

Random walks generated by equilibrium contact processes

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Abstract

We consider dynamic random walks where the nearest neighbour jump rates are determined by an underlying supercritical contact process in equilibrium. This has previously been studied by den Hollander and dos Santos (arXiv:1209.1511). We show the CLT for such a random walk, valid for all supercritical infection rates for the contact process environment.

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1 Introduction

In this note we consider a random motion $(X_t)_{t \geq 0}$ in \mathbb{Z} generated by a supercritical one dimensional contact process $(\xi_t)_{t \geq 0}$ in upper equilibrium $\bar{\nu}$. We suppose that the motion $(X_t)_{t \geq 0}$ performs nearest neighbour jumps with rate depending on the local values of ξ_t : there exist $r_0 < \infty$ and functions g_1 and g_{-1} that depend only on the spins within r_0 of the origin so that for all $t, i = \pm 1$,

$$P(X_{t+h} = X_t + i | X_s, \xi_s, s \leq t) = hg_i(\theta_{X_t} \circ \xi_t) + o(h)$$

as $h \rightarrow 0$, where $(\theta_y \circ \xi)(x) = \xi(y+x)$ for all x, y . By contrast, the evolution of process X does not affect that of the contact process ξ .

Remark: For simplicity we take X to be a nearest neighbour random walk and also ξ to be a nearest neighbour symmetric contact process. The approach and result given here extend without difficulty to random walks whose jumps are finite range (with all jump rates being appropriate shifts of cylinder functions of ξ). Equally with a bit more care the arguments can be adapted to deal with finite range contact processes.

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Using the standard notation (see Section 2), with λ standing for the infection rate and $\lambda_c \in (0, \infty)$ for the critical parameter of the standard one dimensional contact process, we may state our result as follows:

Theorem 1.1. *For all $\lambda > \lambda_c$ and any non trivial (i.e. non identically zero) g_i as above, there exist $\mu \in \mathbb{R}$ and $\alpha > 0$ so that, as $t \rightarrow +\infty$,*

$$\frac{X_t - \mu t}{\alpha \sqrt{t}} \xrightarrow{D} N(0, 1).$$

This result has already been shown for λ large in the case of $r_0 = 0$, see [6] and [7], by a nice regeneration argument. We exploit in this article the strong regeneration properties of $(\xi_t)_{t \geq 0}$ but in a different way, though we also embed an i.i.d. sequence of r.v.s in our process. To our knowledge the first central limit theorem for the contact process was due to [5] who considered the position of the rightmost occupied site for a one sided supercritical contact process. A beautiful alternative proof was produced by [8] (who wrote his approach explicitly for oriented percolation). The central limit proof of [6] is in this tradition.

We suppose that the process ξ is generated by a Harris system as is usual. Details will be supplied in the next section.

The process X is generated by a Poisson process N^X of rate $M' > \|g_1\|_\infty + \|g_{-1}\|_\infty$ and associated i.i.d. uniform $[0, 1]$ r.v.s $\{U_i\}_{i \geq 1}$: if $t \in N^X$ is the i 'th Poisson point, then a jump from X_{t-} to $X_{t-} - 1$ occurs only if $U_i \in [0, g_{-1}(\theta_{X_{t-}} \circ \xi_{t-})/M']$ and jumps to $X_{t-} + 1$ only if $U_i \in [1 - g_1(\theta_{X_{t-}} \circ \xi_{t-})/M', 1]$.

Thus, irrespective of the behaviour of $(\xi_t)_{t \geq 0}$ over a time interval I , if $N^X \cap I = \emptyset$ then X makes no jumps over time interval I . We now fix throughout the paper $M > M'$.

In the following we will use the expression dynamic random walk to denote a pair (ξ, X) which evolve according to these stipulated rules. We will say that a pair (ξ, X) is a piecewise dynamic random walk if it evolves according to the given rules on time intervals $[\beta_i, \beta_{i+1})$ with β_i increasing to infinity, but may have a global jump in the pair (ξ, X) at times β_i . This will be clear in Section 5.

2 A reminder on the contact process

The contact process with parameter $\lambda > 0$ on a connected graph $G = (V, E)$ is a continuous-time Markov process $(\xi_t)_{t \geq 0}$ with state space $\{0, 1\}^V$ and generator

$$\Omega f(\xi) = \sum_{x \in V} (f(\phi_x \xi) - f(\xi)) + \lambda \cdot \sum_{e \in E} (f(\phi_e \xi) - f(\xi)), \tag{2.1}$$

where f is any local function on $\{0, 1\}^V$ and, given $x \in V$ and $\{y, z\} \in E$, we define $\phi_x \xi, \phi_{\{y, z\}} \xi \in \{0, 1\}^V$ by

$$\phi_x \xi(w) = \begin{cases} 0 & \text{if } w = x; \\ \xi(w) & \text{otherwise;} \end{cases} \quad \phi_{\{y, z\}} \xi(w) = \begin{cases} \max(\xi(y), \xi(z)) & \text{if } w \in \{y, z\}; \\ \xi(w) & \text{otherwise.} \end{cases}$$

Given $A \subseteq V$, we write $(\xi_t^A)_{t \geq 0}$ to denote the contact process started from the initial configuration that is equal to 1 at vertices of A and 0 at other vertices. When we write (ξ_t) , with no superscript, the initial configuration will either be clear from the context or unimportant. We often abuse notation and identify configurations $\xi \in \{0, 1\}^V$ with the corresponding sets $\{x \in V : \xi(x) = 1\}$.

The contact process is a model for the spread of an infection in a population. Vertices of the graph (sometimes referred to as *sites*) represent individuals. In a configuration $\xi \in \{0, 1\}^V$, individuals in state 1 are said to be *infected*, and individuals in state 0 are

healthy. Pairs of individuals that are connected by edges in the graph are in proximity to each other in the population. The generator (2.1) gives two types of transition for the dynamics. First, infected individuals *heal* with rate 1. Second, given two individuals in proximity so that one is infected and the other is not, with rate λ there occurs a *transmission*, as a consequence of which both individuals end up infected.

The configuration $\underline{0} \in \{0, 1\}^V$ that is equal to zero at all vertices is a trap for (ξ_t) . For certain choices of the underlying graph G and the parameter λ , it may be the case that the probability of the event $\{\underline{0} \text{ is never reached}\}$ is positive even if the process starts from finitely many infected sites. In fact, whether or not this probability is positive does not depend on the set of initially infected sites, as long as this set is nonempty and finite. We say that the process *survives* if this probability is positive; otherwise we say that the process *dies out*. Survival or not depends on the value of the parameter λ . As is intuitive, there is a value λ_c (depending on G) so that there is survival above λ_c and nonsurvival below. Moreover, $0 < \lambda_c < \infty$ when G is an infinite connected graph of bounded degree.

We now recall the graphical construction of the contact process and its self-duality property. Fix a graph $G = (V, E)$ and $\lambda > 0$. We take the following family of independent Poisson point processes on $(-\infty, \infty)$:

$$\begin{aligned} (D^x) : x \in V & \text{ with rate } 1; \\ (N^e) : e \in E & \text{ with rate } \lambda. \end{aligned}$$

Let H denote a realization of all these processes. Given $x, y \in V$, $s \leq t$, we say that x and y are connected by an *infection path in H* (and write $(x, s) \leftrightarrow (y, t)$ in H) if there exist times $t_0 = s < t_1 < \dots < t_k = t$ and vertices $x_0 = x, x_1, \dots, x_{k-1} = y$ such that

- $D^{x_i} \cap (t_i, t_{i+1}) = \emptyset$ for $i = 0, \dots, k-1$;
- $\{x_i, x_{i+1}\} \in E$ for $i = 0, \dots, k-2$;
- $t_i \in N^{x_{i-1}, x_i}$ for $i = 1, \dots, k-1$.

Such a collection will be called a *path* from (x, s) to (y, t) (here and elsewhere, we drop the dependence on H if a Harris system is given). Points of the processes (D^x) are called *death marks* and points of (N^e) are *links*; infection paths are thus paths that traverse links and do not touch death marks. H is called a *Harris system*; we often omit the dependence on H . For $A, B \subseteq V$, we write $A \times \{s\} \leftrightarrow B \times \{t\}$ if $(x, s) \leftrightarrow (y, t)$ for some $x \in A, y \in B$. We analogously write $A \times \{s\} \leftrightarrow (y, t)$ and $(x, s) \leftrightarrow B \times \{t\}$. Finally, given set $C \subseteq V \times (-\infty, \infty)$, we write $A \times \{s\} \leftrightarrow B \times \{t\}$ *inside C* if there is an infection path from a point in $A \times \{s\}$ to a point in $B \times \{t\}$ which is entirely contained in C .

Given $A \subseteq V$, put

$$\xi_t^A(x) = \mathbb{1}_{\{A \times \{0\} \leftrightarrow (x, t)\}} \text{ for } x \in V, t \geq 0 \tag{2.2}$$

(here and in the rest of the paper, $\mathbb{1}$ denotes the indicator function). It is well-known that the process $(\xi_t^A)_{t \geq 0} = (\xi_t^A(H))_{t \geq 0}$ thus obtained has the same distribution as that defined by the infinitesimal generator (2.1). The advantage of (2.2) is that it allows us to construct in the same probability space versions of the contact processes with all possible initial distributions. From this joint construction, we also obtain the *attractiveness* property of the contact process: if $A \subseteq B \subseteq V$, then $\xi_t^A(H) \subseteq \xi_t^B(H)$ for all t . From now on, we always assume that the contact process is constructed from a Harris system. In discussing dynamic random walks, it will be understood that the Poisson process N^X and associated uniform random variables also are part of the Harris system.

Now fix $A \subseteq V$, $t \in \mathbb{R}$ and a Harris system H . Let us define the *dual process* $(\hat{\xi}_s^{A,t})_{0 \leq s < \infty}$ by

$$\hat{\xi}_s^{A,t}(y) = \mathbb{1}_{\{(y, t-s) \leftrightarrow A \times \{t\} \text{ in } H\}}.$$

If $A = \{x\}$, we write $(\hat{\xi}_s^{x,t})$. This process satisfies two important properties. First, the distribution of $(\hat{\xi}_s^{x,t} \ s \geq 0)$ is the same as that of a contact process with same initial configuration. Second, it satisfies the *duality equation*

$$\xi_t^A \cap B \neq \emptyset \text{ if and only if } A \cap \hat{\xi}_t^{B,t} \neq \emptyset. \tag{2.3}$$

In particular,

$$\xi_t^1(x) = 1 \text{ if and only if } \hat{\xi}_t^{x,t} \neq \emptyset, \tag{2.4}$$

where (ξ_t^1) is the process started from full occupancy.

Also (for $\lambda > \lambda_c$) if we put $\xi_0(x) = 1$ if and only if $\hat{\xi}_0^{x,0}$ never dies out, then configuration ξ_0 has the upper equilibrium distribution $\bar{\nu}$.

We will talk of a contact process $\{\xi_t\}_{t \geq 0}$ restricted to $R \subseteq V \times \mathbb{R}$ to mean the contact process generated by Harris system paths that are entirely contained in R . This is interpreted to signify that $\xi_t(x) = 0$ for each $(x, t) \notin R$. We remark that if R_1 and R_2 are disjoint, then conditional upon initial configurations, two contact processes restricted respectively to R_1 and R_2 are independent. When necessary we use the notation ξ^R or $\xi^{A,R}$ to denote contact processes restricted to space time regions R , with A standing for the initial configuration.

We use the suffix t to denote contact processes run from time t .

From now on we will consider the supercritical contact process ($\lambda > \lambda_c$) on the integer lattice, $V = \mathbb{Z}$ with E the set of nearest neighbour edges.

We now recall classical results about the contact process on the line. The proposition below can be found in [2] or Theorem 3.23, chapter VI of [9].

Proposition 2.1. *There exists a constant $c_1 \in (0, \infty)$ so that for τ the stopping time equal to the first hitting time of \emptyset for the process, we have*

$$(i) \quad P^{\xi_0}(\tau < \infty) < \frac{1}{c_1} e^{-c_1 \sum_x \xi_0(x)}$$

and

$$(ii) \quad P^{\xi_0}(t < \tau < \infty) < \frac{1}{c_1} e^{-c_1 t},$$

for any configuration ξ_0 .

One important consequence of (ii) above (indeed of the slightly weaker version when ξ_0 has only one occupied site) is the fact that if instead of considering as occupied (at a given time) sites whose dual survives forever, we consider sites whose dual survives to large time t , then the resulting configuration has a distribution very close to equilibrium.

We have that a contact process $(\xi_t^x)_{t \geq 0}$ has exponentially small (in t) chance of surviving until time t but subsequently dying out. Furthermore by considering large deviations of the rightmost descendant r_t^x and leftmost descendant l_t^x (see [4] and Theorem 3.23, of [9]), we have that

Lemma 2.2. *There exists $h_1 > 0$ so that*

$$P(H(t)) < \frac{1}{h_1} e^{-h_1 t}$$

for the event $H(t)$: $\xi_t^x \neq \emptyset$ but either $|\xi_t^x \cap (x, x+t)| < h_1 t$ or $|\xi_t^x \cap (x-t, x)| < h_1 t$, where $|\cdot|$ refers to the cardinality.

Remark: The second statement in Proposition 2.1 follows at once from the first and the above lemma.

The classical renormalization argument that compares the contact process with supercritical oriented percolation, see for instance the proof of [9, Corollary VI.3.22] and classical contour arguments for oriented percolation (see e.g. [1]) give the following

Lemma 2.3. Given $\beta \in [0, 1)$, γ and η both strictly greater than β , there exists constant c_2 so that for $(\xi_t : t \geq 0)$ the contact process restricted to rectangle $[0, L] \times [0, T]$, one has: if

(i) ξ_0 has no gaps of size L^β

and (ii) the dual $\hat{\xi}_{T-L^\gamma}^{x,T}$ has cardinality at least L^η ,

then the conditional probability that $\xi_T(x) = 1$ is at least

$$1 - e^{-(c_2 L^{\eta-\beta})}$$

for L sufficiently large.

Similarly we arrive at

Proposition 2.4. Given finite positive K , there exists a constant $c_3 = c_3(K) \in (0, \infty)$ so that for l sufficiently large if ξ^R is a contact process restricted to $[0, Kl] \times [0, l]$ so that ξ_0^R has no vacant subinterval of $[0, Kl]$ of length v , then

$$P(\xi_t^R \equiv 0 \text{ on an interval } I \subseteq [0, Kl] \text{ with } |I| \geq v) \leq \frac{Kl}{c_3} (e^{-c_3 v} + e^{-c_3 l/v}).$$

In particular we have the following.

Corollary 2.5. There exists a constant $c_4 \in (0, +\infty)$ so that for all n large, if $x \in (-n^2 2^n, n^2 2^n)$ and ξ_t is a configuration with no $n^{3/2}$ vacant intervals on $[-2n^2 2^n, 2n^2 2^n]$, then outside a set of probability at most $n^9 e^{-c_4 \log^{3/2}(n)}$, the configuration ξ_{t+n^4} has no gap of size $\log^{3/2}(n)$ within n^9 of x .

Simple large deviations estimates for the rightmost particle for a one-sided initial configuration give the following:

Lemma 2.6. There exists a constant $c_5 \in (0, \infty)$ so that for a given Harris system, the chance that there is a path from $(-\infty, 0) \times (0, T)$ to $(RT, \infty) \times (0, T)$ is less than $e^{-c_5 RT}$ for all $R > \frac{1}{c_5}$, $T > 1$.

We also state the following general result, which is shown through basic techniques:

Lemma 2.7. There exists a constant $\tilde{c} > 0$ so that, if the contact process ξ is in upper equilibrium $\bar{\nu}$, then for all n large

$$P(\exists t \leq 2.2^{3n}, |x| \leq 3.2^{4n} \text{ so that } \xi_t \equiv 0 \text{ on } (x, x + n^{3/2})) \leq \frac{1}{2} e^{-\tilde{c} n^{3/2}}.$$

3 An approximate equilibrium

Consider a Harris system on $\mathbb{Z} \times (-\infty, 0)$ and define ξ' and ξ as follows:

$\xi'(x) = 1$ if and only if $(\hat{\xi}_s^x)_{s \geq 0}$ the dual restricted to some space time region R_x satisfies some condition C_x .

$\xi(x) = 1$ if and only if $\hat{\xi}^x$ survives forever. In particular ξ is in equilibrium $\bar{\nu}$.

Writing p_x for the probability that $\xi'(x) \neq \xi(x)$, it is clear that if $\sum_x p_x \leq \frac{1}{2}$, then ξ' has a law equal to the equilibrium distribution conditioned on an event of reasonable probability.

We now get to define R_x and C_x adapted to scale 2^n . We first fix h_1 the positive constant of Lemma 2.2.

A) For $|x| \leq n^9$:

$R_x = \mathbb{Z} \times (-t(n), 0)$ where $t(n) = \log^4(n)/2$ and C_x is the event that at time $t(n)$, $\hat{\xi}^x \cap [-n^9, n^9]$ has size at least $h_1 t(n)$.

B) For $|x| > n^9$:

$R_x = \mathbb{Z} \times (-\infty, 0)$ and C_x the condition of surviving forever.

To verify the condition

$$\sum_x p_x < 1/2,$$

for n large, we first note that the summands for $|x| > n^9$ are zero. Secondly we have by (i) in Proposition 2.1, and taking c_1 the positive constant in that statement:

$$\sum_{|x| \leq n^9} P(\xi'(x) = 1, \xi(x) = 0) \leq \frac{1}{c_1} \left((2n^9 + 1)e^{-c_1 h_1 \log^4(n)/2} \right)$$

which converges to zero as n tends to infinity.

For the term $\sum_x P(\xi'(x) = 0, \xi(x) = 1)$ summed over $|x| \leq n^9$, we apply Lemma 2.2 to get the required bounds.

We now alter this definition for $|x| > n^9$. The objective is to define a configuration which is essentially the same as above but which is independent of certain rectangles of the Harris system. The "cost" of losing the global closeness to equilibrium by changing the values far away is small compared to the independence gained. We replace condition B) with

B') for $|x| > n^9$, we set $\xi'(x) = 1$.

It is to be noted that with this amended definition the configuration ξ' is independent of the Harris system on $\mathbb{Z} \times (-\infty, -t(n))$

We let $\nu(= \nu(n))$ be the distribution of the configuration ξ' with rules given in A) and B') above, conditioned on the event that for all $x \in [-n^9, n^9]$, $\xi'(x) = 1$ whenever $\hat{\xi}^x$ survives till time $t(n)$.

4 A regeneration time

The purpose of this section is to describe a regeneration time $\sigma = \sigma(n, T)$ associated to a space and time scale 2^n and a stopping time T (also called n order regeneration time). We remark that stopping times in this paper will always refer to the natural filtration of our Harris system (plus some auxiliary random variables). In this sense also, the regeneration time will be a stopping time occurring after the stopping time T . The construction will be such that at time σ a random configuration ξ'_σ will be produced so that

- (i) ξ'_σ has distribution ν ($= \nu(n)$ as in Section 3) relative to X_σ ,
- (ii) with very high probability $\xi'_\sigma(x) = \xi_\sigma(x)$ for $|x - X_\sigma| \leq n^9$.

Remark: Of course given $\xi'_\sigma(x) \equiv 1$ for $|x - X_\sigma| > n^9$, we cannot have $\xi'_\sigma = \xi_\sigma$. The idea is that in subsequent evolution of a dynamic random walk (ξ', X') with $X'_\sigma = X_\sigma$ we have $X'_s = X_s$ with very high probability. See Lemma 4.3.

We will suppose n is fixed and drop it from notation for our regeneration time $\sigma = \sigma(n, T)$.

The time σ is obtained via a series of *runs*. Each run will probably be aborted before completion but if it doesn't then, as far as evolution on a scale of 2^n is concerned, the process will start from a given distribution (which will weakly depend upon n). In the following T will be a stopping time bounded by 2^{3n} . (This can be extended to times in $[0, 2^{Kn}]$ for K large but fixed.) For notational reasons we take (T, X_T) to be $(0, 0)$. We begin a run at time t (for the first run $t = 0$) by considering the joint (ξ, X) process on time interval $(t, t + n^4 + \log^4(n))$. If the run is aborted, then we try a subsequent "run" at time $t + n^4 + \log^4(n)$ and so on until a complete run is obtained.

A run consists of at most five stages. A (complete) run will produce $\sigma = t + n^4 + \log^4(n)$, either if the first stage is a failure or if all five stages succeed, in which case we say that the run is successful. The latter case will be good from the point of view of (ii) above while the first case will be bad (but mercifully of small probability). If the run is successful, then the distribution of $\xi_{t+n^4+\log^4(n)}$ shifted by $X_{t+n^4+\log^4(n)}$ will be ν at least on interval $(-n^9, n^9)$.

As we shall now see: the first stage should succeed with very large probability (for T, X_T not too large as explained below). The second stage will succeed with probability of order $e^{-M' \log^4(n)}$. The next two will succeed with high probability (for n large), and the fifth with a reasonable probability, each given the success of the previous stages.

- The first stage consists simply of seeing if there is a vacant gap of size $n^{3/2}$ for ξ_t on $(X_t - 2n^2 2^n, X_t + 2n^2 2^n)$. If so we conclude the run and designate $\sigma = t + n^4 + \log^4(n)$. We put X'_σ equal to X_σ and ξ'_σ to be a random configuration independent of the natural Harris system for (ξ, X) so that shifted by X_σ , ξ'_σ has distribution ν . Technically this gives a complete run (since it has established σ) but of course this case has severed any link between ξ and ξ' and will be treated as a "disaster". It is however easy to see that the chance this occurs for $t \in [0, 2^{3n}]$ is bounded as $e^{-\tilde{c}n^{3/2}}$ for some universal positive constant \tilde{c} if our stopping time T is less than 2^{3n} (see Lemma 2.7). Thus the contribution to the various integrals considered in later sections will be negligible. If there are no $n^{3/2}$ gaps we describe the first stage of the run as a success.

- The second stage is a success if (recalling the notation set in the introduction) we have

- 1) $N^X(t + n^4) - N^X(t) < Mn^4$ and
- 2) $N^X(t + n^4 + \log^4(n)) - N^X(t + n^4) = 0$.

We remark that the first condition is satisfied with probability tending to one as n becomes large, while the second condition has probability exactly $e^{-M' \log^4(n)}$. The first condition implies that whatever the contact process might be, X moves less than Mn^4 on the time interval $(t, t + n^4)$ and is constant on the time interval $(t + n^4, t + n^4 + \log^4(n))$. As with subsequent stages, if this is a failure we let the process run up until time $t + n^4 + \log^4(n)$ in order to regain the Markov property.

- Given the second stage is a success, we pass to the next, and require that on the interval ¹ $(X(t + n^4) - n^9, X(t + n^4) + n^9)$ there be no gaps of size $\log^{3/2}(n)$ for ξ_{t+n^4} .

We note here that as ξ_t has no $n^{3/2}$ gaps the chance of this event, given also a successful second stage, is close to one by Corollary 2.5.

- For the fourth stage we construct $\xi'_{t+n^4+\log^4(n)}$ according to the n level specifications from the given Harris system shifted spatially by $X(t + n^4 + \log^4(n))$ and temporally by $t_1 = t + n^4 + \log^4(n)$:

For $|x| \leq n^9$, $\xi'_{t+n^4+\log^4(n)}(X(t+n^4+\log^4(n))+x) = 1$ if and only if the dual $\hat{\xi}^{x+X(t+n^4+\log^4(n)), t_1}$ survives for time $t(n) = \log^4(n)/2$ satisfying condition C_x (Sec. 3) suitably displaced.

For $|x| > n^9$ we set $\xi'_{t+n^4+\log^4(n)}(X(t + n^4 + \log^4(n)) + x) = 1$.

The fourth stage is successful if

$$\xi'_{t+n^4+\log^4(n)}(x) \geq \tilde{\xi}_{t+n^4+\log^4(n)}(x) \quad \text{for all } x,$$

where $\tilde{\xi}_{t+n^4+\log^4(n)}$ is given by $\tilde{\xi}_{t+n^4+\log^4(n)}(x) = 1$ if and only if the dual $\hat{\xi}^{x, t_1}$ survives for time $t(n) = \log^4(n)/2$ (or equivalently $\tilde{\xi}$ is the contact process defined on the Harris system from time $t + n^4 + \log^4(n)/2$, starting with full occupancy). By Lemma 2.2, the probability of success at the fourth step tends to one as n tends to infinity. Indeed, it amounts to prove that for all $|x| \leq n^9$ one has $\xi'_{t+n^4+\log^4(n)}(X(t + n^4 + \log^4(n)) + x) = 1$

¹When the notation would be too clumsy, we write $X(t)$ instead of X_t .

whenever $\hat{\xi}^{x+X(t+n^4+\log^4(n)),t_1} = 1$, and the calculation is essentially given in Section 3. It is to be noted that this condition relies on a Harris system disjoint from (and independent of) the Harris systems observed in stage 2.

• Finally for the fifth stage we note that, provided that the requisite stages have been successfully passed, the conditional chance that for every x such that $|X_{t+n^4} - x| \leq n^9$ and $\xi'_{t+n^4+\log^4(n)}(x) = 1$ one has

$$\xi'_{t+n^4+\log^4(n)}(x) = \xi_{t+n^4+\log^4(n)}(x)$$

is at least $3/4$ for n large, as an easy calculation using Lemma 2.3 shows. This conditional probability will depend on the random configuration $\xi'_{t+n^4+\log^4(n)}$ as well as the configuration ξ_{t+n^4} . Let us denote it by $p(\xi'_{t+n^4+\log^4(n)}, \xi_{t+n^4})$. Having introduced an auxiliary independent uniform random variable U associated to the “run” (enlarging the probability space if necessary), we then say that the run is (globally) a success if

$$\xi'_{t+n^4+\log^4(n)}(x) = \xi_{t+n^4+\log^4(n)}(x) \quad \text{for every } x \text{ as above}$$

and $U \leq \frac{3/4}{p(\xi'_{t+n^4+\log^4(n)}, \xi_{t+n^4})}$.

Using this randomization procedure we see that, conditionally on success, the distribution of $\xi_{t+n^4+\log^4(n)}$ shifted by $X_{t+n^4+\log^4(n)}$ and restricted to the interval $[-n^9, n^9]$ coincides with ν restricted to $[-n^9, n^9]$.

Notation: For a stopping time T , we let $\sigma_T = \sigma(n, T)$ denote the end time of the first successful run after beginning the runs at time T :

$\sigma_T = \inf\{T+k(n^4+\log^4(n)) : \text{a complete run is initiated at time } T+(k-1)(n^4+\log^4(n))\}$. We say σ_T is the T regeneration. We say that a *disaster* occurs at σ_T if $\xi'_{\sigma_T}(x) \neq \xi_{\sigma_T}(x)$ for some x within n^9 of X_{σ_T} . The arguments given above imply the following.

Lemma 4.1. *There exists a positive constant c so that for any initial configuration ξ_T at time T , the probability of a complete run is at least $\frac{c}{n}e^{-M'\log^4(n)}$.*

Proposition 4.2. *There exists constant $c_6 \in (0, \infty)$ so that for n large and any stopping time, T , for the filtration (\mathcal{F}_t) determined by the (ξ, X) process, σ_T , the first time for a complete run starting at T , satisfies*

$$P(\sigma_T > T + n^8 e^{M'\log^4(n)} | \mathcal{F}_T) < e^{-c_6 n^3} \quad \text{a.s.}$$

Moreover, the probability that there is a stopping time $T \leq 2^{3n}$ such that the run completed at σ_T is not successful is bounded from above by $e^{-\tilde{c}n^{3/2}}$.

Proof. Since each run takes an interval of time of length at most $n^4 + \log^4(n) \leq 2n^4$, the proof of the first statement follows at once from Lemma 4.1 by repeated application of the Markov property. The second statement follows from the first and Lemma 2.7, together with simple estimates on the Poisson process. \square

The next result is crucial for our regeneration arguments as it implies that replacing at regeneration times our configuration ξ_σ by a regenerated ξ'_σ does not change the evolution of X , thus facilitating an i.i.d. structure for X .

Lemma 4.3. *Consider two dynamic random walks (ξ, X) and (ξ', X') run with the same Harris system and so that*

(i) $X_0 = X'_0$

(ii) $\xi_0(x) = \xi'_0(x)$ for $|x - X_0| \leq n^9$, elsewhere $\xi_0(x) \leq \xi'_0(x)$.

Then, for a suitable positive constant c_8 , outside of probability $e^{-c_8 n^2}$ we have either

a) ξ_0 has an $n^{3/2}$ gap within 2^{4n} of X_0 , or

b) $X_{n^4} = X'_{n^4}$ and $\xi_{n^4}(x) = \xi'_{n^4}(x)$ for $|x - X_{n^4}| \leq 2^{3n}$.

Proof. It is only necessary to show the inequality holding for n sufficiently large, so in the following we will take n to be large enough that a finite number of asymptotic inequalities hold. It suffices to consider initial configurations (ξ_0, X_0) and (ξ'_0, X'_0) as in (i) and (ii) such that ξ_0 has no $n^{3/2}$ gaps within 2^{4n} of X_0 , and to find two "bad" events B_1 and B_2 , with appropriately small probabilities conditioned on such ξ_0 , and such that if neither B_1 nor B_2 occur, then

$$\xi_{n^4}(x) = \xi'_{n^4}(x) \text{ for all } x \in [X_{n^4} - 2^{3n}, X_{n^4} + 2^{3n}] \tag{4.1}$$

and

$$X_{n^4} = X'_{n^4}. \tag{4.2}$$

The first event, B_1 , is simply taken as $\cup_{x \in [X_0 - 2^{4n}/2, X_0 + 2^{4n}/2]} B_1^x$, where

$$B_1^x = \left\{ \hat{\xi}_{n^4}^{x, n^4} \neq \emptyset, \hat{\xi}_{n^4}^{x, n^4} \cap \xi_0 \cap [X_0 - 2^{4n}, X_0 + 2^{4n}] = \emptyset \right\}. \tag{4.3}$$

We clearly have $B_1^x \subseteq C_1^x \cup C_2^x$, where

$$C_1^x = \left\{ \hat{\xi}_{n^4/2}^{x, n^4} \neq \emptyset, |\hat{\xi}_{n^4/2}^{x, n^4} \cap (x, x + n^4/2)| \leq h_1 n^4/2 \right\}$$

and

$$C_2^x = \left\{ |\hat{\xi}_{n^4/2}^{x, n^4} \cap (x, x + n^4/2)| > h_1 n^4/2, \hat{\xi}_{n^4}^{x, n^4} \cap \xi_0 = \emptyset \right\}.$$

The event C_1^x is independent of ξ_0 , and by Lemma 2.2, $P(C_1^x) \leq \frac{1}{h_1} e^{-h_1 n^4/2}$. We now remark that for n large $(x, x + n^4/2) \subseteq [X_0 - 2^{4n}, X_0 + 2^{4n}]$ for all x in the range of interest, and so ξ_0 will have no $n^{3/2}$ vacant intervals in this interval. Also by Lemma 2.3, $P^{\xi_0}(C_2^x) \leq e^{-c_2 h_1 n^{5/2}/2}$, uniformly over x and ξ_0 under the given condition of no gaps, where P^{ξ_0} refers to the conditional probability given ξ_0 . Therefore uniformly for all such ξ_0

$$P^{\xi_0}(B_1) \leq (2^{4n} + 1) \left(e^{-c_2 h_1 n^{5/2}/2} + \frac{1}{h_1} e^{-h_1 n^4/2} \right)$$

which is less than $\frac{1}{2} e^{-n^2}$ for n large.

As for the second bad event, its complement B_2^c is given by

$$B_2^c = \{ X'_{n^4} = X_{n^4} \in [X_0 - n^5, X_0 + n^5] \}. \tag{4.4}$$

Before estimating $P^{\xi_0}(B_2)$ uniformly over ξ_0 as above, notice that the desired properties (4.1), (4.2) hold on $B_1^c \cap B_2^c$. This is automatic for (4.2). On the other hand, (4.1) follows from (4.3) and (4.4) once we take (ii) into account.

First we have by simple tail estimates for Poisson random variables,

$$P^{\xi_0} \left(\sup_{0 \leq s \leq n^4} (|X_s - X_0| \vee |X'_s - X'_0|) > n^5 \right) \leq e^{-cn^5}$$

for some strictly positive c depending on M' but not on n . So it remains to argue that X_{n^4} and X'_{n^4} must be equal with very large probability. To do this it will suffice to show that (outside a set of very small probability),

$$\xi_s(x) = \xi'_s(x) \text{ for all } 0 \leq s \leq n^4, \text{ for all } x \in [X_0 - n^5 - r_0, X_0 + n^5 + r_0] \tag{4.5}$$

where r_0 was defined in the first paragraph. We divide up $[X_0 - n^5, X_0 + n^5]$ into disjoint intervals $\{I_i\}_{i \in J}$ of cardinality n^5 . Let the collection of indices of intervals entirely to the left of $[X_0 - n^5 - r_0, X_0 + n^5 + r_0]$ be J_a , while J_b denotes the collection of indices for intervals entirely to the right. For any $i \in J$, let D_i be the event that there is a path

from $(x, 0)$ to some (y, n^4) entirely contained in space time rectangle $I_i \times [0, n^4]$ for some x with $\xi_0(x) = \xi'_0(x) = 1$. We note first that the events D_i are conditionally independent given ξ_0 and that by our restriction on the size of gaps for ξ_0 we have that given ξ_0 each D_i has a conditional probability bounded away from zero in a way that depends on λ but not on n . From this it follows that uniformly over ξ_0 without $n^{3/2}$ gaps as above, we have

$$P^{\xi_0}(\cap_{i \in J_a} D_i^c) + P^{\xi_0}(\cap_{i \in J_b} D_i^c) \leq e^{-c'n^4}.$$

But (4.5) holds at once on the set $\cup_{i \in J_a} D_i \cap \cup_{i \in J_b} D_i$, and we are done. □

5 Existence of normalizing constants

In this section we wish to use our coupling time to establish the existence of μ and α so that as $n \rightarrow \infty$

$$E\left(\frac{X_{2^n}}{2^n}\right) \rightarrow \mu \quad \text{and} \quad \frac{1}{2^n}E(X_{2^n} - 2^n\mu)^2 \rightarrow \alpha^2.$$

Considering (ξ', X') associated to a regeneration time $\sigma = \sigma(n, T)$ for $T = 0$ as in the previous section, and $\nu = \nu(n)$ the measure defined in Section 3 we will need:

Lemma 5.1. *There exists a constant $c_{10} < \infty$ so that for all n ,*

$$\left| E\left(\frac{X_{2^n}}{2^n}\right) - E^{\nu(n)}\left(\frac{X_{2^n}}{2^n}\right) \right| < \frac{c_{10}n^8 e^{M' \log^4(n)}}{2^n}.$$

Proof. Let $\sigma = \sigma(n, 0)$ be the n order regeneration time for time 0 and X' the dynamic random walk resulting from σ . Let D denote the event that either

- (i) $\sigma > n^8 e^{M' \log^4(n)}$ or
- (ii) $X_{\sigma+2^n} - X_\sigma \neq X'_{\sigma+2^n} - X'_\sigma$.

By Lemmas 4.3 and 2.7 and Proposition 4.2 we have that $P(D) < e^{-\tilde{c}n^{3/2}} + e^{-c_8 n^2} + e^{-c_6 n^3}$. But

$$\begin{aligned} & \left| E(X_{2^n}) - E^{\nu(n)}(X_{2^n}) \right| = \left| E(X_{2^n}) - E(X'_{\sigma+2^n} - X'_\sigma) \right| \\ & \leq E(|X_{2^n} - (X_{\sigma+2^n} - X_\sigma)| \mathbb{1}_{D^c}) + E(|X_{2^n} - (X'_{\sigma+2^n} - X'_\sigma)| \mathbb{1}_D). \end{aligned}$$

The first term is bounded by $E(N^X(n^8 e^{M' \log^4(n)}) + N^X(2^n + n^8 e^{M' \log^4(n)}) - N^X(2^n))$, which is $2M'n^8 e^{M' \log^4(n)}$. For the term containing $\mathbb{1}_D$ we use the Cauchy Schwarz inequality to conclude the proof. □

Lemma 5.2. *There exists a constant $c_{11} < \infty$ so that for all n ,*

$$\left| E\left(\frac{X_{2^n}}{2^n}\right) - E\left(\frac{X_{2^{n+1}}}{2^{n+1}}\right) \right| < \frac{c_{11}n^8 e^{M' \log^4(n)}}{2^n}.$$

Proof. We begin by now taking $\tilde{\sigma} = \sigma(n, 2^n)$ to be the regeneration time after 2^n , and argue as in the proof of Lemma 5.1, where we now take the event D that either

- (i) $\tilde{\sigma} - 2^n > n^8 e^{M' \log^4(n)}$ or
- (ii) $X_{\tilde{\sigma}+2^n} - X_{\tilde{\sigma}} \neq X'_{\tilde{\sigma}+2^n} - X'_{\tilde{\sigma}}$.

As in the proof of Lemma 5.1, D has a very small probability. Proceeding then as in that proof, and since $E(X'_{\tilde{\sigma}+2^n} - X'_{\tilde{\sigma}}) = E^{\nu(n)}(X_{2^n})$ we can write

$$\begin{aligned} EX_{2^{n+1}} &= EX_{2^n} + E(X_{\tilde{\sigma}} - X_{2^n}) + E(X_{\tilde{\sigma}+2^n} - X_{\tilde{\sigma}}) + E(X_{2^{n+1}} - X_{\tilde{\sigma}+2^n}) \\ &= 2EX_{2^n} + O(n^8 e^{M' \log^4(n)}) + E(X_{\tilde{\sigma}} - X_{2^n}) + E(X_{2^{n+1}} - X_{\tilde{\sigma}+2^n}) \\ &= 2EX_{2^n} + O(n^8 e^{M' \log^4(n)}) \end{aligned}$$

where in the second equality, we have also used Lemma 5.1. From this the lemma follows. □

This immediately begets

Corollary 5.3. *There exists $\mu \in (-\infty, \infty)$ so that*

$$\lim_{n \rightarrow \infty} \frac{E(X_{2^n})}{2^n} = \mu.$$

Remark: It is not difficult to see that $E(X_t/t)$ converges to μ , see for instance the proof of Lemma 5.8.

We now look for a bound for $E\left(\frac{X_{2^{2n}}}{2^{2n}} - \mu\right)^2$.

As we have seen $\mu = \lim_{n \rightarrow \infty} \frac{E(X_{2^n})}{2^n}$ exists. Furthermore by Lemma 5.1 for

$$\mu_n = \frac{E^{\nu(n)}(X_{2^n})}{2^n} \text{ we have } |\mu_n - \mu| \leq \frac{c_{10}n^8 e^{M' \log^4(n)}}{2^n}.$$

Given a dynamic random walk (ξ, X) and a scale n , we define a sequence of renewal points $\beta_i, i \geq 1$ as follows: let $\beta_1 = \sigma(0)$, the regeneration time for the stopping time 0. Subsequently for $i \geq 1$, we define β_{i+1} so that $\beta_{i+1} - \beta_i$ is the regeneration time for the stopping time 2^n for the dynamic random walk (ξ^i, X^i) so that

- (i) (ξ^i, X^i) is generated by the Harris system temporally shifted by β_i ;
- (ii) for $|x - X_{\beta_{i+1} - \beta_i}^i| \leq n^9$, $\xi_0^{i+1}(x) = (\xi^i)_{\beta_{i+1} - \beta_i}'(x)$; elsewhere $\xi_0^{i+1}(x) = 1$;
- (iii) $X_0^{i+1} = X_{\beta_{i+1} - \beta_i}^i$.

We wish to only deal with "good" i , where we say that i is good if all $0 \leq j < i$ are good and if

- (a) $\beta_{i+1} - \beta_i \leq 2^n + n^8 e^{M' \log^4(n)}$
- (b) $N^X(\beta_i) - N^X(\beta_{i-1}) < 2n^2 2^n$, where for notational completeness, we take $\beta_0 = 0$ and (ξ^0, X^0) to be the original dynamic random walk (ξ, X) .

Setting $S = \inf\{i : i \text{ is not good}\}$, we define the random variables $Z_i, i \geq 1$ by

- for $j < S$, $Z_j = X_{\beta_{j+1} - \beta_j}^j - X_0^j$,
- for $j \geq S$, Z_j are taken from an independent i.i.d. sequence of random variables with the distribution that of Z_1 conditioned on 1 being good.

We note that unless a disaster occurs at some stage β_j (i.e. $\xi_0^j(x) \neq \xi_{\beta_j - \beta_{j-1}}^{j-1}(x)$ for some x within n^9 of $X_{\beta_j - \beta_{j-1}}^{j-1}$) we have for each $i, \xi_0^i \geq \xi_{\beta_i}$. Thus we have via Proposition 4.2 and Lemmas 4.3 and 2.7, and simple estimate with the Poisson distribution, that outside a set of probability $2^n(e^{-\tilde{c}n^{3/2}} + e^{-c_8n^2} + e^{-cn^2} + e^{-c_6n^3})$, for some $c > 0$ that depends on M' , all $i \leq 2^n$ are good and for such i we have

$$X_{\beta_i} - X_{\beta_1} = \sum_{k=1}^{i-1} Z_k.$$

Take R to be the integer part of $\frac{2^{2n}}{2^n + n^8 e^{M' \log^4(n)}} - 2$ and let us define

$$Y_{2^{2n}} = X_{\beta_1} + \sum_{j=1}^R Z_j + (X_{2^{2n}} - X_{\beta_{R+1}}) \equiv \sum_{j=1}^R Z_j + F_n.$$

As noted, outside probability $2^n(e^{-\tilde{c}n^{3/2}} + e^{-c_8n^2} + e^{-cn^2} + e^{-c_6n^3})$, $Y_{2^{2n}} = X_{2^{2n}}$.

Now, by techniques already employed for Lemma 5.1, it is easy to see that for suitable universal c_{12} , we have

$$|E(Z_1) - 2^n \mu| \leq c_{12} n^8 e^{M' \log^4(n)}$$

and so we may write (with Z'_j equal to Z_j minus its expectation)

$$Y_{2^{2n}} - 2^{2n}\mu = \sum_{j=1}^R Z'_j + F'_n,$$

where $E((F'_n)^2) \leq cn^{16}e^{2M' \log^4(n)}2^{2n}$ for universal c . From this and the obvious bound $E(Z_j^2) \leq \bar{c}2^{2n}$ we obtain, for some universal C , that

$$E(Y_{2^{2n}} - 2^{2n}\mu)^2 \leq C2^{3n}.$$

We finally use the elementary identity

$$E(X_{2^{2n}} - 2^{2n}\mu)^2 = E(Y_{2^{2n}} - 2^{2n}\mu)^2 + E(((X_{2^{2n}} - 2^{2n}\mu)^2 - (Y_{2^{2n}} - 2^{2n}\mu)^2)I_{X_{2^{2n}} \neq Y_{2^{2n}}})$$

and Cauchy Schwarz to conclude

Lemma 5.4. *There exists universal constant c_{13} so that for all positive integer n , $E((X_{2^{2n}} - 2^{2n}\mu)^2) \leq c_{13}2^{3n}$.*

We can now prove

Proposition 5.5. *For X as defined above, there exists $\alpha \in (0, \infty)$ so that as $n \rightarrow \infty$*

$$2^n E\left(\frac{X_{2^n}}{2^n} - \mu\right)^2 \rightarrow \alpha^2.$$

Proof. The proof that the limit is strictly positive is given below in Proposition 5.7. For the existence, we write as before $X_{2^{n+1}} - 2^{n+1}\mu$ as $(X_{2^n} - 2^n\mu) + Y_1 + Z_1 + Y_2$, where Y_1 is the increment of X over time $[2^n, \sigma]$ with $\sigma = \sigma(n, 2^n)$ being the time of the regeneration after time 2^n . $Z_1 = X'_{\sigma+2^n} - X'_\sigma - E(X'_{\sigma+2^n} - X'_\sigma)$ and Y_2 is defined via the above equality. Then we have

$$\begin{aligned} E((X_{2^{n+1}} - 2^{n+1}\mu)^2) &= E((X_{2^n} - 2^n\mu)^2) + E(Z_1^2) + 2E((X_{2^n} - 2^n\mu)Z_1) \\ &+ E((Y_1 + Y_2)^2) + 2E((Y_1 + Y_2)Z_1) \\ &+ 2E((Y_1 + Y_2)(X_{2^n} - 2^n\mu)). \end{aligned}$$

By our choice of Z_1 we have $2E((X_{2^n} - 2^n\mu)Z_1) = 0$, while by Cauchy Schwarz and Lemma 5.2 we have, for n large, (and some finite K not depending on n)

$$|E((Y_1 + Y_2)^2) + 2E((Y_1 + Y_2)Z_1) + 2E((Y_1 + Y_2)(X_{2^n} - 2^n\mu))| \leq K2^{\frac{3n}{4}}n^8e^{M' \log^4(n)}.$$

It simply remains to check that (increasing K if necessary) $|EZ_1^2 - E((X_{2^n} - 2^n\mu)^2)| \leq Kn^8e^{M' \log^4(n)}2^{\frac{3n}{4}}$ to see that

$$|E((X_{2^{n+1}} - 2^{n+1}\mu)^2) - 2E((X_{2^n} - 2^n\mu)^2)| \leq c2^{\frac{3n}{4}}n^8e^{M' \log^4(n)}, \tag{5.1}$$

from which we obtain the existence of limit of $2^{-n}E((X_{2^n} - 2^n\mu)^2)$. □

We now adapt the previous argument to give bounds on $E((X_{2^{n+1}} - 2^{n+1}\mu)^4)$. As before we write

$$X_{2^{n+1}} - 2^{n+1}\mu = X_{2^n} - 2^n\mu + Y_1 + Z_1 + Y_2 = X_{2^n} - 2^n\mu + Z_1 + Y.$$

So we can write $(X_{2^{n+1}} - 2^{n+1}\mu)^4$ as

$$(X_{2^n} - 2^n\mu)^4 + Z_1^4 + 6(X_{2^n} - 2^n\mu)^2Z_1^2 + 4Z_1^3(X_{2^n} - 2^n\mu) + 4Z_1(X_{2^n} - 2^n\mu)^3 + W,$$

where the random variable W is defined by the above equality. Now $E(Z_1(X_{2^n} - 2^n\mu)^3) = 0$ and $E((X_{2^n} - 2^n\mu)^2 Z_1^2) = E((X_{2^n} - 2^n\mu)^2)E(Z_1^2) = 2^{2n}\alpha^4(1+O(1))$, while $|E(Z_1^3(X_{2^n} - 2^n\mu))| = |EZ_1^3||E(X_{2^n} - 2^n\mu)| \leq (EZ_1^4)^{\frac{3}{4}}|E(X_{2^n} - 2^n\mu)| \leq (EZ_1^4)^{\frac{3}{4}}Kn^8e^{M'\log^4(n)}$. On the other hand,

$$E(W) = E(Y^4) + 6E(Y^2V^2) + 4E(Y^3V) + 4E(YV^3)$$

for $V = X_{2^n} - 2^n\mu + Z_1$. Using Holder's inequality, we see that

$$E(W) \leq K(EY^4)^{\frac{1}{4}} \left((E(X_{2^n} - 2^n\mu)^4)^{\frac{3}{4}} + (EZ_1^4)^{\frac{3}{4}} + (EY^4)^{\frac{3}{4}} \right).$$

In the same way we have

$$E(Z_1^4) \leq E\left((X_{2^n} - 2^n\mu)^4\right) \left(1 + \frac{K}{2^{n/4}}\right)$$

for some universal finite K . Putting all together and setting $V_n = \frac{E[(X_{2^n} - 2^n\mu)^4]}{2^{2n}}$ we see that,

$$V_{n+1} \leq \frac{V_n}{2} \left(1 + \frac{K}{2^{n/4}}\right) + 6\frac{\alpha^4}{4}(1 + O(1)) + \frac{V_n^{3/4}}{2^{n/2+2}}Kn^8e^{M'\log^4(n)} + \frac{Kn^{32}e^{4M'\log^4(n)}}{2^{2n+2}},$$

so that V_n satisfies the simpler recursion

$$V_{n+1} \leq \frac{V_n}{2} \left(1 + \frac{K'}{2^{n/4}}\right) + K',$$

for suitable constant K' , and we get

Lemma 5.6. *For the process X and μ as in Corollary 5.3*

$$\sup_n \frac{E(X_{2^n} - 2^n\mu)^4}{2^{2n}} < \infty.$$

We now wish to prove that α is strictly positive.

Proposition 5.7. *The constant α defined above is strictly positive.*

Proof. In the proof of Proposition 5.5 (see (5.1)) we showed that

$$2^{n+1}E\left(\left(\frac{X_{2^{n+1}}}{2^{n+1}} - \mu\right)^2\right) = 2^nE\left(\left(\frac{X_{2^n}}{2^n} - \mu\right)^2\right) + O(n^8e^{M'\log^4(n)}2^{-n/4}).$$

Given this, we see at once that it suffices to show that there exists $\beta < 1/4$ so that for each n_0 there exists $n_1 \geq n_0$ so that

$$E\left((X_{2^{n_1}} - 2^{n_1}\mu)^2\right) \geq 2^{n_1(1-\beta)}.$$

To do this we introduce a new regeneration time σ' similar to the regeneration time of order n but with two additional stages added into the "runs". We first choose a $j \in \{-1, 1\}$ so that $\|g_j\|_\infty \neq 0$. Without loss of generality this will be $j = 1$. We define a run beginning at a Markov time t . If the first five stages are successful then at time $t + n^4 + \log^4(n)$ the process ξ (relative to X) is in approximate equilibrium, $\nu(n)$ at least close to X . So we have with $b_2(> 0)$ probability that $g_1 > b_2$ on the configuration ξ shifted by X .

The sixth and seventh stages are motivated by a desire to create a "regeneration time" σ' so that the distribution of ξ relative to position X is (essentially) the same

irrespective of whether X has advanced by zero or by one during a certain time interval. This will add uncertainty into the system thus increasing the "variance".

The primary sixth stage event is that on time interval $[t + n^4 + \log^4(n), t + n^4 + \log^4(n) + 1]$, we have that either N^X is constant or increases by one, and the uniform random variable associated to the single Poisson point is in $[1 - b_2/M', 1]$.

Thus on this event during time interval $[t + n^4 + \log^4(n), t + n^4 + \log^4(n) + 1]$, X either advances by one or stays fixed. Our task is to show that the process will forget which.

To this end, we also require that on this time interval there is no point in $N^{x,y}$ where one of $\{x, y\}$ is in $[X(t + n^4 + \log^4(n)) - r_0, X(t + n^4 + \log^4(n)) + r_0]$ and the other is outside. We also require that N^X is constant on the time interval $[t + n^4 + \log^4(n) + 1, t + n^4 + \log^4(n) + 1 + \log^4(n)]$, and that at time $t + n^4 + \log^4(n) + 1$, the process ξ has no $\log^{5/4}(n)$ gaps on the interval $[X(\tilde{t}(t, n)) - n^9, X(\tilde{t}(t, n)) + n^9]$, where we write (for shortness) $\tilde{t}(t, n) \stackrel{def}{=} t + n^4 + 2 \log^4(n) + 1$.

Then we define a configuration $\gamma'_{\tilde{t}(t, n)}$ as with our definition of σ :

For $|x - X(t + n^4 + \log^4(n) + 1)| \leq n^5$ we choose C_x to be the condition that at time $\log^4(n)/2$, the dual $\hat{\xi}^{x, \tilde{t}(t, n)}$ has at least $h_1 \log^4(n)/2$ occupied sites in the spatial interval $[X(t + n^4 + \log^4(n) + 1) - n^9, X(t + n^4 + \log^4(n) + 1) + n^9]$.

We require that for no x in the above interval do we have $\hat{\xi}^{x, \tilde{t}(t, n)}$ survives for time $\log^4(n)/2$ but $\gamma'_{\tilde{t}(t, n)}(x) = 0$.

Finally, for the seventh stage we simply introduce (just as in stage 4 for σ) an auxiliary uniform random variable U . We can show via simple arguments that $\gamma'_{\tilde{t}(t, n)} = \xi_{\tilde{t}(t, n)}$ on $[X(\tilde{t}(t, n)) - n^9, X(\tilde{t}(t, n)) + n^9]$ with probability $q = q(\gamma'_{\tilde{t}(t, n)}, \xi_{\tilde{t}(t, n)})$ which will be at least $\frac{3}{4}$.

The last stage (and hence the "run") will be a success if this occurs and if $U \leq \frac{3}{4q}$.

Then relative to $X(t + n^4 + \log^4(n) + 1) = X(t + n^4 + 2 \log^4(n) + 1)$ (and independently of $X(t + n^4 + \log^4(n) + 1) - X(t + n^4 + \log^4(n))$) we have that $\xi_{X(t + n^4 + 2 \log^4(n) + 1)}$ has distribution $\nu = \nu(n)$.

As before we produce mostly failures but will with high probability produce a success before time $2^{n/9}$. We then let the process restart the series of runs and continue. It is the easy to see that for n large

$$E((X_{2^n} - 2^n \mu)^2) \geq 2^{n7/8}.$$

By the first paragraph this concludes the proof. □

We finish this section with a technical result

Lemma 5.8. *There exists a constant c_{13} so that for all n ,*

$$\sup_{t \leq 2^n} E((X_t - t\mu)^2) \leq c_{13} 2^n.$$

Proof. We need only consider $t \in (2^{n-1}, 2^n)$. If $t \in (2^{n-1}, 2^{n-1} + 2 \cdot 2^{n/4})$, it is easy to see that $E((X_t - t\mu)^2) \leq K 2^n$ for universal K so we need only treat $t \in (2^{n-1} + 2 \cdot 2^{n/4}, 2^n)$. In this case we can find $n - 1 = n_1 > n_2 > \dots > n_r$ so that $n_r \geq n/4 - 1$ and

$$t - \sum_{k=1}^r 2^{n_k} \in (2^{n/4}/4, 2^{2n/4}).$$

Given these n_k we construct regeneration times σ_k and processes (ξ^k, X^k) in the manner used in the proof of Lemma 5.4 so that

- (i) process (ξ^0, X^0) is our given dynamic random walk (ξ, X) ,
- (ii) for $k \geq 1$, σ_k is the n_k order regeneration for process (ξ^{k-1}, X^{k-1}) after $\sigma_{k-1} + 2^{n_{k-1}}$ (replacing $\sigma_0 + 2^{n_0}$ by 0 if $k = 1$).

The resulting process $((\xi^{k-1})'_s, (X^{k-1})'_s)_{s \geq \sigma_k}$ is written (ξ^k, X^k) .

We introduce the following notation:

$$V_i = X^{i-1}(\sigma_i) - X^{i-1}(\sigma_{i-1} + 2^{n_{i-1}}) - (\sigma_i - \sigma_{i-1} - 2^{n_{i-1}})\mu, \quad i = 1, 2, \dots, r,$$

$$Y_i = X^i(\sigma_i + 2^{n_i}) - X^i(\sigma_i) - 2^{n_i}\mu, \quad i = 1, 2, \dots, r,$$

(where we again take $\sigma_0 + 2^{n_0}$ as zero) and

$$Z = X_t - t\mu - \sum_{k=1}^r (V_i + Y_i).$$

It is not necessary in the definition of Z to assume that $\sigma_{n_r} + 2^{n_r}$ is less than t , though the probability that it is not will be less than $e^{-c_6 n^3 / 4^3}$ for n large (see Proposition 4.2).

Further let $W = \sum_{k=1}^r V_i$. Then we have that

$$E((X_t - t\mu)^2) = E\left(\left(\sum_{k=1}^r Y_i + W + Z\right)^2\right) \leq 3E\left(\left(\sum_{k=1}^r Y_i\right)^2\right) + 3E(W^2) + 3E(Z^2).$$

It is easily seen that for some K universal $E(W^2)$ and $E(Z^2)$ are both bounded by $K2^{n/2}$. For the other part of the bound, recall that by Proposition 5.5 we have that for n large (and hence $n/4 - 1$ large), for each i

$$E(Y_i^2) \leq 22^{n_i} \alpha^2.$$

It then follows by Minkowski inequality

$$E\left(\left(\sum_{k=1}^r Y_i\right)^2\right) \leq 2\alpha^2 \left(\sum_{i=1}^{n-1} 2^{i/2}\right)^2 \leq K2^{n-1}.$$

□

Corollary 5.9. *The statement in Lemma 5.8 applies as well for the dynamic random walk (ξ, X) assumed to start with ξ distributed as $\nu(n)$ and $X_0 = 0$.*

Proof. Indeed it remains to notice that the same proof works, while the only difference regards the process (ξ^0, X^0) in the first step of the above proof. □

Proposition 5.10. *Consider a process $(X_{\sigma+t} - X_\sigma - t\mu)_{t \leq 2^n}$ where σ is an n order regeneration time. For each $\gamma > 0$, there exists $c_\gamma > 0$ so that for all n large*

$$P\left(\sup_{s \leq 2^n} |X_{\sigma+s} - X_\sigma - s\mu| \geq 2^{\frac{n(1+\gamma)}{2}}\right) \leq 2P(|X_{\sigma+2^n} - X_\sigma - 2^n\mu| \geq \frac{1}{2}2^{\frac{n}{2}(1+\gamma)}) \leq \frac{1}{c_\gamma}2^{-n\gamma}.$$

Proof. Let $T = \inf\{s > 0: |X_{\sigma+s} - X_\sigma - s\mu| \geq 2^{\frac{n(1+\gamma)}{2}}\} \wedge 2^n$. This is a stopping time for the Harris system filtration. If $2^n - T < n^9 e^{M' \log^4(n)}$, then there is hardly anything to prove and so we suppose otherwise. At time T we begin runs concluding in an n order regeneration σ . We put $Z = X'_{2^n} - X'_\sigma - (2^n - \sigma)\mu$ and define random variable W by

$$X_{2^n} - 2^n\mu = (X_T - T\mu) + ((X_\sigma - X_T) - (\sigma - T)\mu) + Z + W,$$

so

$$|X_{2^n} - 2^n\mu| \geq |X_T - T\mu| - |(X_\sigma - X_T) - (\sigma - T)\mu| - |Z| - |W|,$$

$$\geq 2^{\frac{n(1+\gamma)}{2}} - |(X_\sigma - X_T) - (\sigma - T)\mu| - |Z| - |W|.$$

By elementary bounds on regeneration times and Poisson process tail probabilities and Proposition 4.2, we have that outside probability $2e^{-c_6 n^3}$ for n large

$$|(X_\sigma - X_T) - (\sigma - T)\mu| \leq Mn^9 e^{M' \log^4(n)} \ll 2^{\frac{n(1+\gamma)}{2}}.$$

Secondly, given information up to T , the term Z is equal in distribution to $X_s - s\mu$ for $s = 2^n - \sigma$ where the (ξ, X) process begins with ξ in distribution $\nu(n)$ at least when restricted to the sites within n^9 of $X_0 = 0$. We may then apply Corollary 5.9 to see that for suitable universal constant K

$$P\left(|Z| \geq 2^{\frac{n(1+\gamma)}{2}}/4\right) \leq K2^{-n\gamma}. \tag{5.2}$$

Thus we obtain (at least for large n), using the usual bounds as before for the probability that the random variable W is zero,

$$\begin{aligned} P\left(\sup_{s \leq 2^n} |X_{\sigma+s} - X_\sigma - s\mu| \geq 2^{\frac{n(1+\gamma)}{2}}\right) \\ \leq 2P(|X_{2^n} - 2^n\mu| \geq 2^{\frac{n(1+\gamma)}{2}}/4), \end{aligned}$$

and we are done. □

6 Proof of Theorem 1.1

Given this we can establish our invariance principle. We consider $2^n \leq t < 2^{n+1}$ and choose scale $2^{\frac{n}{2}(1+\beta)} = 2^{n_1}$ for $0 < \beta < 1$. We can apply Proposition 4.2 to show that if we define n_1 order regeneration times σ_k recursively, as with the proof of Lemma 5.4, so that σ_k is the time of the first regeneration for process (ξ^{k-1}, X^{k-1}) after starting runs at time $\sigma_{k-1} + 2^{n_1}$, and

$$(\xi_{\sigma_k}^k, X_{\sigma_k}^k) = (\xi^{k-1}, X^{k-1})'$$

then we have with high probability that for all $\sigma_k < t$ that

$$X_{\sigma_k+2^{n_1}} - X_{\sigma_k} = X_{\sigma_k+2^{n_1}}^k - X_{\sigma_k}^k.$$

We decompose the motion $(X_s)_{s \leq t}$ into its increments over an alternating series of intervals $I_1, J_1, I_2, J_2, \dots$, where $I_k = [\sigma_{k-1} + 2^{n_1}, \sigma_k]$ for n_1 regeneration times σ_k , and $J_k = [\sigma_k, \sigma_k + 2^{n_1}]$.

We have that $\sigma_k \geq t$ for $k = k_0 = \lceil \frac{t}{2^{n_1}} \rceil$ and (outside very small probability) $\sigma_k < t$ for $k = k_1 = \lfloor \frac{t}{2^{n_1} + n^8 e^{M' \log^4(n)}} \rfloor$.

Thus via the usual invariance principle and Berry Esseen bounds (see e.g. [2]) we have

$$(A) \sum_{k=1}^{k_1} \frac{(X_{\sigma_k+2^{n_1}} - X_{\sigma_k} - \mu 2^{n_1})}{\sqrt{t}} \xrightarrow{D} N(0, \alpha^2),$$

$$(B) \sum_{k=1}^{k_0} \frac{(X_{\sigma_k+2^{n_1}} - X_{\sigma_k} - \mu 2^{n_1})}{\sqrt{t}} \xrightarrow{D} N(0, \alpha^2),$$

and

$$(C) W_t \stackrel{def}{=} \sup_{k_1 \leq k' \leq k_0} \left| \sum_{k=k_1}^{k'} \frac{(X_{\sigma_k+2^{n_1}} - X_{\sigma_k} - 2^{n_1}\mu)}{\sqrt{t}} \right| \xrightarrow{pr} 0.$$

Furthermore we have that with probability tending to 1 at $n \rightarrow \infty$ (with the usual convention for $\sigma_0 + 2^{n_1}$):

$$W_t^1 \stackrel{def}{=} \sum_{k=1}^{k_0} \left| \frac{(X_{\sigma_k} - X_{\sigma_{k-1}+2^k} - (\sigma_k - \sigma_{k-1} + 2^{n_1})\mu)}{\sqrt{t}} \right| \leq 2Kn^8 e^{M' \log^4(n)} \frac{2^{n \frac{(1-\beta)}{2}}}{2^{\frac{n}{2}}},$$

which then tends to zero as $n \rightarrow \infty$.

Furthermore by Proposition 5.10 we have for $k_2 = \sup\{k : \sigma_k < t\}$, that with probability that tends to one as $n \rightarrow \infty$

$$W_t^2 \stackrel{def}{=} \sup_{\sigma_{k_2} \leq s \leq \sigma_{k_2} + 2^{n_1}} \left| \frac{(X_s - X_{\sigma_{k_2}} - (s - \sigma_{k_2})\mu)}{\sqrt{t}} \right| \leq \frac{2^{\frac{n(1+\beta)(1+\gamma)}{4}}}{2^{\frac{n}{2}}} < 2^{-n\epsilon}$$

for $\frac{(1+\beta)(1+\gamma)}{2} < 1$ and $\epsilon = 1 - \frac{(1+\beta)(1+\gamma)}{2}$.

Then we have

$$\left| \frac{X_t - t\mu}{\sqrt{t}} - \sum_{k=1}^{k_1} \frac{(X_{\sigma_k + 2^{n_1}} - X_{\sigma_k} - 2^{n_1}\mu)}{\sqrt{t}} \right| \leq W_t + W_t^1 + W_t^2,$$

and the desired convergence follows. □

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