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The iPod Model

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Abstract

We introduce a Voter Model variant, inspired by social evolution of musical preferences. In our model, agents have preferences over a set of songs and upon meeting update their own preferences incrementally towards those of the other agents they meet. Using the spectral gap of an associated Markov chain, we give a geometry dependent result on the asymptotic consensus time of the model.

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1 Introduction

The terminology of Finite Markov Information Exchange (FMIE) models has been introduced [1] [3] as a catch-all for the interpretation of Interacting Particle Systems (IPS) models as stochastic social dynamics. Many important and classical models fit under this two-level framework; the bottom level a meeting model among agents, and the top level an information exchange algorithm performed at each meeting.

For classic IPS models, such as the Voter Model, with a simple meeting algorithm the FMIE perspective is perhaps unnecessary. Coupling and comparison to random walks, among other methods, suffice[2]. In this paper however, we will introduce and study a (much) generalized Voter Model - inspired by the evolution of musical preferences among a group of friends - as an FMIE process.

1.1 The iPod Model

Here we introduce the iPod FMIE model. The underlying framework of the stochastic process is a weighted graphs $\mathfrak{G}$ on N vertices. We will refer to each vertex as an agent and occasionally to our vertex set as I. Associated to the edges are symmetric meeting rates $\nu_{i,j}$ for $1 \leq i \neq j \leq N$. We assume that all meeting rates are normalized, i.e.

$$\sum_j \nu_{i,j} = 1$$

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for all agents i.

Each agent i is equipped at each time t with a probability measure $X_t(i)$ on $\{1, 2, \ldots, \sigma\}$ which we will reference by its distribution $X_t^k(i)$ for $1 \leq k \leq \sigma$.

We consider σ as a fixed number of songs and $X_t^k(i)$ the preference of agent i at time t for song k. The stochastic process X_t updates over time as follows. Between every pair of agents i, j we associate a Poisson process with rate $\nu_{i,j}$ whose times we refer to as meetings between i and j. At a meeting time t between agents i and j, each agent picks a song σ_i and σ_j independently and distributed according to $X_{t-}(i)$ and $X_{t-}(j)$.

We interpret this as each agent choosing a song to play to the other agent based on their preferences. After agent i hears the song chosen by j he updates his preferences according to

$$X_t^{\sigma_j}(i) = (1-\eta)X_{t-}^{\sigma_j}(i) + \eta$$

and

$$X_t^k(i) = (1-\eta)X_{t-}^k(i)$$

for all other $k \neq \sigma_j$. Here $0 < \eta < 1$ is a fixed interaction parameter. Agent j updates her preferences similarly. It is immediate that if $X_{t-}(i)$ is a probability measure then so is $X_t(i)$. Note that we are implicitly working with cadlag paths.

Analogous to results on the consensus time of the Voter Model - for instance [7] or more generally [12] - in this paper we will estimate the fixation time (to be defined) of the iPod process. Interestingly, again similar to the Voter Model our proof will explore a connection between this process and the Wright-Fisher diffusion [7].

A special feature of the model (Proposition 2.3) is that the average (over agents) preference for a given song evolves as a martingale, analogous to the total proportion of agents with a given opinion on the voter model. This distinguishes the iPod model from many other variants of the voter model that have been studied [4].

Similar models, but with *unidirectional* updating of opinions, have been studied in the context of language evolution [11]. Our bounds (on the analogous consensus time as a function of the spectral gap) in our setting are sharper by a factor of $\ln(N)$, but we are unsure whether our methods would apply in their setting.

1.2 Fixation Time

We will be focused on estimating the fixation time T_{fix} of the iPod process. Every time two agents meet at least one distinct song is played between them and so at least one of the σ songs is played infinitely often. Given that only one song is played infinitely often, we define T_{fix} to be the last time any other song is played.

We note that T_{fix} is not a stopping time and a priori could be infinite, i.e. if more than one song is played infinitely often. However, we will show that this is not the case and in fact T_{fix} has finite expectation, the bounding of which will be our primary goal.

Theorem 1.1. There exists a constant $C(\eta)$ so that from any initial configuration of σ songs, the fixation time T_{fix} has expectation

$$\mathbb{E}\, T_{\text{fix}} \leq C(\eta)\frac{N}{\lambda},$$

where λ is the spectral gap of $\mathfrak{G}$.

The spectral gap λ of reversible Markov chain is interpreted as its asymptotic rate of convergence to its stationary distribution, and can be defined by the second eigenvalue of the chain's transition matrix [10]. In our setting, we define the spectral gap λ in terms of the edge weights $\nu_{i,j}$. First, for any function $f\colon I \to \mathbb{R}$ we define the Dirichlet form $\varepsilon(f, f)$ by

$$\varepsilon(f, f) = \sum_{i,j} \frac{\nu_{i,j}}{2N} \left(f(i) - f(j)\right)^2.$$

The spectral gap λ is then defined in our context by the extremal characterization

$$\lambda = \inf_{f\colon I \to \mathbb{R} \mid \mathrm{Var}(f) \neq 0} \frac{\varepsilon(f,f)}{\mathrm{Var}(f)}.$$

There is extensive literature [10] giving order of magnitude bounds on the $N \to \infty$ asymptotic behaviour of λ_N for particular families of N-vertex graphs. For such families, Theorem 1.1 gives an order of magnitude upper bound on the asymptotic fixation time, for fixed η and σ. Similar results are known relating λ to the time of 'voting completion' in the classical Voter model [6]. We will show (Theorem 7.1) the tightness of this bound in the case of a particular special family of graphs.

2 Projection on a Single Song

We begin by focusing on the projection of our system to a single song. For some fixed (arbitrary) song k we will consider only $X^k(i)$ which we will write simply as $x(i)$ dropping the k. When two agents i, j meet, each independently chooses to either play song k or not; with probability $x(i)$ and $x(j)$ respectively. Writing $\mathrm{Ber}(x(i))$ and $\mathrm{Ber}(x(j))$ for independent Bernoulli variables with given success parameters, we see that if i and j meet at time t then

$$x_t(i) = (1-\eta)x_{t-}(i) + \eta\,\mathrm{Ber}(x_{t-}(j)),$$

with $x(j)$ updating similarly. At such a meeting, for all other agents $k \neq i, j$, $x(k)$ remains unchanged.

This implies that the evolution of any given song can be considered separately from the others - though not independently. We will therefore focus first on the FMIE system $\{x_t(i)\}_{i \in I, t \geq 0}$ evolving as above and then later return to the original multi-song model. The primary object of study in our one song model will be the average preference for the song, written

$$M_t = \sum_{i \in I} \frac{x_t(i)}{N}.$$

Our goal in this section will be to estimate M_t^2 using martingales. We will use the shorthand $x_t = \{x_t(i)\colon 1 \leq i \leq N\}$ for the configuration at time t. In particular, we will often use x_0 for an arbitrary initial configuration. By comparison, we will use X_t (respectively X_0) for a configuration of the multi-song model.

We will begin by analysing a few quantities derived from x_t.

2.1 Derived Quantities

For ease of notation we will occasionally drop t. Our primary object of study will be the (L^1) average of the preferences $x(i)$, denoted M_t which is introduced above. We will repeatedly make use of the following lem on the step sizes of M_t.

Lemma 2.1. If t is a meeting time then

$$|M_t - M_{t-}| \leq \frac{2\eta}{N}.$$

Proof. If agent i is involved in a meeting at t, then either

$$x_t(i) = (1-\eta)x_{t-}(i) \text{ or } x_t(i) = (1-\eta)x_{t-}(i) + \eta,$$

and so

$$|x_t(i) - x_{t-}(i)| \leq \eta.$$

As only two agents are involved in any meeting, our bound follows easily. □

As a warm-up for the more complicated quantities to appear later, we begin by showing that M_t evolves as a continuous time martingale. We here implicitly use the filtration $\mathfrak{F}_t$ generated by $\{x_t(i)\}_{i\in I, t\geq 0}$. Also, note that we may clearly assume that almost surely meeting times between agents are unique and that the set of meeting times has no accumulation point.

We will make use of the process dynamics notation

$$\mathbb{E}\left(dA_t|\mathfrak{F}_{t-}\right) = (\text{resp. } \geq, \leq) B_t dt$$

to mean that

$$A_t - A_0 - \int_0^t B_r dr$$

is a martingale (respectively submartingale, supermartingale). Clearly this notation is compatible with arithmetic operations. To calculate a process's dynamics, we make repeated use of the following lem, the proof of which is straightforward.

Lemma 2.2. Let A_t be a function of the $x_t(i)$. Then

$$\mathbb{E}\left(dA_t|\mathfrak{F}_{t-}\right) = \sum_{i,j} \nu_{i,j}\, \mathbb{E}\left(A_t - A_{t-}|i \text{ and } j \text{ meet at } t\right) dt$$

In particular, for the average preference M_t we have the following dynamics.

Proposition 2.3. With respect to the filtration $\mathfrak{F}_t$, M_t is a continuous time martingale.

Proof. To begin we note that since $\mathbb{E}\,\mathrm{Ber}(x_t(j)) = x_t(j)$ we have that

$$\mathbb{E}\left(x_t(i)|i \text{ and } j \text{ meet at time } t, \mathfrak{F}_{t-}\right) = (1-\eta)x_{t-}(i) + \eta x_{t-}(j),$$

and similarly for $x_t(j)$. Summing both we find that

$$\mathbb{E}\left(x_t(i) + x_t(j)|i \text{ and } j \text{ meet at time } t, \mathfrak{F}_{t-}\right) = x_{t-}(i) + x_{t-}(j).$$

As only $x(i)$ and $x(j)$ change at such a time t, this gives us that

$$\mathbb{E}\left(M_t|i \text{ and } j \text{ meet at time } t, \mathfrak{F}_{t-}\right) = M_{t-},$$

which clearly implies that

$$\mathbb{E}\left(dM_t|\mathfrak{F}_{t-}\right) = 0,$$

i.e. M_t is a martingale. □

We next look at the process dynamics of M_t^2. To do so we introduce the quantity Q_t given by

$$Q_t = \sum_{i\in I} \frac{x_t(i)(1-x_t(i))}{N}.$$

In particular we use Lemma 2.2 to calculate the following.

Proposition 2.4. The variation M_t^2 satisfies

$$\mathbb{E}\left(dM_t^2|\mathfrak{F}_{t-}\right) = \frac{2\eta^2}{N} Q_t dt.$$

Proof. As before, we begin by calculating that for $k \neq i, j$, since $x(k)$ does not change after a meeting between i and j that:

$$\mathbb{E}\left(x_t(k)(x_t(i)+x_t(j))|i \text{ and } j \text{ meet at time } t, \mathfrak{F}_{t-}\right) = x_{t-}(k)(x_{t-}(i)+x_{t-}(j)).$$

Next we calculate that

$$\begin{aligned}&\mathbb{E}(x_t^2(i)|i \text{ and } j \text{ meet at } t, \mathfrak{F}_{t-})\\ &\qquad = (1-\eta)^2 x_{t-}^2(i) + 2\eta(1-\eta)x_{t-}(i)x_{t-}(j) + \eta^2 x_{t-}(j),\end{aligned}$$

and similarly for $x^2(j)$. Finally we have that

$$\begin{aligned}&\mathbb{E}(x_t(i)x_t(j)|i \text{ and } j \text{ meet at } t, \mathfrak{F}_{t-})\\ &\qquad = (1-\eta)^2 x_{t-}(i)x_{t-}(j) + \eta(1-\eta)[x_{t-}^2(i)+x_{t-}^2(j)] + \eta^2 x_{t-}(i)x_{t-}(j).\end{aligned}$$

Putting this all together we find that

$$\begin{aligned}&\mathbb{E}((\sum_i x_t(i))^2|i \text{ and } j \text{ meet at } t, \mathfrak{F}_{t-})\\ &\qquad = (\sum_i x_{t-}(i))^2 + \eta^2(x_{t-}(i) - x_{t-}^2(i) + x_{t-}(j) - x_{t-}^2(j)).\end{aligned}$$

Using Lemma 2.2, summing over i, j and normalizing by N^2 we find that

$$\mathbb{E}\left(dM_t^2|\mathfrak{F}_{t-}\right) = \frac{2\eta^2}{N}Q_t dt.$$

□

Instead of M^2, we will often be more concerned with $M_t(1-M_t)$. As M_t is a martingale, from Proposition 2.4 we easily have that

$$\mathbb{E}\left(dM_t(1-M_t)|\mathfrak{F}_{t-}\right) = -\frac{2\eta^2}{N}Q_t dt.$$

A central tool for the study of the underlying Markov Chain on $\mathfrak{G}$ is the Dirichlet form ε. We recall that the Dirchilet form $\varepsilon(f,f)$ for a function $f\colon I \to \mathbb{R}$ is defined as

$$\varepsilon(f,f) = \sum_{i,j} \frac{\nu_{ij}}{2N}(f(i)-f(j))^2.$$

We will write $\varepsilon(x_t, x_t)$ for the Dirichlet form of the function $i \mapsto x_t(i)$.

The main fact that we will need about the Dirichlet form is its relationship to the spectral gap. We recall the definition of the spectral gap of a Markov Chain is given by

$$\lambda = \inf_{f\colon I\to\mathbb{R} \,|\, \mathrm{Var}(f)\neq 0} \frac{\varepsilon(f,f)}{\mathrm{Var}(f)},$$

where $\mathrm{Var}(f)$ is the variance of the function $f(i)$ with respect to the uniform measure on I. A simple but important fact we make repeated use of is that $0 < \lambda \leq 1$.

Following Lemma 2.2 we can calculate dQ.

Proposition 2.5. The sum Q_t satisfies

$$\mathbb{E}\left(dQ_t|\mathfrak{F}_{t-}\right) = 4\eta(1-\eta)\varepsilon(x_t,x_t)dt - 2\eta^2 Q_t dt,$$

as well as

$$\mathbb{E}\left(dQ_t|\mathfrak{F}_t\right) \geq 4\lambda\eta(1-\eta)M_t(1-M_t)dt - \left(2\eta^2 + 4\lambda\eta(1-\eta)\right)Q_t dt.$$

Proof. We begin by noting that $Q_t = M_t - \sum_i \frac{x_t^2(i)}{N}$ and so

$$\mathbb{E}\left(dQ_t|\mathfrak{F}_{t-}\right) = \mathbb{E}\left(d\left(\sum_i \frac{x_t^2(i)}{N}\right)|\mathfrak{F}_{t-}.\right)$$

We have from Proposition 2.4 that

$$\begin{aligned}&\mathbb{E}(x_t^2(i)|i \text{ and } j \text{ meet at } t, \mathfrak{F}_{t-})\\ &\qquad = (1-\eta)^2 x_{t-}^2(i) + 2\eta(1-\eta)x_{t-}(i)x_{t-}(j) + \eta^2 x_{t-}(j).\end{aligned}$$

When agents i and j meet, only $x(i)$ and $x(j)$ change and so

$$\begin{aligned}&\mathbb{E}(Q_t - Q_{t-}|i \text{ and } j \text{ meet at } t, \mathfrak{F}_{t-})\\ &\qquad = -\mathbb{E}\left(\frac{x_t^2(i) - x_{t-}^2(i)}{N} + \frac{x_t^2(j) - x_{t-}^2(j)}{N}|i \text{ and } j \text{ meet at } t, \mathfrak{F}_{t-}\right)\\ &\qquad = (2\eta - \eta^2)\frac{x_{t-}^2(i) + x_{t-}^2(j)}{N} - 4\eta(1-\eta)\frac{x_{t-}(i)x_{t-}(j)}{N}\\ &\qquad - \eta^2\frac{x_{t-}(j) + x_{t-}(i)}{N}\\ &\qquad = \frac{4\eta(1-\eta)}{2N}(x_{t-}(i) - x_{t-}(j))^2\\ &\qquad - \frac{\eta^2}{N}(x_{t-}(i)(1 - x_{t-}(i)) + x_{t-}(j)(1 - x_{t-}(j))).\end{aligned}$$

Summing over i and j our first equation for dQ_t is done. The second is an immediate consequence of the first using the identity

$$\varepsilon(x, x)_t \geq \lambda \operatorname{Var}(x)_t = \lambda(M_t(1 - M_t) - Q_t).$$

□

2.2 Within an small Neighborhood

Next we focus our attention on M_t stuck within the neighborhood $(M_0 - \epsilon, M_0 + \epsilon)$ for some small (unspecified for now) ϵ. Let τ be the escape time of the interval, i.e.

$$\tau = \inf\{t \geq 0 \colon M_t \notin (M_0 - \epsilon, M_0 + \epsilon)\},$$

and ς any stopping time with $\varsigma \leq \tau$ almost surely.

For ease of notation in this section we will often write $\mathbb{E}$ for $\mathbb{E}_{x_0}$ - that is the expectation starting from some initial condition x_0, perhaps with some (to be specified) condition on M_0.

Our main goal now is to give a lower bound on the quadratic variation of M_t until time ς.

2.3 A Lower Bound

First we look for a bound on the heterozygosity $M_t(1 - M_t)$. We will make repeated use of the following calculus exercise.

Lemma 2.6. For a fixed x_0, if

$$\epsilon \leq \frac{x_0(1 - x_0)}{2}$$

and $x_0 - \epsilon \leq x \leq x_0 + \epsilon$ then

$$x(1 - x) \geq \frac{1}{2}x_0(1 - x_0).$$

Using our process dynamics calculations we may now begin to bound ς.

Lemma 2.7. There exist positive constants $C(\eta), D(\eta)$ so that

$$\mathbb{E}\int_0^{\varsigma} Q_r dr \geq C(\eta)\lambda M_0(1-M_0)\,\mathbb{E}\,\varsigma - D(\eta)(\mathbb{E}\,Q_{\varsigma} - Q_0).$$

Proof. First we recall that from Proposition 2.5 we have a submartingale

$$Y_t = Q_t - Q_0 - 4\lambda\eta(1-\eta)\int_0^t M_r(1-M_r)dr + (2\eta^2 + 4\lambda\eta(1-\eta))\int_0^t Q_r dr.$$

The Optional Stopping Theorem shows $\mathbb{E}\,Y_{\varsigma} \geq \mathbb{E}\,Y_0 = 0$, so

$$\begin{aligned}
\mathbb{E}\,Q_{\varsigma} - Q_0 &+ \left(2\eta^2 + 4\lambda\eta(1-\eta)\right)\int_0^{\varsigma} Q_r dr \\
&\geq 4\lambda\eta(1-\eta)\,\mathbb{E}\int_0^{\varsigma} M_r(1-M_r)dr \\
&\geq 4\lambda\eta(1-\eta)\,\mathbb{E}\int_0^{\varsigma} \frac{1}{2}M_0(1-M_0)dr \text{ by Lemma 2.6} \\
&\geq 2\lambda\eta(1-\eta)M_0(1-M_0)\,\mathbb{E}\,\varsigma.
\end{aligned}$$

Next, we note that since $\lambda \leq 1$

$$2\eta^2 + 4\lambda\eta(1-\eta) \leq 4\eta - 2\eta^2.$$

Substituting this in and rearranging the inequality

$$(4\eta - 2\eta^2)\,\mathbb{E}\int_0^{\varsigma} Q_r dr \geq 2\lambda\eta(1-\eta)M_0(1-M_0)\,\mathbb{E}\,\varsigma - \mathbb{E}\,Q_{\tau} + Q_0$$

and so

$$\mathbb{E}\int_0^{\tau} Q_r dr \geq C(\eta)\lambda M_0(1-M_0)\,\mathbb{E}\,\tau - D(\eta)(\mathbb{E}\,Q_{\tau} - Q_0)$$

for $C(\eta) = \frac{2\eta(1-\eta)}{4\eta-2\eta^2}$ and $D(\eta) = \frac{1}{4\eta-2\eta^2}$. □

Using this we are ready for our lower bound.

Lemma 2.8. There exist positive constants $A(\eta), B(\eta)$ with

$$\mathbb{E}\,M_{\varsigma}^2 - M_0^2 \geq \frac{1}{N}\left(A(\eta)\lambda M_0(1-M_0)\,\mathbb{E}\,\varsigma - B(\eta)\left(\mathbb{E}\,Q_{\varsigma} - \mathbb{E}\,Q_0\right)\right).$$

Proof. Proposition 2.4 shows

$$M_t^2 - M_0^2 - \frac{2\eta^2}{N}\int_0^t Q_r dr$$

is a martingale. The Optional Stopping Theorem and Lemma 2.7 show

$$\mathbb{E}\,M_{\varsigma}^2 - M_0^2 \geq \frac{2\eta^2}{N}\left(C(\eta)\lambda M_0(1-M_0)\,\mathbb{E}\,\varsigma - D(\eta)(\mathbb{E}\,Q_{\varsigma} - Q_0)\right),$$

which finishes our proof. □

3 Escaping an small Neighborhood

Write M_t^k for the family of martingales given by the average preferences for songs $1 \leq k \leq \sigma$ and $\mathbf{M}_t$ for

$$\mathbf{M}_t = (M_t^1, \ldots, M_t^\sigma)$$

which lives in the simplex

$$\mathbb{S}_\sigma = \{(x_1, \ldots, x_\sigma) \colon \sum_{i=1}^{\sigma} x_i = 1 \text{ and } x_i \geq 0 \text{ for all } i\}.$$

Define $\phi \colon \mathbb{R}^\sigma \to \mathbb{R}$ by

$$\phi(x_1, \ldots, x_\sigma) = \sum_{i=1}^{\sigma} x_i(1 - x_i),$$

and $\phi_* \colon \mathbb{R}^\sigma \to \mathbb{R}$ by

$$\phi_*(x_1, \ldots, x_\sigma) = \min_{1 \leq i \leq \sigma} x_i(1 - x_i).$$

We now consider the escape time of the martingale $\mathbf{M}_t$ from an ϵ-ball around $\mathbf{M}_0$. Define

$$\tau = \inf\{t \geq 0 \colon |\mathbf{M}_t - \mathbf{M}_0| \geq \epsilon\}.$$

In particular, we will consider ϵ sufficiently small, that is, ϵ satisfying

$$\frac{\eta}{4N} \leq \epsilon \leq \frac{\phi_*(\mathbf{M}_0)}{2}. \tag{3.1}$$

The importance of the lower bound will be clear later.

For simplicity of notation, write

$$\mathbf{Q}_t = \sum_{i=1}^{\sigma} Q_t^i,$$

where Q_t^i is the quantity as before for song i. Our main result in this section is the following.

Proposition 3.1. There exists a constant $A(\eta)$ such that, for $\phi_*(\mathbf{M}_0)$ and ϵ satisfying Equation (3.1) we have

$$\mathbb{E}\,\tau \leq A(\eta) \frac{N\epsilon^2}{\lambda\phi(\mathbf{M}_0)} + \frac{B(\eta)}{\lambda}\,\mathbb{E}(\mathbf{Q}_\tau - \mathbf{Q}_0).$$

We will prove Proposition 3.1 by considering the quadratic variation of the martingale $\mathbf{M}_t$, that is $\langle \mathbf{M} \rangle_t$ given by

$$\langle \mathbf{M} \rangle_t = \sum_{i=1}^{\sigma} \left(M_t^i - M_0^i\right)^2.$$

We will need the following easy lem on the step sizes of $\mathbf{M}_t$, which can be proven similarly to Lemma 2.1.

Lemma 3.2. For any time t we have

$$|\mathbf{M}_t - \mathbf{M}_{t-}| \leq \frac{3\eta}{N}.$$

We now give our proof of Proposition 3.1.

Proof. First, for each song i, by Lemma 2.8 we have

$$\mathbb{E}\left(M_\tau^i - M_0^i\right)^2 \geq \frac{A(\eta)\lambda}{N} M_0^i(1-M_0^i)\,\mathbb{E}\,\tau - \frac{B(\eta)}{N}\left(\mathbb{E}\,Q_\tau^i - \mathbb{E}\,Q_0^i\right),$$

where Q_t^i is the corresponding term for song i.

Combining these, we therefore have that

$$\mathbb{E}\langle \mathbf{M}\rangle_\tau \geq \frac{A(\eta)\lambda}{N}\phi(\mathbf{M}_0)\,\mathbb{E}\,\tau - \frac{B(\eta)}{N}\sum_i \left(\mathbb{E}\,Q_\tau^i - \mathbb{E}\,Q_0^i\right).$$

For the upper bound of $\mathbb{E}\langle \mathbf{M}\rangle_t$, by Lemma 3.2 and the definition of τ we have that

$$\begin{aligned}\langle \mathbf{M}\rangle_\tau &\leq \left(\epsilon + \frac{3\eta}{N}\right)^2 \\ &\leq 13^2\epsilon^2\end{aligned}$$

using assumption that $\epsilon \geq \frac{\eta}{4N}$.

Combining these two facts

$$13^2\epsilon^2 \geq \frac{A(\eta)\lambda}{N}\phi(\mathbf{M}_0)\,\mathbb{E}\,\tau - \frac{B(\eta)}{N}\,\mathbb{E}(\mathbf{Q}_\tau - \mathbf{Q}_0)$$

from which our claim follows easily

□

4 Approaching the Boundary

Our goal in this section is to apply Proposition 3.1 to give an upper bound on the first time S that $\mathbf{M}_t$ approaches an extreme point of $\mathbb{S}_\sigma$. From any initial configuration, define the stopping time S by

$$S = \inf\{t \geq 0\colon M_t^i \geq 1 - \frac{\eta}{2N} \text{ for some song } i.\}.$$

Of course at S, all other songs $j \neq i$ must have

$$M_S^j \leq \frac{\eta}{2N}.$$

The main result on S is the following.

Proposition 4.1. There exists $A(\eta)$ so that from any initial configuration X_0

$$\mathbb{E}_{X_0} S \leq A(\eta)\frac{N}{\lambda}\phi(\mathbf{M}_0).$$

We begin by approximating S by a sequence of stopping times, then recall some basic facts about the Wright-Fisher diffusion, and finally use a coupling argument to estimate the stopping times.

4.1 The Sequence τ_k

We will consider the series of stopping times τ_k for the martingale $\mathbf{M}_t$ defined inductively as follows. Let $\tau_0 = 0$ and for $k \geq 1$ define τ_k

$$\tau_k = \inf\left\{t \geq \tau_{k-1}\colon |\mathbf{M}_t - \mathbf{M}_{\tau_{k-1}}| \geq \frac{\phi_*(\mathbf{M}_{\tau_{k-1}})}{2}\right\},$$

that is the first time after τ_{k-1} that $\mathbf{M}_t$ exits the ball of radius $\frac{\phi_*(\mathbf{M}_{\tau_{k-1}})}{2}$ around $\mathbf{M}_{\tau_{k-1}}$.

Using Proposition 3.1, we have the following bound on the expectation of the increments of our stopping times.

Lemma 4.2. There exists a constant $A(\eta)$ so that, from any initial X_0, if $K \geq k$ then for all songs i:

$$\mathbb{E}\left(\tau_k - \tau_{k-1} | F_{\tau_{k-1}}\right) \leq A(\eta) \frac{N}{\lambda} \frac{\phi_*^2(\mathbf{M}_{\tau_{k-1}})}{\phi(\mathbf{M}_{\tau_{k-1}})} + \frac{B(\eta)}{\lambda} \mathbb{E}(\mathbf{Q}_{\tau_k} - \mathbf{Q}_{\tau_{k-1}}).$$

Proof. This is an immediate application of Proposition 3.1 and the Strong Markov property at time τ_{k-1}. □

We will see that the first term in Lemma 4.2 matches the equivalent estimate for a certain Brownian diffusion.

4.2 The Wright-Fisher Diffusion

We will now make use of some basic facts about the neutral σ-allele Wright-Fisher diffusion

$$\mathbf{W}_t = (W_t^1, \ldots, W_t^\sigma),$$

taking values in the simplex $\mathbb{S}_\sigma$. An excellent introduction to the WF diffusion and its place in genetics can be found in [9]. In the classical Voter model, the Wright-Fisher diffusion appears as a limit of the voter density process [5]. Here we take a slightly different approach and embed our finite process $\mathbf{M}_t$ directly into the diffusion.

To begin our comparison between the iPod model and the WF diffusion, we first need a bound on the escape time of the WF process from a small ball.

Lemma 4.3. From any initial $\mathbf{W}_0 = w_0$, for $\epsilon < \frac{\phi_*(w_0)}{2}$ the first escape τ of $\mathbf{W}_t$ from the ϵ-ball about w_0 satisfies

$$\mathbb{E}_{w_0} \tau \geq \frac{\epsilon^2}{2\phi(w_0)}.$$

Proof. This follows immediately from a standard calculation of the quadratic variation of $\mathbf{W}_t$ and the fact that for $\epsilon < \frac{\phi_*(w_0)}{2}$, if x is in ϵ-ball around w_0 we have

$$\phi(x) \geq \frac{\phi(w_0)}{2}$$

by Lemma 2.6. □

We will also need the classical bound for the absorption time of the σ-allele Wright-Fisher diffusion, that is

$$T_{\text{abs}} = \inf\{t \geq 0 \colon W_t^i = 1 \text{ for some } i\}.$$

Lemma 4.4. (Theorem 8.2 [9]) Starting from $\mathbf{W}_0 = x$, we have

$$\mathbb{E}_x T_{\text{abs}} = -2 \sum_{i=1}^{\sigma} (1 - x_i) \ln(1 - x_i)$$

An immediate corollary to Lemma 4.4, by an application of Jensen's inequality, is the fact that $\mathbb{E}_x T_{\text{abs}}$ is uniformly bounded above, for all σ and x, by 1.

4.3 The Comparison Calculation

Let $\mathbf{W}_t$ be a σ-allele Wright-Fisher diffusion started at $\mathbf{W}_0 = \mathbf{M}_0$. The discrete martingale $\{\mathbf{M}_{\tau_k}\}_{k\geq 0}$ is clearly square integrable and so [8] we can find a sequence of stopping times $\tilde{\tau}_k$ for $\mathbf{W}_t$ so that

$$\{\mathbf{M}_{\tau_k}\}_{k\geq 0} =^d \{\mathbf{W}_{\tilde{\tau}_k}\}_{k\geq 0}. \tag{4.1}$$

We will focus on the first time that $\mathbf{M}_t$ approaches one of the extreme points of $\mathbb{S}_\sigma$. Recall the stopping time S defined by

$$S = \inf\{t \geq 0 \colon M_t^i \geq 1 - \frac{\eta}{2N} \text{ for some } i\}, \tag{4.2}$$

and let

$$K = \inf\{k \geq 0 \colon \tau_k \geq S\}.$$

Martingale arguments give us the following.

Lemma 4.5. $K < \infty$ almost surely.

Proof. This follows immediately from the fact that before τ_K we have

$$\frac{\phi_*}{2}(\mathbf{M}_t) \geq \frac{\eta}{8N}$$

and so the discrete time martingale $\mathbf{M}_0, \mathbf{M}_{\tau_1}, \mathbf{M}_{\tau_2}, \ldots$ has step sizes

$$|\mathbf{M}_{\tau_k} - \mathbf{M}_{\tau_{k-1}}| \geq \frac{\eta}{8N}.$$

Thus on the bounded region $\mathbb{S}_\sigma$, K must occur after finitely many steps almost surely. □

Next, let $\tilde{K}$ be the equivalent index for $\mathbf{W}_t$, that is

$$\tilde{K} = \inf\{k \geq 0 \colon \mathbf{W}^i_{\tilde{\tau}_k} \geq 1 - \frac{\eta}{2N} \text{ for some } i\}.$$

For $\mathbf{M}_t$ we have clearly that

$$S \leq \tau_K. \tag{4.3}$$

Furthermore, if $\mathbf{M}_0$ is in the interior of $\mathbb{S}_\sigma$ then so is $\mathbf{M}_t$ for all $t \geq 0$ as a non-zero preference $X^k(i)$ for some song k by an agent i decreases geometrically and so is never actually zero. Thus, $\mathbf{M}_{\tau_k}$ is also in the interior of $\mathbb{S}_\sigma$ and therefore so must be $\mathbf{W}_{\tilde{\tau}_{\tilde{K}}}$ by their equivalence in distribution. Therefore,

$$\tilde{\tau}_{\tilde{K}} \leq T_{\text{abs}},$$

as for $t \geq T_{\text{abs}}$, $\mathbf{W}_t$ has already absorbed and is constant.

Lemma 4.6. The hitting times $\tilde{\tau}_k, k \geq 1$ satisfies

$$\mathbb{E}\left(\tilde{\tau}_k - \tilde{\tau}_{k-1} | F_{\tilde{\tau}_{k-1}}\right) \geq \frac{1}{8}\phi(\mathbf{W}_{\tau_{k-1}}).$$

Proof. Note that starting at $w_0 = \mathbf{W}_{\tilde{\tau}_{k-1}}$, the time $\tilde{\tau}_k$ can only occur after $\mathbf{W}_t$ leaves the ball of radius $\frac{1}{2}\phi_*(w_o)$ around w_0 as $\mathbf{W}_{\tilde{\tau}_k}$ is already outside this interval and W_t is continuous. Write τ for the first exit time of this interval. Applying the Strong Markov Property we see that

$$\begin{aligned}\mathbb{E}\left(\tilde{\tau}_k - \tilde{\tau}_{k-1} | F_{\tilde{\tau}_{k-1}}\right) &\geq \mathbb{E}_{w_0}\left(\tau | F_{\tilde{\tau}_{k-1}}\right) \\ &\geq \frac{1}{8}\frac{\phi_*^2(\mathbf{W}_{\tau_{k-1}})}{\phi(\mathbf{W}_{\tau_{k-1}})} \text{ by Lemma 4.3,}\end{aligned}$$

completing our proof. □

We are now ready to prove Proposition 4.1.

Proof. We recall by Equation (4.3), $\mathbb{E}\, S \le \mathbb{E}\, \tau_K$ and so we will focus on bounding $\mathbb{E}\, \tau_K$. As $K < \infty$ almost surely by Lemma 4.5 we have that

$$\mathbb{E}\, \tau_K = \mathbb{E}\left(\sum_{k=1}^{\infty} (\tau_k - \tau_{k-1}) 1_{K \ge k} \right).$$

By Lemma 4.2

$$\mathbb{E}\left(\tau_k - \tau_{k-1} | F_{\tau_{k-1}}\right) \le A(\eta) \frac{N}{\lambda} \frac{\phi_*^2(\mathbf{M}_{\tau_{k-1}})}{\phi(\mathbf{M}_{\tau_{k-1}})} + \frac{B(\eta)}{\lambda}\, \mathbb{E}(\mathbf{Q}_{\tau_k} - \mathbf{Q}_{\tau_{k-1}}),$$

for some constants $A(\eta), B(\eta)$ depending only on η. Therefore we can calculate using the Strong Markov property that

$$\begin{aligned}
\mathbb{E}\left((\tau_k - \tau_{k-1}) 1_{K \ge k}\right) &= \mathbb{E}\left(\mathbb{E}\left((\tau_k - \tau_{k-1}) 1_{K \ge k} | F_{\tau_{k-1}}\right)\right) \\
&= \mathbb{E}\left(1_{K \ge k}\, \mathbb{E}_{x_{\tau_{k-1}}}\left(\tau_k - \tau_{k-1}\right)\right) \\
&\le A(\eta) \frac{N}{\lambda}\, \mathbb{E}\left(1_{K \ge k} \frac{\phi_*^2(\mathbf{M}_{\tau_{k-1}})}{\phi(\mathbf{M}_{\tau_{k-1}})}\right) + \frac{B(\eta)}{\lambda}\, \mathbb{E}(\mathbf{Q}_{\tau_k} - \mathbf{Q}_{\tau_{k-1}}),
\end{aligned}$$

using that $1_{K \ge k} \le 1$ for the second term.

From Equation (4.1) $\{\mathbf{M}_{\tau_k}\}_{k \ge 0}$ and $\{\mathbf{W}_{\tilde{\tau}_k}\}_{k \ge 0}$ are equivalent in distribution, so

$$\mathbb{E}\left(1_{K \ge k} \frac{\phi_*^2(\mathbf{M}_{\tau_{k-1}})}{\phi(\mathbf{M}_{\tau_{k-1}})}\right) = \mathbb{E}\left(1_{\tilde{K} \ge k} \frac{\phi_*^2(\mathbf{W}_{\tau_{k-1}})}{\phi(\mathbf{W}_{\tau_{k-1}})}\right).$$

By Lemma 4.3

$$\frac{1}{8} \frac{\phi_*^2(\mathbf{W}_{\tau_{k-1}})}{\phi(\mathbf{W}_{\tau_{k-1}})} \le \mathbb{E}\left(\tilde{\tau}_k - \tilde{\tau}_{k-1} | F_{\tilde{\tau}_{k-1}}\right),$$

so we can calculate

$$\begin{aligned}
\mathbb{E}\left(1_{K \ge k} \frac{\phi_*^2(\mathbf{M}_{\tau_{k-1}})}{\phi(\mathbf{M}_{\tau_{k-1}})}\right) &\le \left(\mathbb{E}\, 1_{\tilde{K} \ge k} 8\, \mathbb{E}\left(\tilde{\tau}_k - \tilde{\tau}_{k-1} | F_{\tilde{\tau}_{k-1}}\right)\right) \\
&= 8\, \mathbb{E}\left(\mathbb{E}\left((\tilde{\tau}_k - \tilde{\tau}_{k-1}) 1_{\tilde{K} \ge k} | F_{\tilde{\tau}_{k-1}}\right)\right) \\
&= 8\, \mathbb{E}\left((\tilde{\tau}_k - \tilde{\tau}_{k-1}) 1_{\tilde{K} \ge k}\right).
\end{aligned}$$

Therefore we see that

$$\begin{aligned}
\mathbb{E}\, \tau_K &\le 8 \frac{N}{\lambda} A(\eta) \sum_{k \ge 0} \mathbb{E}\left((\tilde{\tau}_k - \tilde{\tau}_{k-1}) 1_{\tilde{K} \ge k}\right) + \frac{B(\eta)}{\lambda} \sum_{k \ge 0} \mathbb{E}(\mathbf{Q}_{\tau_k} - \mathbf{Q}_{\tau_{k-1}}) \\
&\le 8 \frac{N}{\lambda} A(\eta)\, \mathbb{E}\, \tilde{\tau}_{\tilde{K}} + \frac{B(\eta)}{\lambda} \\
&\le 8 \frac{N}{\lambda} A(\eta)\, \mathbb{E}\, T_{\text{abs}} + \frac{B(\eta)}{\lambda},
\end{aligned}$$

since the sum

$$\sum_{k \ge 0} \mathbb{E}(\mathbf{Q}_{\tau_k} - \mathbf{Q}_{\tau_{k-1}})$$

is telescoping and $\mathbf{Q}_t$ is bounded by 1.

By Equation (4.3) we have $S \le \tau_K$ and using Lemma 4.4 to bound $\mathbb{E}\, T_{\text{abs}}$ we can conclude that

$$\begin{aligned}\mathbb{E}\, S &\le 8\frac{N}{\lambda} A(\eta)\, \mathbb{E}_{\mathbf{W}_0}\, T_{\text{abs}} + \frac{B(\eta)}{\lambda}\\ &= A'(\eta)\frac{N}{\lambda}\phi(\mathbf{W}_0)\\ &= A'(\eta)\frac{N}{\lambda}\phi(\mathbf{M}_0)\end{aligned}$$

for some other constant $A'(\eta)$ since the first term dominates for $N >> 0$. Our conclusion follows. □

5 The Fixation Time

We are now ready to prove our bound on the fixation time of the general iPod model with σ songs. We recall that for each agent i, we write their preference for song k by $X_t^k(i)$.

We begin by estimating the fixation time given that the preference M_t^k for some (fixed but arbitrary) song k has approached the boundary 1. Specifically, we will consider starting from an initial configuration X_0 with

$$M_0^k \ge 1 - \frac{\eta}{2N},$$

or equivalently $S = 0$ for the stopping time S as above. Of course all other songs $j \neq k$ then have

$$M_0^j \le \frac{\eta}{2N}.$$

As long as M_t^k is near 1, the fixation time T_{fix} can only be the last time any song other than k plays. Projecting on k, this is the last time one of the Bernoulli trials for k has failed. We begin by showing that from such an initial configuration, T_{fix} has with positive probability already occurred.

Proposition 5.1. From an initial configuration X_0 with $M_0^k \ge 1 - \frac{\eta}{2N}$, we have

$$\mathbb{P}_{X_0}\left(T_{\text{fix}} = 0\right) \ge \frac{1}{2}$$

Proof. We will consider the stopping time R, the first time any song other than k plays. Before R, each $X^k(i)$ can only increase. Therefore at time R - without loss of generality, a meeting of agents i and j - if another song is played by only one of i, j then

$$\begin{aligned}X_R^k(i) + X_R^k(j) &= (1-\eta)(X_{R-}(i) + X_{R-}(j)) + \eta && (5.1)\\ &\le 2(1-\eta) + \eta && (5.2)\\ &= 2 - \eta. && (5.3)\end{aligned}$$

If both agents play a different song, then $X^k(i) + X^k(j)$ is even smaller at R.

This then implies that on $\{R < \infty\}$

$$M_R^k \le 1 - \frac{\eta}{N}.$$

Now, applying the Optional Stopping Theorem to $R \wedge t$, we find that

$$1 - \frac{\eta}{2N} \leq M_0 \tag{5.4}$$
$$= \mathbb{E}\, M^k_{R\wedge t} \tag{5.5}$$
$$= \mathbb{E}\left(M^k_R 1_{R\leq t} + M^k_t 1_{t<R}\right) \tag{5.6}$$
$$\leq (1 - \frac{\eta}{N})(1 - \mathbb{P}\,(t < R)) + 1\,\mathbb{P}\,(t < R)\,. \tag{5.7}$$

Solving for $\mathbb{P}\,(t < R)$ we find that

$$\mathbb{P}\,(t < R) \geq \frac{1}{2}.$$

As this is true for arbitrary t, we have $\mathbb{P}\,(R = \infty) \geq \frac{1}{2}$ from which our result follows. □

Next we need to consider what happens when M^k_t approaches 1, but the song k fails to play at a meeting.

Proposition 5.2. Consider the stopping time R given by

$$R = \inf\{t \geq 0\colon \text{ some song other than k plays at } t\}.$$

From any initial configuration $M^k_0 \geq 1 - \frac{\eta}{2N}$, we have

$$\mathbb{E}\,(R 1_{R<\infty}) \leq \frac{1}{8\eta}.$$

Proof. Let T_n, $1 \leq n < \infty$ be the n-th meeting time. We first define

$$\tilde{R} = \inf\{n \geq 0\colon \text{ some song other than k plays at } T_n\}.$$

We will calculate how M^k_t changes after the first meeting time, given that song k is played by both agents at the meeting time T_1.

If agents i and j meet and both play k at T_1 then

$$X^k_{T_1}(i) = (1 - \eta)X^k_0(i) + \eta$$

and similarly for $X^k(j)$. So given that i and j meet and play k

$$M^k_{T_1} = M^k_0 - \frac{\eta(X^k_0(i) + X^k_0(j))}{N} + \frac{2\eta}{N}.$$

Summing over pairs of agents we find that

$$\begin{aligned}
&\mathbb{E}\left(M^k_{T_1} |\text{ both agents play k at } T_1, \mathfrak{F}_0\right) \\
&= \sum_{i,j} \mathbb{E}\left(M^k_{T_1} |\text{ i meets j, both play k at } T_1, \mathfrak{F}_0\right) \mathbb{P}\,(\text{ i meets j at } T_1 | \mathfrak{F}_0) \\
&= \sum_{i,j} \frac{\nu_{ij}}{N}\, \mathbb{E}\left(M^k_0 - \frac{\eta(X^k_0(i) + X^k_0(j) - 2)}{N} |\text{ i \& j both play k at } T_1, \mathfrak{F}_0\right) \\
&= \sum_{i,j} \frac{\nu_{ij}}{N}(M^k_0 - \frac{\eta(X^k_0(i) + X^k_0(j) - 2)}{N}) \\
&= M^k_0 + \frac{2\eta}{N} - \sum_{i,j} \frac{\nu_{ij}}{N} \frac{\eta(X^k_0(i) + X^k_0(j))}{N} \\
&= M^k_0 + \frac{2\eta}{N} - \frac{2\eta M^k_0}{N} \\
&= (1 - \frac{2\eta}{N})M^k_0 + \frac{2\eta}{N}.
\end{aligned}$$

By the same calculation we find that

$$\mathbb{E}\left(M^k_{T_2}|\text{both agents play k at } T_2, \mathfrak{F}_{T_1}\right) = (1 - \frac{2\eta}{N})M^k_{T_1} + \frac{2\eta}{N}$$

and so

$$\begin{aligned}
&\mathbb{E}\left(M^k_{T_2}|\text{ both agents play k at } T_1 \text{ and } T_2, \mathfrak{F}_0\right)\\
&\quad = (1 - \frac{2\eta}{N})\left((1 - \frac{2\eta}{N})M^k_0 + \frac{2\eta}{N}\right) + \frac{2\eta}{N}\\
&\quad = (1 - \frac{2\eta}{N})^2 M^k_0 + 1 - (1 - \frac{2\eta}{N})^2\\
&\quad = 1 - (1 - \frac{2\eta}{N})^2(1 - M^k_0).
\end{aligned}$$

Continuing the same easy inductive calculation we find that

$$\mathbb{E}\left(M^k_{T_n}|\tilde{S} > n, \mathfrak{F}_0\right) = 1 - (1 - \frac{2\eta}{N})^n(1 - M^k_0).$$

Next, we need to know the chance of some song other than k being played at time T_n given $M^k_{T_{n-1}}$. We will need the identity

$$1 - xy \le (1 - x) + (1 - y)$$

for $x, y \le 1$ - which follows easily from $1 + (1 - x)(1 - y) \ge 1$. Using that, and that the probability of at least one of i, j not playing k is $1 - X^k(i)X^k(j)$, we have

$$\begin{aligned}
&\mathbb{P}\left(\text{ A song other than k is played at } T_n|M^k_{T_{n-1}}\right)\\
&\quad = \sum_{i,j}\frac{\nu_{ij}}{N}\,\mathbb{P}\left(\text{ Another song is played at } T_n|M^k_{T_{n-1}}, \text{i meets j at } T_n\right)\\
&\quad = \sum_{i,j}\frac{\nu_{ij}}{N}\left(1 - X^k_{T_{n-1}}(i)X^k_{T_{n-1}}(j)\right)\\
&\quad \le \sum_{i,j}\frac{\nu_{ij}}{N}\left(1 - X^k_{T_{n-1}}(i) + 1 - X^k_{T_{n-1}}(j)\right)\\
&\quad \le 2(1 - M^k_{T_{n-1}}).
\end{aligned}$$

Therefore we have that

$$\begin{aligned}
&\mathbb{P}\left(\tilde{R} = n|\mathfrak{F}_0\right)\\
&\quad = \mathbb{P}\left(\tilde{R} > n - 1, \text{Another song is played at } T_n|\mathfrak{F}_0\right)\\
&\quad \le \mathbb{P}\left(\text{Another song is played at } T_n|\tilde{R} > n - 1, \mathfrak{F}_0\right)\\
&\quad \le \mathbb{E}\left(2(1 - M^k_{T_{n-1}})|\tilde{R} > n - 1, \mathfrak{F}_0\right)\\
&\quad = 2(1 - \frac{2\eta}{N})^{n-1}(1 - M^k_0).
\end{aligned}$$

For the first inequality here we used the simple bound

$$\mathbb{P}(A \cap B) \le \mathbb{P}(A|B).$$

This allows us to calculate that

$$\begin{aligned}\mathbb{E}\left(\tilde{R}1_{\tilde{R}<\infty}|\mathfrak{F}_0\right) &= \sum_{n\geq 0} n\,\mathbb{P}\left(\tilde{R}=n|\mathfrak{F}_0\right)\\ &\leq \sum_{n\geq 0} n2(1-\frac{2\eta}{N})^{n-1}(1-M_0^k)\\ &= 2(1-M_0^k)\sum_{n\geq 0} n(1-\frac{2\eta}{N})^{n-1}\\ &\leq \frac{\eta}{2N}\frac{N^2}{4\eta^2}\\ &= \frac{N}{8\eta},\end{aligned}$$

using our assumption that $M_0^k \geq 1-\frac{\eta}{2N}$ and the Taylor series expansion

$$\sum_{n\geq 0} nx^{n-1} = \frac{1}{(1-x)^2},$$

for $|x|<1$.

Our result then follows since meetings occur independently at rate $\frac{1}{N}$ and so

$$\mathbb{E}\left(R1_{R<\infty}|\mathfrak{F}_0\right) = \frac{1}{N}\,\mathbb{E}\left(\tilde{R}1_{\tilde{R}<\infty}|\mathfrak{F}_0\right).$$

□

We are finally prepared to prove Theorem 1.1.

Proof. We will calculate here an upper bound for

$$\max_{X_0}\mathbb{E}_{X_0} T_{\text{fix}}$$

i.e. the upper bound over all initial configurations X_0.

Let S be as above, i.e. the first time that some song k has $M_t^k \geq 1-\frac{\eta}{2N}$ and let K be that song. Note that this defines K uniquely as $1-\frac{\eta}{2N}\geq\frac{1}{2}$. Let R be stopping time (as above) defined by

$$R = \inf\{t\geq S|\text{ some song other than K is played}\}.$$

We first recall from Proposition 5.1 that at time S, we have

$$\mathbb{P}_{X_S}\left(T_{\text{fix}}=0\right)\geq\frac{1}{2}.$$

Also, at time S, if T_{fix} has not yet occurred, then some song other than K will play again and so $R<\infty$.

From Proposition 4.1 we have that there exists a constant $C(\eta)$ so that from any initial configuration X_0

$$\mathbb{E}_{X_0} S \leq C(\eta)\frac{N}{\lambda}.$$

We then have for any initial X_0:

$$\begin{aligned}
\mathbb{E}_{X_0} T_{\text{fix}} &= \mathbb{E}_{X_0} \mathbb{E}\left((T_{\text{fix}} - S) + S | \mathfrak{F}_S\right) \\
&= \mathbb{E}_{X_0} S + \mathbb{E}_{X_0} \left(\mathbb{E}_{X_S} T_{\text{fix}}\right) \\
&= \mathbb{E}_{X_0} S + \mathbb{E}_{X_0} \left(\mathbb{E}_{X_S} \left(T_{\text{fix}} 1_{T_{\text{fix}}>0}\right)\right) \\
&= \mathbb{E}_{X_0} S + \mathbb{E}_{X_0} \left(\mathbb{E}_{X_S} \left((T_{\text{fix}} - R) 1_{R<\infty} + R 1_{R<\infty}\right)\right) \\
&= \mathbb{E}_{X_0} S + \mathbb{E}_{X_0} \left(\mathbb{E}_{X_S} R 1_{R<\infty}\right) + \mathbb{E}\,\mathbb{E}\left((T_{\text{fix}} - R) 1_{R<\infty} | R\right) \\
&= \mathbb{E}_{X_0} S + \frac{1}{8\eta} + \mathbb{E}\left(1_{R<\infty} \mathbb{E}_{X_R} T_{\text{fix}}\right) \\
&\leq C(\eta) \frac{N}{\lambda} + \frac{1}{8\eta} + \mathbb{E}\left(1_{R<\infty} m\right) \\
&\leq 2C(\eta) \frac{N}{\lambda} + \frac{1}{2} \max_{x_0} \mathbb{E}_{x_0} T_{\text{fix}}.
\end{aligned}$$

Here the $\frac{1}{8\eta}$ is clearly dominated by the first term for $N >> 0$. Therefore, we have that

$$\max_{X_0} \mathbb{E}_{X_0} T_{\text{fix}} \leq 2C'(\eta) \frac{N}{\lambda} + \frac{1}{2} \max_{X_0} \mathbb{E}_{X_0} T_{\text{fix}}$$

for some other constant $C'(\eta)$ and so

$$\mathbb{E}_{X_0} T_{\text{fix}} \leq 4C'(\eta) \frac{N}{\lambda}$$

from which our conclusion follows. □

6 The Interaction Parameter η

Next we consider the asymptotic of our bound with respect to η. Tracing through the steps of our proof of Proposition 4.1, we may actually prove the following improved bound.

Proposition 6.1. There exists a constant C so that from any initial configuration x_0, the first escape time S satisfies

$$\mathbb{E}_{x_0} S \leq \frac{C}{\eta^3 (1-\eta)} \frac{N}{\lambda}.$$

Then, repeating the arguments in Section 5, we may improve our bound in Theorem 1.1 on the expectation of the fixation time T_{fix}.

Theorem 6.2. There exists a constant C so that from any initial X_0 the fixation time T_{fix} satisfies

$$\mathbb{E}\, T_{\text{fix}} \leq \frac{C}{\eta^3 (1-\eta)} \frac{N}{\lambda}.$$

We conj that this can actually be improved to depend on η as $\frac{1}{\eta(1-\eta)}$.

7 The Complete Graph Case

As an example of a geometry in which more can be said than Theorem 1.1, we look at the complete graph K_N on N vertices. Specifically, we have uniform meeting rates between agents, that is $\nu_{ij} = \frac{1}{N-1}$ for all pairs of agents i, j. It is standard fact that the spectral gap $\lambda_{K_N} = 1$ and so Theorem 1.1 shows that the fixation time has

$$\mathbb{E}\, T_{\text{fix}} = O(N).$$

A simple argument will show that this order of magnitude bound is in fact tight.

7.1 A Lower Bound

Throughout this section we assume that there are at least two songs, i.e. $\sigma \geq 2$. To achieve any reasonable lower bound, we need to ignore starting conditions that are likely already at fixation by time $t = 0$. We call an initial configuration **non-trivial** if there exists at least one song k with

$$\frac{1}{2\sigma} \leq M_0^k \leq 1 - \frac{1}{2\sigma},$$

and will consider only non-trivial initial configurations. The choice of the factor of $\frac{1}{2}$ here is of course arbitrary.

Theorem 7.1. There exists a constant $C(\eta, \sigma)$ such that for K_N started from any non-trivial initial configuration, the fixation time T_{fix} has

$$\mathbb{E}\, T_{\text{fix}} \geq C(\eta, \sigma) N.$$

Proof. Recalling Proposition 4.1, first consider any one song and consider its average preference $M_t, t \geq 0$. From the proof of Proposition 2.4

$$\mathbb{E}\left(dM_t(1 - M_t)|\mathfrak{F}_{t-}\right) = -\frac{2\eta^2}{N} Q_t dt,$$

which combined with $Q \leq \frac{1}{4}$ gives that

$$M_t(1 - M_t) - M_0(1 - M_0) + \frac{\eta^2}{2N} t$$

is a sub-martingale.

By assumption, there exists at least one song k with $M_0^k(1 - M_0^k) \geq \frac{1}{4\sigma}$. Let

$$T_2 = \inf_{t \geq 0} \{M_t^k \notin \left(\frac{1}{8\sigma}, 1 - \frac{1}{8\sigma}\right)\},$$

be the first time that M_t^k leaves the interval $\left(\frac{1}{8\sigma}, 1 - \frac{1}{8\sigma}\right)$. Then applying the Optional Stopping Theorem

$$\mathbb{E}\, M_{T_2}^k (1 - M_{T_2}^k) + \frac{\eta^2}{2N} \mathbb{E}\, T_2 \geq M_0(1 - M_0) \geq \frac{1}{4\sigma}.$$

At time T_2, we have

$$M_{T_2}^k (1 - M_{T_2}^k) \leq \frac{1}{8\sigma},$$

and so we can conclude that

$$\mathbb{E}\, T_2 \geq \frac{N}{4\eta^2 \sigma}.$$

To complete the proof, we need only show that the fixation time T_{fix} is with high probability the same order of magnitude as T_2.

Consider the first meeting after time T_2, between some agents i and j. If two different songs are played at that meeting, then by definition T_{fix} must not have yet occurred. The probability that at a meeting at time t that agent i plays song k and j does not, or vis-versa, is

$$X_t^k(i)(1 - X_t^k(j)) + X_t^k(j)(1 - X_t^k(i)).$$

Therefore, on the complete graph, the probability that two different songs play at a meeting at time t is

$$\begin{aligned}\sum_{i\neq j}\binom{N}{2}^{-1}\left(X_t^k(i)(1-X_t^k(j))+X_t^k(j)(1-X_t^k(i))\right)\\ &=\sum_{i\neq j}\frac{X_t^k(i)(1-X_t^k(j))}{N(N-1)}\\ &=\frac{N}{N-1}M_t^k(1-M_t^k)-\sum_i\frac{X_t^k(i)^2}{N(N-1)}\\ &\geq M_t^k(1-M_t^k)-\frac{1}{N-1}.\end{aligned}$$

Recalling Lemma 2.1, at time T_2 we still have

$$M_{T_2}^k\in\left(\frac{1}{8\sigma}-\frac{2\eta}{N},1-\frac{1}{8\sigma}+\frac{2\eta}{N}\right)$$

and so at time T_2 we have

$$M_{T_2}^k(1-M_{T_2}^k)\geq\left(\frac{1}{8\sigma}-\frac{2\eta}{N}\right)^2$$

Thus, the probability at time T_2 that fixation has occurred is bounded by

$$\begin{aligned}\mathbb{P}_{X(T_2)}(T_{\text{fix}}\geq 0)&\geq M_{T_2}^k(1-M_{T_2}^k)-\frac{1}{N-1}\\ &\geq\left(\frac{1}{8\sigma}-\frac{2\eta}{N}\right)^2-\frac{1}{N-1}.\end{aligned}$$

Applying the Strong Markov property, we can conclude that

$$\begin{aligned}\mathbb{E}\,T_{\text{fix}}&\geq\mathbb{E}\,T_2 1(T_{\text{fix}}\geq T_2)\\ &=\mathbb{E}\,T_2\,\mathbb{E}(1(T_{\text{fix}}\geq T_2)|T_2)\\ &=\mathbb{E}\,T_2\left(\left(\frac{1}{8\sigma}-\frac{2\eta}{N}\right)^2-\frac{1}{N-1}\right)\\ &\geq\frac{N}{4\eta^2\sigma}\left(\left(\frac{1}{8\sigma}-\frac{2\eta}{N}\right)^2-\frac{1}{N-1}\right)\end{aligned}$$

finishing the proof.

□

8 Further Directions

We conclude by presenting a few possible further directions for research on the iPod model.

8.1 Improve the Fixation Time Bound

Heuristically, from any initial configuration the processes X_t^k mixes on a time scale of the order of the relaxation time λ^{-1}. Then, for any song k, when $x_t(i)\approx M_t$ we have $Q_t\approx M_t(1-M_t)$ and so

$$\mathbb{E}(dM_t(1-M_t)|F_{t-})\approx-\frac{2\eta^2}{N}M_t(1-M_t)dt.$$

Following through the same embedding and comparison arguments, we then find a fixation time of $O(N)$. Therefore we conj that for any initial configuration

$$\mathbb{E}\, T_{\text{fix}} = O(\lambda^{-1} + N) = O(\max(\lambda^{-1}, N).$$

8.2 Lower Bound and Improved Coupling

The heart of our proof of an upper bound for $\mathbb{E}\, T_{\text{fix}}$ in Theorem 1.1 is the approximate lower bound

$$\mathbb{E}\left(d\langle \mathbf{M} \rangle_t | \mathfrak{F}_{t-}\right) \gtrsim \frac{\lambda}{N} \phi(\mathbf{M}_t) dt$$

which enables the comparison to the Wright-Fisher diffusion. Jensen's inequality applied to Proposition 2.4 gives the easy upper bound

$$\mathbb{E}\left(d\langle \mathbf{M} \rangle_t | \mathfrak{F}_{t-}\right) \leq \frac{2\eta^2}{N} \phi(\mathbf{M}) dt$$

which gives a lower bound for $\mathbb{E}\, T_{\text{fix}}$ of order N, which is likely not tight. A better approximate upper bound - matching the order of magnitude of the lower bound - would allow a direct coupling of $\mathbf{M}_t$ to the σ-allele Wright-Fisher diffusion, at least in the $N \to \infty$ limit, analogous to results in [5].

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