f\left(\varphi_x(t)\right)\big)=\left(b^*\nabla f\right)\left(\varphi_x(t)\right),$$

we shall say that a function f is non-decreasing (resp. constant, concave, etc.) along the flow if $t\mapsto f(\varphi_x(t))$ is non-decreasing (resp. constant, etc.) for all $x\in\Omega$; in other word if $b^*\nabla f\geq0$ (resp. $=0$, etc.).

Conditionally to the event $Z_k=x$, the inter-jump time $T_k=S_{k+1}-S_k$ has a density

$$p_x(t)=\lambda\left(\varphi_x(t)\right)e^{-\int_0^t\lambda(\varphi_x(s))ds}$$

on $\mathbb{R}^+$. We assume the inter-jump times are $a.s.$ finite (which is clear if $\liminf_{t\to\infty} \lambda(\varphi_x(t)) > 0$ for all x), and define

$$Kf(x) = \int_0^{+\infty} f(\varphi_x(t))\, p_x(t)dt = \mathbb{E}\left(f(\varphi_x(T_k))|Z_k = x\right).$$

Then $P = KQ$ is the transition operator for the chain Z.

Transferring properties from X to Z, or the converse, is far from obvious. In fact it is quite easy to find counter-examples for which one is ergodic and not the other (see examples 34.28 and 34.33 of [22]). In [21] this problem is solved with the definition of another embedded chain by adding observation points at constant rate. That being said, in the following we won't delve into this issue, and simply assume Z has a unique invariant law μ_e (which can often be proved under conditions of irreducibility, aperiodicity and existence of a Lyapunov function). In this case we can express μ from μ_e:

Lemma 3.3 (**Theorem 34.31 of [22], p.123**). *Assume* $C = \mu_e K\left(\frac{1}{\lambda}\right) = \mu_e\left[\int_0^\infty e^{-\int_0^t \lambda(\varphi_x(s))ds}dt\right] < \infty$. *Then*

$$\mu f = C^{-1}\mu_e K\left(\frac{f}{\lambda}\right).$$

In other words, $\mu = \nu_e \widetilde{K}$ *where*

$$\begin{aligned} \widetilde{K}f(x) &= \frac{1}{K(\frac{1}{\lambda})(x)} K\left(\frac{f}{\lambda}\right)(x) \\ \nu_e f &= \frac{1}{C}\mu_e\left[fK\left(\frac{1}{\lambda}\right)\right]. \end{aligned}$$

In the following we will always assume the condition $C < \infty$ holds, so that ν_e and $\widetilde{K}$ are well defined.

Here is our plan: from Lemma 3.2, we may establish a Beckner's inequality for μ_e by proving the operator P is contractive. By perturbative results on functional inequalities (see [17] or Appendix) this may give an inequality for ν_e. Finally, again from Lemma 3.2, we may transfer the inequality from ν_e to μ by proving the operator $\widetilde{K}$ is confining.

The rest of this section will thus enlighten some general facts which will later help us (mostly in dimension 1) prove K and $\widetilde{K}$ are confining. It is strongly inspired by the work of Chafaï, Malrieu and Paroux [18], in which a log-Sobolev inequality is proved for the invariant measure of the embedded chain of a particular PDMP, the TCP with linear rate (see Example 4.4).

Recall we assumed $\lambda > 0$, so that

$$t \mapsto \Lambda_x(t) := \int_0^t \lambda(\varphi_x(u))\, du$$

is invertible for all $x \in \Omega$. Moreover since we assumed the jump times are *a.s.* finite, necessarily, for all $x \in \Omega$, $\Lambda_x(t) \to \infty$ as $t \to \infty$. Remark that

$$\Lambda_{\varphi_x(s)}(t) = \Lambda_x(t+s) - \Lambda_x(s) \tag{3.6}$$

which yields both

$$b(x)^*\nabla_x(\Lambda_x(t)) = \left.\frac{d}{ds}\right|_{s=0}\left(\Lambda_{\varphi_x(s)}(t)\right) = \lambda(\varphi_x(t)) - \lambda(x) \tag{3.7}$$

and, taking $u = \Lambda_{\varphi_x(s)}(t)$ in $t+s = \Lambda_x^{-1}\left(\Lambda_{\varphi_x(s)}(t) + \Lambda_x(s)\right)$,

$$\Lambda_{\varphi_x(s)}^{-1}(u) \quad = \quad \Lambda_x^{-1}(u + \Lambda_x(s)) - s. \tag{3.8}$$

If $X_0 = x$ and if T_x is the next time of jump then

$$E = \int_0^{T_x} \lambda(\phi_x(u))\, du$$

is independent from X_0, and has a standard exponential law. In other words $T_x \overset{dist}{=} \Lambda_x^{-1}(E)$, and $T_{\varphi_x(t)} \overset{dist}{=} \Lambda_{\varphi_x(t)}^{-1}(\Lambda_x(T_x))$.

Lemma 3.4. *If λ is non-decreasing along the flow, then for all $x \in \Omega$ and $t > 0$, the law of $T_{\varphi_x(t)}$ is the image of the law of T_x by a 1-Lipschitz function.*

Proof. Let $x \in \Omega$ and $t > 0$. For $s > 0$ we note $G(s) = \Lambda_{\varphi_x(t)}^{-1}(\Lambda_x(s))$, so that $T_{\varphi_x(t)} \overset{dist}{=} G(T_x)$. From $\frac{d}{du}(\Lambda_x(u)) = \lambda(\varphi_x(u))$, we get

$$\begin{aligned} G'(s) &= \frac{\lambda(\varphi_x(s))}{\lambda\left(\varphi_{\varphi_x(t)}\left(\Lambda_{\varphi_x(t)}^{-1}(\Lambda_x(s))\right)\right)} \\ &= \frac{\lambda(\varphi_x(s))}{\lambda(\varphi_x(t + G(s)))} \end{aligned}$$

From the relation (3.8) and the fact that Λ_x (hence Λ_x^{-1}) is non-decreasing,

$$t + G(s) = \Lambda_x^{-1}(\Lambda_x(s) + \Lambda_x(t)) \geq \Lambda_x^{-1}(\Lambda_x(s)) = s.$$

Thus $\lambda(\varphi_x(s)) \leq \lambda(\varphi_x(t + G(s)))$ and $|G'(s)| \leq 1$. $\square$

The assumption that the jump rate is non-decreasing along the flow is natural in several applications where the role of the jump mechanism is to counteract a deterministic trend (growth/fragmentation models for cells [13], TCP dynamics [18], etc.). In this context, the more the system is driven away by the flow, the more it is likely to jump. From a mathematical point of view, thanks to Lemma 3.4, a Beckner's inequality for the law $K(x)$ may be transferred to $K(\varphi_x(t))$ for all $t > 0$.

In fact this is also true for $\widetilde{K}$. Let $\widetilde{T}_x$ be a random variable on $\mathbb{R}_+$ with density $\frac{e^{-\Lambda_x(t)}}{\int_0^\infty e^{-\Lambda_x(w)}dw}$, so that

$$\widetilde{K}f(x) = \mathbb{E}\left[f\left(\varphi_x\left(\widetilde{T}_x\right)\right)\right].$$

Lemma 3.5. *If λ is non-decreasing along the flow, then for all $x \in \Omega$ and $t > 0$, the law of $\widetilde{T}_{\varphi_x(t)}$ is the image of the law of $\widetilde{T}_x$ by a 1-Lipschitz function.*

Proof. We will prove Lemma 3.4 applies here. Indeed the law of $\widetilde{T}_{\varphi_x(t)}$ is the law of $\widetilde{T}_x - t$ conditionally to the event $\widetilde{T}_x > t$, exactly as the law of $T_{\varphi_x(t)}$ is the law of $T_x - t$ conditionally to the event $T_x > t$. We need to find a jump rate which defines $\widetilde{T}_x$ as the jump time of a Markov process.

Let $e^{-V(s)}ds$ be a positive probability density on $\mathbb{R}_+$, assume V is convex and let

$$r(t) \quad = \quad \frac{e^{-V(t)}}{\int_t^\infty e^{-V(s)}ds}.$$

Note that $r(t) = \frac{d}{dt}\left(-\ln \int_t^\infty e^{-V(s)}ds\right)$, so that

$$e^{-\int_0^t r(s)ds} \quad = \quad \int_t^\infty e^{-V(s)}ds.$$

Differentiating this equality yields

$$r(t)e^{-\int_0^t r(s)ds} \quad = \quad e^{-V(t)}.$$

We want to prove r is non-decreasing. From the convexity of V,

$$r(t) = \frac{e^{-V(t)}}{\int_t^\infty e^{-V(s)}ds} = \frac{\int_t^\infty V'(s)e^{-V(s)}ds}{\int_t^\infty e^{-V(s)}ds} \geq V'(t).$$

As a consequence,

$$r'(t) = r(t)\left(r(t) - V'(t)\right) \geq 0.$$

In the case of $\widetilde{T}_x$, if λ is non-decreasing along the flow then $V(t) = \Lambda_x(t) - \ln \int_0^\infty e^{-\Lambda_x(w)}dw$ is convex, so that the corresponding r is non-decreasing and Lemma 3.4 applies. □

Lemma 3.6. *For all $f \in \mathcal{A}$, $x \in \Omega$,*

$$b(x)^*\nabla\left(Kf\right)(x) \quad = \quad \lambda(x)K\left(\frac{b^*\nabla f}{\lambda}\right)(x).$$

In particular if λ is non-decreasing along the flow, $|b^*\nabla\left(Kf\right)| \leq K|b^*\nabla f|$.

Proof. From the representation

$$Kf(x) = \mathbb{E}\left(f\left(\varphi_x(T_x)\right)\right) = \mathbb{E}\left(f\left(\varphi_x\left(\Lambda_x^{-1}(E)\right)\right)\right),$$

we compute (recall $f \in \mathcal{A}$ is smooth and compactly supported)

$$\begin{aligned}
b(x)^*\nabla\left(Kf\right)(x) &= \left.\frac{d}{ds}\right|_{s=0}\left(Kf\left(\varphi_x(s)\right)\right)\\
&= \mathbb{E}\left(\left.\frac{d}{ds}\right|_{s=0} f\left[\varphi_{\varphi_x(s)}\left(\Lambda^{-1}_{\varphi_x(s)}(E)\right)\right]\right)\\
&= \mathbb{E}\left(\left.\frac{d}{ds}\right|_{s=0} f\left[\varphi_x\left(s + \Lambda^{-1}_{\varphi_x(s)}(E)\right)\right]\right)\\
&= \mathbb{E}\left(\left.\frac{d}{ds}\right|_{s=0} f\left[\varphi_x\left(\Lambda_x^{-1}\left(E + \Lambda_x(s)\right)\right)\right]\right) \qquad \text{(from Relation (3.8))}\\
&= \mathbb{E}\left(\Lambda_x'(0)\left(\Lambda_x^{-1}\right)'(E)\left(b^*\nabla f\right)\left[\varphi_x\left(\Lambda_x^{-1}(E)\right)\right]\right)\\
&= \mathbb{E}\left(\frac{\lambda(x)}{\lambda\left(\varphi_x\left(\Lambda_x^{-1}(E)\right)\right)}\left(b^*\nabla f\right)\left[\varphi_x\left(\Lambda_x^{-1}(E)\right)\right]\right).
\end{aligned}$$

If λ is non-decreasing along the flow, $\lambda\left(\varphi_x(t)\right) \geq \lambda(x)$ for all $t \geq 0$. □

Lemma 3.7. *Let $h(x) = \int_0^\infty e^{-\Lambda_x(u)}du$. Then for all $f \in \mathcal{A}$, $x \in \Omega$,*

$$b(x)^*\nabla\left(\widetilde{K}f\right)(x) = \frac{\widetilde{K}(hb^*\nabla f)(x)}{h(x)}.$$

In particular if λ is non-decreasing along the flow $|b^*\nabla\widetilde{K}f|(x) \leq \widetilde{K}|b^*\nabla f|(x)$.

Proof.

$$\widetilde{K}f(x) = \mathbb{E}\left[f\left(\varphi_x\left(\widetilde{T}_x\right)\right)\right].$$

Note that $F_x(t) = \int_0^t \frac{e^{-\Lambda_x(s)}}{\int_0^\infty e^{-\Lambda_x(w)}dw} ds$ the cumulative function of $\widetilde{T}_x$ is invertible. Let U be a uniform random variable on $[0,1]$. Then

$$\begin{aligned}
\widetilde{K}f(x) &= \mathbb{E}\left[f\left(\varphi_x\left(F_x^{-1}(U)\right)\right)\right] \\
\Rightarrow \quad b(x)^*\nabla\widetilde{K}f(x) &= \mathbb{E}\left[\frac{d}{ds}\bigg|_{s=0} f\left(\varphi_x\left(s+F^{-1}_{\varphi_x(s)}(U)\right)\right)\right] \\
&= \mathbb{E}\left[\left(1+b(x)^*\nabla_x\left(F_x^{-1}(U)\right)\right)(b^*\nabla f)\left(\varphi_x\left(F_x^{-1}(U)\right)\right)\right],
\end{aligned}$$

If $u \in [0,1]$, from $\nabla_x\left(F_x\left(F_x^{-1}(u)\right)\right) = \nabla_x(u) = 0$ we get

$$b(x)^*\nabla_x\left(F_x^{-1}(u)\right) = \frac{-b(x)^*\nabla_x(F_x)\left(F_x^{-1}(u)\right)}{F_x'\left(F_x^{-1}(u)\right)}. \tag{3.9}$$

On the first hand $F_x'(t) = \frac{e^{-\Lambda_x(t)}}{\int_0^\infty e^{-\Lambda_x(w)}dw}$. On the other hand from Equality (3.7) we compute

$$\begin{aligned}
b(x)^*\nabla_x(F_x)(t) &= \frac{\int_0^t \left(\lambda(x)-\lambda\left(\phi_x(s)\right)\right)e^{-\Lambda_x(s)}ds}{\int_0^\infty e^{-\Lambda_x(w)}dw} + F_x(t)\frac{\int_0^\infty\left(\lambda\left(\phi_x(w)\right)-\lambda(x)\right)e^{-\Lambda_x(w)}dw}{\int_0^\infty e^{-\Lambda_x(w)}dw} \\
&= \lambda(x)F_x(t) + \frac{\left[e^{-\Lambda_x(s)}\right]_0^t}{\int_0^\infty e^{-\Lambda_x(w)}dw} - \lambda(x)F_x(t) - F_x(t)\frac{\left[e^{-\Lambda_x(\omega)}\right]_0^\infty}{\int_0^\infty e^{-\Lambda_x(w)}dw} \\
&= \frac{-1+e^{-\Lambda_x(t)}+F_x(t)}{\int_0^\infty e^{-\Lambda_x(w)}dw}.
\end{aligned}$$

Relation (3.9) yields

$$\begin{aligned}
1+b(x)^*\nabla_x\left(F_x^{-1}(u)\right) &= 1-\frac{-1+e^{-\Lambda_x\left(F_x^{-1}(u)\right)}+F_x\left(F_x^{-1}(u)\right)}{e^{-\Lambda_x\left(F_x^{-1}(u)\right)}} \\
&= e^{\Lambda_x(t)}(1-F_x(t)).
\end{aligned}$$

with $t = F_x^{-1}(u)$. Thanks to Equation (3.6),

$$\begin{aligned}
e^{\Lambda_x(t)}(1-F_x(t)) &= e^{\Lambda_x(t)}\int_t^\infty \frac{e^{-\Lambda_x(s)}}{\int_0^\infty e^{-\Lambda_x(v)}dv}ds \\
&= \frac{\int_0^\infty e^{-\Lambda_x(w+t)+\Lambda_x(t)}dw}{\int_0^\infty e^{-\Lambda_x(v)}dv} \\
&= \frac{\int_0^\infty e^{-\Lambda_{\varphi_x(t)}(w)}dw}{\int_0^\infty e^{-\Lambda_x(v)}dv}.
\end{aligned}$$

Bringing the pieces together, we have proved

$$\begin{aligned}
b(x)^*\nabla\left(\widetilde{K}f\right)(x) &= \mathbb{E}\left[\frac{h\left(\varphi_x\left(F_x^{-1}(U)\right)\right)}{h(x)}(b^*\nabla f)\left(\varphi_x\left(F_x^{-1}(U)\right)\right)\right] \\
&= \frac{\widetilde{K}\left(hb^*\nabla f\right)(x)}{h(x)}
\end{aligned}$$

When λ is non-decreasing along the flow, from (3.7), $x \mapsto \Lambda_x(t)$ is non-decreasing along the flow for all $t \geq 0$, and $h\left(\varphi_x(t)\right) \leq h(x)$. $\square$

4 Examples

We refer to the Appendix to check the general assumptions of the introduction (and especially $f \in \mathcal{A} \Rightarrow P_t f \in \mathcal{A}$) hold in the following examples.

4.1 The TCP with constant rate

A simple yet instructive example on $\mathbb{R}_+$ is the TCP with constant rate of jump with generator

$$Lf(x) = f'(x) + \lambda \left(\mathbb{E}\left(f(Rx)\right) - f(x)\right)$$

where R is a random variable on $[0,1)$ and $\lambda > 0$ is constant. It is a simple growth/fragmentation model, or may be obtained by renormalizing a pure fragmentation model (cf. [28] for instance). In [35, 32], ergodicity is proved and it is shown the moments of the invariant measure μ are all finite.

Applying Theorem 2.2 with $J_b = 0$, $M = \mathbb{E}\left(R^2\right)$ and $a = 1$, we get

Proposition 4.1. *for all $f \in \mathcal{A}$,*

$$|(P_t f)'|^2 \leq e^{-\lambda\left(1-\mathbb{E}\left(R^2\right)\right)t} P_t |f'|^2.$$

Corollary 2.3 then yields a contraction at rate $\lambda\left(1 - \mathbb{E}\left(R^2\right)\right)$ of the Wasserstein distance $\mathcal{W}_1(\nu_1 P_t, \nu_2 P_t)$. In fact by coupling two processes starting at different points to have the same jump times and the same factor R at each jump, one get that for any $p \geq 1$, the $\mathcal{W}_p$ distance decays at rate $\lambda p^{-1}\left(1 - \mathbb{E}\left(R^p\right)\right)$ (see [18]), and those rates are optimal (see [36]). In particular $\lambda\left(1 - \mathbb{E}\left(R^2\right)\right)$ is the rate of decay of $\mathcal{W}_2^2$ (which in turn implies Proposition 4.1).

Let

$$Kf = \int_0^\infty f(x+s)\lambda e^{-\lambda s} ds.$$

Obviously $(Kf)' = K(f')$. Moreover the exponential law $\mathcal{E}(1)$ satisfies a Poincaré inequality $\mathcal{B}(2,4)$, so that by the change of variable $z \mapsto z/\lambda$, $\mathcal{E}(\lambda)$ satisfies $\mathcal{B}(2, 4\lambda^{-2})$. Finally, the law $K(x)$ is the image of $\mathcal{E}(\lambda)$ by the translation $u \mapsto u + x$, which is a 1-Lipschitz transformation. As a conclusion,

Lemma 4.2. *The operator K is $(4\lambda^{-2}, 1, 2)$-confining.*

As far as the jump operator $Qf(x) = \mathbb{E}\left(f(Rx)\right)$ is concerned, we have already used the sub-commutation

$$\left((Qf)'\right)^2 \leq \mathbb{E}\left(R^2\right) Q(f')^2.$$

However a local Poincaré inequality (3.4) for $Q(x)$ would mean $\forall f \in \mathcal{A},\ x > 0$,

$$\begin{array}{rrcl} & \mathbb{E}\left(f^2(Rx)\right) - \left(\mathbb{E}\left(f(Rx)\right)\right)^2 & \leq & c\mathbb{E}\left[\left(f'(Rx)\right)^2\right] \\ \Leftrightarrow & \mathbb{E}\left(g_x^2(R)\right) - \left(\mathbb{E}\left(g_x(R)\right)\right)^2 & \leq & \dfrac{c}{x^2}\mathbb{E}\left[\left(g_x'(R)\right)^2\right] \end{array}$$

with $g_x(r) = f(rx)$. This implies the law of R satisfies $\mathcal{B}\left(2, cx^{-2}\right)$ for all $x > 0$, hence $\mathcal{B}\left(2, 0\right)$, which means R is deterministic. Indeed, when R is deterministic, the local inequality always holds:

Lemma 4.3. *If $R = \delta$ a.s. with a constant $\delta \in [0,1)$ then Q is $(0, \delta^2, p)$-contractive.*

When R is random, what prevents to straightforwardly use our argument is the possibility of arbitrarily little concentrated jump, for instance with uniform law on $(0,x)$ for any x. It's a shame because if, say, R is uniform on $\left(0, \frac{1}{2}\right)$, it means when the process jumps it is at least divided by 2 but can be even much more contracted. In particular its invariant measure should be more concentrated near zero than the process with $R = \frac{1}{2}$ a.s. for which, as we will see, the invariant measure satisfies a Poincaré inequality. This illustrates a limit of our procedure.

Proposition 4.4. *If $R = \delta$ is deterministic then μ satisfies the Poincaré inequality*

$$\forall f \in \mathcal{A}, \qquad \mu(f - \mu f)^2 \quad \leq \quad \frac{4}{\lambda^2(1-\delta^2)}\mu(f')^2.$$

As a consequence,

$$\forall f \in \mathcal{A}, \qquad \mu(P_t f - \mu f)^2 \quad \leq \quad \frac{4e^{-\lambda(1-\delta^2)t}}{\lambda(1-\delta^2)}\mu(f')^2.$$

Proof. Since K and Q are confining, from Lemma 3.2, $P = KQ$ is $(4\lambda^{-2}\delta^2, \delta^2, 2)$-confining and μ_e the invariant measure of the embedded chain satisfies $\mathcal{B}\left(2, \frac{4\delta^2}{\lambda^2(1-\delta^2)}\right)$. From Lemma 3.3, $\mu = \mu_e K$ and so by Lemma 3.2 again μ satisfies $\mathcal{B}\left(2, \frac{4}{\lambda^2(1-\delta^2)}\right)$. The second inequality is a consequence of this Poincaré inequality and of Proposition 4.1. $\square$

In fact in this example the spectrum of the generator in $L^2(\mu)$ is explicit: there are polynomial eigenfunctions, and since the tail of μ is exponential, polynomials are dense in $L^2(\mu)$ and these eigenfunctions are the only ones in $L^2(\mu)$. The eigenvalues are $l_k = \lambda(\mathbb{E}\left[R^k\right] - 1)$ with $k \in \mathbb{Z}^+$. The convergence rate of the L^2-norm obtained in Proposition 4.4 for a deterministic R appears to be $\frac{1}{2}|l_2|$ and not the spectral gap $|l_1|$, and of course

$$\frac{1}{2}|l_2| = \lambda\mathbb{E}\left[(1-R)\frac{1+R}{2}\right] \leq \lambda\mathbb{E}(1-R) = |l_1|.$$

Nevertheless $\frac{1}{2}|l_1| \leq \frac{1}{2}|l_2|$ so we get the right rate up to a factor $1/2$.

4.2 The storage model

Let U be a positive random variable with all moments being finite, and consider the generator on $\mathbb{R}_+$

$$Lf(x) = -xf'(x) + \lambda\left(\mathbb{E}\left[f(x+U)\right] - f(x)\right).$$

This is, in a sense, the converse of the TCP: the jumps send the process away from 0 and the flow brings it back. Applying Theorem 2.2 with $M = 1$, $a = 1$ and $J_b = -1$, we get

$$|\nabla P_t f|^2 \leq e^{-2t} P_t |\nabla f|^2. \tag{4.1}$$

Besides in this case it is easy to obtain a Wasserstein decay, as the distance s between two processes starting at different points and coupled to have the same jump times and the same jump sizes U at each jump satisfies $s' = -s$, and such a decay implies (4.1) (see [31]; the converse is not clear, since P_t is a mix of a Dirac mass and a smooth density).

To prove a Beckner's inequality, the same problem arises as in the previous example with a random R: here the law $K(x)$, namely the law of $e^{-T}x$ with T an exponential random variable, can be as little concentrated as possible when x goes to infinity, so that K does not satisfy a local Beckner's inequality (3.4).

4.3 The TCP with increasing rate

Consider the generator on $\mathbb{R}_+$

$$Lf(x) \quad = \quad f'(x) + \lambda(x)\left(f(\delta x) - f(x)\right). \tag{4.2}$$

We have already studied the constant rate case. Before tackling the case of $\lambda(x) = x$, we consider in this section an intermediate difficulty, with the following assumptions:

λ is smooth non-decreasing, all its derivatives grow at most polynomially at infinity, $\lambda(0) = \lambda_* > 0$, and $\ln \lambda$ is a κ-Lipschitz function. Let $\beta = \frac{2\kappa^2}{1-\delta^2}$, so that

$$\begin{aligned}\frac{(\lambda')^2}{\beta\lambda} - \lambda\left(1-\delta^2\right) &= \frac{\lambda\left(1-\delta^2\right)}{2}\left(\frac{(\lambda')^2}{\lambda^2\kappa^2} - 2\right) \\ &\leq -\frac{\lambda_*\left(1-\delta^2\right)}{2}.\end{aligned}$$

In other word, Inequality (2.2) holds with $\eta = -\frac{\lambda_*\left(1-\delta^2\right)}{2}$ and $a = 1$. To apply Theorem 2.4, we also need to prove a Poincaré inequality.

Lemma 4.5. *The operators*

$$Kf(x) = \int_0^\infty f(x+t)\lambda(x+t)e^{-\int_0^t \lambda(x+s)ds}dt$$

and

$$\widetilde{K}f(x) = \int_0^\infty f(x+t)\frac{e^{-\int_0^t \lambda(x+s)ds}}{\int_0^\infty e^{-\int_0^u \lambda(x+s)ds}}dt$$

are $\left(\frac{4}{\lambda_*^2}, 1, 2\right)$*-confining.*

Proof. The sub-commutation (3.3) is a direct consequence of Lemmas 3.6 and 3.7, since the rate of jump is non-decreasing and $b = 1$. On the other hand $K(x)$ (resp $\widetilde{K}(x)$) is the law of $x + T_x$ (resp. $x + \widetilde{T}_x$) which is from Lemma 3.4 the image by a 1-Lipschitz function of T_0 (resp. $\widetilde{T}_0$). Thus we only need to prove the inequality holds for $K(0)$ and $\widetilde{K}(0)$.

For the case of $K(0)$, denote by $F(t) = 1 - e^{-\Lambda_0(t)}$ the cumulative function of T_0. Then, if E is a standard exponential random variable,

$$\begin{aligned}T_0 &\overset{dist}{=} F^{-1}\left(1-e^{-E}\right) \\ &= \Lambda_0^{-1}(E).\end{aligned}$$

Since Λ_0^{-1} is a non-decreasing concave function with $\left(\Lambda_0^{-1}\right)'(0) = \frac{1}{\lambda_*}$, T_0 is a $\frac{1}{\lambda_*}$-Lipschitz transformation of E, whose law satisfies the Poincaré inequality $\mathcal{B}(2,4)$.

In Lemma 3.5 we saw the cumulative function of $\widetilde{T}_0$ is $t \mapsto 1 - e^{-\int_0^t r(s)ds}$ with an increasing function r defined by

$$r(t) = \frac{e^{-\Lambda_0(t)}}{\int_t^\infty e^{-\Lambda_0(s)}ds}.$$

The previous argument shows $\widetilde{T}_0$ is a $\frac{1}{r(0)}$-Lipschitz transformation of E, and

$$r(0) = \frac{1}{\int_0^\infty e^{-\Lambda_0(s)}ds} \geq \frac{1}{\int_0^\infty e^{-\lambda_* s}ds} = \lambda_*.$$

□

Remark: in fact if moreover $\lambda(x) \geq k(1+x)^q$ for some $k > 0$ and $q \in [0,1]$, the laws of T_0 and $\widetilde{T}_0$ satisfy some generalized Poincaré inequality $\mathcal{I}(\alpha, c)$ with $\alpha = \frac{2q}{q+1}$ (see [8, Theorem 3] and [17]), or in other words the Beckner's inequalities $\mathcal{B}\left(p, c(p-1)^{\alpha-1}\right)$ for all $p \in (1,2]$. By the previous arguments, K and $\widetilde{K}$ are $\left(c(p-1)^{\alpha-1}, 1, p\right)$-confining for all $p \in (1,2]$.

Corollary 4.6. *The invariant measure μ of the process satisfies a Poincaré inequality $\mathcal{B}(2,c)$ for some explicit $c > 0$.*

Proof. It is clear the jump operator Q is $(0,\delta^2,2)$-contractive, so that from Lemma 3.2, $P = KQ$ is $\left(\frac{4\delta^2}{\lambda_*^2},\delta^2,2\right)$-contractive, and μ_e the invariant measure of the embedded chain associated with the process satisfies a Poincaré inequality $\mathcal{B}\left(2,\frac{4\delta^2}{\lambda_*^2(1-\delta^2)}\right)$. Let

$$h(x) = K\left(\frac{1}{\lambda}\right)(x) = \int_0^\infty e^{-\int_0^s \lambda(x+u)du}ds.$$

It is a non-increasing function with $h(0) \leq \int_0^\infty e^{-\lambda_* s}ds = \frac{1}{\lambda_*}$. In order to prove the perturbation ν_e of μ_e, defined by $\nu_e(f) = \frac{1}{\mu_e(h)}\mu_e(fh)$, satisfies a Poincaré inequality, we will use Lemma 4.16, which requires an upper bound on the median m_e of μ_e. Note that it is possible to couple a process X with rate λ and a process Z with constant rate λ_* so that, if they start at the same point, the first one will always stay below the second one: suppose such a coupling (X,Z) has been defined up to a jump time T_k of X. Then both process increase linearly up to the next jump time T_{k+1} of X. At time T_{k+1}, X jumps, but Z jumps only with probability $\frac{\lambda_*}{\lambda(X_{T_k}+T_k)}$, else it does not move. In other words the jump part of the generator of Z is thought as

$$\lambda_*\left(f(\delta x) - f(x)\right) = \lambda(x)\left(\left(\frac{\lambda_*}{\lambda(x)}f(\delta x) + \left(1-\frac{\lambda_*}{\lambda(x)}\right)f(x)\right) - f(x)\right).$$

Such a coupling proves m_e is less than the median of the invariant law of the process with constant rate λ_*. Let Z_∞ be a random variable with this invariant law, so that, if E is a standard exponential random variable,

$$\begin{aligned} Z_\infty &\overset{dist.}{=} \delta\left(Z_\infty + \frac{1}{\lambda_*}E\right) \\ \Rightarrow \qquad (1-\delta)\mathbb{E}(Z_\infty) &= \frac{\delta}{\lambda_*}. \end{aligned}$$

Hence from Markov's inequality, $m_e \leq \frac{2\delta}{\lambda_*(1-\delta)}$. Finally, from Lemma 4.16, ν_e satisfies a Poincaré inequality with constant

$$c' = \frac{32\delta^2}{\lambda_*^3(1-\delta^2)h\left(\frac{2\delta}{\lambda_*(1-\delta)}\right)},$$

and since $\widetilde{K}$ is confining, from Lemma 3.2, $\mu = \nu_e\widetilde{K}$ satisfies such an inequality with constant

$$c = \frac{4\delta^2}{\lambda_*^2} + c'.$$

$\square$

Remark: if, again, $\lambda(x) \geq k(1+x)^q$ for some $k>0$ and $q\in[0,1]$, these arguments prove the invariant measure satisfies a generalized Poincaré inequality $\mathcal{I}(\alpha,c)$ for some $c>0$ and $\alpha = \frac{2q}{q+1}$. Thus the invariant measure inherits the concentration properties of the law of the jump time T_0: the logarithm of its density tail is (at most) of order $-x^{q+1}$.

Let $(P_t)_{t\geq 0}$ be the semi-group associated to the generator (4.3) and for $f\in\mathcal{A}$ let $W_t = \mu\left((P_tf)'\right)^2$ and $V_t = \mu\left(P_tf - \mu f\right)^2$. We have proved Theorem 2.4 holds:

Corollary 4.7. *If λ is increasing with $\lambda(0) = \lambda_* > 0$ and $\ln\lambda$ is κ-Lipschitz then*

$$W_t + \beta V_t \leq (W_0 + \beta V_0)e^{-\frac{\eta}{1+\beta c}t}.$$

with c given by Corollary 4.6 and

$$\begin{aligned}\eta &= \frac{\lambda_*(1-\delta^2)}{2}\\ \beta &= \frac{2\kappa^2}{1-\delta^2}.\end{aligned}$$

4.4 The TCP with linear rate

In this section,

$$Lf(x) = f'(x) + x\left(f(\delta x) - f(x)\right), \tag{4.3}$$

where $\delta \in [0,1)$, and we will prove Proposition 1.3. We keep the general notations for $(P_t)_{t\geq 0}$, Q, λ and μ (for the proof of ergodicity, see [26]), and write $\text{Ent}f = \mu\left(f^2 \ln f^2\right) - \mu(f^2)\ln\mu(f^2)$.

In the first instance, from Theorem 2.4, Proposition 1.3 is proven in Section 4.4.1 under the additional assumption that the invariant law satisfies some weighted functional inequalities. These weighted inequalities are equivalent to non-weighted inequalities for the invariant measure of a twisted process, and the latter may be established thanks to the tools of Section 3. More precisely, in Section 4.4.2, we prove that the transition operator of the embedded chain corresponding to the twisted process is contractive, which implies its invariant law satisfies a log-Sobolev inequality, and in Section 4.4.3 we transfer this inequality to the continuous-time process via perturbative arguments.

4.4.1 Decay of the gradient, given the weighted functional inequalities

Recall Theorem 2.4 is based on a balance condition on the way the space is contracted or expanded by the drift and the jumps. Here, the deterministic motion is just a translation at constant speed: the flow is isometric. On the other hand the jumps mechanism do contract the space, but there are few jumps in the vicinity of the origin, and thus a condition as (1.8) cannot hold uniformly in $x > 0$ with $\eta > 0$. An idea is to consider a metric different from the euclidian one which is uniformly contracted for all $x > 0$. This metric can be equivalent to the euclidian one for x away from 0, but near 0, it should distend the distances, so that the deterministic flow $\phi_x(t) = x + t$ contracts the new metric (this is reminiscent of the construction of the Lyapunov function $\tilde{V}$ in [7, Section 3]) .

As we saw on Section 2, working with another metric is equivalent to working with weighted gradients, namely considering the condition (2.2) with $a \neq 1$. To cope with the rate of jump that vanishes at the origin, we will apply Theorem 2.4 with a weight a that behaves linearly near 0. More precisely, let

$$\begin{aligned}a(x) &= 1 - e^{-x},\\ \phi_a(f) &= a|f'|^2,\\ W_t &= \mu\left(\phi_a\left(P_t f\right)\right).\end{aligned}$$

Lemma 4.8. *Suppose μ satisfies the weighted Poincaré inequality*

$$\forall f \in \mathcal{A}, \qquad \mu\left(f - \mu f\right)^2 \leq c\mu\left(\phi_a(f)\right)$$

for some $c > 0$, and let

$$\theta = \frac{\sqrt{5}-1}{2} + \ln\left(\frac{3+\sqrt{5}}{2}\right) \simeq 1.58.$$

Then for all $\beta > ((1-\delta)\theta)^{-1}$*,* $t>0$ *and* $f \in \mathcal{A}$*,*

$$W_t \leq e^{-\frac{(1-\delta)\theta - \frac{1}{\beta}}{1+\beta c}t} (1+\beta c) W_0.$$

Proof. Note that a is a concave function, so that

$$a(\delta x) = a(\delta x + (1-\delta)0) \geq \delta a(x) + (1-\delta)a(0) = \delta a(x).$$

Therefore

$$\phi_a(Qf)(x) = a(x)\delta^2 |f'(\delta x)|^2 \leq \delta a(\delta x)|f'(\delta x)|^2 = \delta Q(\phi_a(f))(x).$$

To apply Theorem 2.4 we thus have to bound below

$$\frac{a'(x)}{a(x)} + x(1-\delta) - \frac{a(x)}{x\beta} \quad \geq \quad (1-\delta)\left(\frac{1}{e^x-1}+x\right) - \frac{1}{\beta}.$$

The function $g(x) = \frac{1}{e^x-1} + x$ goes to $+\infty$ at 0 and $+\infty$ and admits a unique positive critical point for which

$$\begin{aligned} e^x &= (e^x-1)^2 \\ \Rightarrow \qquad x &= \ln\left(\frac{3+\sqrt{5}}{2}\right). \end{aligned}$$

Hence for all $x>0$, $g(x) \geq g\left(\ln\left(\frac{3+\sqrt{5}}{2}\right)\right) = \theta$ and Theorem 2.4 holds with $\eta = (1-\delta)\theta - \frac{1}{\beta}$. □

Corollary 4.9. *Suppose* μ *satisfies the weighted inequalities, for all* $f \in \mathcal{A}$*,*

$$\begin{aligned} \mu(f-\mu f)^2 &\leq c_1 \mu(\phi_a(f)), \\ \mathit{Ent} f &\leq c_2 \mu(\phi_a(f)) \end{aligned} \tag{4.4}$$

for some $c_1, c_2 > 0$*, and let* θ *be such as defined in Lemma 4.8. Then for all* $\beta > ((1-\delta)\theta)^{-1}$*,* $t>0$ *and* $f \in \mathcal{A}$*,*

$$\mathit{Ent}P_t f \leq c_2 e^{-\frac{(1-\delta)\theta - \frac{1}{\beta}}{1+\beta c_1}t} (1+\beta c_1) \mu(f')^2.$$

Proof. From Lemma 4.8 and the fact $a \leq 1$,

$$\mathrm{Ent}P_t f \leq c_2 W_t \leq c_2 e^{-\frac{(1-\delta)\theta - \frac{1}{\beta}}{1+\beta c_1}t}(1+\beta c_1) W_0 \leq c_2 e^{-\frac{(1-\delta)\theta - \frac{1}{\beta}}{1+\beta c_1}t}(1+\beta c_1)\mu(f')^2.$$

□

Thus, to prove Proposition 1.3, it only remains to prove a weighted log-Sobolev inequality holds. Let

$$\psi(x) = \int_0^x \frac{1}{\sqrt{a(y)}} dy.$$

It is a concave, non-decreasing, one-to-one function. If Z is a random variable with law μ and $Y = \psi(Z)$, then

$$\begin{aligned} & \mathbb{E}\left(f^2(Z)\ln f^2(Z)\right) - \mathbb{E}\left(f^2(Z)\right)\ln \mathbb{E}\left(f^2(Z)\right) &\leq\ & c\mathbb{E}\left(a(Z)(f')^2(Z)\right) \\ \Leftrightarrow\quad & \mathbb{E}\left(g^2(Y)\ln g^2(Y)\right) - \mathbb{E}\left(g^2(Y)\right)\ln \mathbb{E}\left(g^2(Y)\right) &\leq\ & c\mathbb{E}\left((g')^2(Y)\right) \end{aligned}$$

with $g(y) = f\left(\psi^{-1}(y)\right)$. As a consequence we will study the Markov process $\psi(X) = (\psi(X_t))_{t\geq 0}$, where $X = (X_t)_{t\geq 0}$ has generator (4.3), and prove a classical non-weighted log-Sobolev for the invariant measure of this twisted process, which will imply the weighted log-Sobolev assumed in Corollary 4.9.

4.4.2 Confining operators for the twisted process

The jump kernel of $\psi(X)$ is

$$Q_\psi g(z) = g\left(\psi\left(\delta\psi^{-1}(z)\right)\right).$$

Let K_ψ and $\widetilde{K}_\psi$ be the operators defined in Section 3.2 corresponding to the process $\psi(X)$.

Lemma 4.10. *For all $g \in \mathcal{A}$,*

$$\begin{array}{rcl} |(Q_\psi g)'| & \leq & \sqrt{\delta} Q_\psi |g'| \\ |(K_\psi g)'| & \leq & K_\psi |g'| \\ |(\widetilde{K}_\psi g)'| & \leq & \widetilde{K}_\psi |g'|. \end{array}$$

Proof. Recall $a(\delta x) \geq \delta a(x)$ for all $x \geq 0$, and so

$$\begin{array}{rcl} (Q_\psi g)'(z) & = & \delta\left(\psi^{-1}\right)'(z)\psi'\left(\delta\psi^{-1}(z)\right) Q_\psi g'(z) \\ & = & \dfrac{\delta\left(\psi^{-1}\right)'(z)}{\sqrt{a\left(\delta\psi^{-1}(z)\right)}} Q_\psi g'(z) \\ & \leq & \dfrac{\sqrt{\delta}\left(\psi^{-1}\right)'(z)}{\sqrt{a\left(\psi^{-1}(z)\right)}} Q_\psi |g'|(z) \\ & = & \sqrt{\delta} Q_\psi |g'|(z). \end{array}$$

On the other hand the vector field associated to $\psi(X)$ is $b_\psi(z) = \frac{1}{\sqrt{a(\psi^{-1}(z))}}$, and the rate of jump is non-decreasing along the flow. Hence, according to Lemma 3.6,

$$b_\psi |(K_\psi g)'| \quad \leq \quad K_\psi\left(b_\psi |g'|\right)$$

(and according to Lemma 3.7, the same goes for $\widetilde{K}_\alpha$). Note that the support of both probability measures $K_\psi(z)$ and $\widetilde{K}_\psi(z)$ is $[z, \infty]$, and that b_ψ is non-increasing along the flow, so that

$$|(K_\psi g)'|(z) \leq \frac{K_\psi\left(b_\psi |g'|\right)(z)}{b_\psi(z)} \leq K_\psi\left(|g'|\right)(z)$$

(and the same goes for $\widetilde{K}_\psi$). □

Lemma 4.11. *For any $z > 0$, the law $K_\psi(z)$ (resp. $\widetilde{K}_\psi(z)$) can be obtained from $K_\psi(0)$ (resp. $\widetilde{K}_\psi(0)$) through a 1-Lipschitz transformation.*

Proof. Let T_x be the first time of jump of X starting from x. According to Lemma 3.4, there exists a 1-Lipschitz function G such that $T_x \overset{dist}{=} G(T_0)$. Note that $K_\psi\left(\psi(x)\right)$ is the law of $\psi\left(x + T_x\right)$. Let $H(z) = \psi\left(x + G\left(\psi^{-1}(z)\right)\right)$, so that $\psi(x + T_x) \overset{dist}{=} H\left(\psi(T_0)\right)$. We compute

$$\begin{array}{rcl} |H'(z)| & = & |G'\left(\psi^{-1}(z)\right)\left(\psi^{-1}\right)'(z)\psi'\left(x + G\left(\psi^{-1}(z)\right)\right)| \\ & \leq & \dfrac{\psi'\left(x + G\left(\psi^{-1}(z)\right)\right)}{\psi'\left(\psi^{-1}(z)\right)}. \end{array}$$

Now ψ is concave, and in the proof of Lemma 3.4 we have seen that $x + G(s) \geq s$ for all $s \geq 0$; hence $|H'(z)| \leq 1$ for all $z \geq 0$.

Similarly, let $\widetilde{T}_x$ be a random variable on $\mathbb{R}_+$ with density $\frac{e^{-\int_0^t (x+s)ds}}{\int_0^\infty e^{-\int_0^u (x+s)ds} du}$, so that $\widetilde{K}_\psi\left(\psi(x)\right)$ is the law of $\psi\left(x + \widetilde{T}_x\right)$. From Lemma 3.5 there exists a 1-Lipschitz function $\widetilde{G}$ such that $\widetilde{T}_x \overset{dist}{=} \widetilde{G}(\widetilde{T}_0)$, and the previous argument concludes. □

Lemma 4.12. *Both $K_\psi(0)$ and $\widetilde{K}_\psi(0)$ satisfy $\mathcal{B}(1,4)$.*

Proof. If T_0 is the first time of jump of X starting from 0 then $K_\psi(0)$ is the law of $\psi(T_0)$. For any $f \in \mathcal{A}$,

$$\begin{aligned} K_\psi f(0) &= \int_0^\infty f(\psi(u))\, u e^{-\frac{u^2}{2}} du \\ &= \int_0^\infty f(z)\, e^{-\frac{\left(\psi^{-1}(z)\right)^2}{2} + \ln \psi^{-1}(z) + \frac{1}{2}\ln\left(a\left(\psi^{-1}(z)\right)\right)} dz. \end{aligned}$$

On the other hand, if N is a standard Gaussian variable then $\widetilde{K}_\psi(0)$ is the law of $\psi(|N|)$, and for all $f \in \mathcal{A}$

$$\begin{aligned} \widetilde{K}_\psi f(0) &= \int_0^\infty f(\psi(u)) \left(\frac{\pi}{2}\right)^{-\frac{1}{2}} e^{-\frac{u^2}{2}} du \\ &= \int_0^\infty f(z)\, e^{-\frac{\left(\psi^{-1}(z)\right)^2}{2} + \frac{1}{2}\ln\left(a\left(\psi^{-1}(z)\right)\right) - \frac{1}{2}\ln\left(\frac{\pi}{2}\right)} dz. \end{aligned}$$

For $\varepsilon \in \{0,1\}$, let $V_\varepsilon(z) = \frac{1}{2}\left(\psi^{-1}(z)\right)^2 - \varepsilon \ln \psi^{-1}(z) - \frac{1}{2}\ln\left(a\left(\psi^{-1}(z)\right)\right)$; we want to prove V_ε is strictly convex. Writing $x = \psi^{-1}(z)$, we compute $\partial_z(x) = \sqrt{a(x)}$ and

$$\begin{aligned} V_\varepsilon'(z) &= \sqrt{a(x)}\left(x - \frac{\varepsilon}{x} - \frac{a'(x)}{2a(x)}\right) \\ V_\varepsilon''(z) &= \frac{a'(x)}{2}\left(x - \frac{\varepsilon}{x} - \frac{a'(x)}{2a(x)}\right) + a(x)\left(1 + \frac{\varepsilon}{x^2} - \frac{a''(x)}{2a(x)} + \frac{1}{2}\left(\frac{a'(x)}{a(x)}\right)^2\right) \\ &= \varepsilon\left(\frac{a(x)}{x^2} - \frac{a'(x)}{2x}\right) + \frac{a'(x)x}{2} + \frac{\left(a'(x)\right)^2}{4a(x)} + a(x) - \frac{1}{2}a''(x). \end{aligned}$$

As a first step, note that $V_1''(z) \geq V_0''(z)$: indeed, $V_1''(z) - V_0''(z) = \frac{j(x)}{x^2}$ with

$$\begin{aligned} j(y) &= a(y) - \frac{y}{2}a'(y) \\ \Rightarrow \quad j'(y) &= \frac{1}{2}a'(y) - \frac{y}{2}a''(y) \ > \ 0 \end{aligned}$$

(since a is non-decreasing and concave). Since $j(0) = 0$, it implies $j(y) \geq 0$ for all $y \geq 0$, in other words $V_1''(z) \geq V_0''(z)$. On the other hand,

$$\begin{aligned} V_0''(z) &\geq a(x) - \frac{1}{2}a''(x) \\ &\geq \frac{1}{2}. \end{aligned}$$

As a consequence, both $K_\psi(0)$ and $\widetilde{K}_\psi(0)$ satisfy $\mathcal{B}(1,4)$ (see for instance [4, Theorem 5.4.7], applied to the diffusion with generator $\partial_x^2 - V_\varepsilon'\partial_x$). □

To sum up the consequences of the previous results,

Corollary 4.13.

1. *The operators K_ψ and $\widetilde{K}_\psi$ are $(4,1,1)$-confining and the operator Q_ψ is $(0,\sqrt{\delta},1)$-contractive.*
2. *The invariant measure ν_ψ of the embedded chain associated to $\psi(X)$ satisfies $\mathcal{B}\left(1, \frac{4\sqrt{\delta}}{1-\sqrt{\delta}}\right)$.*

Proof. The sub-commutation property has been showed in Lemma 4.10, and the local inequality is clear for Q_ψ which is deterministic, and is a consequence of Lemma 4.11 and 4.12 for K_ψ and $\widetilde{K}_\psi$.

From Lemma 3.2, the transition operator of the embedded chain associated to $\psi(X)$, $P_\psi = K_\psi Q_\psi$, is $(4\sqrt{\delta}, \sqrt{\delta}, 1)$-confining, conclusion follows again from Lemma 3.2. $\square$

4.4.3 Perturbation and conclusion

The last step of our procedure is the study of a perturbation of ν_ψ. Since the rate of jump of $Z = \psi(X)$ at point z is $\lambda_\psi(z) = \psi^{-1}(z)$ and the operator K_ψ is such that $K_\psi f(\psi(x)) = \mathbb{E}\left(f\left(\psi(x+T_x)\right)\right)$, according to Lemma 3.3, we need to investigate the perturbation of ν_ψ by the function g defined by

$$\begin{aligned} g\left(\psi(x)\right) &= K_\psi\left(\frac{1}{\lambda_\psi}\right)\left(\psi(x)\right) \\ &= \mathbb{E}\left(\frac{1}{x+T_x}\right). \end{aligned}$$

Lemma 4.14. *The function g is decreasing, and $\ln g$ is $\sqrt{\frac{2}{\pi}}$-Lipschitz.*

Proof. Let

$$h(x) = \mathbb{E}\left(\frac{1}{x+T_x}\right) = \int_0^\infty e^{-\int_0^t (x+u)du} dt = \int_0^\infty e^{-\frac{t^2}{2} - xt} dt,$$

so that $g(z) = h\left(\psi^{-1}(z)\right)$. Since h is decreasing and ψ^{-1} is increasing, g is decreasing. Moreover, as $|\left(\ln g\right)'(z)| = \sqrt{a\left(\psi^{-1}(s)\right)}|(\ln h)'\left(\psi^{-1}(z)\right)|$ and $a \leq 1$, it is sufficient to prove $\ln h$ is $\sqrt{\frac{2}{\pi}}$-Lipschitz. Since $h' < 0$ and $h'' > 0$, $(\ln h)'$ is negative and increasing: for all $x \geq 0$,

$$0 \quad \geq \quad (\ln h)'(x) \quad \geq \quad \frac{h'(0)}{h(0)} \quad = \quad -\sqrt{\frac{2}{\pi}}.$$

$\square$

To apply to ν_ψ and g the perturbation Lemma 4.16 of the Appendix, we need to bound $g(m_\psi)$, where m_ψ is the median of ν_ψ, and $\nu_\psi(g^{-1})$. In fact, note that ν_ψ, which is the invariant measure of the embedded chain associated to the process $\psi(X)$, is also the image through the function ψ of μ_e the invariant measure of the embedded chain associated to the initial process X. In particular if m_e is the median of μ_e then $m_\psi = \psi(m_e)$. Keeping the notation $h(x) = g\left(\psi(x)\right)$, we have $g(m_\psi) = h(m_e)$ and $\nu_\psi(g^{-1}) = \mu_e(h^{-1})$.

Lemma 4.15. *We have*

$$\max\left(\frac{h(0)}{h(m_e)}, \mu_e\left(h^{-1}\right)\right) \quad \leq \quad 3\left(1 + \frac{\delta}{\sqrt{1-\delta^2}}\right).$$

Proof. Recall that, keeping the notations of Section 3.2, if T_x is the first time of jump of the process starting from x and E is a standard exponential variable, then $T_x \overset{dist}{=} \Lambda_x^{-1}(E)$. In the present case $\Lambda_x(t) = \int_0^t (x+u)du$, so that $T_x \overset{dist}{=} \sqrt{x^2+2E} - x$. In particular if Y is a random variable with measure μ_e, $Y \overset{dist}{=} \delta\sqrt{Y^2+2E}$, so that

$$(1-\delta^2)\mathbb{E}\left(Y^2\right) = 2\delta^2\mathbb{E}(E) = 2\delta^2.$$

From this,

$$\mathbb{P}\left(Y \geq t\right) \ \leq \ \frac{\delta^2}{(1-\delta^2)t^2},$$

which implies

$$m_e \leq \frac{\sqrt{2}\delta}{\sqrt{1-\delta^2}}.$$

Moreover

$$h(x) = \mathbb{E}\left(\frac{1}{x+T_x}\right) \geq \frac{1}{x+2}\mathbb{P}\left(T_x \leq 2\right) \geq \frac{1}{x+2}\mathbb{P}\left(T_0 \leq 2\right) = \frac{1}{x+2}\left(1-e^{-2}\right).$$

Hence

$$\frac{h(0)}{h(m_e)} \leq \sqrt{\frac{\pi}{2}} \times \frac{m_e+2}{1-e^{-2}} \leq 3\left(1+\frac{\delta}{\sqrt{1-\delta^2}}\right).$$

Finally, if Y is a random variable with law μ_e,

$$\mu_e\left(h^{-1}\right) = \mathbb{E}\left(\frac{1}{h(Y)}\right) \leq \frac{1}{1-e^{-2}}\left(2+\sqrt{\mathbb{E}\left(Y^2\right)}\right) \leq 3\left(1+\frac{\delta}{\sqrt{1-\delta^2}}\right).$$

□

We can now bring the pieces together.

Proof of Proposition 1.3. We have proved in Corollary 4.13 that ν_ψ satisfies a log-Sobolev inequality. From Lemmas 4.14, 4.15 and 4.16, the perturbation ν_g of ν_ψ defined by $\nu_g f = \frac{1}{\nu_\psi(g)}\nu_\psi(gf)$ also satisfies such an inequality. From Lemma 3.3, the invariant measure of $\psi(X)$ is $\nu_g \widetilde{K}_\psi$, and it also satisfies a log-Sobolev inequality since $\widetilde{K}_\psi$ is confining (Corollary 4.13). It means μ, the invariant measure of X, satisfies a weighted log-Sobolev inequality

$$\mu\left(f^2\ln f^2\right) - \mu(f^2)\ln\mu(f^2) \leq c\mu\left(a|f'|^2\right).$$

The conditions of Corollary 4.9 are fulfilled, and Proposition 1.3 is proved. □

Appendix

The general assumptions are satisfied in Section 4

Consider the TCP process on $\mathbb{R}_+$ with generator

$$Lf(x) = f'(x) + \lambda(x)\left(\mathbb{E}f(Rx) - f(x)\right). \tag{4.5}$$

where R is a random variable in $[0,1)$ and λ is smooth positive increasing and all its derivatives grow at most polynomially at infinity. This covers all the TCP processes studied in Section 4. Starting at x, the process necessarily remains during a time t in $[0, x+t]$, on which λ is bounded, which implies there can't be infinitely many jumps in a finite time and the process is defined for all times. When $f(x) = x^p$ for $p \geq 1$, $Lf \leq -\rho_p f + C_p$ for some $\rho_p, C_p > 0$, and moreover starting from a compact $\mathcal{K}$, it is clear the transition density $\mathcal{P}(X_t = y \mid X_0 = x)$ for $t > 1$ is uniformly bounded with respect to $x \in \mathcal{K}$ and $y \in [1,2]$, which implies ergodicity ([6]) and all the moments of the invariant measure are finite. The condition $C < \infty$ in Lemma 3.3 is clearly satisfied.

The set $\mathcal{A}$ of $\mathcal{C}^\infty$ function whose derivatives grow at most polynomially at infinity is clearly fixed by L and by $Qf = \mathbb{E}f(Rx)$. If $f \in \mathcal{C}^\infty$, then so is $P_t f$ (see [14, Theorem VII.5, p.111]). Differentiating $\partial_t P_t f(x) = LP_t f(x)$ with respect to x, we obtain $\partial_t\partial_x^k P_t f = L\partial_x^k P_t f + [\partial_x^k, L]P_t f$, where $[,]$ stands for the Poisson bracket. According to [14, Theorem VII.10, p.117], this yields

$$\partial_x^k P_t f = P_t\partial_x^k f + \int_0^t P_{t-s}[\partial_x^k, L]P_s f. \tag{4.6}$$

The assumptions on λ and R imply there exist $C_k, N_k > 0$ such that $|[\partial_x^k, L]g|(x) \leq C_k(1+x)^{N_k}\sup_{y\leq x}\sum_{j<k}|\partial_x^j g(y)|$ for any smooth g. Suppose f and all its derivatives grow at most polynomially at infinity. If $|f| \leq c(1+x)^n$, since starting from $X_0 = x$ necessarily $X_t \in [0, x+t]$, then $|P_t f| \leq c(1+x+t)^n \leq c_t(1+x)^n$ where $t \mapsto c_t$ is locally bounded. By induction on k in Equation 4.6, the same argument proves that if $f \in \mathcal{A}$ then $P_t f \in \mathcal{A}$.

For the storage process with generator

$$Lf(x) = -xf'(x) + \lambda\left(\mathbb{E}\left[f(x+U)\right] - f(x)\right)$$

with $\mathbb{E}(U^p) < \infty\ \forall p \geq 0$, we can adapt the previous arguments, but in fact in this case $X_t = e^{-t}X_0 + V_t$ where V_t does not depend on X_0, but only on the Poisson process N_t which defines the jump times and on the sequence $(U_k)_{k\geq 1}$ of jump sizes. Since $V_t \leq \sum_{i=1}^{N_t} U_i$, all the moments of V_t are finite, which allow to differentiate under the integral sign to directly obtain $\partial_x^k P_t f = e^{-kt}P_t\partial_x^k f$. If $|\partial_x^k f| \leq c(1+x)^n$ then $|P_t\partial_x^k f| \leq c\mathbb{E}\left(1 + (x+V_t)^n\right) \leq c_t(1+x)^n$, hence P_t fixes $\mathcal{A}$, and the cases of L and $Qf = \mathbb{E}\left[f(x+U)\right]$ are similar.

Monotonous perturbation on the half-line

Let ν be a probability measure on $\mathbb{R}_+$ with a positive smooth density (still denoted by ν), and g be a positive smooth function on $\mathbb{R}_+$ such that $\nu(g) = 1$. We define ν_g, the perturbation of ν by g, by $\nu_g(f) = \nu(fg)$ for all bounded f. Let m be the median of ν, defined by $\nu\left([0,m]\right) = \frac{1}{2}$.

The aim of this section is to prove the following:

Lemma 4.16. *Suppose g is non-increasing and $g(0) := \lim_{x\to 0} g(x) \neq \infty$.*

1. *If ν satisfies the Poincaré inequality $\mathcal{B}(2, c_1)$, then ν_g satisfies $\mathcal{B}(2, c_2)$ with*

$$c_2 = 8\frac{g(0)}{g(m)}c_1.$$

2. *If $\ln g$ is κ-Lipschitz and ν satisfies the log-Sobolev inequality $\mathcal{B}(1, c_1)$ then ν_g satisfies $\mathcal{B}(1, c_2)$ with for all $\varepsilon \in (0,1)$*

$$c_2 \leq \left(\frac{2}{1-\varepsilon} + 8\frac{g(0)}{g(m)}\left(2 + \frac{\frac{c_1\kappa^2}{2} + \varepsilon\ln\nu\left(g^{1-\frac{1}{\varepsilon}}\right)}{1-\varepsilon}\right)\right)c_1.$$

Remark: actually as far as point 2 is concerned the monotonicity of g is only needed to get the explicit estimate of c_2: as soon as ν satisfies a log-Sobolev inequality and $\ln g$ is Lipschitz, ν_g satisfies a log-Sobolev inequality (see [2]).

Moreover when ν satisfies a log-Sobolev inequality and $\ln g$ is Lipschitz, $\nu\left(g^\alpha\right)$ is finite for all $\alpha \in \mathbb{R}$ (see [1]), so that c_2 is finite.

proof of point 1. According to Muckenhoupt work (see [4, Theorem 6.2.2 p. 99 and Remark 6.2.3]), a probability with density $h > 0$ satisfies $\mathcal{B}(2,c)$ iff $B_{m_h}(h)$ is finite when m_h is the median of $h(t)dt$ and

$$B_\alpha(h) = \max\left(\sup_{x\in(\alpha,\infty)}\left(\int_x^\infty h(t)dt\int_\alpha^x\frac{1}{h(t)}dt\right), \sup_{x\in(0,\alpha)}\left(\int_0^x h(t)dt\int_x^\alpha\frac{1}{h(t)}dt\right)\right).$$

Furthermore, in that case, the optimal c (namely the smallest c such that $\mathcal{B}(2,c)$ holds) is such that

$$\frac{1}{2}\inf_{\alpha>0} B_\alpha(h) \leq \frac{1}{2}B_{m_h}(h) \leq c \leq 4\inf_{\alpha>0} B_\alpha(h) \leq 4B_{m_h}(h).$$

In the present case, for all $x \geq m$,

$$\begin{aligned}\int_x^\infty g(t)\nu(t)dt\int_m^x \frac{1}{g(t)\nu(t)}dt &\leq \int_x^\infty g(x)\nu(t)dt\int_m^x \frac{1}{g(x)\nu(t)}dt\\ &\leq 2c_1.\end{aligned}$$

and for all $x \leq m$

$$\begin{aligned}\int_0^x g(t)\nu(t)dt\int_x^m \frac{1}{g(t)\nu(t)}dt &\leq \int_0^x g(0)\nu(t)dt\int_x^m \frac{1}{g(m)\nu(t)}dt\\ &\leq 2\frac{g(0)}{g(m)}c_1.\end{aligned}$$

Hence ν_g satisfies $\mathcal{B}(2,c_2)$ with

$$c_2 \leq 4\inf_{\alpha>0} B_\alpha(\nu g) \leq 4B_m(\nu g) \leq 8\frac{g(0)}{g(m)}c_1.$$

□

proof of point 2. Following a computation of Aida and Shigekawa ([2]), we apply the inequality $\mathcal{B}(1,c_1)$, namely

$$\forall f\in\mathcal{A},\qquad \nu\left(f^2\ln f^2\right) \leq c_1\nu(f')^2 + \left(\nu f^2\right)\ln\left(\nu f^2\right),$$

to the function $f\sqrt{g}$, which reads

$$\forall f\in\mathcal{A},\qquad \nu_g\left(f^2\ln f^2\right) + \nu_g\left(f^2\ln g\right) \leq c_1\nu_g\left(f' + \frac{g'}{2g}f\right)^2 + \left(\nu_g f^2\right)\ln\left(\nu_g f^2\right) \tag{4.7}$$

From the inequality $(a+b)^2 \leq 2a^2+2b^2$ and the assumption on $\ln g$,

$$\nu_g\left(f' + \frac{g'}{2g}f\right)^2 \leq 2\nu_g(f')^2 + \frac{\kappa^2}{2}\nu_g(f^2).$$

On the other hand, from the Young inequality $st \leq s\ln s - s + e^t$ applied with $s=\varepsilon f^2$ and $t = -\varepsilon^{-1}\ln\left(\frac{g}{g(0)}\right)$ for any $\varepsilon>0$,

$$\begin{aligned}-\nu_g\left(f^2\ln g\right) &= -\nu_g\left(f^2\ln\left(\frac{g}{g(0)}\right)\right) - \ln g(0)\nu_g\left(f^2\right)\\ &\leq \varepsilon\nu_g\left(f^2\ln f^2\right) - \left(\varepsilon(1-\ln\varepsilon)+\ln g(0)\right)\nu_g\left(f^2\right) + \nu_g\left(\left(\frac{g(0)}{g}\right)^{\frac{1}{\varepsilon}}\right).\end{aligned}$$

Thus Inequality (4.7) yields

$$\begin{aligned}(1-\varepsilon)\nu_g\left(f^2\ln f^2\right) &\leq 2c_1\nu_g(f')^2 + \left(\frac{c_1\kappa^2}{2} - \varepsilon(1-\ln\varepsilon) - \ln g(0)\right)\nu_g\left(f^2\right) + \nu_g\left(\left(\frac{g(0)}{g}\right)^{\frac{1}{\varepsilon}}\right)\\ &\quad +\nu_g(f^2)\ln\nu_g(f^2).\end{aligned}$$

Thanks to Gross' Lemma (2.2 of [29]), this implies (for $\varepsilon<1$)

$$\nu_g\left(f^2\ln f^2\right) - \nu_g(f^2)\ln\nu_g(f^2) \leq \frac{2c_1}{1-\varepsilon}\nu_g(f')^2 + \gamma\nu_g\left(f^2\right) \tag{4.8}$$

with

$$\begin{aligned}\gamma &= \frac{\frac{c_1\kappa^2}{2} - \varepsilon(1-\ln\varepsilon) - \ln g(0)}{1-\varepsilon} + \frac{\varepsilon}{1-\varepsilon}\left(1 + \ln\left(\frac{\nu_g\left(\left(\frac{g(0)}{g}\right)^{\frac{1}{\varepsilon}}\right)}{\varepsilon}\right)\right)\\ &= \frac{\frac{c_1\kappa^2}{2} + \varepsilon\ln\nu_g\left(g^{-\frac{1}{\varepsilon}}\right)}{1-\varepsilon}.\end{aligned}$$

It is classical to retrieve a log-Sobolev inequality from Inequality (4.8) and a Poincaré inequality, thanks to the following inequality (see [23], p.146): if $h = f - \nu_g f$,

$$\nu_g\left(f^2\ln f^2\right) - \nu_g(f^2)\ln\nu_g(f^2) \le \nu_g\left(h^2\ln h^2\right) - \nu_g(h^2)\ln\nu_g(h^2) + 2\nu_g\left(h^2\right).$$

Together with Inequality (4.8) applied to h, and since $h' = f'$,

$$\nu_g\left(f^2\ln f^2\right) - \nu_g(f^2)\ln\nu_g(f^2) \quad\le\quad \frac{2c_1}{1-\varepsilon}\nu_g(f')^2 + (\gamma+2)\nu_g\left((f-\nu_g f)^2\right)$$

Since ν satisfies $\mathcal{B}(1,c_1)$ it also satisfies $\mathcal{B}(2,c_1)$. Thus, according to point 1 of Lemma 4.16, ν_g satisfies $\mathcal{B}\left(2, 8\frac{g(0)}{g(m)}c_1\right)$, which means

$$\nu_g\left(f^2\ln f^2\right) - \nu_g(f^2)\ln\nu_g(f^2) \quad\le\quad \left(\frac{2}{1-\varepsilon} + 8\frac{g(0)}{g(m)}(2+\gamma)\right)c_1\nu_g(f')^2.$$

□

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