Stochastic heat equations with general multiplicative Gaussian noises: Hölder continuity and intermittency

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Abstract

This paper studies the stochastic heat equation with multiplicative noises: $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u\dot{W}$, where $\dot{W}$ is a mean zero Gaussian noise and $u\dot{W}$ is interpreted both in the sense of Skorohod and Stratonovich. The existence and uniqueness of the solution are studied for noises with general time and spatial covariance structure. Feynman-Kac formulas for the solutions and for the moments of the solutions are obtained under general and different conditions. These formulas are applied to obtain the Hölder continuity of the solutions. They are also applied to obtain the intermittency bounds for the moments of the solutions.

Keywords: Fractional Brownian motion; Malliavin calculus; Skorohod integral; Young's integral; stochastic partial differential equations; Feynman-Kac formula; intermittency.
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1 Introduction

In this paper we are interested in the stochastic heat equation in $\mathbb{R}^d$ driven by a general multiplicative centered Gaussian noise. This equation can be written as

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u\dot{W}, \quad t > 0, x \in \mathbb{R}^d, \tag{1.1}$$

with initial condition $u_{0,x} = u_0(x)$, where u_0 is a continuous and bounded function. In this equation, the notation $\dot{W}$ stands for the partial derivative $\frac{\partial^{d+1}W}{\partial t\partial x_1\cdots\partial x_d}$ (or $\frac{\partial^d W}{\partial x_1\cdots\partial x_d}$ when the noise does not depend on time), where W is a random field defined in the next section. We assume that $\dot{W}$ has a covariance of the form

$$\mathbf{E}\left[\dot{W}_{t,x}\dot{W}_{s,y}\right] = \gamma(s - t)\,\Lambda(x - y),$$

where γ and Λ are general nonnegative and nonnegative definite (generalized) functions satisfying some integrability conditions. The product appearing in the above equation (1.1) can be interpreted as an ordinary product of the solution $u_{t,x}$ times the noise $\dot{W}_{t,x}$ (which is a distribution). In this case the evolution form of the equation will involve a Stratonovich integral (or pathwise Young integral). The product in (1.1) can also be interpreted as a Wick product (defined in the next section) and in this case the solution satisfies an evolution equation formulated by using the Skorohod integral. We shall consider both of these formulations.

There has been a widespread interest in the model (1.1) in the recent past, with several motivations for its study:

• It is one of the basic stochastic partial differential equations (PDEs) one might wish to solve, either by extending Itô's theory [16, 41] or by pathwise techniques [10, 22]. These developments are also related to Zakai's equation from filtering theory.

• It appears naturally in homogenization problems for PDEs driven by highly oscillating stationary random fields (see [20, 24, 28] and references therein). Notice that in this case limit theorems are often obtained through a Feynman-Kac representation of the solution to the heat equation.

• Equation (1.1) is also related to the KPZ growth model through the Cole-Hopf's transform. In this context, definitions of the equation by means of renormalization and rough paths techniques have been recently investigated in [21, 23].

• There is a strong connection between equation (1.1) and the partition function of directed and undirected continuum polymers. This link has been exploited in [33, 42] and is particularly presented in [1], where basic properties of an equation of type (1.1) are translated into corresponding properties of the polymer.

• The multiplicative stochastic heat equation exhibits concentration properties of its energy. This interesting phenomenon is referred to as *intermittency* for the solution process u to (1.1) (see e.g [12, 13, 14, 18, 31]), and as a *localization* property for the polymer measure [7]. The intermittency property for our model is one of the main results of the current paper, and will be developed later in the introduction.

• Finally, the large time behavior of equation (1.1) also provides some information on the random operator $Lu = \Delta u + \dot{W}u$. A sample of the related Lyapunov exponent literature is given by [8, 37].

Being so ubiquitous, the model (1.1) has thus obviously been the object of numerous studies.

Indeed, when the noise $\dot{W}$ is white in time and colored in space, that is, when γ is the Dirac delta function $\delta_0(x)$, there is a huge literature devoted to our linear stochastic heat equation. Notice that in this case the stochastic integral involving $\dot{W}$ is interpreted in an extended Itô sense. Starting with the seminal paper by Dalang [15], these equations, even with more general nonlinearities (namely $u\dot{W}$ in (1.1) is replaced by $\sigma(u)\dot{W}$ for a general nonlinear function σ), have received a lot of attention. In this context, the existence and uniqueness of a solution is guaranteed by the integrability condition

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\xi|^2} < \infty, \tag{1.2}$$

where μ is the Fourier transform of Λ. This condition is sharp, in the sense that it is also necessary in the case of an additive noise.

Recently, there also has been a growing interest in studying equation (1.1) when the noise is colored in time. Unlike the case where the noise is white in time, one can no longer make use of the martingale structure of the noise, and just making sense of the equation offers new challenges. Recent progresses for some specific Gaussian noises include [4, 27, 29] by means of stochastic analysis methods, and [10, 22, 17] using rough paths arguments.

As mentioned above, we shall focus in this article on intermittency properties for the stochastic heat equation (1.1). There exist several ways to express this phenomenon, heuristically meaning that the process u concentrates into a few very high peaks. However, all the definitions involve two functions $\{a(t); t \geq 0\}$ and $\{\ell(k); k \geq 2\}$ such that $\ell(k) \in (0, \infty)$ and:

$$\ell(k) := \limsup_{t \to \infty} \frac{1}{a(t)} \log \left(\mathbf{E} \left[|u_{t,x}|^k \right] \right), \tag{1.3}$$

where we assume that the limit above is independent of x. In this case, we call $a(t)$ the upper Lyapunov rate and $\ell(k)$ the upper Lyapunov exponent. The process u is then called *weakly intermittent* if

$$\ell(2) > 0, \quad \text{and} \quad \ell(k) < \infty \quad \forall \, k \geq 2.$$

The computation of the exact value of Lyapunov exponents is difficult in general. A related property (which corresponds to the intuitive notion of intermittency) requires that for any $k_1 > k_2$ the moment of order k_1 is significantly greater than the moment of order k_2, or otherwise stated:

$$\limsup_{t \to \infty} \frac{\mathbf{E}^{1/k_1}\left[|u_{t,x}|^{k_1}\right]}{\mathbf{E}^{1/k_2}\left[|u_{t,x}|^{k_2}\right]} = \infty \,. \tag{1.4}$$

Most of the studies concerning this challenging property involve a white noise in time, and we refer to [6, 8, 18] for an account on the topic. The recent paper [3] tackles the problem for a fractional noise in time, with some special (though important) examples of spatial covariance structures, within the landmark of Skorohod equations. In this case the results are confined to weak intermittency, with an upper bound on L^k moments obtained invoking hypercontractivity arguments and lower bounds computed only for the L^2 norm.

With all those preliminary considerations in mind, the current paper proposes to study existence-uniqueness results, Feynman-Kac representations, chaos expansions and intermittency results for a very wide class of Gaussian noises $\dot{W}$ (including in particular those considered in [3, 16]), for both Skorohod and Stratonovich type equations (1.1). In particular we obtain some lower bounds for $\ell(k)$ defined by (1.3) for all $k \geq 2$, which are sharp in the sense that they have the same exponential order as the upper bounds.

More specifically, here is a brief description of the results obtained in the current paper:

(i) In the Skorohod case, the mild solution has a formal Wiener chaos expansion, which converges in $L^2(\Omega)$ provided γ is locally integrable and the spectral measure μ of the spatial covariance satisfies condition (1.2). Moreover, the solution is unique. This result (proved in Theorem 3.2) is based on Fourier analysis techniques, and covers the particular examples of the Riesz kernel and the Bessel kernel considered by Balan and Tudor in [4]. Our results also encompass the case of the fractional covariance $\Lambda(x) = \prod_{i=1}^{d} H_i(2H_i - 1)|x_i|^{2H_i-2}$, where $H_i > \frac{1}{2}$ and condition (1.2) is satisfied if and only if $\sum_{i=1}^{d} H_i > d - 1$. This particular structure has been examined in [27].

(ii) Under these general hypothesis to ensure the existence and uniqueness of the solution of Skorohod type one cannot expect to have a Feynman-Kac formula for the solution, but one can establish Feynman-Kac-type formulas for the moments of the solution. The formulas we obtain (see (3.21)), generalize those obtained for the Riesz or the Bessel kernels in [4, 27].

(iii) Under more restrictive integrability assumptions on γ and μ (see Hypothesis 4.1) we derive a Feynman-Kac formula for the solution u to (1.1) in the Stratonovich sense. An immediate application of the Feynman-Kac formula is the Hölder continuity of the solution.

(iv) In the Stratonovich case, we give a notion of solution using two different methodologies. One is based on the Stratonovich integral defined as the limit in probability of the integrals with respect to a regularization of the noise, and another one uses a pathwise approach, weighted Besov spaces and a Young integral approach. We show that the two notions coincide and some existence-uniqueness results which are (to the best of our knowledge) the first link between pathwise and Malliavin calculus solutions to equation (1.1).

(v) Under some further restrictions (see hypothesis at the beginning of Section 6), we obtain some sharp lower bounds for the moments of the solution. Namely, we can

find explicit numbers κ_1 and κ_2 and constants c_j, C_j for $j = 1, 2$ such that

$$C_1 \exp\left(c_1 t^{\kappa_1} k^{\kappa_2}\right) \leq \mathbf{E}\left[|u_{t,x}|^k\right] \leq C_2 \exp\left(c_2 t^{\kappa_1} k^{\kappa_2}\right)$$

for all $t \geq 0$, $x \in \mathbb{R}^d$ and $k \geq 2$.

As it might be clear from the description above, our central object for the study of (1.1) is the Feynman-Kac formula for the solution u or for its moments, which is a very interesting result in its own right. A substantial part of the article is devoted to establish these formulae with optimal conditions on the covariances γ and Λ, including critical cases. Notice that we also handle the case of noises which only depend on the space variable. This situation is usually treated separately in the paper, due to its particular physical relevance.

Here is the organization of the paper. In Section 2, we briefly set up some preliminary material on the Gaussian noises that we are dealing with. We also recall some material from Malliavin calculus. Section 3 is devoted to the stochastic heat equation of Skorohod type. Existence and uniqueness of the mild solutions are obtained, and Feynman-Kac formula for the moments of the solution is established. Section 4 focuses on the Feynman-Kac formula related to equation (1.1) and studies the regularity of the process u^F defined in that way under some conditions on γ and Λ. In section 5 we first prove that the process u^F can really be seen as a solution to the stochastic heat equation interpreted in a mild sense related to Malliavin calculus. However, uniqueness is missing in this general context. Under some slightly more restrictive conditions on the noises, we then study the existence and uniqueness of the mild solution to equation (1.1) using Young integration techniques. Finally, Section 6 is concerned with the bounds for the moments and related intermittency results.

Notations. In the remainder of the article, all generic constants will be denoted by c, C, and their value may vary from different occurrences. We denote by $p_t(x)$ the d-dimensional heat kernel $p_t(x) = (2\pi t)^{-d/2} e^{-|x|^2/2t}$, for any $t > 0$, $x \in \mathbb{R}^d$.

2 Preliminaries

This section is devoted to a further description of the structure of our noise W. We consider first the time dependent case and later the time independent case. We will also provide some basic elements of Malliavin calculus used in the paper.

2.1 Time dependent noise

Let us start by introducing some basic notions on Fourier transforms of functions: the space of real valued infinitely differentiable functions with compact support is denoted by $\mathcal{D}(\mathbb{R}^d)$ or $\mathcal{D}$. The space of Schwartz functions is denoted by $\mathcal{S}(\mathbb{R}^d)$ or $\mathcal{S}$. Its dual, the space of tempered distributions, is $\mathcal{S}'(\mathbb{R}^d)$ or $\mathcal{S}'$. If u is a vector of tempered distributions from $\mathbb{R}^d$ to $\mathbb{R}^n$, then we write $u \in \mathcal{S}'(\mathbb{R}^d, \mathbb{R}^n)$. The Fourier transform is defined with the normalization

$$\mathcal{F}u(\xi) = \int_{\mathbb{R}^d} e^{-\imath\langle\xi,x\rangle} u(x) dx,$$

so that the inverse Fourier transform is given by $\mathcal{F}^{-1}u(\xi) = (2\pi)^{-d}\mathcal{F}u(-\xi)$.

Similarly to [15], on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ we consider a Gaussian noise W encoded by a centered Gaussian family $\{W(\varphi); \varphi \in \mathcal{D}([0, \infty) \times \mathbb{R}^d)\}$, whose covariance structure is given by

$$\mathbf{E}\left[W(\varphi) W(\psi)\right] = \int_{\mathbb{R}_+^2 \times \mathbb{R}^{2d}} \varphi(s, x)\psi(t, y)\gamma(s - t)\Lambda(x - y)dxdydsdt, \qquad (2.1)$$

where $\gamma : \mathbb{R} \to \mathbb{R}_+$ and $\Lambda : \mathbb{R}^d \to \mathbb{R}_+$ are non-negative definite functions and the Fourier transform $\mathcal{F}\Lambda = \mu$ is a tempered measure, that is, there is an integer $m \geq 1$ such that $\int_{\mathbb{R}^d}(1+|\xi|^2)^{-m}\mu(d\xi) < \infty$.

Let $\mathcal{H}$ be the completion of $\mathcal{D}([0,\infty)\times\mathbb{R}^d)$ endowed with the inner product

$$
\begin{aligned}
\langle\varphi,\psi\rangle_{\mathcal{H}} &= \int_{\mathbb{R}_+^2\times\mathbb{R}^{2d}}\varphi(s,x)\psi(t,y)\gamma(s-t)\Lambda(x-y)\,dxdydsdt \qquad (2.2)\\
&= \int_{\mathbb{R}_+^2\times\mathbb{R}^d}\mathcal{F}\varphi(s,\xi)\overline{\mathcal{F}\psi(t,\xi)}\gamma(s-t)\mu(d\xi)\,dsdt,
\end{aligned}
$$

where $\mathcal{F}\varphi$ refers to the Fourier transform with respect to the space variable only. The mapping $\varphi \to W(\varphi)$ defined in $\mathcal{D}([0,\infty)\times\mathbb{R}^d)$ extends to a linear isometry between $\mathcal{H}$ and the Gaussian space spanned by W. We will denote this isometry by

$$
W(\phi) = \int_0^\infty\int_{\mathbb{R}^d}\phi(t,x)W(dt,dx)
$$

for $\phi \in \mathcal{H}$. Notice that if ϕ and ψ are in $\mathcal{H}$, then $\mathbf{E}\left[W(\phi)W(\psi)\right] = \langle\phi,\psi\rangle_{\mathcal{H}}$. Furthermore, $\mathcal{H}$ contains the class of measurable functions ϕ on $\mathbb{R}_+\times\mathbb{R}^d$ such that

$$
\int_{\mathbb{R}_+^2\times\mathbb{R}^{2d}}|\phi(s,x)\phi(t,y)|\,\gamma(s-t)\Lambda(x-y)\,dxdydsdt < \infty\,. \qquad (2.3)
$$

We shall make a standard assumption on the spectral measure μ, which will prevail until the end of the paper.

Hypothesis 2.1. *The measure μ satisfies the following integrability condition:*

$$
\int_{\mathbb{R}^d}\frac{\mu(d\xi)}{1+|\xi|^2} < \infty. \qquad (2.4)
$$

Let us now recall some of the main examples of stationary covariances, which will be our guiding examples in the remainder of the paper.

Example 2.2. One of the most popular spatial covariances is given by the so-called Riesz kernel, for which $\Lambda(x) = |x|^{-\eta}$ and $\mu(d\xi) = c_{\eta,d}|\xi|^{-(d-\eta)}\,d\xi$. We refer to this kind of noise as a spatial η-Riesz noise. In this case, Hypothesis 2.1 is satisfied whenever $0 < \eta < 2$, which will be our standing assumption.

Example 2.3. We shall also handle the space white noise case, namely $\Lambda = \delta_0$ (notice that in this case Λ is not a function but a measure) and $\mu = $ Lebesgue. This noise satisfies Hypothesis 2.1 only in dimension 1.

Example 2.4. The spatial covariance given by the so-called Bessel kernel is defined by

$$
\Lambda(x) = \int_0^\infty w^{\frac{\eta-d}{2}}e^{-w}e^{-\frac{|x|^2}{4w}}\,dw\,.
$$

In this case $\mu(d\xi) = c_{\eta,d}(1+|\xi|^2)^{-\frac{\eta}{2}}\,d\xi$ and Hypothesis 2.1 is satisfied if $\eta > d-2$.

Example 2.5. An example of time covariance γ that has received a lot of attention is the case of a one-dimensional Riesz kernel, which corresponds to the fractional Brownian motion. Suppose that $\gamma(t) = H(2H-1)|t|^{2H-2}$ with $\frac{1}{2} < H < 1$ and W is a noise with this time covariance and a spatial covariance Λ. For any $t \geq 0$ and any $\varphi \in C_c^\infty(\mathbb{R}^d)$, the function $\mathbf{1}_{[0,t]}\varphi$ belongs to the space $\mathcal{H}$, and we can define $W_t(\varphi) := W(\mathbf{1}_{[0,t]}\varphi)$. Then, for any fixed $\varphi \in \mathcal{D}(\mathbb{R}^d)$, the stochastic process $\{c_\varphi^{-1/2}W_t(\varphi); t \geq 0\}$ is a fractional Brownian motion with Hurst parameter H, where

$$
c_\varphi = \int_{\mathbb{R}^d}|\mathcal{F}\varphi(\xi)|^2\mu(d\xi).
$$

That is $\mathbf{E}\left[W_t(\varphi)W_s(\varphi)\right] = R_H(s,t)c_\varphi$, where for each $H \in (0,1)$ we have:

$$R_H(s,t) = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right).$$

Example 2.6. In the same way, the spatial fractional covariance is given by $\Lambda(x) = \prod_{i=1}^d H_i\,(2H_i - 2)|x_i|^{2H_i-2}$, where $\frac{1}{2} < H_i < 1$ for $i = 1,\ldots,d$. The Fourier transform of Λ is $\mu(d\xi) = C_H \prod_{i=1}^d |\xi_i|^{1-2H_i}d\xi$, where C_H is a constant depending on the parameters H_i. Then an easy calculation shows that when $\sum_{i=1}^d H_i > d - 1$, Hypothesis 2.1 holds.

If W is a noise with fractional space and time covariances, with Hurst parameters H_0 in time, and $H_1,\ldots,H_d$ in space, then we can write formally $W(\varphi)$ as the distributional integral $\int_{\mathbb{R}_+ \times \mathbb{R}^d} \varphi(t,x)\dot{W}_H(t,x)dtdx$, where $\dot{W}_H(t,x)$ is the formal partial derivative $\frac{\partial^{d+1}W_H}{\partial t \partial x_1 \cdots \partial x_d}(t,x)$ and W_H is centered Gaussian random field which is a fractional Brownian motion on each coordinate, that is,

$$\mathbf{E}\left[W_H(s,x)W_H(t,y)\right] = R_{H_0}(s,t)\prod_{i=1}^d R_{H_i}(x_i,y_i), \quad s,t \geq 0, x,y \in \mathbb{R}^d.$$

2.2 Time independent noise

In this case we consider a zero mean Gaussian family $W = \{W(\varphi); \varphi \in \mathcal{D}(\mathbb{R}^d)\}$, defined in a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, with covariance

$$\mathbf{E}\left[W(\varphi)W(\psi)\right] = \int_{\mathbb{R}^{2d}} \varphi(x)\psi(y)\Lambda(x-y)\,dxdy, \tag{2.5}$$

where, as before, $\Lambda : \mathbb{R}^d \to \mathbb{R}_+$ is a non-negative definite function whose Fourier transform μ is a tempered measure. In this case $\mathcal{H}$ is the completion of $\mathcal{D}(\mathbb{R}^d)$ endowed with the inner product

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \int_{\mathbb{R}^{2d}} \varphi(x)\psi(y)\Lambda(x-y)dxdy = \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}\mu(d\xi). \tag{2.6}$$

The mapping $\varphi \to W(\varphi)$ defined in $\mathcal{D}(\mathbb{R}^d)$ extends to a linear isometry between $\mathcal{H}$ and the Gaussian space spanned by W, denoted by

$$W(\phi) = \int_{\mathbb{R}^d} \phi(x)W(dx)$$

for $\phi \in \mathcal{H}$. If ϕ and ψ are in $\mathcal{H}$, then $\mathbf{E}\left[W(\phi)W(\psi)\right] = \langle \phi, \psi \rangle_{\mathcal{H}}$ and $\mathcal{H}$ contains the class of measurable functions ϕ on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^{2d}} |\phi(x)\phi(y)|\,\Lambda(x-y)\,dxdy < \infty. \tag{2.7}$$

2.3 Elements of Malliavin calculus

Consider first the case of a time dependent noise. We will denote by D the derivative operator in the sense of Malliavin calculus. That is, if F is a smooth and cylindrical random variable of the form

$$F = f(W(\phi_1),\ldots,W(\phi_n)),$$

with $\phi_i \in \mathcal{H}$, $f \in C_p^\infty(\mathbb{R}^n)$ (namely f and all its partial derivatives have polynomial growth), then DF is the $\mathcal{H}$-valued random variable defined by

$$DF = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(W(\phi_1),\ldots,W(\phi_n))\phi_j.$$

The operator D is closable from $L^2(\Omega)$ into $L^2(\Omega; \mathcal{H})$ and we define the Sobolev space $\mathbb{D}^{1,2}$ as the closure of the space of smooth and cylindrical random variables under the norm

$$\|DF\|_{1,2} = \sqrt{\mathbf{E}[F^2] + \mathbf{E}[\|DF\|_{\mathcal{H}}^2]}.$$

We denote by δ the adjoint of the derivative operator given by the duality formula

$$\mathbf{E}\left[\delta(u)F\right] = \mathbf{E}\left[\langle DF, u \rangle_{\mathcal{H}}\right], \tag{2.8}$$

for any $F \in \mathbb{D}^{1,2}$ and any element $u \in L^2(\Omega; \mathcal{H})$ in the domain of δ. The operator δ is also called the *Skorohod integral* because in the case of the Brownian motion, it coincides with an extension of the Itô integral introduced by Skorohod. We refer to Nualart [40] for a detailed account of the Malliavin calculus with respect to a Gaussian process. If DF and u are almost surely measurable functions on $\mathbb{R}_+ \times \mathbb{R}^d$ verifying condition (2.3), then the duality formula (2.8) can be written using the expression of the inner product in $\mathcal{H}$ given in (2.2)

$$\mathbf{E}\left[\delta(u)F\right] = \mathbf{E}\left[\int_{\mathbb{R}_+^2 \times \mathbb{R}^{2d}} D_{s,x}F\, u_{t,y}\, \gamma(s-t)\, \Lambda(x-y)\, dsdtdxdy\right]. \tag{2.9}$$

Let us recall three other classical relations in stochastic analysis, which will be used in the paper:

(i) Divergence type formula. For any $\phi \in \mathcal{H}$ and any random variable F in the Sobolev space $\mathbb{D}^{1,2}$, we have

$$FW(\phi) = \delta(F\phi) + \langle DF, \phi \rangle_{\mathcal{H}}. \tag{2.10}$$

(ii) A duality relationship. Given a random variable $F \in \mathbb{D}^{2,2}$ and two elements $h, g \in \mathcal{H}$, the duality formula (2.8) implies

$$\mathbf{E}\left[F\, W(h)W(g)\right] = \mathbf{E}\left[\langle D^2F, h \otimes g \rangle_{\mathcal{H}^{\otimes 2}}\right] + \mathbf{E}\left[F\right] \langle h, g \rangle_{\mathcal{H}}. \tag{2.11}$$

(iii) Definition of the Wick product of a random and a Gaussian element. If $F \in \mathbb{D}^{1,2}$ and h is an element of $\mathcal{H}$, then Fh is Skorohod integrable and, by definition, the Wick product equals the Skorohod integral of Fh

$$\delta(Fh) = F \diamond W(h). \tag{2.12}$$

When handling the stochastic heat equation in the Skorohod sense we will make use of chaos expansions, and we should give a small account on this notion. For any integer $n \geq 0$ we denote by $\mathbf{H}_n$ the nth Wiener chaos of W. We recall that $\mathbf{H}_0$ is simply $\mathbb{R}$ and for $n \geq 1$, $\mathbf{H}_n$ is the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_n(W(h)); h \in \mathcal{H}, \|h\|_{\mathcal{H}} = 1\}$, where H_n is the nth Hermite polynomial. For any $n \geq 1$, we denote by $\mathcal{H}^{\otimes n}$ (resp. $\mathcal{H}^{\odot n}$) the nth tensor product (resp. the nth symmetric tensor product) of $\mathcal{H}$. Then, the mapping $I_n(h^{\otimes n}) = H_n(W(h))$ can be extended to a linear isometry between $\mathcal{H}^{\odot n}$ (equipped with the modified norm $\sqrt{n!}\|\cdot\|_{\mathcal{H}^{\otimes n}}$) and $\mathbf{H}_n$.

Consider now a random variable $F \in L^2(\Omega)$ measurable with respect to the σ-field $\mathcal{F}^W$ generated by W. This random variable can be expressed as

$$F = \mathbf{E}[F] + \sum_{n=1}^{\infty} I_n(f_n), \tag{2.13}$$

where the series converges in $L^2(\Omega)$, and the elements $f_n \in \mathcal{H}^{\odot n}$, $n \geq 1$, are determined by F. This identity is called the Wiener-chaos expansion of F.

The Skorohod integral (or divergence) of a random field u can be computed by using the Wiener chaos expansion. More precisely, suppose that $u = \{u_{t,x}; (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ is a random field such that for each (t,x), $u_{t,x}$ is an $\mathcal{F}^W$-measurable and square integrable random variable. Then, for each (t,x) we have a Wiener chaos expansion of the form

$$u_{t,x} = \mathbf{E}[u_{t,x}] + \sum_{n=1}^{\infty} I_n(f_n(\cdot, t, x)). \tag{2.14}$$

Suppose also that

$$\mathbf{E}\left[\int_0^\infty \int_0^\infty \int_{\mathbb{R}^{2d}} |u_{t,x} \, u_{s,y}| \, \gamma(s-t)\Lambda(x-y) \, dxdydsdt\right] < \infty.$$

Then, we can interpret u as a square integrable random function with values in $\mathcal{H}$ and the kernels f_n in the expansion (2.14) are functions in $\mathcal{H}^{\otimes(n+1)}$ which are symmetric in the first n variables. In this situation, u belongs to the domain of the divergence (that is, u is Skorohod integrable with respect to W) if and only if the following series converges in $L^2(\Omega)$

$$\delta(u) = \int_0^\infty \int_{\mathbb{R}^d} u_{t,s} \, \delta W_{t,x} = W(\mathbf{E}[u]) + \sum_{n=1}^{\infty} I_{n+1}(\widetilde{f}_n(\cdot, t, x)), \tag{2.15}$$

where $\widetilde{f}_n$ denotes the symmetrization of f_n in all its $n+1$ variables.

The operators D and δ can be introduced in a similar way in the time independent case. If DF and u are almost surely measurable functions on $\mathbb{R}^d$ verifying condition (2.7), then formula (2.8) can be written using the expression of the inner product in $\mathcal{H}$ given in (2.6):

$$\mathbf{E}[\delta(u)F] = \mathbf{E}\left[\int_{\mathbb{R}^{2d}} D_x F \, u(y) \, \Lambda(x-y) \, dxdy\right]. \tag{2.16}$$

3 Equation of Skorohod type

The first part of this section is devoted to the study of the following d-dimensional stochastic heat equation with the time dependent multiplicative Gaussian noise W introduced in Section 2.1, where the product is understood in the Wick sense (see (2.12)):

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u \diamond \frac{\partial^{d+1} W}{\partial t \partial x_1 \cdots \partial x_d}, \tag{3.1}$$

the initial condition being a continuous and bounded function $u_0(x)$. This equation is formal and below we provide a rigorous definition of a mild solution using the Skorohod integral. The main objective of this section is to show that the mild solution exists and is unique in $L^2(\Omega)$, assuming that the spectral measure μ satisfies Hypothesis 2.1. This is proved by showing that the formal Wiener chaos expansion which defines the solution converges in $L^2(\Omega)$. In a second part of this section we obtain a Feynman-Kac formula for the moments of the solution. In the last part we will extend these results to the case where the noise is time independent.

3.1 Existence and uniqueness of a solution via chaos expansions

Recall that we denote by $p_t(x)$ the d-dimensional heat kernel $p_t(x) = (2\pi t)^{-d/2} e^{-|x|^2/2t}$, for any $t > 0$, $x \in \mathbb{R}^d$. For each $t \geq 0$ let $\mathcal{F}_t$ be the σ-field generated by the random variables $W(\varphi)$, where φ has support in $[0,t] \times \mathbb{R}^d$. We say that a random field $u_{t,x}$ is adapted if for each (t,x) the random variable $u_{t,x}$ is $\mathcal{F}_t$-measurable. We define the solution of equation (3.1) as follows.

Definition 3.1. *An adapted random field $u = \{u_{t,x}; t \geq 0, x \in \mathbb{R}^d\}$ such that $\mathbf{E}[u_{t,x}^2] < \infty$ for all (t,x) is a mild solution to equation (3.1) with initial condition $u_0 \in C_b(\mathbb{R}^d)$, if for any $(t,x) \in [0,\infty) \times \mathbb{R}^d$, the process $\{p_{t-s}(x-y)u_{s,y}\mathbf{1}_{[0,t]}(s); s \geq 0, y \in \mathbb{R}^d\}$ is Skorohod integrable, and the following equation holds*

$$u_{t,x} = p_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)u_{s,y}\,\delta W_{s,y}. \tag{3.2}$$

Suppose now that $u = \{u_{t,x}; t \geq 0, x \in \mathbb{R}^d\}$ is a solution to equation (3.2). Then according to (2.13), for any fixed (t,x) the random variable $u_{t,x}$ admits the following Wiener chaos expansion

$$u_{t,x} = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t, x)), \tag{3.3}$$

where for each (t,x), $f_n(\cdot, t, x)$ is a symmetric element in $\mathcal{H}^{\otimes n}$. Thanks to (2.15) and using an iteration procedure, one can then find an explicit formula for the kernels f_n for $n \geq 1$

$$f_n(s_1, x_1, \ldots, s_n, x_n, t, x) = \frac{1}{n!} p_{t - s_{\sigma(n)}}(x - x_{\sigma(n)}) \cdots p_{s_{\sigma(2)} - s_{\sigma(1)}}(x_{\sigma(2)} - x_{\sigma(1)}) p_{s_{\sigma(1)}} u_0(x_{\sigma(1)}),$$

where σ denotes the permutation of $\{1, 2, \ldots, n\}$ such that $0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t$ (see, for instance, equation (4.4) in [27], where this formula is established in the case of a noise which is white in space). Then, to show the existence and uniqueness of the solution it suffices to show that for all (t,x) we have

$$\sum_{n=0}^{\infty} n! \|f_n(\cdot, t, x)\|_{\mathcal{H}^{\otimes n}}^2 < \infty. \tag{3.4}$$

Theorem 3.2. *Suppose that μ satisfies Hypothesis 2.1 and γ is locally integrable. Then relation (3.4) holds for each (t,x). Consequently, equation (3.1) admits a unique mild solution in the sense of Definition 3.1.*

Proof. Fix $t > 0$ and $x \in \mathbb{R}^d$. Set $f_n(s, y, t, x) = f_n(s_1, y_1, \ldots, s_n, y_n, t, x)$. We have the following expression

$$
\begin{aligned}
&n! \|f_n(\cdot, t, x)\|_{\mathcal{H}^{\otimes n}}^2 \\
= \quad &n! \int_{[0,t]^{2n}} \int_{\mathbb{R}^{2nd}} f_n(s, y, t, x) f_n(r, z, t, x) \prod_{i=1}^n \Lambda(y_i - z_i) \prod_{i=1}^n \gamma(s_i - r_i)\, dydzdsdr \\
\leq \quad &n! \|u_0\|_\infty^2 \int_{[0,t]^{2n}} \int_{\mathbb{R}^{2nd}} g_n(s, y, t, x) g_n(r, z, t, x) \prod_{i=1}^n \Lambda(y_i - z_i) \prod_{i=1}^n \gamma(s_i - r_i)\, dydzdsdr
\end{aligned}
$$

where $dx = dx_1 \cdots dx_n$, the differentials dy, ds and dr are defined similarly and

$$g_n(s, y, t, x) = \frac{1}{n!} p_{t - s_{\sigma(n)}}(x - y_{\sigma(n)}) \cdots p_{s_{\sigma(2)} - s_{\sigma(1)}}(y_{\sigma(2)} - y_{\sigma(1)}). \tag{3.5}$$

Set now $\mu(d\xi) \equiv \prod_{i=1}^n \mu(d\xi_i)$. Using the Fourier transform and Cauchy-Schwarz, we

obtain

$$n!\|f_n(\cdot, t, x)\|_{\mathcal{H}^{\otimes n}}^2$$

$$\leq \quad n!\|u_0\|_\infty^2 \int_{[0,t]^{2n}} \int_{\mathbb{R}^{nd}} \mathcal{F}g_n(s, \cdot, t, x)(\xi)\overline{\mathcal{F}g_n(r, \cdot, t, x)(\xi)}\mu(d\xi)\prod_{i=1}^n \gamma(s_i - r_i)dsdr$$

$$\leq \quad n!\|u_0\|_\infty^2 \int_{[0,t]^{2n}} \left(\int_{\mathbb{R}^{nd}} |\mathcal{F}g_n(s, \cdot, t, x)(\xi)|^2 \mu(d\xi)\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^{nd}} |\mathcal{F}g_n(r, \cdot, t, x)(\xi)|^2 \mu(d\xi)\right)^{\frac{1}{2}}$$

$$\times \prod_{i=1}^n \gamma(s_i - r_i)dsdr,$$

and thus, thanks to the basic inequality $ab \leq \frac{1}{2}(a^2 + b^2)$ and the fact that γ is locally integrable, this yields:

$$n!\|f_n(\cdot, t, x)\|_{\mathcal{H}^{\otimes n}}^2 \quad \leq \quad n!\|u_0\|_\infty^2 \int_{[0,t]^{2n}} \int_{\mathbb{R}^{nd}} |\mathcal{F}g_n(s, \cdot, t, x)(\xi)|^2 \mu(d\xi)\prod_{i=1}^n \gamma(s_i - r_i)dsdr$$

$$\leq \quad C^n n!\|u_0\|_\infty^2 \int_{[0,t]^{2n}} \int_{\mathbb{R}^{nd}} |\mathcal{F}g_n(s, \cdot, t, x)(\xi)|^2 \mu(d\xi)ds, \qquad (3.6)$$

where $C = 2\int_0^t \gamma(r)dr$. Furthermore, it is readily checked from expression (3.5) that there exists a constant $C > 0$ such that the Fourier transform of g_n satisfies

$$|\mathcal{F}g_n(s, \cdot, t, x)(\xi)|^2 = \frac{C^n}{(n!)^2} \prod_{i=1}^n e^{-(s_{\sigma(i+1)} - s_{\sigma(i)})|\xi_{\sigma(i)} + \cdots + \xi_{\sigma(1)}|^2},$$

where we have set $s_{\sigma(n+1)} = t$. As a consequence,

$$(n!)^2 \int_{\mathbb{R}^{nd}} |\mathcal{F}g_n(s, \cdot, t, x)(\xi)|^2 \mu(d\xi)$$

$$\leq \quad C^n \prod_{i=1}^n \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} e^{-(s_{\sigma(i+1)} - s_{\sigma(i)})|\xi_{\sigma(i)} + \eta|^2} \mu(d\xi_{\sigma(i)})$$

$$= \quad C^n \prod_{i=1}^n \sup_{\eta \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} \frac{e^{-\frac{|x_{\sigma(i)}|^2}{4(s_{\sigma(i+1)} - s_{\sigma(i)})}}}{(4\pi(s_{\sigma(i+1)} - s_{\sigma(i)}))^{\frac{d}{2}}} e^{\iota x_{\sigma(i)} \cdot \eta} \Lambda(x_{\sigma(i)})dx_{\sigma(i)} \right|$$

$$\leq \quad C^n \prod_{i=1}^n \int_{\mathbb{R}^d} e^{-(s_{\sigma(i+1)} - s_{\sigma(i)})|\xi_{\sigma(i)}|^2} \mu(d\xi_{\sigma(i)}), \qquad (3.7)$$

where we invoke the fact that $|e^{\iota x_{\sigma(i)} \cdot \eta}| = 1$ to get rid of the supremum in η. Therefore, from relations (3.6) and (3.7) we obtain

$$n!\|f_n(\cdot, t, x)\|_{\mathcal{H}^{\otimes n}}^2 \leq \|u_0\|_\infty^2 C^n \int_{\mathbb{R}^{nd}} \int_{T_n(t)} \prod_{i=1}^n e^{-(s_{i+1} - s_i)|\xi_i|^2} \, ds \, \mu(d\xi), \qquad (3.8)$$

where we denote by $T_n(t)$ the simplex

$$T_n(t) = \{0 < s_1 < \cdots < s_n < t\}. \qquad (3.9)$$

Let us now estimate the right hand side of (3.7): making the change of variables $s_{i+1} - s_i = w_i$ for $1 \leq i \leq n-1$, and $t - s_n = w_n$, and denoting $dw = dw_1 dw_2 \cdots dw_n$, we end up with

$$n!\|f_n(\cdot, t, x)\|_{\mathcal{H}^{\otimes n}}^2 \leq \|u_0\|_\infty^2 C^n \int_{\mathbb{R}^{nd}} \int_{S_{t,n}} e^{-\sum_{i=1}^n w_i|\xi_i|^2} dw \prod_{i=1}^n \mu(d\xi_i),$$

where $S_{t,n} = \{(w_1, \ldots, w_n) \in [0, \infty)^n : w_1 + \cdots + w_n \leq t\}$. We also split the contribution of μ in the following way: fix $N \geq 1$ and set

$$C_N = \int_{|\xi| \geq N} \frac{\mu(d\xi)}{|\xi|^2}, \quad \text{and} \quad D_N = \mu\{\xi \in \mathbb{R}^d : |\xi| \leq N\}. \tag{3.10}$$

By Lemma 3.3 below, we can write

$$n! \|f_n(\cdot, t, x)\|_{\mathcal{H}^{\otimes n}}^2 \leq \|u_0\|_\infty^2 C^n \sum_{k=0}^n \binom{n}{k} \frac{t^k}{k!} D_N^k (2C_N)^{n-k}. \tag{3.11}$$

Next we choose a sufficiently large N such that $2CC_N < 1$, which is possible because of condition (2.4). Using the inequality $\binom{n}{k} \leq 2^n$ for any positive integers n and $0 \leq k \leq n$, we have

$$\sum_{n=0}^\infty n! \|f_n(\cdot, t, x)\|_{\mathcal{H}^{\otimes n}}^2 \leq \|u_0\|_\infty^2 \sum_{n=0}^\infty C^n \sum_{k=0}^n \binom{n}{k} \frac{t^k}{k!} D_N^k (2C_N)^{n-k}$$

$$\leq \|u_0\|_\infty^2 \sum_{k=0}^\infty \sum_{n=k}^\infty C^n 2^n \frac{t^k}{k!} D_N^k (2C_N)^{n-k} = \|u_0\|_\infty^2 \sum_{k=0}^\infty \frac{t^k}{k!} D_N^k (2C_N)^{-k} \sum_{n=k}^\infty (2CC_N)^n$$

$$\leq \|u_0\|_\infty^2 \frac{1}{1 - 2CC_N} \sum_{k=0}^\infty \frac{t^k D_N^k (2C_N)^{-k} (2CC_N)^k}{k!} < \infty.$$

This proves the theorem. $\qquad\square$

Next we establish the lemma that is used in the proof of Theorem 3.2.

Lemma 3.3. *Let μ satisfy the condition (2.4). For any $N > 0$ we let D_N and C_N be given by (3.10). Then we have*

$$\int_{\mathbb{R}^{nd}} \int_{S_{t,n}} e^{-\sum_{i=1}^n w_i |\xi_i|^2} dw \prod_{i=1}^n \mu(d\xi_i) \leq \sum_{k=0}^n \binom{n}{k} \frac{t^k}{k!} D_N^k (2C_N)^{n-k}.$$

Proof. By our assumption (2.4), C_N is finite for all positive N. Let I be a subset of $\{1, 2, \ldots, n\}$ and $I^c = \{1, 2, \ldots, n\} \setminus I$. Then we have

$$\int_{\mathbb{R}^{nd}} \int_{S_{t,n}} \prod_{i=1}^n e^{-w_i |\xi_i|^2} dw\, \mu(d\xi)$$

$$= \int_{\mathbb{R}^{nd}} \int_{S_{t,n}} \prod_{i=1}^n e^{-w_i |\xi_i|^2} (\mathbf{1}_{\{|\xi_i| \leq N\}} + \mathbf{1}_{\{|\xi_i| > N\}})\, dw\, \mu(d\xi)$$

$$= \sum_{I \subset \{1,2,\ldots,n\}} \int_{\mathbb{R}^{nd}} \int_{S_{t,n}} \prod_{i \in I} e^{-w_i |\xi_i|^2} \mathbf{1}_{\{|\xi_i| \leq N\}} \times \prod_{j \in I^c} e^{-w_j |\xi_j|^2} \mathbf{1}_{\{|\xi_j| \geq N\}}\, dw\, \mu(d\xi).$$

For the indices i in the set I we estimate $e^{-w_j |\xi_j|^2}$ by 1. Then, using the inclusion

$$S_{t,n} \subset S_t^I \times S_t^{I^c},$$

where $S_t^I = \{(w_i, i \in I) : w_i \geq 0, \sum_{i \in I} w_i \leq t\}$ and $S_t^{I^c} = \{(w_i, i \in I^c) : w_i \geq 0, \sum_{i \in I^c} w_i \leq t\}$ we obtain

$$\int_{\mathbb{R}^{nd}} \int_{S_{t,n}} \prod_{i=1}^n e^{-w_i |\xi_i|^2} dw\, \mu(d\xi)$$

$$\leq \sum_{I \subset \{1,2,\cdots,n\}} \int_{\mathbb{R}^{nd}} \int_{S_t^I \times S_t^{I^c}} \prod_{i \in I} \mathbf{1}_{\{|\xi_i| \leq N\}} \times \prod_{j \in I^c} e^{-w_j |\xi_j|^2} \mathbf{1}_{\{|\xi_j| \geq N\}}\, dw\, \mu(d\xi).$$

Furthermore, one can bound the integral over $S_t^{I^c}$ in the following way

$$\int_{S_t^{I^c}} \prod_{j \in I^c} e^{-w_j |\xi_j|^2}\, dw \le \int_{[0,t]^{I^c}} \prod_{j \in I^c} e^{-w_j |\xi_j|^2}\, dw = \prod_{j \in I^c} \frac{1 - e^{-t|\xi_j|^2}}{|\xi_j|^2} \le \prod_{j \in I^c} \frac{1}{|\xi_j|^2}.$$

We can thus bound $\int_{\mathbb{R}^{nd}} \int_{S_{t,n}} \prod_{i=1}^{n} e^{-w_i |\xi_i|^2}\, dw\, \mu(d\xi)$ by:

$$\sum_{I \subset \{1,2,\cdots,n\}} \frac{t^{|I|}}{|I|!} \big(\mu\{\xi \in \mathbb{R}^d : |\xi| \le N\} \big)^{|I|} 2^{|I^c|} \int_{|\xi_j| > N, \forall j \in I^c} \prod_{j \in I^c} \frac{\mu(d\xi_j)}{|\xi_j|^2}$$

$$= \sum_{I \subset \{1,2,\cdots,n\}} \frac{t^{|I|}}{|I|!} D_N^{|I|} (2C_N)^{|I^c|} = \sum_{k=0}^{n} \binom{n}{k} \frac{t^k}{k!} D_N^k (2C_N)^{n-k},$$

which is our claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.2 Feynman-Kac formula for the moments

Our next objective is to find a formula for the moments of the mild solution to equation (3.1). For any $\delta > 0$, we define the function $\varphi_\delta(t) = \frac{1}{\delta} \mathbf{1}_{[0,\delta]}(t)$ for $t \in \mathbb{R}$. Then, $\varphi_\delta(t) p_\varepsilon(x)$ provides an approximation of the Dirac delta function $\delta_0(t,x)$ as ε and δ tend to zero.

We set

$$\dot{W}_{t,x}^{\varepsilon,\delta} = \int_0^t \int_{\mathbb{R}^d} \varphi_\delta(t - s) p_\varepsilon(x - y) W(ds, dy). \tag{3.12}$$

Now we consider the approximation of equation (3.1) defined by

$$\frac{\partial u_{t,x}^{\varepsilon,\delta}}{\partial t} = \frac{1}{2} \Delta u_{t,x}^{\varepsilon,\delta} + u_{t,x}^{\varepsilon,\delta} \diamond \dot{W}_{t,x}^{\varepsilon,\delta}. \tag{3.13}$$

We recall that the Wick product $u_{t,x}^{\varepsilon,\delta} \diamond \dot{W}_{t,x}^{\varepsilon,\delta}$ is well defined as a square integrable random variable provided the random variable $u_{t,x}^{\varepsilon,\delta}$ belongs to the space $\mathbb{D}^{1,2}$ (see (2.12)), and in this case we have

$$u_{s,y}^{\varepsilon,\delta} \diamond \dot{W}_{s,y}^{\varepsilon,\delta} = \int_0^s \int_{\mathbb{R}^d} \varphi_\delta(s - r) p_\varepsilon(y - z) u_{s,y}^{\varepsilon,\delta} \delta W_{r,z}. \tag{3.14}$$

Furthermore, the mild or evolution version of (3.13) is

$$u_{t,x}^{\varepsilon,\delta} = p_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x - y) u_{s,y}^{\varepsilon,\delta} \diamond \dot{W}_{s,y}^{\varepsilon,\delta} ds\, dy. \tag{3.15}$$

Substituting (3.14) into (3.15), and formally applying Fubini's theorem yields

$$u_{t,x}^{\varepsilon,\delta} = p_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} \left(\int_0^t \int_{\mathbb{R}^d} p_{t-s}(x - y) \varphi_\delta(s - r) p_\varepsilon(y - z) u_{s,y}^{\varepsilon,\delta} ds\, dy \right) \delta W_{r,z}. \tag{3.16}$$

This leads to the following definition.

Definition 3.4. *An adapted random field $u^{\varepsilon,\delta} = \{u_{t,x}^{\varepsilon,\delta}; t \ge 0, x \in \mathbb{R}^d\}$ is a mild solution to equation (3.13) if for each $(r,z) \in [0,t] \times \mathbb{R}^d$ the integral*

$$Y_{r,z}^{t,x} = \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x - y) \varphi_\delta(s - r) p_\varepsilon(y - z) u_{s,y}^{\varepsilon,\delta} ds\, dy$$

exists and $Y^{t,x}$ is a Skorohod integrable process such that (3.16) holds for each (t,x).

Notice that according to relation (2.8), the above definition is equivalent to saying that $u_{t,x}^{\varepsilon,\delta} \in L^2(\Omega)$, and for any random variable $F \in \mathbb{D}^{1,2}$, we have

$$\mathbf{E}\left[F u_{t,x}^{\varepsilon,\delta}\right] = \mathbf{E}[F] p_t u_0(x) + \mathbf{E}\left[\langle Y^{t,x}, DF\rangle_{\mathcal{H}}\right]. \tag{3.17}$$

In order to derive a Feynman-Kac formula for the moment of order $k \geq 2$ of the solution to equation (3.1) we need to introduce k independent d-dimensional Brownian motions B^j, $j = 1, \ldots, k$, which are independent of the noise W driving the equation. We shall study the probabilistic behavior of some random variables with double randomness, and we thus introduce some additional notation:

Notation 3.5. *We denote by* $\mathbf{P}, \mathbf{E}$ *the probability and expectation with respect to the annealed randomness concerning the couple* (B, W), *where* $B = (B^1, \ldots, B^k)$, *while we set respectively* $\mathbf{E}_B$ *and* $\mathbf{E}_W$ *for the expectation with respect to one randomness only.*

With this notation in mind, define

$$u_{t,x}^{\varepsilon,\delta} = \mathbf{E}_B\left[\exp\left(W(A_{t,x}^{\varepsilon,\delta}) - \frac{1}{2}\alpha_{t,x}^{\varepsilon,\delta}\right)\right], \tag{3.18}$$

where

$$A_{t,x}^{\varepsilon,\delta}(r,y) = \frac{1}{\delta}\left(\int_0^{\delta \wedge (t-r)} p_\varepsilon(B_{t-r-s}^x - y)ds\right)\mathbf{1}_{[0,t]}(r), \quad \text{and} \quad \alpha_{t,x}^{\varepsilon,\delta} = \|A_{t,x}^{\varepsilon,\delta}\|_{\mathcal{H}}^2, \tag{3.19}$$

for a standard d-dimensional Brownian motion B independent of W. Then one can prove that $u_{t,x}^{\varepsilon,\delta}$ is a mild solution to equation (3.13) in the sense of Definition 3.4. The proof is similar to the proof of Proposition 5.2 in [27] and we omit the details.

The next theorem asserts that the random variables $u_{t,x}^{\varepsilon,\delta}$ have moments of all orders, uniformly bounded in ε and δ, and converge to the mild solution of equation (3.1), which is unique by Theorem 3.2, as δ and ε tend to zero. Moreover, it provides an expression for the moments of the mild solution of equation (3.1).

Theorem 3.6. *Suppose* γ *is locally integrable and* μ *satisfies Hypothesis 2.1. Then for any integer* $k \geq 1$ *we have*

$$\sup_{\varepsilon,\delta} \mathbf{E}\left[|u_{t,x}^{\varepsilon,\delta}|^k\right] < \infty, \tag{3.20}$$

the limit $\lim_{\varepsilon\downarrow 0} \lim_{\delta\downarrow 0} u_{t,x}^{\varepsilon,\delta}$ *exists in* L^p *for all* $p \geq 1$, *and it coincides with the mild solution* u *of equation (3.1). Furthermore, we have for any integer* $k \geq 2$

$$\mathbf{E}\left[u_{t,x}^k\right] = \mathbf{E}_B\left[\prod_{i=1}^k u_0(B_t^i + x) \exp\left(\sum_{1 \leq i < j \leq k}\int_0^t\int_0^t \gamma(s-r)\Lambda(B_s^i - B_r^j)dsdr\right)\right], \tag{3.21}$$

where $\{B^j; j = 1, \ldots, k\}$ *is a family of* d-*dimensional independent standard Brownian motions independent of* W.

Proof. To simplify the proof we assume that u_0 is identically one. Fix an integer $k \geq 2$. Using (3.18) we have

$$\mathbf{E}\left[\left(u_{t,x}^{\varepsilon,\delta}\right)^k\right] = \mathbf{E}_W\left[\prod_{j=1}^k \mathbf{E}_B\left[\exp\left(W(A_{t,x}^{\varepsilon,\delta,B^j}) - \frac{1}{2}\alpha_{t,x}^{\varepsilon,\delta,B^j}\right)\right]\right],$$

where for any $j = 1, \ldots, k$, $A_{t,x}^{\varepsilon,\delta,B^j}$ and $\alpha_{t,x}^{\varepsilon,\delta,B^j}$ are evaluations of (3.19) using the Brownian motion B^j. Therefore, since $W(A_{t,x}^{\varepsilon,\delta,B^j})$ is a Gaussian random variable conditionally on

B, we obtain

$$
\begin{aligned}
\mathbf{E}\left[\left(u_{t,x}^{\varepsilon,\delta}\right)^k\right] &= \mathbf{E}_B\left[\exp\left(\frac{1}{2}\|\sum_{j=1}^k A_{t,x}^{\varepsilon,\delta,B^j}\|_{\mathcal{H}}^2 - \frac{1}{2}\sum_{j=1}^k \alpha_{t,x}^{\varepsilon,\delta,B^j}\right)\right] \\
&= \mathbf{E}_B\left[\exp\left(\frac{1}{2}\|\sum_{j=1}^k A_{t,x}^{\varepsilon,\delta,B^j}\|_{\mathcal{H}}^2 - \frac{1}{2}\sum_{j=1}^k \|A_{t,x}^{\varepsilon,\delta,B^j}\|_{\mathcal{H}}^2\right)\right] \\
&= \mathbf{E}_B\left[\exp\left(\sum_{1\le i<j\le k}\langle A_{t,x}^{\varepsilon,\delta,B^i}, A_{t,x}^{\varepsilon,\delta,B^j}\rangle_{\mathcal{H}}\right)\right].
\end{aligned}
\tag{3.22}
$$

Let us now evaluate the quantities $\langle A_{t,x}^{\varepsilon,\delta,B^i}, A_{t,x}^{\varepsilon,\delta,B^j}\rangle_{\mathcal{H}}$ above: by the definition of $A_{t,x}^{\varepsilon,\delta,B^i}$, for any $i\ne j$ we have

$$
\langle A_{t,x}^{\varepsilon,\delta,B^i}, A_{t,x}^{\varepsilon,\delta,B^j}\rangle_{\mathcal{H}} = \int_0^t\int_0^t\int_{\mathbb{R}^d}\mathcal{F}A_{t,x}^{\varepsilon,\delta,B^i}(u,\cdot)(\xi)\overline{\mathcal{F}A_{t,x}^{\varepsilon,\delta,B^j}}(v,\cdot)(\xi)\gamma(u-v)\mu(d\xi)dudv.
\tag{3.23}
$$

On the other hand, for $u\in[0,t]$ we can write

$$
\begin{aligned}
\mathcal{F}A_{t,x}^{\varepsilon,\delta,B^i}(u,\cdot)(\xi) &= \frac{1}{\delta}\int_0^{\delta\wedge(t-u)}\mathcal{F}p_\varepsilon(B_{t-u-s}^i+x-\cdot)(\xi)ds \\
&= \frac{1}{\delta}\int_0^{\delta\wedge(t-u)}\exp\left(-\frac{\varepsilon^2|\xi|^2}{2}+\imath\left\langle\xi, B_{t-u-s}^i+x\right\rangle\right)ds.
\end{aligned}
$$

Thus

$$
\langle A_{t,x}^{\varepsilon,\delta,B^i}, A_{t,x}^{\varepsilon,\delta,B^j}\rangle_{\mathcal{H}}
\tag{3.24}
$$
$$
= \int_{\mathbb{R}^d}\left(\int_0^t\int_0^t\left(\frac{1}{\delta^2}\int_0^{\delta\wedge v}\int_0^{\delta\wedge u}e^{\imath\left\langle\xi, B_{u-s_1}^i-B_{v-s_2}^j\right\rangle}ds_1ds_2\right)\gamma(u-v)dudv\right)\times e^{-\varepsilon^2|\xi|^2}\mu(d\xi),
$$

and we divide the proof in several steps.

Step 1: We claim that,

$$
\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}\langle A_{t,x}^{\varepsilon,\delta,B^i}, A_{t,x}^{\varepsilon,\delta,B^j}\rangle_{\mathcal{H}} = \int_0^t\int_0^t\gamma(u-v)\Lambda(B_u^i-B_v^j)dudv,
\tag{3.25}
$$

where the convergence holds in $L^1(\Omega)$. Notice first that the right-hand side of equation (3.25) is finite almost surely because

$$
\mathbf{E}_B\left[\int_0^t\int_0^t\gamma(u-v)\Lambda(B_u^i-B_v^j)dudv\right] = \int_0^t\int_0^t\int_{\mathbb{R}^d}\gamma(u-v)e^{-\frac{1}{2}(u+v)|\xi|^2}\mu(d\xi)dudv
$$

and we show that this is finite making the change of variables $x=u-v$, $y=u+v$, and using our hypothesis on γ and μ like in the proof of Theorem 3.2.

In order to show the convergence (3.25) we first let δ tend to zero. Then, owing to the continuity of B and applying some dominated convergence arguments to (3.24), we obtain the following limit almost surely and in $L^1(\Omega)$

$$
\lim_{\delta\downarrow 0}\langle A_{t,x}^{\varepsilon,\delta,B^i}, A_{t,x}^{\varepsilon,\delta,B^j}\rangle_{\mathcal{H}} = \int_{\mathbb{R}^d}\left(\int_0^t\int_0^t e^{\imath\left\langle\xi, B_u^i-B_v^j\right\rangle}\gamma(u-v)dudv\right)e^{-\varepsilon^2|\xi|^2}\mu(d\xi).
\tag{3.26}
$$

Finally, it is easily checked that the right-hand side of (3.26) converges in $L^1(\Omega)$ to the right-hand side of (3.25) as ε tends to zero, by means of a simple dominated convergence argument again.

Step 2: For notational convenience, we denote by B and $\widetilde{B}$ two independent d-dimensional Brownian motions, and $\mathbf{E}$ will denote here the expectation with respect to both B and $\widetilde{B}$. We claim that for any $\lambda > 0$

$$\sup_{\varepsilon,\delta} \mathbf{E}\left[\exp\left(\lambda \left\langle A_{t,x}^{\varepsilon,\delta,B}, A_{t,x}^{\varepsilon,\delta,\widetilde{B}}\right\rangle_{\mathcal{H}}\right)\right] < \infty\,. \tag{3.27}$$

Indeed, starting from (3.24), making the change of variables $u - s_1 \to u$, $v - s_2 \to v$, assuming $\delta \le t$, and using Fubini's theorem, we can write

$$\left\langle A_{t,x}^{\varepsilon,\delta,B}, A_{t,x}^{\varepsilon,\delta,\widetilde{B}}\right\rangle_{\mathcal{H}} = \frac{1}{\delta^2}\int_0^\delta \int_0^\delta \int_0^{t-s_1}\int_0^{t-s_2}\int_{\mathbb{R}^d} \exp\left(-\imath(B_u - \widetilde{B}_v)\cdot \xi\right)$$
$$\times \exp(-\varepsilon|\xi|^2)\gamma(u+s_1-v-s_2)\,\mu(d\xi)\,dudvds_1ds_2\,.$$

We now control the moments of $\langle A_{t,x}^{\varepsilon,\delta,B}, A_{t,x}^{\varepsilon,\delta,\tilde{B}}\rangle_{\mathcal{H}}$ in order to reach exponential integrability:

$$\left\langle A_{t,x}^{\varepsilon,\delta,B}, A_{t,x}^{\varepsilon,\delta,\widetilde{B}}\right\rangle_{\mathcal{H}}^n = \frac{1}{\delta^{2n}}\int_{O_{\delta,n}}\int_{\mathbb{R}^{dn}}\exp\left(-\imath\sum_{l=1}^n (B_{u_l} - \widetilde{B}_{v_l})\cdot \xi_l\right)$$
$$\times\, e^{-\varepsilon\sum_{l=1}^n |\xi_l|^2}\prod_{l=1}^n \gamma(u_l + s_l - v_l - \widetilde{s}_l)\,\mu(d\xi)\,dsd\widetilde{s}dudv, \quad (3.28)$$

where $\mu(d\xi) = \prod_{l=1}^n \mu(d\xi_l)$, the differentials $ds, d\widetilde{s}, du, dv$ are defined similarly, and

$$O_{\delta,n} = \{(s,\widetilde{s},u,v);\, 0 \le s_l, \widetilde{s}_l \le \delta,\, 0 \le u_l \le t - s_l,\, 0 \le v_l \le t - \widetilde{s}_l,\, \text{for all } 1 \le l \le n\}\,.$$

Moreover, we have:

$$\mathbf{E}\left[\exp\left(-\imath\sum_{l=1}^n (B_{u_l} - \widetilde{B}_{v_l})\cdot \xi_l\right)\right] = \exp\left(-\frac{1}{2}\mathbf{Var}\left(\sum_{l=1}^n (B_{u_l} - \widetilde{B}_{v_l})\cdot \xi_l\right)\right) \tag{3.29}$$
$$= \exp\left(-\frac{1}{2}\sum_{1\le i,j\le n}(u_i \wedge u_j + v_i \wedge v_j)\xi_i\cdot\xi_j\right)\,.$$

Taking into account the fact that γ is locally integrable, this yields

$$\mathbf{E}\left[\left\langle A_{t,x}^{\varepsilon,\delta,B}, A_{t,x}^{\varepsilon,\delta,\widetilde{B}}\right\rangle_{\mathcal{H}}^n\right]$$
$$\le C^n\int_{[0,t]^{2n}}\int_{\mathbb{R}^{dn}}\exp\left(-\frac{1}{2}\sum_{1\le i,j\le n}(s_i\wedge s_j + \widetilde{s}_i \wedge \widetilde{s}_j)\xi_i\cdot\xi_j\right)\mu(d\xi)dsd\widetilde{s}$$
$$\le C^n\int_{\mathbb{R}^{dn}}\int_{[0,t]^n}\exp\left(-\sum_{1\le i,j\le n}(s_i\wedge s_j)\,\xi_i\cdot\xi_j\right)ds\mu(d\xi)\,.$$

Since

$$\int_{\mathbb{R}^{dn}}\exp\left(-\sum_{1\le i,j\le n}(s_i\wedge s_j)\xi_i\cdot\xi_j\right)\mu(d\xi)$$

is a symmetric function of $s_1, s_2, \dots, s_n$, we can restrict our integral to $T_n(t) = \{0 < s_1 < s_2 < \cdots < s_n < t\}$. Hence, using the convention $s_0 = 0$, we have

$$\mathbf{E}\left[\left\langle A_{t,x}^{\varepsilon,\delta,B}, A_{t,x}^{\varepsilon,\delta,\widetilde{B}}\right\rangle_{\mathcal{H}}^n\right] \le C^n n! \int_{\mathbb{R}^{dn}}\int_{T_n(t)}\exp\left(-\sum_{1\le i,j\le n}(s_i\wedge s_j)\xi_i\cdot\xi_j\right)ds\mu(d\xi) \quad (3.30)$$
$$= C^n n! \int_{\mathbb{R}^{dn}}\int_{T_n(t)}\exp\left(-\sum_{i=1}^n (s_i - s_{i-1})|\xi_i + \cdots + \xi_n|^2\right)ds\mu(d\xi)\,.$$

Thus, using the same argument as in the proof of the estimate (3.7), we end up with

$$
\mathbf{E}\left[\left\langle A_{t,x}^{\varepsilon,\delta,B}, A_{t,x}^{\varepsilon,\delta,\widetilde{B}}\right\rangle_{\mathcal{H}}^n\right] \leq C^n n! \int_{T_n(t)} \prod_{i=1}^n \left(\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} e^{-(s_i - s_{i-1})|\xi_i + \eta|^2} \mu(d\xi)\right) ds
$$

$$
\leq C^n n! \int_{T_n(t)} \prod_{i=1}^n \left(\int_{\mathbb{R}^d} e^{-(s_i - s_{i-1})|\xi_i|^2} \mu(d\xi_i)\right) ds .
$$

Making the change of variable $w_i = s_i - s_{i-1}$, the above integral is equal to

$$
C^n n! \int_{S_{t,n}} \int_{\mathbb{R}^{dn}} \prod_{i=1}^n e^{-w_i|\xi_i|^2} \mu(d\xi) dw \leq C^n n! \sum_{k=0}^n \binom{n}{k} \frac{t^k}{k!} D_N^k (2C_N)^{n-k},
$$

where we have resorted to Lemma 3.3 for the last inequality. Therefore,

$$
\frac{1}{n!}\mathbf{E}\left[\left\langle A_{t,x}^{\varepsilon,\delta,B}, A_{t,x}^{\varepsilon,\delta,\widetilde{B}}\right\rangle_{\mathcal{H}}^n\right] \leq C^n \sum_{k=0}^n \binom{n}{k} \frac{t^k}{k!} D_N^k (2C_N)^{n-k},
$$

which is exactly the right hand side of (3.11). Thus, along the same lines as in the proof of Theorem 3.2, we get

$$
\mathbf{E}\left[\exp\left(\lambda \left\langle A_{t,x}^{\varepsilon,\delta,B}, A_{t,x}^{\varepsilon,\delta,\widetilde{B}}\right\rangle_{\mathcal{H}}\right)\right] = \sum_{n=0}^\infty \frac{\lambda^n}{n!} \mathbf{E}\left[\left\langle A_{t,x}^{\varepsilon,\delta,B}, A_{t,x}^{\varepsilon,\delta,\widetilde{B}}\right\rangle_{\mathcal{H}}^n\right] < \infty,
$$

which completes the proof of (3.27).

Step 3: Starting from (3.22), (3.25) and (3.27) we deduce that $\mathbf{E}[(u_{t,x}^{\varepsilon,\delta})^k]$ converges as δ and ε tend to zero to the right-hand side of (3.21). On the other hand, we can also write

$$
\mathbf{E}\left[u_{t,x}^{\varepsilon,\delta} u_{t,x}^{\varepsilon',\delta'}\right] = \mathbf{E}_B\left[\exp\left(\langle A_{t,x}^{\varepsilon,\delta,B^1}, A_{t,x}^{\varepsilon',\delta',B^2}\rangle_{\mathcal{H}}\right)\right] .
$$

As before we can show that this converges as $\varepsilon, \delta, \varepsilon', \delta'$ tend to zero. So, $u_{t,x}^{\varepsilon,\delta}$ converges in L^2 to some limit $v_{t,x}$, and the limit is actually in L^p, for all $p \geq 1$. Moreover, $\mathbf{E}[v_{t,x}^k]$ equals to the right hand side of (3.21). Finally, letting δ and ε tend to zero in equation (3.17) we get

$$
\mathbf{E}[F v_{t,x}] = \mathbf{E}[F] + \mathbf{E}\left[\langle DF, vp_{t-\cdot}(x - \cdot)\rangle_{\mathcal{H}}\right]
$$

which implies that the process v is the solution of equation (3.1), and by the uniqueness of the solution we have $v = u$. $\qquad\square$

Remark 3.7. If the space dimension is 1, we can consider equation (3.1) assuming that the time covariance function is $\gamma(t) = H(2H - 1)|t|^{2H-2}$, $\frac{1}{2} < H < 1$, and the noise is white in space, which means $\Lambda(x)$ is the Dirac delta function $\delta_0(x)$. The integral form of this Gaussian noise is a two-parameter process which is a Brownian motion in space and a fractional Brownian motion with Hurst parameter H in time. This equation has been studied in [27], where the existence of a unique mild solution has been proved, and the following formula for the moments of the solution has been obtained

$$
\mathbf{E}\left[u_{t,x}^k\right]
$$
$$
= \mathbf{E}_B\left[\prod_{i=1}^k u_0(B_t^i + x) \exp\left(\alpha_H \sum_{1 \leq i < j \leq k} \int_0^t \int_0^t |s - r|^{2H-2} \delta_0(B_s^i - B_r^j) ds dr\right)\right], \quad (3.31)
$$

where $\alpha_H = H(2H - 1)$. Notice that in the above expression the exponent is a sum of weighted intersection local times.

3.3 Time independent noise

In this section we consider the following stochastic heat equation in the Skorohod sense driven by the multiplicative time independent noise introduced in Section 2.2:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u \diamond \frac{\partial^d W}{\partial x_1 \cdots \partial x_d}\,. \tag{3.32}$$

The notion of mild solution based on the Skorohod integral is similar to Definition 3.1.

Definition 3.8. *An adapted random field $u = \{u_{t,x}; t \geq 0, x \in \mathbb{R}^d\}$ such that $\mathbf{E}[u_{t,x}^2] < \infty$ for all (t,x) is a mild solution to equation (3.32) with initial condition $u_0 \in C_b(\mathbb{R}^d)$, if for any $0 \leq s \leq t, x \in \mathbb{R}^d$, the process $\{p_{t-s}(x-y)u_{s,y}; y \in \mathbb{R}^d\}$ is Skorohod integrable in the sense given by relation (2.16), and the following equation holds:*

$$u_{t,x} = p_t u_0(x) + \int_0^t \left(\int_{\mathbb{R}^d} p_{t-s}(x-y)u_{s,y}\delta W_y \right) ds. \tag{3.33}$$

Suppose that $u = \{u_{t,x}; t \geq 0, x \in \mathbb{R}^d\}$ is a mild solution to equation (3.32). Then for any fixed (t,x), the random variable $u_{t,x}$ admits the following Wiener chaos expansion:

$$u_{t,x} = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t,x)), \tag{3.34}$$

where for each (t,x), $f_n(\cdot,t,x)$ is a symmetric element in $\mathcal{H}^{\otimes n}$. Notice that here the space $\mathcal{H}$ contains functions of the space variable y only. Using an iteration procedure similar to the one described at Section 3.1, one can find the explicit formula for the kernels f_n for $n \geq 1$:

$$f_n(x_1,\ldots,x_n,t,x)$$
$$= \frac{1}{n!} \int_{[0,t]^n} p_{t-s_{\sigma(n)}}(x - x_{\sigma(n)}) \cdots p_{s_{\sigma(2)}-s_{\sigma(1)}}(x_{\sigma(2)} - x_{\sigma(1)})\, p_{s_{\sigma(1)}} u_0(x_{\sigma(1)})\, ds_1 \cdots ds_n\,,$$

where σ denotes the permutation of $\{1,2,\ldots,n\}$ such that $0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t$. Then, to show the existence and uniqueness of the solution it suffices to show that for all (t,x) we have

$$\sum_{n=0}^{\infty} n! \|f_n(\cdot,t,x)\|_{\mathcal{H}^{\otimes n}}^2 < \infty\,. \tag{3.35}$$

Theorem 3.9. *Assume that μ satisfies Hypothesis 2.1. Then (3.35) holds for each (t,x) and equation (3.32) has a unique mild solution.*

The proof of this theorem is analogous to the proof of Theorem 3.2 and is omitted for sake of conciseness. As in the previous subsection, we can deduce the following moment formula for the solution to equation (3.32).

$$\mathbf{E}\left[u_{t,x}^k\right] = \mathbf{E}_B\left[\prod_{i=1}^{k} u_0(B_t^i + x)\exp\left(\sum_{1 \leq i < j \leq k} \int_0^t \int_0^t \Lambda(B_s^i - B_r^j)dsdr\right)\right], \tag{3.36}$$

where B^i, $i = 1,\ldots,k$, are d-dimensional independent Brownian motions.

4 Feynman-Kac functional

In this section we construct a candidate solution for equation (1.1) using a suitable version of Feynman-Kac formula. The construction has been inspired by the approach developed in [29] for the case of fractional noises. We will establish the existence and Hölder continuity properties of the Feynman-Kac functional.

4.1 Construction of the Feynman-Kac functional

We first consider the time dependent noise introduced in Section 2.1, and later we deal with the time independent noise introduced in Section 2.2.

4.1.1 Time dependent noise

Suppose first that W is the time dependent noise introduced in Section 2.1. If the initial condition of equation (1.1) is a continuous and bounded function u_0, analogously to [29] we define

$$u_{t,x} = \mathbf{E}_B \left[u_0(B_t^x) \exp \left(\int_0^t \int_{\mathbb{R}^d} \delta_0(B_{t-r}^x - y)W(dr, dy) \right) \right], \tag{4.1}$$

where B^x is a d-dimensional Brownian motion independent of W and starting at $x \in \mathbb{R}^d$. Our first goal is thus to give a meaning to the functional

$$V_{t,x} = \int_0^t \int_{\mathbb{R}^d} \delta_0(B_{t-r}^x - y)W(dr, dy) \tag{4.2}$$

appearing in the exponent of the Feynman-Kac formula (4.1). To this aim, like in the case of the formula for moments (see (3.12)), we will proceed by approximation. Namely, we will approximate V by the process

$$V_{t,x}^\varepsilon = \int_0^t \int_{\mathbb{R}^d} p_\varepsilon(B_{t-r}^x - y)W(dr, dy), \quad \varepsilon > 0, \tag{4.3}$$

which is well defined as a Wiener integral for a fixed path of the Brownian motion B. The convergence of the approximation V^ε is obtained in the next proposition, for which we need to impose the following conditions on the function γ and the measure μ.

Hypothesis 4.1. *There exists a constant $0 < \beta < 1$ such that for any $t \in \mathbb{R}$,*

$$0 \le \gamma(t) \le C_\beta |t|^{-\beta} \tag{4.4}$$

for some constant $C_\beta > 0$, and the measure μ satisfies

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\xi|^{2-2\beta}} < \infty. \tag{4.5}$$

Proposition 4.2. *Let $V_{t,x}^\varepsilon$ be the functional defined in (4.3) and suppose that Hypothesis 4.1 holds. Then for fixed $t \ge 0$ and $x \in \mathbb{R}^d$, the random variable $V_{t,x}^\varepsilon$ converges in $L^2(\Omega)$ towards a functional denoted by $V_{t,x}$. Moreover, conditioned by B, $V_{t,x}$ is a Gaussian random variable with mean 0 and variance*

$$\mathbf{Var}_W(V_{t,x}) = \int_0^t \int_0^t \gamma(r - s)\Lambda(B_r - B_s)drds. \tag{4.6}$$

Proof. Our first goal is to find

$$\lim_{\varepsilon_1, \varepsilon_2 \to 0} \mathbf{E} \left[V_{t,x}^{\varepsilon_1} V_{t,x}^{\varepsilon_2} \right]. \tag{4.7}$$

To this aim, we set $A_{t,x}^\varepsilon(r, y) = p_\varepsilon(B_{t-r}^x - y)\mathbf{1}_{[0,t]}(r)$. Then

$$
\begin{aligned}
\mathbf{E} \left[V_{t,x}^{\varepsilon_1} V_{t,x}^{\varepsilon_2} \right] &= \mathbf{E} \left[W \left(A_{t,x}^{\varepsilon_1} \right) W \left(A_{t,x}^{\varepsilon_2} \right) \right] = \mathbf{E}_B \left[\left\langle A_{t,x}^{\varepsilon_1}, A_{t,x}^{\varepsilon_2} \right\rangle_{\mathcal{H}} \right] \\
&= \mathbf{E}_B \left[\int_0^t \int_0^t \int_{\mathbb{R}^d} \mathcal{F}A_{t,x}^{\varepsilon_1}(u, \cdot)(\xi) \overline{\mathcal{F}A_{t,x}^{\varepsilon_2}}(v, \cdot)(\xi)\gamma(u - v)\mu(d\xi)dudv \right].
\end{aligned}
$$

Furthermore, we can write for $u \le t$

$$\mathcal{F}A_{t,x}^{\varepsilon_1}(u, \cdot)(\xi) = \mathcal{F}p_{\varepsilon_1}(B_{t-u}^x - \cdot)(\xi) = e^{-\frac{1}{2}\varepsilon_1^2|\xi|^2 + \iota\langle\xi, B_u^x\rangle},$$

and thus

$$\left\langle A_{t,x}^{\varepsilon_1}, A_{t,x}^{\varepsilon_2}\right\rangle_{\mathcal{H}} = \int_{\mathbb{R}^d}\left(\int_{[0,t]^2} e^{\imath\langle\xi,\, B_v-B_u\rangle}\gamma(u-v)dudv\right)e^{-\frac{1}{2}(\varepsilon_1^2+\varepsilon_2^2)|\xi|^2}\mu(d\xi). \qquad (4.8)$$

This yields

$$\begin{aligned}
\mathbf{E}\left[V_{t,x}^{\varepsilon_1}V_{t,x}^{\varepsilon_2}\right] &= \mathbf{E}_B\left[\left\langle A_{t,x}^{\varepsilon_1}, A_{t,x}^{\varepsilon_2}\right\rangle_{\mathcal{H}}\right] \\
&= \int_{\mathbb{R}^d}\left(\int_{[0,t]^2} e^{-\frac{1}{2}|\xi|^2|v-u|}\gamma(u-v)dudv\right)e^{-\frac{1}{2}(\varepsilon_1^2+\varepsilon_2^2)|\xi|^2}\mu(d\xi). \quad (4.9)
\end{aligned}$$

Set now

$$\sigma_t^2 := \int_{\mathbb{R}^d}\left(\int_{[0,t]^2} e^{-\frac{1}{2}|\xi|^2|v-u|}\gamma(u-v)dudv\right)\mu(d\xi).$$

It is easily seen by direct integration and by using the hypothesis (4.4) that

$$\int_{[0,t]^2} e^{-\frac{1}{2}|\xi|^2|v-u|}\gamma(u-v)dudv \leq c_\beta\int_{[0,t]^2} e^{-\frac{1}{2}|\xi|^2|v-u|}|u-v|^{-\beta}dudv \leq \frac{c}{1+|\xi|^{2-2\beta}}.$$

Thus

$$\sigma_t^2 \leq c\int_{\mathbb{R}^d}\frac{\mu(d\xi)}{1+|\xi|^{2-2\beta}}\,,$$

which is a finite quantity by hypothesis (4.5). As a consequence, for every sequence ε_n converging to zero, $V_{t,x}^{\varepsilon_n}$ converges in L^2 to a limit denoted by $V_{t,x}$ which does not depend on the choice of the sequence ε_n. Finally, by a similar argument, we show (4.6). This completes the proof of the proposition. $\qquad\square$

Remark 4.3. We could also regularize the noise in time, and define

$$V_{t,x}^{\varepsilon,\delta} = W(A_{t,x}^{\varepsilon,\delta}), \qquad (4.10)$$

where $A_{t,x}^{\varepsilon,\delta}$ has been introduced in (3.19). Then it is easy to check that $V_{t,x}^{\varepsilon,\delta}$ converges as δ tends to zero in $L^2(\Omega)$ to $V_{t,x}^\varepsilon$.

In order to give a meaning to formula (4.1) we need to establish the existence of exponential moments for $V_{t,x}$. To complete this task, we need the following lemma.

Lemma 4.4. *Suppose that Hypothesis 4.1 holds. Then for any $\varepsilon > 0$ there exists a constant C_ε such that for any $v > 0$ we have:*

$$\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-\frac{v}{2}|\xi-\eta|^2}\mu(d\xi) \leq C_\varepsilon + \frac{\varepsilon}{v^{1-\beta}}. \qquad (4.11)$$

Proof. The fact that the left hand side of (4.11) is uniformly bounded in η is proven similarly to (3.7), but is included here for sake of readability. Indeed, consider $\eta \in \mathbb{R}^d, v > 0$ and define a function $\varphi_\eta : \mathbb{R}^d \to \mathbb{R}_+$ by $\varphi_\eta(\xi) = e^{-\frac{v}{2}|\xi-\eta|^2}$. Then according to Parseval's identity we have

$$\int_{\mathbb{R}^d}\varphi_\eta(\xi)\,\mu(d\xi) = c\int_{\mathbb{R}^d}\mathcal{F}\varphi_\eta(x)\,\Lambda(x)dx = c\int_{\mathbb{R}^d} v^{-d/2}e^{-\frac{|x|^2}{2v}}e^{\imath\langle x,\eta\rangle}\Lambda(x)dx.$$

We now use the fact that Λ is assumed to be nonnegative in order to get the following uniform bound in η

$$\int_{\mathbb{R}^d}\varphi_\eta(\xi)\,\mu(d\xi) \leq c\int_{\mathbb{R}^d} v^{-d/2}e^{-\frac{|x|^2}{2v}}\Lambda(x)dx = \int_{\mathbb{R}^d}\varphi_0(\xi)\,\mu(d\xi) = \int_{\mathbb{R}^d} e^{-\frac{v}{2}|\xi|^2}\mu(d\xi).$$

To estimate the right-hand side of the above inequality we introduce a constant $M > 0$, whose exact value is irrelevant for our computations, and let $c_{M,1} = \mu(B(0, M))$, where $B(0, M)$ stands for the ball of radius M centered at 0 in $\mathbb{R}^d$. Then the trivial bound $e^{-\frac{v}{2}|\xi|^2} \le 1$ yields

$$\int_{\mathbb{R}^d} e^{-\frac{v}{2}|\xi|^2} \mu(d\xi) \le c_{M,1} + \int_{|\xi|>M} e^{-\frac{v}{2}|\xi|^2} \mu(d\xi).$$

Invoking the fact that the function $x \mapsto x^{1-\beta} e^{-x}$ is bounded on $\mathbb{R}_+$, we thus get

$$\int_{|\xi|>M} e^{-\frac{v}{2}|\xi|^2} \mu(d\xi) \le \frac{c_2}{v^{1-\beta}} \int_{|\xi|>M} \frac{\mu(d\xi)}{|\xi|^{2-2\beta}} \le \frac{c_2}{v^{1-\beta}} \int_{|\xi|>M} \frac{\mu(d\xi)}{1 + |\xi|^{2-2\beta}}.$$

Summarizing the above, we have obtained that

$$\int_{\mathbb{R}^d} e^{-\frac{v}{2}|\xi-\eta|^2} \mu(d\xi) \le c_{M,1} + \frac{c_2}{v^{1-\beta}} \int_{|\xi|>M} \frac{\mu(d\xi)}{1 + |\xi|^{2-2\beta}},$$

uniformly in $\eta \in \mathbb{R}^d$. Our claim is thus obtained by choosing M large enough so that $c_2 \int_{|\xi|>M} \frac{\mu(d\xi)}{1+|\xi|^{2-2\beta}} \le \varepsilon$, which is possible by hypothesis (4.5). $\qquad\square$

The following elementary integration result will also be crucial for the moment estimates we deduce later.

Lemma 4.5. *Let $\alpha \in (-1+\varepsilon, 1)^m$ with $\varepsilon > 0$ and set $|\alpha| = \sum_{i=1}^m \alpha_i$. Recall (see (3.9)) that $T_m(t) = \{(r_1, r_2, \ldots, r_m) \in \mathbb{R}^m : 0 < r_1 < \cdots < r_m < t\}$. Then there is a constant κ such that*

$$J_m(t, \alpha) := \int_{T_m(t)} \prod_{i=1}^m (r_i - r_{i-1})^{\alpha_i} dr \le \frac{\kappa^m t^{|\alpha|+m}}{\Gamma(|\alpha| + m + 1)},$$

where by convention, $r_0 = 0$.

Proof. Using identities on Beta functions and a recursive algorithm we can snow that

$$J_m(t, \alpha) = \frac{\prod_{i=1}^m \Gamma(\alpha_i + 1) t^{|\alpha|+m}}{\Gamma(|\alpha| + m + 1)},$$

and the result follows thanks to the fact that the Γ function is bounded on $(\varepsilon, 2)$. $\qquad\square$

With these preliminary results in hand, we can now prove the exponential integrability of the random variable $V_{t,x}$ defined in Proposition 4.2.

Theorem 4.6. *Let $V_{t,x}$ be the functional defined in Proposition 4.2, and assume Hypothesis 4.1. Then for any $\lambda \in \mathbb{R}$ and $T > 0$, we have $\sup_{t \in [0,T],\, x \in \mathbb{R}^d} \mathbf{E}[\exp(\lambda V_{t,x})] < \infty$. In particular, the functional (4.1) is well defined.*

Proof. Fix $t > 0$ and $x \in \mathbb{R}^d$. Conditionally to B, the random variable $V_{t,x}$ is Gaussian and centered. From (4.6), we obtain

$$\mathbf{E}\left[\exp(\lambda V_{t,x})\right] = \mathbf{E}_B\left[\exp\left(\frac{\lambda^2}{2} \int_0^t \int_0^t \gamma(r-s)\Lambda(B_r - B_s) dr ds\right)\right] = \mathbf{E}_B\left[\exp\left(\frac{\lambda^2}{2} Y\right)\right],$$

where

$$Y = \int_0^t \int_0^t \gamma(r-s)\Lambda(B_r - B_s) dr ds.$$

In order to show that $\mathbf{E}\left[\exp(\lambda Y)\right] < \infty$ for any $\lambda \in \mathbb{R}$, we are going to use an elaboration of a method introduced by Le Gall [34] (see also [27, 29]). With respect to those

contributions, our case requires a careful handling of the weights Λ and γ. Notice in particular that in our general setting we do not have scaling properties, and some additional work is necessary to overcome this difficulty.
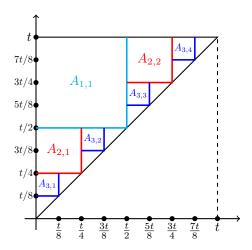
Le Gall's method starts from the following construction: for $n \geq 1$ and $k = 1, \ldots, 2^{n-1}$ we set

$$J_{n,k} := \left[\frac{(2k-2)\,t}{2^n}, \frac{(2k-1)\,t}{2^n} \right), \quad I_{n,k} := \left[\frac{(2k-1)\,t}{2^n}, \frac{2k\,t}{2^n} \right), \quad \text{and} \quad A_{n,k} := J_{n,k} \times I_{n,k}.$$

Notice then that $\{A_{n,k}; \, n \geq 1, k = 1, \ldots, 2^{n-1}\}$ is a partition of the simplex $T_2(t)$, and in addition $I_{n,k-1} \cap I_{n,k} = \varnothing$ and $J_{n,k-1} \cap J_{n,k} = \varnothing$ (see Figure 1 for an illustration). We can thus write

$$Y = \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} a_{n,k}, \quad \text{where} \quad a_{n,k} = \int_{A_{n,k}} \gamma(r - s) \Lambda(B_r - B_s) dr ds \,.$$

Figure 1: Le Gall's partition of $T_2(t)$ into disjoint rectangles of decreasing area.

Observe that for fixed n the random variables $\{a_{n,k}; \, k = 1, \ldots, 2^{n-1}\}$ are independent, owing to the fact that they depend on the increments of B on disjoint sets. Now, thanks to the fact that $J_{n,k} \cap I_{n,k} = \varnothing$, for all $(r,s) \in A_{n,k}$ we can decompose $B_r - B_s$ into $(B_r - B_{\frac{(2k-1)\,t}{2^n}}) - (B_s - B_{\frac{(2k-1)\,t}{2^n}})$, where the two pieces of the difference are independent Brownian motions. Thus the following identity in law holds true:

$$\{B_r - B_s; \, (r,s) \in A_{n,k}\} \stackrel{(d)}{=} \left\{ B_{r - \frac{(2k-1)\,t}{2^n}} - \widetilde{B}_{s - \frac{(2k-1)\,t}{2^n}}; \, (r,s) \in A_{n,k} \right\},$$

where B and $\widetilde{B}$ are two independent Brownian motions. With an additional change of variables $r - \frac{(2k-1)t}{2^n} \mapsto r$ and $\frac{(2k-1)t}{2^n} - s \mapsto s$, this easily yields the following identity

$$
\begin{aligned}
a_{n,k} &\stackrel{(d)}{=} \int_{A_{n,k}} \gamma(r + s) \, \Lambda \left(B_{\frac{(2k-1)t}{2^n} + r} - \widetilde{B}_{\frac{(2k-1)t}{2^n} - s} \right) \, ds dr \\
&\stackrel{(d)}{=} \int_0^{\frac{t}{2^n}} \int_0^{\frac{t}{2^n}} \gamma(r + s) \, \Lambda(B_r + \widetilde{B}_s) \, ds dr := a_n.
\end{aligned}
$$

Summarizing the considerations above, we have found that

$$Y = \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} a_{n,k}, \tag{4.12}$$

where for each $n \geq 1$ the collection $\{a_{n,k}; \ k = 1, \ldots, 2^{n-1}\}$ is a family of independent random variables such that

$$a_{n,k} \stackrel{(d)}{=} a_n, \quad \text{with} \quad a_n = \int_0^{\frac{t}{2^n}} \int_0^{\frac{t}{2^n}} \gamma(r+s) \, \Lambda(B_r + \widetilde{B}_s) \, dsdr,$$

where $B, \widetilde{B}$ are two independent Brownian motions. Notice that the transformation of $B_r - B_s$ into $B_r + \widetilde{B}_s$ we have achieved is essential for our future computations. Indeed, it will be translated into some singularities $(r-s)^{-1}$ in a neighborhood of 0 in $\mathbb{R}_+^2$ becoming some more harmless singularities of the form $(r+s)^{-1}$. The proof is now decomposed in several steps.

Step 1. First we need to estimate the moments of the random variable a_n. We claim that for any $\varepsilon > 0$ there exist constants $C_{\varepsilon,1} > 0$ and $C_2 > 0$ (which depend on t) such that

$$\mathbf{E}[a_n^m] \leq C_{\varepsilon,1} m! \left(\frac{C_2 \varepsilon}{2^n} \right)^m. \tag{4.13}$$

In order to show (4.13), we first write

$$\mathbf{E}\left[a_n^m\right] = \int_{[0,\frac{t}{2^n}]^m} \int_{[0,\frac{t}{2^n}]^m} \prod_{i=1}^m \gamma(r_i + s_i) \mathbf{E}\left[\prod_{i=1}^m \Lambda(B_{r_i} + \widetilde{B}_{s_i}) \right] dsdr \, .$$

Let p_B be the joint density of $(B_{r_1} + \widetilde{B}_{s_1}, \ldots, B_{r_m} + \widetilde{B}_{s_m})$, which is a Schwartz function. Hence, using the Fourier transform and the same considerations as for (3.29), we get

$$\mathbf{E}\left[\prod_{i=1}^m \Lambda(B_{r_i} + \widetilde{B}_{s_i}) \right] = \int_{\mathbb{R}^{md}} \prod_{i=1}^m \Lambda(x_i) p_B(x) dx = \int_{\mathbb{R}^{md}} e^{-\frac{1}{2} \sum_{i,j=1}^m \xi_i \cdot \xi_j (r_i \wedge r_j + s_i \wedge s_j)} \prod_{i=1}^m \mu(d\xi_i) \, .$$

We now proceed as in the proof of Theorem 3.6, with an additional care in the computation of terms. Thanks to our assumption (4.4) on γ and the basic inequality $a + b \geq 2\sqrt{ab}$ for nonnegative a, b, we have

$$\mathbf{E}\left[a_n^m\right] = \int_{[0,\frac{t}{2^n}]^m} \int_{[0,\frac{t}{2^n}]^m} \int_{\mathbb{R}^{md}} e^{-\frac{1}{2} \sum_{i,j=1}^m \xi_i \cdot \xi_j (r_i \wedge r_j + s_i \wedge s_j)} \prod_{i=1}^m \mu(d\xi_i) \prod_{i=1}^m \gamma(r_i + s_i) dsdr$$

$$\leq (2^{-\beta} C_\beta)^m \int_{[0,\frac{t}{2^n}]^m} \int_{[0,\frac{t}{2^n}]^m} \int_{\mathbb{R}^{md}} e^{-\frac{1}{2} \sum_{i,j=1}^m \xi_i \cdot \xi_j (r_i \wedge r_j + s_i \wedge s_j)} \prod_{i=1}^m \mu(d\xi_i) \prod_{i=1}^m (r_i s_i)^{-\frac{\beta}{2}} dsdr,$$

and thus, invoking Cauchy-Schwarz inequality with respect to the measure $\prod_{i=1}^m (r_i s_i)^{-\frac{\beta}{2}} drds$, we end up with

$$\mathbf{E}\left[a_n^m\right] \leq (2^{-\beta} C_\beta)^m \int_{[0,\frac{t}{2^n}]^m} \int_{[0,\frac{t}{2^n}]^m} \left(\int_{\mathbb{R}^{md}} e^{-\sum_{i,j=1}^m \xi_i \cdot \xi_j (r_i \wedge r_j)} \prod_{i=1}^m \mu(d\xi_i) \right)^{\frac{1}{2}}$$

$$\times \left(\int_{\mathbb{R}^{md}} e^{-\sum_{i,j=1}^m \xi_i \cdot \xi_j (s_i \wedge s_j)} \prod_{i=1}^m \mu(d\xi_i) \right)^{\frac{1}{2}} \prod_{i=1}^m (r_i s_i)^{-\frac{\beta}{2}} dsdr.$$

Since in the above expression, both integrals with respect to the measure $\prod_{i=1}^{m} \mu(d\xi_i)$ are symmetric functions of the r_i's and s_i's, we can restrict the integral to the region $T_m(\frac{t}{2^n})$, where $T_m(t)$ has been defined in (3.9). Therefore, similarly to (3.30) and with the convention $r_0 = 0$, we obtain that for any $\varepsilon > 0$ the expectation $\mathbf{E}[a_n^m]$ is bounded by

$$(2^{-\beta}C_\beta)^m (m!)^2 \left(\int_{T_m(\frac{t}{2^n})} \left(\int_{\mathbb{R}^{md}} e^{-\sum_{i=1}^{m}(r_i - r_{i-1})|\xi_i + \cdots + \xi_m|^2} \prod_{i=1}^{m} \mu(d\xi_i) \right)^{\frac{1}{2}} \prod_{i=1}^{m} |r_i|^{-\frac{\beta}{2}} dr \right)^2$$

$$\leq (2^{-\beta}C_\beta)^m (m!)^2 \left(\int_{T_m(\frac{t}{2^n})} \prod_{i=1}^{m} \left(C_\varepsilon + \frac{\varepsilon}{(r_i - r_{i-1})^{1-\beta}} \right)^{\frac{1}{2}} \prod_{i=1}^{m} (r_i - r_{i-1})^{-\frac{\beta}{2}} dr \right)^2,$$

where we have used Lemma 4.4 and we have bounded $r_i^{-\frac{\beta}{2}}$ by $(r_i - r_{i-1})^{-\frac{\beta}{2}}$. We now resort to the inequality $(a+b)^{\frac{1}{2}} \leq a^{\frac{1}{2}} + b^{\frac{1}{2}}$ in order to get

$$\begin{aligned}
\mathbf{E}[a_n^m] &\leq (2^{-\beta}C_\beta)^m (m!)^2 \left(\int_{T_m(\frac{t}{2^n})} \prod_{i=1}^{m} \left(\sqrt{C_\varepsilon} + \frac{\sqrt{\varepsilon}}{(r_i - r_{i-1})^{\frac{1-\beta}{2}}} \right) \prod_{i=1}^{m} (r_i - r_{i-1})^{-\frac{\beta}{2}} dr \right)^2 \\
&= (2^{-\beta}C_\beta)^m (m!)^2 \left(\int_{T_m(\frac{t}{2^n})} \sum_{\theta \in \{0,1\}^m} \prod_{i=1}^{m} C_\varepsilon^{\frac{\theta_i}{2}} \varepsilon^{\frac{1}{2}(1-\theta_i)} (r_i - r_{i-1})^{\frac{\beta-1}{2}(1-\theta_i) - \frac{\beta}{2}} dr \right)^2.
\end{aligned}$$

Hence, a direct application of Lemma 4.5 shows that there exists a positive constant K such that

$$\mathbf{E}[a_n^m] \leq K^m (m!)^2 \left(\sum_{l=0}^{m} \binom{m}{l} C_\varepsilon^{\frac{l}{2}} \varepsilon^{\frac{m-l}{2}} \frac{\left(\frac{t}{2^n}\right)^{\frac{(1-\beta)l}{2} + \frac{m}{2}}}{\Gamma(\frac{(1-\beta)l}{2} + \frac{m}{2} + 1)} \right)^2.$$

We now simply bound $\binom{m}{l}$ by 2^m and recall that $(x/3)^x \leq \Gamma(x+1) \leq x^x$ for $x \geq 0$. Allowing the constant K to change from line to line, this yields

$$\begin{aligned}
\mathbf{E}[a_n^m] &\leq K^m (m!)^2 \left(\sum_{l=0}^{m} C_\varepsilon^{\frac{l}{2}} \varepsilon^{\frac{m-l}{2}} \frac{\left(\frac{t}{2^n}\right)^{\frac{(1-\beta)l}{2}} \left(\frac{t}{2^n}\right)^{\frac{m}{2}}}{(m!)^{\frac{1}{2}} (l!)^{\frac{1-\beta}{2}}} \right)^2 \\
&\leq K^m m! \varepsilon^m \left(\frac{t}{2^n} \right)^m \left(\sum_{l=0}^{\infty} \frac{C_\varepsilon^{\frac{l}{2}} \varepsilon^{-\frac{l}{2}} t^{\frac{1-\beta}{2}l}}{(l!)^{\frac{1-\beta}{2}}} \right)^2.
\end{aligned}$$

This completes the proof of (4.13) with $C_{\varepsilon,1} = \left(\sum_{l=0}^{\infty} C_\varepsilon^{\frac{l}{2}} \varepsilon^{-\frac{l}{2}} t^{\frac{1-\beta}{2}l} (l!)^{\frac{\beta-1}{2}} \right)^2$, which is finite because this series is convergent, and $C_2 = Kt$.

Step 2. We now start from relation (4.13) and prove the finiteness of exponential moments for the random variable Y. It turns out that centering is useful in this context, and we thus define $\bar{a}_{n,k} = a_{n,k} - \mathbf{E}[a_{n,k}]$. Then $\mathbf{E}[\bar{a}_{n,k}] = 0$, and for any integer $m \geq 2$ notice that:

$$\mathbf{E}\left[(\bar{a}_{n,k})^m\right] \leq 2^{m-1} \left(\mathbf{E}\left[a_{n,k}^m\right] + (\mathbf{E}[a_{n,k}])^m \right) \leq 2^m \mathbf{E}[a_{n,k}^m].$$

Also recall that $a_{n,k} \stackrel{(d)}{=} a_n$. Thus, using (4.13)

$$\begin{aligned}
\mathbf{E}\left[\exp(\lambda \bar{a}_{n,k})\right] &= 1 + \sum_{m=2}^{\infty} \frac{\lambda^m}{m!} \mathbf{E}\left[(\bar{a}_{n,k})^m\right] \leq 1 + \sum_{m=2}^{\infty} \frac{(2\lambda)^m}{m!} \mathbf{E}\left[(a_{n,k})^m\right] \\
&\leq 1 + \sum_{m=2}^{\infty} C_{\varepsilon,1} \left(\frac{2C_2 \lambda \varepsilon}{2^n} \right)^m.
\end{aligned}$$

Now choose and fix ε such that $C_2\lambda\varepsilon 2^{-n+1} \le \frac{1}{2}$, and we obtain the bound

$$\mathbf{E}\left[\exp(\lambda\overline{a}_{n,k})\right] \le 1 + \frac{C_{\varepsilon,2}\lambda^2}{2^{2n}}, \qquad (4.14)$$

for some positive constant $C_{\varepsilon,2}$. Next we choose $0 < h < 1$, define $b_N = \prod_{j=2}^{N}\left(1 - 2^{-h(j-1)}\right)$, and notice that $\lim_{N\to\infty} b_N = b_\infty > 0$. Then, by Hölder's inequality, for all $N \ge 2$ we have

$$\mathbf{E}\left[\exp\left(\lambda b_N \sum_{n=1}^{N}\sum_{k=1}^{2^{n-1}}\overline{a}_{n,k}\right)\right] \le \left[\mathbf{E}\left[\exp\left(\lambda b_{N-1}\sum_{n=1}^{N-1}\sum_{k=1}^{2^{n-1}}\overline{a}_{n,k}\right)\right]\right]^{1-2^{-h(N-1)}}$$
$$\times \left[\mathbf{E}\left[\exp\left(\lambda b_N 2^{h(N-1)}\sum_{k=1}^{2^{N-1}}\overline{a}_{N,k}\right)\right]\right]^{2^{-h(N-1)}},$$

and taking into account the independence of the $\{a_{N,k}; k \le 2^{N-1}\}$ plus the identity $a_{N,k} \overset{(d)}{=} a_N$, the above expression is equal to $A_N B_N$ with:

$$A_N = \left(\mathbf{E}\left[\exp\left(\lambda b_{N-1}\sum_{n=1}^{N-1}\sum_{k=1}^{2^{n-1}}\overline{a}_{n,k}\right)\right]\right)^{1-2^{-h(N-1)}}$$
$$B_N = \left(\mathbf{E}\left[\exp\left(\lambda b_N 2^{h(N-1)}\overline{a}_N\right)\right]\right)^{2^{(1-h)(N-1)}}.$$

We now appeal to the estimate (4.14) and the elementary inequality $1 + x \le e^x$, valid for any $x \in \mathbb{R}$. This yields

$$B_N \le \left(1 + C_{\varepsilon,2}b_N^2 2^{-2N}\lambda^2 2^{2h(N-1)}\right)^{2^{(1-h)(N-1)}} \le \exp\left(C_{\varepsilon,3}2^{-N(1-h)}\right),$$

for some positive constant $C_{\varepsilon,3}$. Notice that this is where the centering argument on $a_{n,k}$ is crucial. Indeed, without centering we would get a factor 2^{-N} instead of 2^{-2N} in the left hand side of the above expression, and B_N would not be uniformly bounded. Thus, recursively we get

$$\mathbf{E}\left[\exp\left(\lambda b_N \sum_{n=1}^{N}\sum_{k=1}^{2^{n-1}}\overline{a}_{n,k}\right)\right] \le \exp\left(\sum_{n=2}^{N}C2^{-n(1-h)}\right)\mathbf{E}\left[\exp(\overline{a}_{1,1})\right] < \infty.$$

Recalling now from (4.12) that $Y - \mathbf{E}[Y] = \sum_{n=1}^{\infty}\sum_{k=1}^{2^{n-1}}\overline{a}_{n,k}$ and applying Fatou's lemma, we finally get

$$\mathbf{E}\left[\exp\left(\lambda b_\infty(Y - \mathbf{E}[Y])\right)\right] < \infty,$$

which completes the proof. $\qquad\qquad\square$

Our next result is an approximation result for the Feynman-Kac functional which will be used in the next section (see Theorem 5.3). Towards this aim, for any $\varepsilon, \delta > 0$ we define

$$u_{t,x}^{\varepsilon,\delta} = \mathbf{E}\left[u_0(B_t^x)\exp(V_{t,x}^{\varepsilon,\delta})\right], \qquad (4.15)$$

where $V_{t,x}^{\varepsilon,\delta} = W(A_{t,x}^{\varepsilon,\delta})$ and $A_{t,x}^{\varepsilon,\delta}$ is defined in (3.24).

Proposition 4.7. *For any $p \geq 2$ and $T > 0$ we have*

$$\lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbf{E}\left[|u_{t,x}^{\varepsilon,\delta} - u_{t,x}|^p\right] = 0. \tag{4.16}$$

Proof. First, we recall that (see Proposition 4.2 and Remark 4.3) for any fixed $t \geq 0$ and $x \in \mathbb{R}^d$ the random variable $V_{t,x}^{\varepsilon,\delta}$ converges in $L^2(\Omega)$ to $V_{t,x}$ if we let first δ tend to zero and later ε tend to zero. Then in order to show (4.16) it suffices to check that for any $\lambda \in \mathbb{R}$

$$\sup_{\varepsilon,\delta} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbf{E}\left[\exp\left(\lambda V_{t,x}^{\varepsilon,\delta}\right)\right] < \infty. \tag{4.17}$$

Taking first the expectation with respect to the noise W yields

$$\mathbf{E}\left[\exp\left(\lambda V_{t,x}^{\varepsilon,\delta}\right)\right] = \mathbf{E}_B\left[\exp\left(\frac{\lambda^2}{2}\|A_{t,x}^{\varepsilon,\delta}\|_{\mathcal{H}}^2\right)\right].$$

Expanding the exponential into a power series, we will need to bound the moments of the random variable $\|A_{t,x}^{\varepsilon,\delta}\|_{\mathcal{H}}^2$. To do this, we use formula (3.28) with $B = \widetilde{B}$ and $\varepsilon = \varepsilon'$, $\delta = \delta'$. Computing the mathematical expectation of this expression, we end up with:

$$\mathbf{E}\left[\left\|A_{t,x}^{\varepsilon,\delta}\right\|_{\mathcal{H}}^{2n}\right] = \frac{1}{\delta^{2n}} \int_{O_{\delta,n}} \int_{\mathbb{R}^{dn}} \exp\left(-\frac{1}{2}\sum_{i,j=1}^n \mathbf{E}_B[(B_{u_i}-B_{v_i})(B_{u_j}-B_{v_j})]\langle\xi_i,\xi_j\rangle\right)$$
$$\times e^{-\varepsilon\sum_{l=1}^n |\xi_l|^2}\prod_{l=1}^n \gamma(u_l+s_l-v_l-\widetilde{s}_l)\,\mu(d\xi)\,dsd\widetilde{s}dudv.$$

Thanks to the estimate

$$\sup_{0\leq\delta\leq 1}\frac{1}{\delta^2}\int_0^\delta\int_0^\delta |u+s-v-r|^{-\beta}dsdr \leq c_{T,\beta}|u-v|^{-\beta}, \tag{4.18}$$

which holds for any $u,v\in[0,T]$, and owing to assumption (4.4), we get

$$\mathbf{E}\left[\left\|A_{t,x}^{\varepsilon,\delta}\right\|_{\mathcal{H}}^{2n}\right] \leq c_{T,\beta}^n \mathbf{E}_B\left[\left|\int_0^t\int_0^t |u-v|^{-\beta}\Lambda(B_u-B_v)dudv\right|^n\right]. \tag{4.19}$$

It is now readily checked that (4.17) follows from (4.19) and Theorem 4.6. $\square$

4.1.2 Time independent noise

Suppose that W is the time independent noise introduced in Section 2.2. The Feynman-Kac functional is defined as

$$u_{t,x} = \mathbf{E}\left[u_0(B_t^x)\exp\left(\int_0^t\int_{\mathbb{R}^d}\delta_0(B_r^x-y)W(dy)dr\right)\right], \tag{4.20}$$

where $B^x = \{B_t+x, t\geq 0\}$ is a d-dimensional Brownian motion independent of W, starting from x and $u_0\in C_b(\mathbb{R}^d)$ is the initial condition.

As in the case of a time dependent noise, to give a meaning to this functional for every $t>0$, $x\in\mathbb{R}^d$ and $\varepsilon>0$ we introduce the family of random variables

$$V_{t,x}^\varepsilon = \int_0^t\int_{\mathbb{R}^d} p_\varepsilon(B_r^x-y)W(dy)dr,$$

Then, if the spectral measure of the noise μ satisfies condition (2.4), the family $V_{t,x}^{\varepsilon}$ converges in L^2 to a limit denoted by

$$V_{t,x} = \int_0^t \int_{\mathbb{R}^d} \delta_0(B_r^x - y) W(dy) dr \,. \tag{4.21}$$

Conditional on B, $V_{t,x}$ is a Gaussian random variable with mean 0 and variance

$$\mathbf{Var}_W(V_{t,x}) = \int_0^t \int_0^t \Lambda(B_r - B_s) dr ds \,. \tag{4.22}$$

Furthermore, for any $\lambda \in \mathbb{R}$, we have $\mathbf{E}\left[\exp(\lambda V_{t,x})\right] < \infty$. These properties can be obtained using the same arguments as in the time dependent case and we omit the details.

4.2 Hölder continuity of the Feynman-Kac functional

In this subsection, we establish the Hölder continuity in space and time of the the Feynman-Kac functional given by formulas (4.1) and (4.20). These regularity properties will hold under some additional integrability assumptions on the measure μ. To simplify the presentation we will assume that $u_0 = 1$, and as usual we separate the time dependent and independent cases.

4.2.1 Time dependent noise

For the case of a time dependent noise, we will make use of the following condition in order to ensure Hölder type regularities.

Hypothesis 4.8. *Let W be a space-time stationary Gaussian noise with covariance structure encoded by γ and Λ. We assume that condition (4.4) in Hypothesis 4.1 holds for some $\beta > 0$ and the spectral measure μ satisfies*

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\xi|^{2(1-\beta-\alpha)}} < \infty$$

for some $\alpha \in (0, 1-\beta)$.

Theorem 4.9. *Assume Hypothesis 4.8. Let u be the process introduced by relation (4.1) with $u_0 = 1$, namely:*

$$u_{t,x} = \mathbf{E}_B\left[\exp\left(V_{t,x}\right)\right], \quad \text{where} \quad V_{t,x} = \int_0^t \int_{\mathbb{R}^d} \delta_0(B_{t-r}^x - y) W(dr, dy). \tag{4.23}$$

Then u admits a version which is (γ_1, γ_2)-Hölder continuous on any compact set of the form $[0,T] \times [-M,M]^d$, with any $\gamma_1 < \frac{\alpha}{2}$, $\gamma_2 < \alpha$ and $T, M > 0$.

Proof. Owing to standard considerations involving Kolmogorov's criterion, it is sufficient to prove the following bound for all large p and $s, t \in [0,T]$, $x, y \in \mathbb{R}^d$ with $T > 0$:

$$\mathbf{E}\left[|u_{t,x} - u_{s,y}|^p\right] \leq c_{p,T}\left(|t-s|^{\frac{\alpha p}{2}} + |x-y|^{\alpha p}\right). \tag{4.24}$$

We now focus on the proof of (4.24). From the elementary relation $|e^x - e^y| \leq (e^x + e^y)|x-y|$, valid for $x, y \in \mathbb{R}$ and applying the Cauchy-Schwarz inequality it follows

$$
\begin{aligned}
\mathbf{E}\left[|u_{t,x} - u_{s,y}|^p\right] &= \mathbf{E}_W\left[|\mathbf{E}_B\left[\exp\left(V_{t,x}\right)\right] - \mathbf{E}_B\left[\exp\left(V_{s,y}\right)\right]|^p\right] \\
&\leq \mathbf{E}_W\left\{\mathbf{E}_B^p\left[\left(\exp(V_{t,x}) + \exp(V_{s,y})\right)|V_{t,x} - V_{s,y}|\right]\right\} \\
&\leq \mathbf{E}_W^{1/2}\left\{\mathbf{E}_B^p\left[\left(\exp(V_{t,x}) + \exp(V_{s,y})\right)^2\right]\right\} \mathbf{E}_W^{1/2}\left\{\mathbf{E}_B^p\left[|V_{t,x} - V_{s,y}|^2\right]\right\}.
\end{aligned} \tag{4.25}
$$

We now resort to our exponential bound of Theorem 4.6 for $V_{t,x}$, Minkowsky inequality and the relation between L^p and L^2 moments for Gaussian random variables in order to obtain:

$$\mathbf{E}\left[|u_{t,x} - u_{s,y}|^p\right] \leq c_p \left[\mathbf{E}\left[|V_{t,x} - V_{s,y}|^2\right]\right]^{p/2}.$$

We now evaluate the right hand side of this inequality.

Let us start by studying a difference of the form $V_{t,x} - V_{t,y}$, for $t \in (0,T]$ and $x, y \in \mathbb{R}$. The variance of $V_{t,x} - V_{t,y}$ conditioned by B can be computed as in (4.6) and we can write

$$\mathbf{E}\left[|V_{t,x} - V_{t,y}|^2\right]$$

$$= 2\mathbf{E}_B\left[\iint_0^t \int_0^t \gamma(r-s)\Lambda(B_r - B_s)drds - \int_0^t \int_0^t \gamma(r-s)\Lambda(B_r - B_s + x - y)drds\right]$$

$$= 2\int_0^t \int_0^t \int_{\mathbb{R}^d} \gamma(r-s)\left(1 - \cos\langle\xi, x-y\rangle\right)e^{-\frac{1}{2}|r-s||\xi|^2}\mu(d\xi)drds.$$

Using condition (4.4) and the estimate $|1 - \cos\langle\xi, x-y\rangle| \leq |\xi|^{2\alpha}|x-y|^{2\alpha}$, where $0 < \alpha < 1 - \beta$, yields

$$\mathbf{E}\left[|V_{t,x} - V_{t,y}|^2\right] \leq C|x-y|^{2\alpha}\int_0^t \int_0^t \int_{\mathbb{R}^d} |r-s|^{-\beta}e^{-\frac{1}{2}|r-s||\xi|^2}|\xi|^{2\alpha}\mu(d\xi)drds.$$

Finally, as in the proof of Proposition 4.2, Hypothesis 4.8 implies

$$\int_0^T \int_0^T \int_{\mathbb{R}^d} |r-s|^{-\beta}e^{-\frac{1}{2}|r-s||\xi|^2}|\xi|^{2\alpha}\mu(d\xi)drds < \infty,$$

and thus $\mathbf{E}[|V_{t,x} - V_{t,y}|^2] \leq C|x-y|^{2\alpha}$.

The evaluation of the variance of $V_{t,x} - V_{s,x}$, with $0 \leq s < t \leq T$, $x \in \mathbb{R}^d$ goes along the same lines. Indeed, we write $\mathbf{E}[|V_{t,x} - V_{s,x}|^2] \leq 2(A_1 + A_2)$, with

$$A_1 = \mathbf{E}\left[\left|\int_s^t \int_{\mathbb{R}^d} \delta_0(B_{t-r}^x - y)W(dr, dy)\right|^2\right]$$

$$A_2 = \mathbf{E}\left[\left|\int_0^s \int_{\mathbb{R}^d} \left(\delta_0(B_{t-r}^x - y) - \delta_0(B_{s-r}^x - y)\right)W(dr, dy)\right|^2\right].$$

For the term A_1, computing the variance as in (4.6) and using condition (4.4), we obtain

$$A_1 = \mathbf{E}_B\left[\int_0^{t-s} \int_0^{t-s} \gamma(u-v)\Lambda(B_u - B_v)\,dudv\right]$$

$$\leq C_\beta \int_0^{t-s} \int_0^{t-s} \int_{\mathbb{R}^d} |u-v|^{-\beta}e^{-\frac{1}{2}|u-v||\xi|^2}\mu(d\xi)dudv$$

$$\leq C(t-s)\int_0^{t-s} \int_{\mathbb{R}^d} u^{-\beta}e^{-\frac{1}{2}u|\xi|^2}\mu(d\xi)du.$$

Then, Hypothesis 4.8 allows us to write

$$\int_{\mathbb{R}^d} e^{-\frac{1}{2}u|\xi|^2}\mu(d\xi) = C_1 + u^{\alpha+\beta-1}\int_{|\xi|>1} |\xi|^{2(\alpha+\beta-1)}\mu(d\xi)$$

for any $\alpha < 1 - \beta$, which leads to the bound $A_1 \leq C(t-s)^{1+\alpha}$.

The term A_2 can be handled as follows: as in (4.6) we write:

$$A_2 = \mathbf{E}_B\left[\int_0^s \int_0^s \gamma(u-v)\left[\Lambda(B_{t-u} - B_{t-v}) + \Lambda(B_{s-u} - B_{s-v})\right.\right.$$

$$\left.\left. - 2\Lambda(B_{t-u} - B_{s-v})\right]dudv\right], \quad (4.26)$$

and changing to Fourier coordinates, this yields:

$$A_2 \leq 2 \int_0^s \int_0^s \gamma(u-v) \int_{\mathbb{R}^d} \left| e^{-\frac{1}{2}|u-v||\xi|^2} - e^{-\frac{1}{2}|t-s-u+v||\xi|^2} \right| \mu(d\xi) du dv. \qquad (4.27)$$

Using the estimate $|e^{-x} - e^{-y}| \leq (e^{-x} + e^{-y})|x-y|^\alpha$, for any $0 < \alpha < 1 - \beta$ and $x, y \geq 0$ and condition (4.4), we obtain

$$A_2 \leq C|t-s|^\alpha \int_0^s \int_0^s \int_{\mathbb{R}^d} |u-v|^{-\beta} \left(e^{-\frac{1}{2}|u-v||\xi|^2} + e^{-\frac{1}{2}|t-s-u+v||\xi|^2} \right) |\xi|^{2\alpha} \mu(d\xi) du dv.$$

Then, in order to achieve the bound $A_2 \leq |t-s|^\alpha$, it suffices to prove that

$$\int_0^s \int_0^s |u-v|^{-\beta} \int_{\mathbb{R}^d} e^{-\frac{1}{2}|t-s-u+v||\xi|^2} |\xi|^{2\alpha} \mu(d\xi) du dv$$

is uniformly bounded for $0 \leq s < t \leq T$. We decompose the integral with respect to the measure μ into the regions $\{|\xi| \leq 1\}$ and $\{|\xi| > 1\}$. The integral on $\{|\xi| \leq 1\}$ is clearly bounded because μ is finite on compact sets. Taking into account of the hypothesis 4.8, the integral over $\{|\xi| > 1\}$ can be handled using the estimate

$$\sup_{s,t \in [0,T]} \int_0^s \int_0^s |u-v|^{-\beta} e^{-\frac{1}{2}|t-s-u+v||\xi|^2} du dv \leq C|\xi|^{2\beta - 2}.$$

Putting together our bounds on A_1 and A_2, we have been able to prove that $\mathbf{E}[|V_{t,x} - V_{s,x}|^2] \leq |t-s|^\alpha$. Furthermore, gathering our estimates for $V_{t,x} - V_{t,y}$ and $V_{t,x} - V_{s,x}$, this completes the proof of the theorem. $\qquad \square$

Remark 4.10. The results of Theorem 4.9 do not give the optimal Hölder continuity exponents for the process u defined by (4.1). Another strategy could be implemented, based on the Feynman-Kac representation for the $(2p)$-th moments of u. This method is longer than the one presented here, but should lead to some better estimates of the continuity exponents. We stick to the shorter version of Theorem 4.9 for sake of conciseness, and also because optimal exponents will be deduced from the pathwise results of Section 5 (in particular Proposition 5.24).

4.2.2 Time independent noise

In the case of time independent noise, the next result provides a result on the Hölder continuity of the Feynman-Kac functional defined in (4.20). In this case we impose the following additional integrability condition on μ.

Hypothesis 4.11. *Let W be a spatial Gaussian noise with covariance structure encoded by Λ. Suppose that the spectral measure μ satisfies*

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\xi|^{2(1-\alpha)}} < \infty$$

for some $\alpha \in (0, 1)$.

Theorem 4.12. *Let u be the Feynman-Kac functional defined in (4.20) with $u_0 \equiv 1$, namely:*

$$u_{t,x} = \mathbf{E}_B \left[\exp\left(V_{t,x} \right) \right], \quad \text{where} \quad V_{t,x} = \int_0^t \left(\int_{\mathbb{R}^d} \delta_0(B_r^x - y) W(dy) \right) dr.$$

Then u admits a version which is (γ_1, γ_2)-Hölder continuous on any compact set of the form $[0, T] \times [-M, M]^d$, with any $\gamma_1 < \frac{1+\alpha}{2}$, $\gamma_2 < \alpha$ and $T, M > 0$.

Proof. The proof is similar to the proof of Theorem 4.9 and we omit the details. $\qquad \square$

4.3 Examples

Let us discuss the validity of Hypothesis 4.8 and Hypothesis 4.11 in the examples presented in the introduction. In the case of a time dependent noise we assume that the time covariance has the form $\gamma(x) = |x|^{-\beta}$, $0 < \beta < 1$.

For the Riesz kernel $\Lambda(x) = |x|^{-\eta}$, where $\mu(d\xi) = C_\beta |\xi|^{\eta-d} d\xi$, we already know that Hypothesis 2.1 holds if $\eta < 2$. On the other hand, Hypothesis 4.1, which allows us to define the Feynman-Kac functional in the time dependent case, is satisfied if $\eta < 2 - 2\beta$. For the Hölder continuity, Hypothesis 4.8 holds for any $\alpha \in (0, 1 - \beta - \frac{\eta}{2})$ and Hypothesis 4.11 holds for any $\alpha' \in (0, 1 - \frac{\eta}{2})$. Then, by Theorem 4.9 and 4.12, for any $\alpha \in (0, 1 - \beta - \frac{\eta}{2})$, $\alpha' \in (0, 1 - \frac{\eta}{2})$, assuming $u_0 \equiv 0$, the Feynman-Kac functional (4.1) is Hölder continuous of order α in the space variable and of order $\frac{\alpha}{2}$ in the time variable, and the Feynman-Kac functional (4.20) is Hölder continuous of order α in the space variable and of order $\frac{\alpha'+1}{2}$ in the time variable.

For the Bessel kernel, we know that Hypothesis 2.1 is satisfied when $\eta > d - 2$, and Hypothesis 4.8 holds when $\eta > d + 2\beta - 2$. By Theorem 4.9 and 4.12, for any $\alpha \in (0, \min(\frac{\eta-d}{2} - \beta + 1, 1))$ and $\alpha' \in (0, \min(\frac{\beta-d}{2} + 1, 1))$, assuming $u_0 \equiv 0$, the Feynman-Kac functional (4.1) is Hölder continuous of order α in the space variable and of order $\frac{\alpha}{2}$ in the time variable, the Feynman-Kac functional (4.20) is Hölder continuous of order α' in the space variable and of order $\frac{\alpha'+1}{2}$ in the time variable.

Consider the case of a fractional noise with covariance function $\gamma(t) = H(2H - 1)|t|^{2H-2}$ and $\Lambda(x) = \prod_{i=1}^{d} H_i(2H_i - 2)|x_i|^{2H_i-2}$. We know that Hypothesis 2.1 holds when $\sum_{i=1}^{d} H_i > d - 1$. Moreover, when $\sum_{i=1}^{d} H_i > d - 2H + 1$, Hypothesis 4.8 is satisfied. By Theorem 4.9 and 4.12, for any $\alpha \in (0, \sum_{i=1}^{d} H_i + 2H - d - 1)$ and $\alpha' \in (0, \sum_{i=1}^{d} H_i - d + 1)$, assuming $u_0 \equiv 0$, Feynman-Kac functional (4.1) is Hölder continuous of order α in the space variable and of order $\frac{\alpha}{2}$ in the time variable, which recovers the result in [29]). On the other hand, Feynman-Kac functional (4.20) is Hölder continuous of order α' in the space variable and of order $\frac{\alpha'+1}{2}$ in the time variable.

5 Equation in the Stratonovich sense

In this section we consider the following stochastic heat equation of Stratonovich type with the multiplicative Gaussian noise introduced in Section 2.1:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u \frac{\partial^{d+1} W}{\partial t \partial x_1 \cdots \partial x_d}. \tag{5.1}$$

As in the previous sections, the initial condition is a continuous and bounded function u_0. We will discuss two notions of solution. The first one is based on the Stratonovich integral, which is controlled using techniques of Malliavin calculus and a second one is completely pathwise and is based on Besov spaces. We will show that the Feynman-Kac functional (4.1) is a solution in both senses, and in the pathwise formulation it is the unique solution to equation (5.1).

We will also discuss the case of a time independent multiplicative Gaussian noise introduced in Section 2.2, that is

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u \frac{\partial^d W}{\partial x_1 \cdots \partial x_d}, \tag{5.2}$$

with an initial condition $u_0 \in C_b(\mathbb{R}^d)$. As in the case of a time dependent noise, we will show that the Feynman-Kac functional (4.20) is both a mild Stratonovich solution and a pathwise solution.

5.1 Stratonovich solution

Our aim is to define a notion of solution to equation (5.1) by means of a Russo-Vallois type approach, which happens to be compatible with Malliavin calculus tools. As usual, we divide our study into time dependent and time independent cases.

5.1.1 Time dependent case

Let W be the time dependent noise introduced in Section 2.1. In this case, we make use of the following definition of Stratonovich integral.

Definition 5.1. *Given a random field $v = \{v_{t,x}; t \geq 0, x \in \mathbb{R}^d\}$ such that*

$$\int_0^T \int_{\mathbb{R}^d} |v_{t,x}| \, dx dt < \infty$$

almost surely for all $T > 0$, the Stratonovich integral $\int_0^T \int_{\mathbb{R}^d} v_{t,x} W(dt, dx)$ is defined as the following limit in probability, if it exists:

$$\lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} \int_0^T \int_{\mathbb{R}^d} v_{t,x} \dot{W}_{t,x}^{\varepsilon,\delta} \, dx dt \,,$$

where $\dot{W}_{t,x}^{\varepsilon,\delta}$ is the regularization of W defined in (3.12).

With this definition of integral, we have the following notion of solution for equation (5.1).

Definition 5.2. *A random field $u = \{u_{t,x}; t \geq 0, x \in \mathbb{R}^d\}$ is a mild solution of equation (5.1) with initial condition $u_0 \in C_b(\mathbb{R}^d)$ if for any $t \geq 0$ and $x \in \mathbb{R}^d$ the following equation holds*

$$u_{t,x} = p_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x - y) u_{s,y} \, W(ds, dy), \tag{5.3}$$

where the last term is a Stratonovich stochastic integral in the sense of Definition 5.1.

The next result asserts the existence of a solution to equation (5.3) based on the Feynman-Kac representation.

Theorem 5.3. *Assume Hypothesis 4.1 holds true. Then, the process u defined in (4.1) is a mild solution of equation (5.1), in the sense given by Definition 5.2.*

Proof. We proceed similarly to Section 3.2. Consider the following approximation to equation (5.1)

$$\frac{\partial u^{\varepsilon,\delta}}{\partial t} = \frac{1}{2} \Delta u^{\varepsilon,\delta} + u^{\varepsilon,\delta} \dot{W}_{t,x}^{\varepsilon,\delta}, \tag{5.4}$$

with initial condition u_0, where $\dot{W}_{t,x}^{\varepsilon,\delta}$ is defined in (3.12). From the classical Feynman-Kac formula, we know that

$$u_{t,x}^{\varepsilon,\delta} = \mathbf{E}_B \left[u_0(B_t^x) \exp \left(\int_0^t \dot{W}^{\varepsilon,\delta}(t - s, B_s^x) ds \right) \right]. \tag{5.5}$$

Moreover, thanks to Fubini's theorem, we can write

$$
\begin{aligned}
\int_0^t \dot{W}^{\varepsilon,\delta}(t - s, B_s^x) ds &= \frac{1}{\delta} \int_0^t \left(\int_{(t-s-\delta)_+}^{t-s} \int_{\mathbb{R}^d} p_\varepsilon(B_s^x - y) W(dr, dy) \right) ds \\
&= W(A_{t,x}^{\varepsilon,\delta}) = V_{t,x}^{\varepsilon,\delta} \,,
\end{aligned}
$$

where $A_{t,x}^{\varepsilon,\delta}$ is defined in (3.19) and $V_{t,x}^{\varepsilon,\delta}$ is defined in (4.10). Therefore, the process $u_{t,x}^{\varepsilon,\delta}$ is given by (4.15), and Proposition 4.7 implies that (4.16) holds.

Next we prove that u is a mild solution of equation (5.1) in the sense of Definition 5.2. Taking into account of the definition of the Stratonovich integral, is suffices to show that

$$G^{\varepsilon,\delta} := \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) \left(u_{s,y}^{\varepsilon,\delta} - u_{s,y}\right) \dot{W}_{s,y}^{\varepsilon,\delta} \, dy ds$$

converges in $L^2(\Omega)$ to zero when first δ tends to zero and later ε tends to zero. To this aim, we are going to use the following notation:

$$\psi_{s,y}^{\varepsilon,\delta}(r,z) = \frac{1}{\delta} \mathbf{1}_{[(s-\delta)_+,s]}(r) p_\varepsilon(y-z), \quad \text{and} \quad \widetilde{u}_{s,y}^{\varepsilon,\delta} = u_{s,y}^{\varepsilon,\delta} - u_{s,y}.$$

In particular, notice that $\dot{W}_{s,y}^{\varepsilon,\delta} = W\left(\psi_{s,y}^{\varepsilon,\delta}\right)$. Then,

$$\mathbf{E}\left[\left(G^{\varepsilon,\delta}\right)^2\right] = \int_0^t \int_0^t \int_{\mathbb{R}^{2d}} p_{t-s}(x-y) p_{t-r}(x-z) \mathbf{E}\left[\widetilde{u}_{s,y}^{\varepsilon,\delta} \widetilde{u}_{r,z}^{\varepsilon,\delta} W\left(\psi_{s,y}^{\varepsilon,\delta}\right) W\left(\psi_{r,z}^{\varepsilon,\delta}\right)\right] dy dz ds dr,$$

and the expected value above can be analyzed by integration by parts. Indeed, according to relation (5.5), it is readily checked that $\widetilde{u}_{s,y}^{\varepsilon,\delta} \widetilde{u}_{r,z}^{\varepsilon,\delta} = \mathbf{E}_{B,\widetilde{B}}[Z_{s,y,r,z}^{\varepsilon,\delta}]$, with

$$Z_{s,y,r,z}^{\varepsilon,\delta} = u_0(B_s^y) \left[\exp\left(V_{s,y}^{\varepsilon,\delta,B}\right) - \exp\left(V_{s,y}^B\right)\right] u_0(\widetilde{B}_r^z) \left[\exp\left(V_{r,z}^{\varepsilon,\delta,\widetilde{B}}\right) - \exp\left(V_{r,z}^{\widetilde{B}}\right)\right],$$

and where $B, \widetilde{B}$ designate two independent d-dimensional Brownian motions. Moreover, a straightforward application of Fubini's theorem yields:

$$\mathbf{E}\left[\widetilde{u}_{s,y}^{\varepsilon,\delta} \widetilde{u}_{r,z}^{\varepsilon,\delta} W\left(\psi_{s,y}^{\varepsilon,\delta}\right) W\left(\psi_{r,z}^{\varepsilon,\delta}\right)\right] = \mathbf{E}_{B,\widetilde{B}}\left\{\mathbf{E}_W\left[Z_{s,y,r,z}^{\varepsilon,\delta} W\left(\psi_{s,y}^{\varepsilon,\delta}\right) W\left(\psi_{r,z}^{\varepsilon,\delta}\right)\right]\right\}.$$

We can now invoke formula (2.11) plus some easy computations of Malliavin derivatives in order to get:

$$\mathbf{E}\left[\left(G^{\varepsilon,\delta}\right)^2\right] = A_1 + A_2, \tag{5.6}$$

where

$$A_1 = \int_0^t \int_0^t \int_{\mathbb{R}^{2d}} p_{t-s}(x-y) p_{t-r}(x-z) \mathbf{E}\left[\widetilde{u}_{s,y}^{\varepsilon,\delta} \widetilde{u}_{r,z}^{\varepsilon,\delta}\right] \langle \psi_{s,y}^{\varepsilon,\delta}, \psi_{r,z}^{\varepsilon,\delta} \rangle_{\mathcal{H}} \, dy dz ds dr$$

and

$$A_2 = \int_0^t \int_0^t \int_{\mathbb{R}^{2d}} p_{t-s}(x-y) p_{t-r}(x-z) \mathbf{E}\left[Z_{s,y,r,z}^{\varepsilon,\delta} \Gamma_{s,y,r,z}^{\varepsilon,\delta}\right] dy dz ds dr,$$

with the notation

$$
\begin{aligned}
\Gamma_{s,y,r,z}^{\varepsilon,\delta} &= \langle \psi_{s,y}^{\varepsilon,\delta}, A_{r,z}^{\varepsilon,\delta,\widetilde{B}} - \delta_0(\widetilde{B}_{r-.}^z - \cdot)\rangle_{\mathcal{H}} \langle \psi_{r,z}^{\varepsilon,\delta}, A_{s,y}^{\varepsilon,\delta,B} - \delta_0(B_{s-.}^y - \cdot)\rangle_{\mathcal{H}} \\
&\quad + \langle \psi_{s,y}^{\varepsilon,\delta}, A_{s,y}^{\varepsilon,\delta,B} - \delta_0(B_{s-.}^y - \cdot)\rangle_{\mathcal{H}} \langle \psi_{r,z}^{\varepsilon,\delta}, A_{r,z}^{\varepsilon,\delta,\widetilde{B}} - \delta_0(\widetilde{B}_{z-.}^r - \cdot)\rangle_{\mathcal{H}} \\
&\quad + \langle \psi_{s,y}^{\varepsilon,\delta}, A_{s,y}^{\varepsilon,\delta,B} - \delta_0(B_{y-.}^s - \cdot)\rangle_{\mathcal{H}} \langle \psi_{r,z}^{\varepsilon,\delta}, A_{s,y}^{\varepsilon,\delta,B} - \delta_0(B_{s-.}^y - \cdot)\rangle_{\mathcal{H}} \\
&\quad + \langle \psi_{s,y}^{\varepsilon,\delta}, A_{r,z}^{\varepsilon,\delta,\widetilde{B}} - \delta_0(\widetilde{B}_{z-.}^r - \cdot)\rangle_{\mathcal{H}} \langle \psi_{r,z}^{\varepsilon,\delta}, A_{r,z}^{\varepsilon,\delta,\widetilde{B}} - \delta_0(\widetilde{B}_{z-.}^r - \cdot)\rangle_{\mathcal{H}}
\end{aligned}
$$

According to Proposition 4.7, we know that

$$\lim_{\varepsilon\downarrow 0} \lim_{\delta\downarrow 0} \sup_{s\in[0,T], y\in\mathbb{R}^d} \mathbf{E}\left[|\widetilde{u}_{s,y}^{\varepsilon,\delta}|^2\right] = 0,$$

and with the same arguments as in Proposition 4.7 we can also show that

$$\lim_{\varepsilon\downarrow 0} \lim_{\delta\downarrow 0} \sup_{s,r\in[0,T], y,z\in\mathbb{R}^d} \mathbf{E}\left[|Z_{s,yr,z}^{\varepsilon,\delta}|^2\right] = 0.$$

Therefore, with formula (5.6) in mind, the convergence to zero of $B^{\varepsilon,\delta}$ will follow, provided we show the following quantities are uniformly bounded in $\varepsilon \in (0,1)$ and $\delta \in (0,1)$

$$\theta_1 := \int_0^t \int_0^t \int_{\mathbb{R}^{2d}} p_{t-s}(x-y)p_{t-r}(x-z) \left| \langle \psi_{s,y}^{\varepsilon,\delta}, \psi_{r,z}^{\varepsilon,\delta} \rangle_{\mathcal{H}} \right| dydzdsdr \tag{5.7}$$

and

$$\theta_2 := \int_0^t \int_0^t \int_{\mathbb{R}^{2d}} p_{t-s}(x-y)p_{t-r}(x-z) \left\| \Gamma_{s,y,r,z}^{\varepsilon,\delta} \right\|_2 dydzdsdr, \tag{5.8}$$

where $\|\Gamma_{s,y,r,z}^{\varepsilon,\delta}\|_2$ stands for the norm of $\Gamma_{s,y,r,z}^{\varepsilon,\delta}$ in $L^2(\Omega)$. The remainder of the proof is thus just reduced to an estimation of (5.7) and (5.8).

In order to bound θ_1, we apply the estimate (4.18) and the semigroup property of the heat kernel, which yields

$$\langle \psi_{s,y}^{\varepsilon,\delta}, \psi_{r,z}^{\varepsilon,\delta} \rangle_{\mathcal{H}} = \left(\frac{1}{\delta^2} \int_{(s-\delta)_+}^s \int_{(r-\delta)_+}^r \gamma(u-v)dudv \right) \int_{\mathbb{R}^{2d}} p_\varepsilon(y-z_1)p_\varepsilon(z-z_2)\Lambda(z_1-z_2)dz_1dz_2$$

$$\leq c_{T,\beta}|r-s|^{-\beta} \int_{\mathbb{R}^d} p_{2\varepsilon}(y-z-w)\Lambda(w)dw.$$

Substituting this estimate into (5.7), we obtain

$$\theta_1 \leq c_{T,\beta} \int_0^t \int_0^t \int_{\mathbb{R}^d} p_{2t-s-r+2\varepsilon}(w)\Lambda(w)|r-s|^{-\beta}dw \leq c'_{T,\beta} \int_0^{2t} \int_{\mathbb{R}^d} p_{2t-s}(w)\Lambda(w)dwds < \infty,$$

where we get rid of ε in Fourier mode, similarly to the proof of (3.27).

We now turn to the control of θ_2: we first write, using the estimate (4.18) and the semigroup property of the heat kernel,

$$\langle \psi_{s,y}^{\varepsilon,\delta}, A_{r,z}^{\varepsilon,\delta} \rangle_{\mathcal{H}} = \frac{1}{\delta^2} \int_{(s-\delta)_+}^s \int_{(r-\sigma-\delta)_+}^{r-\sigma} \int_0^r \int_{\mathbb{R}^{2d}} p_\varepsilon(y-z_1)p_\varepsilon(B_\sigma^z - z_2)\gamma(u-v)$$

$$\times \Lambda(z_1-z_2)dz_1dz_2d\sigma dvdu$$

$$\leq c_{T,\beta} \int_0^r \int_{\mathbb{R}^d} p_{2\varepsilon}(y-B_{r-\sigma}^z-w)\Lambda(w)|s-\sigma|^{-\beta}dwd\sigma$$

Invoking again arguments of Fourier analysis, analogous to those in the proof of (3.27), we can show that

$$\mathbf{E}\left[\left| \int_0^r \int_{\mathbb{R}^d} p_{2\varepsilon}(y-B_{r-\sigma}^z-w)\Lambda(w)|s-\sigma|^{-\beta}dwd\sigma \right|^4 \right] \leq \mathbf{E}\left[\left| \int_0^r \Lambda(B_{r-\sigma})|s-\sigma|^{-\beta}d\sigma \right|^4 \right],$$

and

$$\sup_{r,s \in [0,T]} \mathbf{E}\left[\left| \int_0^r \Lambda(B_{r-\sigma})|s-\sigma|^{-\beta}d\sigma \right|^4 \right] < \infty.$$

This implies that $\|\Gamma_{s,y,r,z}^{\varepsilon,\delta}\|_2$, and thus, θ_2, are uniformly bounded. The proof of the theorem is complete. $\qquad\square$

Remark 5.4. Consider the case where the space dimension is 1, $\Lambda(x)$ is the Dirac delta function $\delta_0(x)$ corresponding to the white noise, which in our setting means that condition (4.5) is satisfied with $0 < \beta < \frac{1}{2}$. Then our theorems of Section 4 cover assumption (4.4), with $0 < \beta < \frac{1}{2}$ too, if we interpret the composition $\Lambda(B_r - B_s)$ as a generalized Wiener functional. Notice that in the case of the fractional Brownian motion with Hurst parameter H (that is $\gamma(x) = c_H|x|^{2H-2}$) the condition $0 < \beta < \frac{1}{2}$ means that $H > \frac{3}{4}$. In this case it is already known that the process defined by (4.1) is still a solution to equation (5.1) (see [29]).

5.1.2 Time independent case

Let W be the time independent noise introduced in Section 2.2. We claim that as in the time independent case, the Feynman-Kac functional given by (4.20) is a mild solution to equation (5.2) in the Stratonovich sense.

The Stratonovich integral with respect to the noise W is defined as the limit of the integrals with respect to regularization of the noise.

Definition 5.5. *Given a random field $v = \{v_x; x \in \mathbb{R}^d\}$ such that $\int_{\mathbb{R}^d} |v_x| dx < \infty$ almost surely, the Stratonovich integral $\int_{\mathbb{R}^d} v_x W(dx)$ is defined as the following limit in probability, if it exists:*

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^d} v_x \dot{W}_x^\varepsilon dx \,,$$

where $\dot{W}_x^\varepsilon = \int_{\mathbb{R}^d} p_\varepsilon(x - y) W(dy)$.

With this definition of integral, we have the following notion of solution for equation (5.2).

Definition 5.6. *A random field $u = \{u_{t,x}; t \geq 0, x \in \mathbb{R}^d\}$ is a mild solution of equation (5.2) if we have*

$$u_{t,x} = p_t u_0(x) + \int_0^t \left(\int_{\mathbb{R}^d} p_{t-s}(x - y) u_{s,y} W(dy) \right) ds$$

almost surely for all $t \geq 0$, where the last term is a Stratonovich stochastic integral in the sense of Definition 5.5.

The next result is the existence of a solution based on the Feynman-Kac representation.

Theorem 5.7. *Suppose that μ satisfies (2.4). Then, the process $u_{t,x}$ given by (4.20) is a mild solution of equation (5.2).*

The proof of this theorem is similar to that of Theorem 5.3, and it is omitted.

5.2 Existence and uniqueness of a pathwise solution

In this section we define and solve equations (5.1) and (5.2) in a pathwise manner in $\mathbb{R}^d$, when the noise W satisfies some additional hypotheses. Contrarily to the Stratonovich technology invoked at Section 5.1, the pathwise method yields uniqueness theorems, which will be used in order to identify Feynman-Kac and pathwise solutions. At a technical level, our results will be achieved in the framework of weighted Besov spaces, that we proceed to recall now.

5.2.1 Besov spaces

The definition of Besov spaces is based on Littlewood-Paley theory, which relies on decompositions of functions into spectrally localized blocks. We thus first introduce the following basic definitions.

Definition 5.8. *We call annulus any set of the form $C = \{x \in \mathbb{R}^d : a \leqslant |x| \leqslant b\}$ for some $0 < a < b$. A ball is a set of the form $B = \{x \in \mathbb{R}^d : |x| \leqslant b\}$.*

The localizing functions for the Fourier domain alluded to above are defined as follows.

Notation 5.9. *In the remaining part of this section, we shall use χ, φ to denote two smooth nonnegative radial functions with compact support such that:*

1. *The support of χ is contained in a ball and the support of φ is contained in an annulus C with $a = 3/4$ and $b = 8/3$;*

2. *We have $\chi(\xi) + \sum_{j \geqslant 0} \varphi(2^{-j}\xi) = 1$ for all $\xi \in \mathbb{R}^d$;*

3. *It holds that $supp(\chi) \cap supp(\varphi(2^{-i}\cdot)) = \varnothing$ for $i \geqslant 1$ and if $|i - j| > 1$, then $supp(\varphi(2^{-i}\cdot)) \cap supp(\varphi(2^{-j}\cdot)) = \varnothing$.*

In the sequel, we set $\varphi_j(\xi) := \varphi(2^{-j}\xi)$.

For the existence of χ and φ see [2, Proposition 2.10]. With this notation in mind, the Littlewood-Paley blocks are now defined as follows.

Definition 5.10. *Let $u \in \mathcal{S}'(\mathbb{R}^d)$. We set*

$$\Delta_{-1}u = \mathcal{F}^{-1}(\chi \, \mathcal{F}u), \quad \text{and for } j \geqslant 0 \quad \Delta_j u = \mathcal{F}^{-1}(\varphi_j \, \mathcal{F}u).$$

We also use the notation $S_k u = \sum_{j=-1}^{k-1} \Delta_j u$, valid for all $k \geq 0$.

Observe that one can also write $\Delta_{-1}u = \widetilde{K} * u$ and $\Delta_j u = K_j * u$ for $j \geqslant 0$, where $\widetilde{K} = \mathcal{F}^{-1}\chi$ and $K_j = 2^{jd}\mathcal{F}^{-1}\varphi(2^j\cdot)$. In particular the $\Delta_j u$ are smooth functions for all $u \in \mathcal{S}'(\mathbb{R}^d)$.

In order to handle equations whose space parameter lies in an unbounded domain like $\mathbb{R}^d$, we shall use spaces of weighted Hölder type functions for polynomial or exponential weights, where the weights satisfy some smoothness conditions. In this way we define the following class of weights.

Definition 5.11. *We denote by $\mathcal{W}$ the class of weights $w \in \mathcal{C}_b^\infty(\mathbb{R}^d; \mathbb{R}_+)$ consisting of:*

- *The weights ρ_κ obtained as functions of the form $c\,(1+|x|^\kappa)^{-1}$, with $\kappa \geq 1$, smoothed at 0.*

- *The weights e_λ obtained as functions of the form $c\,e^{-\lambda|x|}$, with $\lambda > 0$, smoothed at 0.*

- *Products of these functions.*

Notice that more general classes of weights are introduced in [44]. We have also tried to stick to the notation given in [24], from which our developments are inspired.

Weighted Besov spaces are sets of functions characterized by their Littlewood-Paley block decomposition. Specifically, their definition is as follows.

Definition 5.12. *Let $w \in \mathcal{W}$ and $\kappa \in \mathbb{R}$. We set*

$$\mathcal{B}_w^\kappa(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d); \|f\|_{w,\kappa} := \sup_{j \geq -1} 2^{j\kappa} \|w\,\Delta_j f\|_{L^\infty} < \infty \right\}. \tag{5.9}$$

We call this space a weighted Besov-Hölder space. When $w = 1$, we just denote the space by $\mathcal{B}^\kappa(\mathbb{R}^d)$, and it corresponds to the usual Besov space $\mathcal{B}_{\infty,\infty}^\kappa(\mathbb{R}^d)$.

Notice that we follow here the terminology of [44]. The weighted Besov-Hölder spaces are well understood objects, and let us recall some basic facts about them.

Proposition 5.13. *Let $w, w_1, w_2 \in \mathcal{W}$, $\kappa \in \mathbb{R}$ and $f \in \mathcal{B}_w^\kappa(\mathbb{R}^d)$. Then the following holds true:*

(i) There exist some positive constants $c_{\kappa,w}^1, c_{\kappa,w}^2$ such that $c_{\kappa,w}^1 \|fw\|_\kappa \leq \|f\|_{w,\kappa} \leq c_{\kappa,w}^2 \|fw\|_\kappa$.

(ii) For $\kappa \in (0,1)$, we have $f \in \mathcal{B}_w^\kappa(\mathbb{R}^d)$ iff fw is a κ-Hölder function.

(iii) If $w_1 < w_2$ we have $\|f\|_{w_1,\kappa} \leq \|f\|_{w_2,\kappa}$.

Proof. Item (i) is borrowed from [44, Chapter 6]. The fact that $\mathcal{B}^\kappa(\mathbb{R}^d)$ coincides with the space of Hölder continuous functions $\mathcal{C}^\kappa(\mathbb{R}^d)$ for $\kappa \in [0,1]$ is shown in [2, Theorem 2.36], and it yields item (ii) thanks to (i). Finally, item (iii) is also taken from [44, Chapter 6]. $\qquad\square$

Let us now state a result about products of distributions which turns out to be useful for our existence and uniqueness result.

Proposition 5.14. *Let w_1, w_2 be two weight functions in $\mathcal{W}$, and $\kappa_1, \kappa_2 \in \mathbb{R}$ such that $\kappa_2 < 0 < \kappa_1$ and $\kappa_1 > |\kappa_2|$ and let $w = w_1 w_2$. Then*

$$(f_1, f_2) \in \mathcal{B}^{\kappa_1}_{w_1} \times \mathcal{B}^{\kappa_2}_{w_2} \longmapsto f_1 f_2 \in \mathcal{B}^{\kappa_2}_w \quad \text{is continuous.} \tag{5.10}$$

Furthermore, the following bound holds true:

$$\|f_1 f_2\|_{\mathcal{B}^{\kappa_2}_w} \leq \|f_1\|_{\mathcal{B}^{\kappa_1}_{w_1}} \|f_2\|_{\mathcal{B}^{\kappa_2}_{w_2}}. \tag{5.11}$$

Finally we label the action of the heat semigroup on functions in weighted Besov spaces.

Proposition 5.15. *Let $w \in \mathcal{W}$, $\kappa \in \mathbb{R}$ and $f \in \mathcal{B}^\kappa_w(\mathbb{R}^d)$. Then for all $t \in [0,\tau]$, $\gamma > 0$ and $\hat\kappa > \kappa$ we have*

$$\|p_t f\|_{w,\hat\kappa} \leq c_{\tau,w,\kappa,\hat\kappa} t^{-\frac{\hat\kappa-\kappa}{2}}\|f\|_{w,\kappa}, \quad \text{and} \quad \|[Id - p_t]f\|_{w,\kappa-2\gamma} \leq c_{\tau,w,\gamma} t^\gamma \|f\|_{w,\kappa}.$$

5.2.2 Notion of solution

In order to give a pathwise definition of solution for equation (5.1), we will replace the noise W by a nonrandom Hölder continuous function in time with values in a Besov space of distributions, denoted by $\mathscr{W}$. We will show later (see Proposition 5.22) that under Hypothesis 4.8, almost surely the mapping $t \to W(\mathbf{1}_{[0,t]}\varphi)$, $\varphi \in \mathcal{D}$, is Hölder continuous with values in this Besov space. We thus label a notation for this kind of space.

Notation 5.16. *Let $\theta \in (0,1)$, $\kappa \in \mathbb{R}$ and $w \in \mathcal{W}$. The space of θ-Hölder continuous functions from $[0,T]$ to a weighted Sobolev space $\mathcal{B}^\kappa_w$ is denoted by $\mathcal{C}^{\theta,\kappa}_{T,w}$. Otherwise stated, we have $\mathcal{C}^{\theta,\kappa}_{T,w} = \mathcal{C}^\theta([0,T]; \mathcal{B}^\kappa_w)$. In order to alleviate notations, we shall write $\mathcal{C}^{\theta,\kappa}_w$ only when the value of T is non ambiguous.*

Now we introduce the pathwise type assumption that we shall make on the multiplicative input distribution $\mathscr{W}$.

Hypothesis 5.17. *We assume that there exist two constants $\theta, \kappa \in (0,1)$ satisfying $\frac{1+\kappa}{2} < \theta < 1$, such that $\mathscr{W} \in \mathcal{C}^{\theta,-\kappa}_{T,\rho_\sigma}$, for any $\sigma > 0$ arbitrarily small.*

We also label some more notation for further use:

Notation 5.18. *For a function $f : [0,T] \to \mathcal{B}$, where $\mathcal{B}$ stands for a generic Banach space, we set $\delta f_{st} = f_t - f_s$ for $0 \leq s \leq t \leq T$. Notice that δ has also been used for Skorohod integrals, but this should not lead to ambiguities since Skorohod integrals won't be used in this section.*

With these preliminaries in hand, we shall combine the following ingredients in order to solve equation (5.1):

• Like the input $\mathscr{W}$, the solution u will live in a space of Hölder functions in time, with values in a weighted Sobolev space of the form $\mathcal{B}^{\kappa_u}_{e_\lambda}$. This allows the use of estimates of Young integration type in order to define integrals involving increments of the form $u\,d\mathscr{W}$.

• We have to take into account of the fact that, when one multiplies the function $u_s \in \mathcal{B}^{\kappa_u}_{e_\lambda}$ by the distribution $\delta \mathscr{W}_{st} \in \mathcal{B}^{-\kappa}_{\rho_\sigma}$, the resulting distribution $u_s \delta \mathscr{W}_{st}$ lies (provided $\kappa_u > \kappa$) into the space $\mathcal{B}^{-\kappa}_{e_\lambda \rho_\sigma}$. This will force us to assume in fact $u_s \in \mathcal{B}^{\kappa_u}_{w_s}$, where the weight $w_s \in \mathcal{W}$ decreases with s.

Let us turn now to the technical part of our task. We first fix positive constants λ, σ and define a weight $w_t = e_{\lambda + \sigma t}$. We shall seek the solution to equation (5.1) in the following space:

$$\mathcal{D}^{\theta_u, \kappa_u}_{\lambda, \sigma} = \left\{ f \in \mathcal{C}([0, T] \times \mathbb{R}^d); \, \|f_t\|_{\mathcal{B}^{\kappa_u}_{w_t}} \leq c_{T, f} \right.$$
$$\left. \text{and} \; \|f_t - f_s\|_{\mathcal{B}^{\kappa_u}_{w_t}} \leq c_f \, |t - s|^{\theta_u} \; \; \forall 0 \leq s < t \leq T \right\}. \quad (5.12)$$

We introduce the Hölder norm in this space by

$$\|f\|_{\mathcal{D}^{\theta_u, \kappa_u}_{\lambda, \sigma}} = \sup_{0 \leq s < t \leq T} \frac{\|f_t - f_s\|_{\mathcal{B}^{\kappa_u}_{w_t}}}{|t - s|^{\theta_u}}. \quad (5.13)$$

We now introduce a pathwise mild formulation for equation (5.1) in the spaces $\mathcal{D}^{\theta_u, \kappa_u}_{\lambda, \sigma}$.

Definition 5.19. *Suppose that $\mathscr{W}$ satisfies Hypothesis 5.17. Let $u \in \mathcal{D}^{\theta_u, \kappa_u}_{\lambda, \sigma}$ for fixed $\lambda, \sigma > 0$, $\theta_u + \theta > 1$ and $\kappa_u \in (\kappa, 1)$. Consider an initial condition $u_0 \in \mathcal{B}^{\kappa_u}_{e_\lambda}$. We say that u is a mild solution to equation*

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + u \frac{\partial \mathscr{W}}{\partial t} \quad (5.14)$$

with initial condition u_0 if it satisfies the following integral equation

$$u_t = p_t u_0 + \int_0^t p_{t-s} \left(u_s \, \mathscr{W}(ds) \right), \quad (5.15)$$

where the product $u \mathscr{W}$ is interpreted in the distributional sense of (5.10) *and the time integral is understood in the Young sense.*

Remark 5.20. Let us specify what we mean by $J^u_t := \int_0^t p_{t-s} \left(u_s \, \mathscr{W}(ds) \right)$ under the conditions of Definition 5.19. First, we should understand J^u_t as

$$J^u_t = \lim_{\varepsilon \to 0} J^{u, \varepsilon}_t, \quad \text{where} \quad J^{u, \varepsilon}_t = \int_0^{t - \varepsilon} p_{t-s} \left(u_s \, \mathscr{W}(ds) \right).$$

The integration on $[0, t - \varepsilon]$ avoids any singularity of p_{t-s} as an operator from $\mathcal{B}^{-\kappa}$ to $\mathcal{B}^{\kappa_u}$, so that $J^{u, \varepsilon}_t$ is defined as a Young integral. This integral is in particular limit of Riemann sums along dyadic partitions of $[0, t]$:

$$J^{u, \varepsilon}_t = \lim_{n \to \infty} \sum_{j=0}^{2^n - 1} p_{t - t^n_j} \left(u_{t^n_j} \, \delta \mathscr{W}_{t^n_j t^n_{j+1}} \right) \mathbf{1}_{[0, t - \varepsilon]}(t^n_{j+1}), \quad \text{where} \quad t^n_j = \frac{jt}{2^n}.$$

We then assume that one can combine the limiting procedures in n and ε (the justification of this step is left to the patient reader), and finally we define:

$$J^u_t = \lim_{n \to \infty} J^{u, n}_t, \quad \text{where} \quad J^{u, n}_t = \sum_{j=0}^{2^n - 1} p_{t - t^n_j} \left(u_{t^n_j} \, \delta \mathscr{W}_{t^n_j t^n_{j+1}} \right). \quad (5.16)$$

Here again, recall that the product $u_{t^n_j} \, \delta \mathscr{W}_{t^n_j t^n_{j+1}}$ is interpreted according to (5.10). This will be our way to understand equation (5.15).

We can now turn to the resolution of the equation in this context.

5.2.3 Resolution of the equation

Our existence and uniqueness result takes the following form:

Theorem 5.21. *Let $\mathscr{W}$ be a Hölder continuous distribution valued function satisfying Hypothesis 5.17, and let λ, σ be two strictly positive constants. Consider an initial condition $u_0 \in \mathcal{B}_{e_\lambda}^{\kappa_u}$. Then:*

(a) *There exist θ_u, κ_u satisfying $\theta_u + \theta > 1$ and $\kappa_u \in (\kappa, 1)$, such that equation (5.15) admits a unique solution in $\mathcal{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}$.*

(b) *The application $(u_0, \mathscr{W}) \mapsto u$ is continuous from $\mathcal{B}_{e_\lambda}^{\kappa_u} \times \mathcal{C}_{T,\rho_\sigma}^{\theta,-\kappa}$ to $\mathcal{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}$.*

Proof. We divide this proof into several steps.

Step 1: Definition of a contracting map. We fix a time interval $[0, \tau]$, where $\tau \leq T$, and along the proof we denote by $\mathcal{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}$ and $\|\cdot\|_{\mathcal{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}}$ the space and the Hölder norm defined in (5.12) and (5.13), respectively, but restricted to the interval $[0, \tau]$.

We consider a map Γ defined on $\mathcal{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}$ by $\Gamma(u) = v$, where v is the function defined by $v := p_t u_0 + J_t^u$ as in Remark 5.20. The proof of our result relies on two steps: (i) Show that Γ defines a map from $\mathcal{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}$ to $\mathcal{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}$, independently of the length of the interval $[0, \tau]$. (ii) Check that Γ is in fact a contraction if τ is made small enough. The two steps hinge on the same type of computations, so that we shall admit point (i) and focus on point (ii) for sake of conciseness.

In order to prove that Γ is a contraction, consider $u^1, u^2 \in \mathcal{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}$, and for $j = 1, 2$ set $v^j = \Gamma(u^j)$. For notational sake, we also set $u^{12} = u^1 - u^2$ and $v^{12} = v^1 - v^2$. Consistently with equation (5.15), v^{12} satisfies the relation

$$v_t^{12} = \int_0^t p_{t-r} \left(u_r^{12} \, \mathscr{W}(dr) \right).$$

Notice that the function v^{12} is in fact defined by relation (5.16). We have admitted point (i) above, which means in particular that we assume that the Riemann sums in (5.16) are converging whenever $u^{12} \in \mathcal{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}$. We now wish to prove that, provided τ is small enough, we have $\|v^{12}\|_{\mathcal{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}} \leq \frac{1}{2}\|u^{12}\|_{\mathcal{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}}$.

Step 2: Study of differences. Let $0 \leq s < t \leq \tau$. We decompose $v_t^{12} - v_s^{12}$ as $L_{st}^1 + L_{st}^2$, with

$$L_{st}^1 = \int_0^s [p_{t-s} - \mathrm{Id}] \, p_{s-v} \left(u_v^{12} \, \mathscr{W}(dv) \right), \quad \text{and} \quad L_{st}^2 = \int_s^t p_{t-v} \left(u_v^{12} \, \mathscr{W}(dv) \right),$$

where the Young integrals with respect to $\mathscr{W}(dv)$ are understood as limit of Riemann sums as in (5.16). We now proceed to the analysis of L_{st}^1 and L_{st}^2.

As in relation (5.16), we write $L_{st}^1 = \lim_{n \to \infty} L_{st}^{1,n}$, where we consider points $s_k^n = 2^{-n} ks$ in the dyadic partition of $[0, s]$ and where we set

$$L_{st}^{1,n} = \sum_{j=0}^{2^n - 1} [p_{t-s} - \mathrm{Id}] \, p_{s-s_j^n} \left(u_{s_j^n}^{12} \, \delta \mathscr{W}_{s_j^n s_{j+1}^n} \right). \tag{5.17}$$

In order to estimate L_{st}^1, let us thus first analyze the quantity $L_{st}^{1,n+1} - L_{st}^{1,n}$. Indeed, it is readily checked that $L_{st}^{1,n+1} - L_{st}^{1,n} = \sum_{j=0}^{2^n - 1} L_{st}^{1,n,j}$, where $L_{st}^{1,n,j}$ is defined by:

$$L_{st}^{1,n,j} = [p_{t-s} - \mathrm{Id}] \, p_{s-s_{2j+1}^{n+1}} \left(u_{s_{2j+1}^{n+1}}^{12} \, \delta \mathscr{W}_{s_{2j+1}^{n+1} s_{2j+2}^{n+1}} \right) - [p_{t-s} - \mathrm{Id}] \, p_{s-s_{2j}^{n+1}} \left(u_{s_{2j}^{n+1}}^{12} \, \delta \mathscr{W}_{s_{2j+1}^{n+1} s_{2j+2}^{n+1}} \right).$$

We now drop the index $n+1$ in the next computations for sake of readability, and write $L_{st}^{1,n,j} = L_{st}^{11,n,j} - L_{st}^{12,n,j}$ with

$$L_{st}^{11,n,j} = [p_{t-s} - \text{Id}]\, p_{s-s_{2j+1}} \left(\delta u_{s_{2j}s_{2j+1}}^{12}\, \delta \mathscr{W}_{s_{2j+1}s_{2j+2}} \right) := [p_{t-s} - \text{Id}]\, \hat{L}_{st}^{11,n,j}$$

$$L_{st}^{12,n,j} = [p_{t-s} - \text{Id}]\, [p_{s_{2j+1}-s_{2j}} - \text{Id}]\, p_{s-s_{2j+1}} \left(u_{s_{2j}}^{12}\, \delta \mathscr{W}_{s_{2j+1}s_{2j+2}} \right) := [p_{t-s} - \text{Id}]\, \hat{L}_{st}^{12,n,j}.$$

We treat again the two terms $L_{st}^{11,n,j}, L_{st}^{12,n,j}$ separately.

Owing to Proposition 5.15, we have

$$\|L_{st}^{11,n,j}\|_{\mathcal{B}_{w_t}^{\kappa_u}} \leq c\,(t-s)^{\theta_u} \|\hat{L}_{st}^{11,n,j}\|_{\mathcal{B}_{w_t}^{\kappa_u+2\theta_u}} \leq \frac{c\,(t-s)^{\theta_u}\, \|\delta u_{s_{2j}s_{2j+1}}^{12}\, \delta \mathscr{W}_{s_{2j+1}s_{2j+2}}\|_{\mathcal{B}_{w_t}^{-\kappa}}}{(s-s_{2j+1})^{\theta_u+\frac{\kappa_u+\kappa}{2}}}$$

Let us now recall the following elementary bound:

$$\varphi_{\alpha,\kappa}(x) := x^\alpha\, e^{-\kappa x} \implies 0 \leq \varphi_{\alpha,\kappa}(x) \leq \frac{c_\alpha}{\kappa^\alpha}, \quad \text{for} \quad x, \alpha, \kappa \in \mathbb{R}_+. \tag{5.18}$$

This entails $w_t \leq c_\sigma (t-t_{2j+1})^{-\sigma} w_{t_{2j+1}} \rho_\sigma$, and according to (5.11) we obtain

$$
\begin{aligned}
\|L_{st}^{11,n,j}\|_{\mathcal{B}_{w_t}^{\kappa_u}} &\leq \frac{c_\sigma\,(t-s)^{\theta_u}\, \|\delta u_{s_{2j}s_{2j+1}}^{12}\, \delta \mathscr{W}_{s_{2j+1}s_{2j+2}}\|_{\mathcal{B}_{w_{s_{2j+1}}\rho_\sigma}^{-\kappa}}}{(s-s_{2j+1})^{\theta_u+\frac{\kappa_u+\kappa}{2}+\sigma}} \\
&\leq \frac{c_\sigma\,(t-s)^{\theta_u}\, \|\delta u_{s_{2j}s_{2j+1}}^{12}\|_{\mathcal{B}_{w_{s_{2j+1}}}^{\kappa_u}} \|\delta \mathscr{W}_{s_{2j+1}s_{2j+2}}\|_{\mathcal{B}_{\rho_\sigma}^{-\kappa}}}{(s-s_{2j+1})^{\theta_u+\frac{\kappa_u+\kappa}{2}+\sigma}} \\
&\leq \frac{c_\sigma\,(t-s)^{\theta_u}\, \|u^{12}\|_{\mathcal{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}} \|\mathscr{W}\|_{\mathcal{C}_{\rho_\sigma}^{\theta,-\kappa}}}{(s-s_{2j+1})^{\theta_u+\frac{\kappa_u+\kappa}{2}+\sigma}} \left(\frac{s}{2^n}\right)^{\theta_u+\theta}.
\end{aligned}
$$

As far as $L_{st}^{12,n,j}$ is concerned, we have as above:

$$\|L_{st}^{12,n,j}\|_{\mathcal{B}_{w_t}^{\kappa_u}} \leq c\,(t-s)^{\theta_u} \|\hat{L}_{st}^{12,n,j}\|_{\mathcal{B}_{w_t}^{\kappa_u+2\theta_u}}. \tag{5.19}$$

We now take an arbitrarily small and strictly positive constant ε and write:

$$
\begin{aligned}
\|\hat{L}_{st}^{12,n,j}\|_{\mathcal{B}_{w_t}^{\kappa_u+2\theta_u}} &\leq (s_{2j+1}-s_{2j})^{1-\theta+\varepsilon} \left\| p_{s-s_{2j+1}} \left(u_{s_{2j}}^{12}\, \delta \mathscr{W}_{s_{2j+1}s_{2j+2}} \right) \right\|_{\mathcal{B}_{w_t}^{\kappa_u+2\theta_u+2(1-\theta+\varepsilon)}} \\
&\leq \frac{(s_{2j+1}-s_{2j})^{1-\theta+\varepsilon}}{(s-s_{2j+1})^{1+\theta_u-\theta+\varepsilon+\frac{\kappa_u+\kappa}{2}}} \|u_{s_{2j}}^{12}\, \delta \mathscr{W}_{s_{2j+1}s_{2j+2}}\|_{\mathcal{B}_{w_t}^{-\kappa}},
\end{aligned}
$$

and thus relation (5.19) entails:

$$\|L_{st}^{12,n,j}\|_{\mathcal{B}_{w_t}^{\kappa_u}} \leq \frac{c_\sigma\,(t-s)^{\theta_u}\, \|u^{12}\|_{\mathcal{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}} \|\mathscr{W}\|_{\mathcal{C}_{\rho_\sigma}^{\theta,-\kappa}}}{(s-s_{2j+1})^{1+\theta_u-\theta+\varepsilon+\frac{\kappa_u+\kappa}{2}+\sigma}} \left(\frac{s}{2^n}\right)^{1+\varepsilon}.$$

Putting together the last two estimates on $L_{st}^{11,n,j}$ and $L_{st}^{12,n,j}$ and choosing $\theta_u = 1 - \theta + \varepsilon$, we thus end up with:

$$\|L_{st}^{1,n,j}\|_{\mathcal{B}_{w_t}^{\kappa_u}} \leq \frac{c_\sigma\,(t-s)^{\theta_u}\, \|u^{12}\|_{\mathcal{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}} \|\mathscr{W}\|_{\mathcal{C}_{\rho_\sigma}^{\theta,-\kappa}}}{(s-s_{2j+1})^{2-2\theta+2\varepsilon+\frac{\kappa_u+\kappa}{2}+\sigma}} \left(\frac{s}{2^n}\right)^{1+\varepsilon}. \tag{5.20}$$

Let us now discuss exponent values: for the convergence of $L_{st}^{1;n}$ we need the condition

$$2 - 2\theta + 2\varepsilon + \frac{\kappa_u+\kappa}{2} + \sigma < 1$$

to be fulfilled. If we choose $\kappa_u = \kappa + 2\varepsilon$, we can recast this condition into $\theta > \frac{1+\kappa}{2} + \frac{3\varepsilon+\sigma}{2}$. Since ε, σ are chosen to be arbitrarily small, we can satisfy this constraint as soon as $\theta > \frac{1+\kappa}{2}$, which was part of our Hypothesis 5.17. For the remainder of the discussion, we thus assume that

$$2 - 2\theta + 2\varepsilon + \frac{\kappa_u + \kappa}{2} + \sigma = 1 - \eta, \quad \text{with} \quad \eta > 0.$$

Step 3: Bound on L_{st}^1. We express $\lim_{n\to\infty} L_{st}^{1,n}$ as $L_{st}^{1,0} + \sum_{n=0}^{\infty}(L_{st}^{1,n+1} - L_{st}^{1,n})$. Now

$$\sum_{n=0}^{\infty} \|L_{st}^{1,n+1} - L_{st}^{1,n}\|_{\mathcal{B}_{w_t}^{\kappa_u}} \leq \sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \|L_{st}^{1,n,j}\|_{\mathcal{B}_{w_t}^{\kappa_u}},$$

and plugging our estimate (5.20), we get that $\sum_{n=0}^{\infty} \|L_{st}^{1,n+1} - L_{st}^{1,n}\|_{\mathcal{B}_{w_t}^{\kappa_u}}$ is bounded by:

$$c_\sigma \|u^{12}\|_{\mathcal{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}} \|\mathscr{W}\|_{\mathcal{C}_{\rho_\sigma}^{\theta,-\kappa}} (t-s)^{\theta_u} \sum_{n=0}^{\infty} \left(\frac{s}{2^n}\right)^\varepsilon \left(\frac{s}{2^n} \sum_{j=0}^{2^n-1} \frac{1}{(s-s_{2j+1})^{1-\eta}}\right).$$

Furthermore, the following uniform bound holds true:

$$\frac{s}{2^n} \sum_{j=0}^{2^n-1} \frac{1}{(s-s_{2j+1})^{1-\eta}} \leq c \int_0^s \frac{dr}{r^{1-\eta}} = c\, s^\eta,$$

and thus

$$\sum_{n=0}^{\infty} \|L_{st}^{1,n+1} - L_{st}^{1,n}\|_{\mathcal{B}_{w_t}^{\kappa_u}} \leq c\, s^\eta (t-s)^{\theta_u} \sum_{n=0}^{\infty} \left(\frac{s}{2^n}\right)^\varepsilon \leq c\, s^{\eta+\varepsilon} (t-s)^{\theta_u},$$

which ensures the convergence of $L_{st}^{1,n}$. Finally, invoking our definition (5.17) plus the fact that $u_0^{12} = 0$, it is readily checked that $L_{st}^{1,0} = 0$. Thus the relation above transfers into:

$$\begin{aligned}
\|L_{st}^1\|_{\mathcal{B}_{w_t}^{\kappa_u}} &\leq c\, s^{\eta+\varepsilon} \|u^{12}\|_{\mathcal{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}} \|\mathscr{W}\|_{\mathcal{C}_{\rho_\sigma}^{\theta,-\kappa}} (t-s)^{\theta_u} \\
&\leq c\, \tau^{\eta+\varepsilon} \|u^{12}\|_{\mathcal{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}} \|\mathscr{W}\|_{\mathcal{C}_{\rho_\sigma}^{\theta,-\kappa}} (t-s)^{\theta_u}.
\end{aligned} \tag{5.21}$$

Step 4: Bound on L_{st}^2. The bound on L_{st}^2 follows along the same lines as for $L_{st}^{1,n}$, and is in fact slightly easier. Let us just mention that we approximate L_{st}^2 by a sequence $L_{st}^{2,n}$ based on the dyadic partition of $[s,t]$, namely $s_j^n = s + j2^{-n}(t-s)$. Like in Step 2, we end up with some terms $L_{st}^{21,n,j}, L_{st}^{22,n,j}$, where

$$L_{st}^{21,n,j} = p_{s-s_{2j+1}}(\delta u_{s_{2j}s_{2j+1}}^{12} \,\delta\mathscr{W}_{s_{2j+1}s_{2j+2}})$$

and

$$L_{st}^{22,n,j} = \left[p_{s_{2j+1}-s_{2j}} - \mathrm{Id}\right] p_{t-s_{2j+1}} \left(u_{s_{2j}}^{12} \,\delta\mathscr{W}_{s_{2j+1}s_{2j+2}}\right).$$

From this decomposition, we leave to the patient reader the task of checking that relation (5.21) also holds true for L_{st}^2.

Step 5: Conclusion. Putting together the last 2 steps, we have been able to prove that for all $0 \leq s < t \leq \tau$ we have

$$\|v_t^{12} - v_s^{12}\|_{\mathcal{B}_{w_t}^{\kappa_u}} \leq c\, \tau^{\eta+\varepsilon} \|u^{12}\|_{\mathcal{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}} \|\mathscr{W}\|_{\mathcal{C}_{\rho_\sigma}^{\theta,-\kappa}} (t-s)^{\theta_u}.$$

Thus, choosing $\tau = (c \|\mathscr{W}\|_{\mathcal{C}_{\rho_\sigma}^{\theta,-\kappa}}/2)^{1/(\varepsilon+\eta)}$, this yields

$$\|v_t^{12} - v_s^{12}\|_{\mathcal{B}_{w_t}^{\kappa_u}} \leq \frac{1}{2} \|u^{12}\|_{\mathcal{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}} (t-s)^{\theta_u},$$

namely the announced contraction property. We have thus obtained existence and uniqueness of the solution to equation (5.15) on $[0, \tau]$. In order to get a global solution on an arbitrary interval, it suffices to observe that all our bounds above do not depend on the initial condition of the solution. One can thus patch solutions on small intervals of constant length τ. The continuity result *(b)* is obtained thanks to the same kind of considerations, and we spare the details to the reader for sake of conciseness.

$\square$

5.2.4 Identification of the Feynman-Kac solution

This section is devoted to the identification of the solution to the stochastic heat equation given by the Feynman-Kac representation formula and the pathwise solution constructed in this section. Calling u^F the Feynman-Kac solution, the global strategy for this identification procedure is the following:

1. Relate the covariance structure (2.1) of the Gaussian noise W to Hypothesis 5.17. We shall see that our Hypothesis 4.8 implies that W satisfies 5.17 almost surely for suitable values of the parameters θ and κ.

2. Prove that u^F coincides with the pathwise solution to (5.15), by means of approximations of the noise W.

We now handle those three problems.

Let us start by establishing the pathwise property of W as a distribution valued function.

Proposition 5.22. *Let W be a centered Gaussian noise defined by μ and γ as in (2.1), satisfying Hypothesis 4.8 for some $0 < \alpha < 1 - \beta$. Then the mapping $(t, \varphi) \to W(\mathbf{1}_{[0,t]}\varphi)$ is almost surely Hölder continuous of order θ in time with values in $\mathcal{B}_{\rho_\sigma}^{-\kappa}$ for arbitrarily small σ and for all $\theta, \kappa \in (0, 1)$ such that $\theta < 1 - \frac{\beta}{2}$ and $\kappa > 1 - \alpha - \beta$. That is, almost surely W satisfies Hypothesis 5.17. Moreover, $\|W\|_{\mathcal{C}_{\rho_\sigma}^{\theta,-\kappa}}$ is a random variable which admits moments of all orders.*

Proof of Proposition 5.22. Fix $\kappa > \kappa' > 1 - \alpha - \beta$. For $q \geq 1$, let us denote the Besov space $\mathcal{B}_{2q,2q,\rho_\sigma}^{-\kappa'}$ by $\mathcal{A}_q$, and recall that the norm on $\mathcal{A}_q$ is given by:

$$\|f\|_{\mathcal{A}_q}^{2q} = \sum_{j \geq -1} 2^{-2qj\kappa'} \|\Delta_j f\|_{L_{\rho_\sigma}^{2q}}^{2q}.$$

We will choose q large enough so that $\mathcal{A}_q \hookrightarrow \mathcal{B}_{\rho_\sigma}^{-\kappa}$, a fact which is ensured by Besov embedding theorems. We will show that almost surely:

$$\|\delta W_{st}\|_{\mathcal{A}_q} \leq Z (t-s)^{\theta}, \tag{5.22}$$

for any $\theta \in (0, 1 - \frac{\beta}{2})$ and the random variable Z admitting moments of all orders. This will complete the proof of the proposition.

To this aim, recall from Section 5.2.1 that $\Delta_j f(x) = [K_j * f](x)$, where $K_j(z) = 2^{jd} K(2^j z)$ and K is the inverse Fourier transform of φ. Otherwise stated, K_j is the

inverse Fourier transform of φ_j. With these preliminary considerations in mind, set $K_{j,x}(y) := K_j(x-y)$ and evaluate:

$$
\begin{aligned}
\mathbf{E}\left[\|\delta W_{st}\|_{\mathcal{A}_q}^{2q}\right] &= \sum_{j \geq -1} 2^{-2qj\kappa'} \int_{\mathbb{R}^d} \mathbf{E}\left[\left|W\left(\mathbf{1}_{[s,t]} \otimes K_{j,x}\right)\right|^{2q}\right] \rho_\sigma^{2q}(x)\,dx \\
&\leq c_q \sum_{j \geq -1} 2^{-2qj\kappa'} \int_{\mathbb{R}^d} \mathbf{E}^q\left[\left|W\left(\mathbf{1}_{[s,t]} \otimes K_{j,x}\right)\right|^2\right] \rho_\sigma^{2q}(x)\,dx \quad (5.23)
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
\mathbf{E}\left[\left|W\left(\mathbf{1}_{[s,t]} \otimes K_{j,x}\right)\right|^2\right] &= \int_{[s,t]^2} \left(\int_{\mathbb{R}^d} |\mathcal{F}K_{j,x}|^2\,\mu(d\xi)\right) \gamma(u-v)\,du\,dv \\
&\leq (t-s)^{2-\beta} \int_{\mathbb{R}^d} \left|\varphi\left(2^{-j}\xi\right)\right|^2 \mu(d\xi). \quad (5.24)
\end{aligned}
$$

Let us introduce the measure $\nu(d\xi) = \mu(d\xi)/(1+|\xi|^{2(1-\alpha-\beta)})$, which is a finite measure on $\mathbb{R}^d$ according to our standing assumption. Also recall from Notation 5.9 that $\mathrm{Supp}(\varphi) \subset \{x \in \mathbb{R}^d : a \leqslant |x| \leqslant b\}$. Hence

$$
\int_{\mathbb{R}^d} \left|\varphi\left(2^{-j}\xi\right)\right|^2 \mu(d\xi) \leq \int_{\mathbb{R}^d} \mathbf{1}_{[0,2^j b]}(|\xi|)\left[1+|\xi|^{2(1-\alpha-\beta)}\right]\nu(d\xi) \leq c_\mu\, 2^{2(1-\alpha-\beta)j}.
$$

Plugging this identity into (5.24) and then (5.23) we end up with the relation $\mathbf{E}[\|\delta W_{st}\|_{\mathcal{A}_q}^{2q}] \leq c_q(t-s)^{(2-\beta)q}$, valid for all $0 \leq s < t \leq T$ and any $q \geq 1$. A standard application of Garsia's and Fernique's lemma then yields relation (5.22), and thus Hypothesis 5.17 . $\qquad \square$

Remark 5.23. In particular, equation (5.14) driven by W admits a unique pathwise solution in $\mathcal{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}$, as in Theorem 5.21, for some $\theta_u > \frac{\beta}{2}$ and $\kappa_u > 1-\alpha-\beta$. Notice here that one obtains (see Theorem 5.3) the existence of a solution to our equation in the Stratonovich sense under Hypothesis 4.1 only. We call this assumption the critical case. In order to get existence and uniqueness of a pathwise solution we have to impose the more restrictive Hypothesis 4.8 with an arbitrarily small constant α, which can be seen as a supercritical situation. This is the price to pay in order to get uniqueness of the solution.

We now turn to the second point of our strategy, namely prove that the Feynman-Kac solution u^F coincides with the unique pathwise solution to equation 5.14 driven by W.

Proposition 5.24. *Let u^F be the random field given by equation (4.1). Assume that W satisfies Hypothesis 4.8. Then there exist $\theta_u > \frac{\beta}{2}$ and $\kappa_u > 1-\alpha-\beta$ such that almost surely u^F belongs to the space $\mathcal{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}$. Moreover, u^F is the pathwise solution to equation (5.14) driven by W.*

Proof. To show that u^F is the pathwise solution to equation 5.14, we use the fact that $u_{t,x}^F$ is the limit in $L^p(\Omega)$ of the approximating sequence $u_{t,x}^{\varepsilon,\delta}$ introduced in (4.15) (see (4.16)) as ε and δ tend to zero, for any $p \geq 1$. On the other hand, it is clear that $u^{\varepsilon,\delta}$ is the pathwise solution to equation (5.14) driven by the trajectories of $W^{\varepsilon,\delta}$

$$
u_t^{\varepsilon,\delta} = p_t u_0 + \int_0^t p_{t-s}\left(u_s^{\varepsilon,\delta}\, W^{\varepsilon,\delta}(ds)\right).
$$

Then, it suffices to take the limit in the above equation to show that u^F is a pathwise solution to equation (5.14) driven by W. In fact, that for two particular sequences $\varepsilon_n \downarrow 0$ and $\delta_n \downarrow 0$ $W^{\varepsilon_n,\delta_n}$ converges to W almost surely in the space $\mathcal{C}_{T,\rho_\sigma}^{\theta,-\kappa}$. This implies (see Theorem 5.21 item (b)) that $u^{\varepsilon_n,\delta_n}$ converges almost surely to a process u in $\mathcal{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}$, which is the pathwise solution to equation 5.14 driven by W. Therefore, $u = u^F$ and this concludes the proof. $\qquad \square$

5.2.5 Time independent case

The case of a time independent noise is obviously easier to handle than the time dependent one. Basically, the Young integration arguments invoked above can be skipped, and they are replaced by Gronwall type lemmas for Lebesgue integration. We won't detail the proofs here, and just mention the main steps for sake of conciseness.

First, the pathwise type assumption we make on the noise W, considered as a distribution on $\mathbb{R}^d$, is the following counterpart of Hypothesis 5.17:

Hypothesis 5.25. *Suppose that $\mathscr{W}$ is a distribution on $\mathbb{R}^d$ such that $\mathscr{W} \in \mathcal{B}_{\rho_\sigma}^{-\kappa}$ with $\kappa \in (0,1)$ and an arbitrarily small constant $\sigma > 0$.*

Another simplification of the time independent case is that one can solve the equation in a space of continuous functions in time (compared to the Hölder regularity we had to consider before), with values in weighted Besov spaces. We thus define the following sets of functions

$$\mathcal{C}_{\lambda,\sigma}^{\kappa_u} = \left\{ f \in \mathcal{C}([0,T] \times \mathbb{R}^d); \|f_t\|_{\mathcal{B}_{w_t}^{\kappa_u}} \le c_f \right\}, \quad \text{where} \quad w_t := e_{\lambda+\sigma t}.$$

With these conventions in hand, we interpret equation (5.2) as a mild equation in the spaces $\mathcal{C}_{\lambda,\sigma}^{\kappa_u}$.

Definition 5.26. *Let $u \in \mathcal{C}_{\lambda,\sigma}^{\kappa_u}$ for $\lambda, \sigma > 0$ and $\kappa_u \in (\kappa, 1)$. Consider an initial condition $u_0 \in \mathcal{B}_{e_\lambda}^{\kappa_u}$. We say that u is a mild solution to equation*

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u\mathscr{W} \tag{5.25}$$

with initial condition u_0 if it satisfies the following integral equation

$$u_t = p_t u_0 + \int_0^t p_{t-s} \left(u_s \mathscr{W} \right) ds, \tag{5.26}$$

where the product $u\mathscr{W}$ is interpreted in the distributional sense.

We can now turn to the resolution of the equation in this context, and the main theorem in this direction is the following.

Theorem 5.27. *Let $\mathscr{W}$ be a distribution satisfying Hypothesis 5.25 and let λ be a strictly positive constant. Then equation (5.26) admits a unique solution in $\mathcal{C}_{\lambda,\sigma}^{\kappa_u}$, in the sense given by Definition 5.26, with $\kappa < \kappa_u < 1$.*

Proof. As in the proof of Theorem 5.21, we focus on the proof of uniqueness, and fix a small time interval $[0,\tau]$. Consider u^1, u^2 two solutions in $\mathcal{C}_{\lambda,\sigma}^{\kappa_u}$ and we set $u^{12} = u^1 - u^2$. Consistently with Definition 5.26, the equation for u^{12} is given by:

$$u_t^{12} = \int_0^t p_{t-s} \left(u_s^{12} \mathscr{W} \right) ds, \tag{5.27}$$

and we wish to prove that $u^{12} \equiv 0$.

Towards this aim, let us bound the Besov norm of u starting from equation (5.27). Owing to Proposition 5.15, we get

$$\|u_t^{12}\|_{\mathcal{B}_{w_t}^{\kappa_u}} \le \int_0^t \|p_{t-s}\left(u_s^{12}\mathscr{W}\right)\|_{\mathcal{B}_{w_t}^{\kappa_u}} ds \le c_{\tau,\lambda,\sigma} \int_0^t (t-s)^{-\frac{(\kappa_u+\kappa)}{2}} \|u_s^{12}\mathscr{W}\|_{\mathcal{B}_{w_t}^{-\kappa}} ds.$$

Along the same lines as in the proof of Theorem 5.21, we now invoke the bound (5.18), which yields $w_t \le c_{\tau,\lambda,\sigma} (t-s)^{-\sigma} w_s \rho_\sigma$. Hence, according to Proposition 5.13 item (iii), we have

$$\|u_t^{12}\|_{\mathcal{B}_{w_t}^{\kappa_u}} \le c_{\lambda,\sigma} \int_0^t (t-s)^{-\frac{(\kappa_u+\kappa)}{2}-\sigma} \|u_s^{12}\mathscr{W}\|_{\mathcal{B}_{w_s\rho_\sigma}^{-\kappa}} ds.$$

Since $\kappa_u > \kappa$, we now apply relation (5.11) with $w_1 = w_s$, $\kappa_1 = \kappa_u$, $w_2 = \rho_\sigma$ and $\kappa_2 = \kappa$. We end up with

$$\|u_t^{12}\|_{\mathcal{B}_{w_t}^{\kappa_u}} \leq c_{\tau,\lambda,\sigma} \|\mathscr{W}\|_{\mathcal{B}_{\rho_\sigma}^{-\kappa}} \int_0^t \frac{\|u_s^{12}\|_{\mathcal{B}_{w_s}^{\kappa_u}}}{(t-s)^{\frac{(\kappa_u+\kappa)}{2}+\sigma}} \, ds \, .$$

Taking into account that $\kappa_u + \kappa < 2$ and σ can be arbitrarily small, our conclusion $u^{12} \equiv 0$ follows easily from a Gronwall type argument. $\qquad\square$

We now state a result which allows to identify the Feynman-Kac and the pathwise solution to our spatial equation. Its proof is omitted for sake of conciseness, since it is easier than in the time dependent case.

Proposition 5.28. *Let W be a spatial Gaussian noise defined by the covariance structure (2.5) and (2.6). Assume that the measure μ satisfies the condition*

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^{2(1-\alpha)}} < \infty, \tag{5.28}$$

for a constant $\alpha \in (0,1)$. Then:

(i) There exists $\kappa \in (0,1)$ such that for any arbitrarily $\sigma > 0$, W has a version in $\mathcal{B}_{\rho_\sigma}^{-\kappa}$ and the random variable $\|W\|_{\mathcal{B}_{\rho_\sigma}^{-\kappa}}$ has moments of all orders, that is the trajectories of W satisfy Hypothesis 5.25. As a consequence, equation (5.26) driven by the trajectories of W admits a unique pathwise solution in $\mathcal{C}_{\lambda,\sigma}^{\kappa_u}$.

(ii) Let u^F be the Feynman-Kac solution to the heat equation given by (4.20). Then almost surely the process u^F lies into $\mathcal{C}_{\lambda,\sigma}^{\kappa_u}$, and it coincides with the unique pathwise solution to equation (5.26).

Remark 5.29. Here again, we see that the Feynman-Kac solution u^F exists under the critical condition $\int_{\mathbb{R}^d}(1+|\xi|^2)^{-1}\mu(d\xi) < \infty$, while the pathwise solution requires the more stringent condition (5.28).

6 Moment estimates

As mentioned in the introduction, intermittency properties for u are characterized by the family of Lyapounov type coefficients $\ell(k)$ defined by (1.3) or by the limiting behavior (1.4). In any case, the intermittency phenomenon stems from an asymptotic study of the moments of u, for large values of k and t. We propose to lead this study in the context of the general Gaussian noises considered in the current paper. Without loss of generality, we may assume that the initial condition is $u_0(x) \equiv 1$.

Notice that delicate results such as limiting behaviors for moments will rely on more specific conditions on the noise W. We are thus going to make use of the following conditions.

Hypothesis 6.1. *There exist positive constants c_0, C_0 and $0 < \beta < 1$, such that*

$$c_0|t|^{-\beta} \leq \gamma(t) \leq C_0|t|^{-\beta}.$$

Hypothesis 6.2. *There exist positive constants c_1, C_1 and $0 < \eta < 2$, such that*

$$c_1|x|^{-\eta} \leq \Lambda(x) \leq C_1|x|^{-\eta}.$$

Hypothesis 6.3. *There exist positive constants c_2, C_2 and $0 < \eta_i < 1$, with $\sum_{i=1}^d \eta_i < 2$, such that*

$$c_2 \prod_{i=1}^d |x_i|^{-\eta_i} \leq \Lambda(x) \leq C_2 \prod_{i=1}^d |x_i|^{-\eta_i}.$$

Clearly, Hypothesis 6.1 and Hypothesis 6.2 generalize the case of Riesz kernels and Hypothesis 6.3 generalizes the case of fractional noises. Notice that under Hypotheses 6.2 or 6.3 the spectral measure μ satisfies the integrability condition (2.4).

Theorem 6.4. *Suppose that γ satisfies Hypothesis 6.1 and Λ satisfies Hypothesis 6.2 or Hypothesis 6.3. Denote*

$$a = \begin{cases} \eta & \text{if Hypothesis 6.2 holds} \\ \sum_{i=1}^{d} \eta_i & \text{if Hypothesis 6.3 holds.} \end{cases}$$

Consider the following two cases:

(i) u *is the solution to the Skorohod equation (3.1) driven by a time dependent noise with time covariance γ and space covariance Λ.*

(ii) u *is the solution to the Stratonovich equation (5.1) driven by a time dependent noise with time covariance γ and space covariance Λ, and we assume that $a < 2 - 2\beta$.*

Then in both of these two cases we have

$$C \exp\left(Ct^{\frac{4-2\beta-a}{2-a}} k^{\frac{4-a}{2-a}} \right) \leq \mathbf{E}\left[u_{t,x}^k \right] \leq C' \exp\left(C't^{\frac{4-2\beta-a}{2-a}} k^{\frac{4-a}{2-a}} \right) \qquad (6.1)$$

for all $t \geq 0$, $x \in \mathbb{R}^d$, $k \geq 2$, where C, C' are constants independent of t and k.

Proof. Let us first discuss the upper bound. For the Skorohod equation, using the chaos expansion and the hypercontractivity property we can derive the upper bound as it has been done in [3]. For the Stratonovich equation, notice first that Hypothesis 4.8 holds because $a < 2 - 2\beta$. Using the Feynmann-Kac formula (4.1) for the solution to equation (5.1), and applying Cauchy-Schwartz inequality yields

$$
\begin{aligned}
\mathbf{E}\left[u_{t,x}^k \right] &= \mathbf{E}_B\left[\exp\left(\sum_{1 \leq i,j \leq k} \int_0^t \int_0^t \gamma(r-s)\Lambda(B_r^i - B_s^j)drds \right) \right] \\
&\leq \left[\mathbf{E}_B\left[\exp\left(2 \sum_{1 \leq i<j \leq k} \int_0^t \int_0^t \gamma(r-s)\Lambda(B_r^i - B_s^j)drds \right) \right] \right]^{\frac{1}{2}} \\
&\quad \times \left[\mathbf{E}_B\left[\exp\left(2 \sum_{i=1}^k \int_0^t \int_0^t \gamma(r-s)\Lambda(B_r^i - B_s^i)drds \right) \right] \right]^{\frac{1}{2}}.
\end{aligned}
$$

In the above expression, the first term is just the square root of the Feynman-Kac formula (3.21) for the moment of order k of the solution of a Skorohod equation with multiplicative noise, with covariances 2γ and 2Λ. For this term we know that we can derive the upper bound (6.1) using the chaos expansion and the hypercontractivity property as it has been done in [3]. For the second factor, using the asymptotic result proved in Proposition 2.1 in [11], we derive the estimate

$$\mathbf{E}^{\frac{k}{2}}\left[\exp\left(2 \int_0^t \int_0^t \gamma(r-s)\Lambda(B_r^1 - B_s^1)drds \right) \right] \leq C^k \exp\left(Ct^{\frac{4-2\beta-a}{2-a}} k \right).$$

Therefore, in this way we can obtain the desired upper bound of $\mathbf{E}\left[u_{t,x}^k \right]$.

Let us now discuss the lower bound. Taking into account again the Feynman-Kac formula (3.21) for the moments of u, it suffices to consider the case of the Skorohod equation (it is readily checked from (3.21) that the moments of u for the Stratonovich equation are greater than those of the Skorohod equation). The argument of the proof

is then based in the small ball probability estimates for Brownian motion. We consider only the case when Λ satisfies the lower bound given in hypothesis Hypothesis 6.2 (Riesz kernel case), since the case Hypothesis 6.3 (fractional noise) is analogous. In this case, we denote by $B_s^{i,l}, l = 1, 2, \ldots, d$ the l-th component of the d-dimensional Brownian motion B_s^i and consider the set

$$A_{\varepsilon,t} = \left\{ \sup_{1 \leq i < j \leq k} \sup_{1 \leq l \leq d} \sup_{0 \leq s,r \leq t} |B_s^{i,l} - B_r^{j,l}| \leq \varepsilon \right\}.$$

Then, owing to formula (3.21) it is easy to see that

$$
\begin{aligned}
\mathbf{E}\left[u_{t,x}\right]^k &\geq \mathbf{E}\left[\exp\left(c \sum_{1 \leq i < j \leq k} \int_0^t \int_0^t |s - r|^{-\beta} |B_s^i - B_r^j|^{-\eta} ds dr\right) \mathbf{1}_{A_{\varepsilon,t}}\right] \quad (6.2) \\
&\geq \mathbf{E}\left[\exp\left(c \sum_{1 \leq i, < j \leq k} \int_0^t \int_0^t |s - r|^{-\beta} \varepsilon^{-\eta} ds dr\right) \mathbf{1}_{A_{\varepsilon,t}}\right] \\
&\geq \exp\left(\frac{c\, k(k-1)}{(2-\beta)(1-\beta)} t^{2-\beta} \varepsilon^{-\eta}\right) \mathbf{P}\left(A_{\varepsilon,t}\right) \geq \exp\left(c\, t^{2-\beta} k^2 \varepsilon^{-\eta}\right) \mathbf{P}\left(A_{\varepsilon,t}\right).
\end{aligned}
$$

The value of the generic constant c might change from line to line. Moreover, notice that

$$\cap_{i=1}^k \cap_{l=1}^d F_{i,l,t} \subset A_{\varepsilon,t}, \quad \text{with} \quad F_{i,l,t} = \left(\sup_{0 \leq s \leq t} |B_s^{i,l}| \leq \frac{\varepsilon}{2}\right).$$

The events $F_{i,l}$ being i.id, we get:

$$\mathbf{P}\left(A_{\varepsilon,t}\right) \geq \mathbf{P}^{kd}\left(F_{\varepsilon,t}\right), \quad \text{with} \quad F_{\varepsilon,t} = \left(\sup_{0 \leq s \leq 1} |b_s| \leq \frac{\varepsilon}{2\sqrt{t}}\right),$$

where b stands for a one dimensional standard Brownian motion. In addition, it is a well known fact (see e.g (1.3) in [35]) that $\mathbf{P}(F_{\varepsilon,t}) \geq C \exp(-\frac{Ct}{\varepsilon^2})$. This entails:

$$\mathbf{E}\left[u_{t,x}\right]^k \geq C \exp\left(c\, t^{2-\beta} k^2 \varepsilon^{-\eta} - \frac{Cdkt}{\varepsilon^2}\right).$$

In order to optimize this expression, we try to equate the two terms inside the exponential above. To this aim, we set

$$\varepsilon = \frac{(2dC)^{\frac{1}{2-\eta}}}{(ckt^{1-\beta})^{\frac{1}{2-\eta}}},$$

to conclude that

$$\mathbf{E}\left[u_{t,x}^k\right] \geq C \exp\left(\frac{c^{\frac{2}{2-\eta}} t^{\frac{4-2\beta-\eta}{2-\eta}} k^{\frac{4-\eta}{2-\eta}}}{2(2Cd)^{\frac{\eta}{2-\eta}}}\right), \quad (6.3)$$

which finishes the proof of (6.1). $\qquad\square$

In the next theorem, the above moment estimates are extended to the time independent case. Its proof is very similar to the proof of Theorem 6.4, and is thus omitted for sake of conciseness.

Theorem 6.5. *Suppose that Λ satisfies Hypothesis 6.2 or Hypothesis 6.3. Set $a = \eta$ if Hypothesis 6.2 holds, and $a = \sum_{i=1}^d \eta_i$ if Hypothesis 6.3 holds. Suppose that u is the solution to the Skorohod equation (3.32) or the Stratonovich equation (5.2) driven by a multiplicative time independent noise with covariance Λ. Then, for any $x \in \mathbb{R}^d$, $k \geq 2$ and $t \geq 0$, we have*

$$C \exp\left(Ct^{\frac{4-a}{2-a}} k^{\frac{4-a}{2-a}}\right) \leq \mathbf{E}\left[u_{t,x}^k\right] \leq C' \exp\left(C' t^{\frac{4-a}{2-a}} k^{\frac{4-a}{2-a}}\right), \quad (6.4)$$

where $C, C' > 0$ are constants independent of t and k.

Remark 6.6. From the proof of the above theorems it is easy to see that if in Hypothesis 6.2, the global lower bound of $\Lambda(x)$, i.e., $c_1|x|^{-\eta} \leq \Lambda(x)$, is replaced by the local lower bound

$$c_1|x|^{-\eta} \leq \Lambda(x), \quad \text{when} \quad |x| \leq R \tag{6.5}$$

for some positive R, or even in a limit form

$$\liminf_{x \to 0} \frac{\Lambda(x)}{|x|^{-\eta}} = c_1 > 0, \tag{6.6}$$

then Theorems 6.4 and 6.5 still hold. The same remark applies to Hypothesis 6.3.

In the following remarks we discuss what happens in the limiting cases $\eta = 0$ and $\beta = 0$.

Remark 6.7. We can consider the case where Λ is an everywhere continuous function. In this case, $\Lambda(x) \leq \Lambda(0)$ for all $x \in \mathbb{R}^d$ and the upper bound in Hypothesis 6.2 is satisfied with $C_1 = \Lambda(0)$ and $\eta = 0$. On the other hand, typically Λ decreases to zero at infinity, as in the standard example $e^{-|x|^2}$, so the lower bound does not hold everywhere. However, the local condition (6.6) holds with again $c_1 = \Lambda(0)$ and $\eta = 0$. As a consequence, Theorem 6.4 can be extended to the case where Λ is everywhere continuous, and we obtain. for the solution of both the Skorohod and Stratonovich equations. the following inequalities

$$C \exp\left(C\Lambda(0)t^{2-\beta}k^2\right) \leq \mathbf{E}\left[u_{t,x}^k\right] \leq C' \exp\left(C'\Lambda(0)t^{2-\beta}k^2\right),$$

where $C, C' > 0$ are constants independent of t and k. We omit the details of the proof. Similarly for Theorem 6.5, where we just formally set $\beta = 0$.

Remark 6.8. Theorem 6.4 holds under the following condition on γ, which is more general than Hypothesis 6.1: there exist positive constants d_0, D_0 and $0 \leq \beta < 1$, such that for all $t > 0$

$$d_0 t^{2-\beta} \leq \int_0^t \int_0^t \gamma(s-r) dr ds \leq D_0 t^{2-\beta}.$$

In particular, when $\beta = 0$, it covers the case when γ is a positive constant.

Finally, when $d = 1$ we can also obtain moment estimates in the case where the space covariance is a Dirac delta function, that is, the noise is white in space.

Theorem 6.9. *Suppose that γ satisfies condition Hypothesis 6.1 and the spatial dimension is 1. Consider two cases:*

(i) *Suppose that u satisfies either the Skorohod equation (3.1) or the Stratonovich equation (5.1) driven by a multiplicative noise with time covariance γ and spatial covariance $\Lambda(x) = \delta_0(x)$. Then, for any $x \in \mathbb{R}^d$, $k \geq 2$ and $t \geq 0$, we have*

$$\exp\left(Ct^{3-2\beta}k^3\right) \leq \mathbf{E}\left[u_{t,x}^k\right] \leq \exp\left(C't^{3-2\beta}k^3\right), \tag{6.7}$$

where $C, C' > 0$ are constants independent of t and k.

(ii) *Suppose that u satisfies either the Skorohod equation (3.32) or the Stratonovich equation (5.2) driven by a time independent multiplicative noise with spatial covariance $\Lambda(x) = \delta_0(x)$. Then, for any $x \in \mathbb{R}^d$, $k \geq 2$ and $t \geq 0$, we have*

$$\exp\left(Ct^3 k^3\right) \leq \mathbf{E}\left[u_{t,x}^k\right] \leq \exp\left(C't^3 k^3\right), \tag{6.8}$$

where $C, C' > 0$ are constants independent of t and k.

Proof. In the Skorohod case with time dependent noise, the moments of $u_{t,x}$ are given by equation (3.31). We will only discuss the lower bound because the upper bound can

be obtained by using chaos expansions as in [3]. We consider the approximation of the Dirac delta function by the heat kernel p_ε, and define

$$I_{t,k,\varepsilon} = \mathbf{E}_B \left[\exp \left(\sum_{1 \le i < j \le k} \int_0^t \int_0^t \gamma(s-r) p_\varepsilon(B_s^i - B_r^j) ds dr \right) \right] . \qquad (6.9)$$

Expanding the exponential and using Fourier analysis as in [27], one can show that $\mathbf{E}\left[u_{t,x}^k\right] \ge I_{t,k,\varepsilon}$, for any $\varepsilon > 0$. For any positive ε, denote

$$A_{k,\varepsilon,t} = \left\{ \max_{1 \le i \le k} \sup_{0 \le s \le t} |B_s^i| \le \sqrt{\varepsilon} \right\} .$$

On the event $A_{k,\varepsilon,t}$ we have $p_\varepsilon(B_s^i - B_r^j) \ge \frac{C}{\sqrt{\varepsilon}}$ for some positive constant C. Therefore, using the lower bound in Hypothesis 6.1, we can write similarly to (6.2):

$$I_{t,k,\varepsilon} \quad \ge \quad \exp\left(c\, k^2 \int_0^t \int_0^t |s-r|^{-\beta} \frac{C}{\sqrt{\varepsilon}} ds dr \right) \mathbf{P}\left(A_{k,\varepsilon,t} \right) .$$

Furthermore, by the scaling property of Brownian motion, $\mathbf{P}\left(A_{k,\varepsilon,t} \right)$ can be written as:

$$\mathbf{P}\left(A_{k,\varepsilon,t} \right) = \mathbf{P}\left(\max_{1 \le i \le k} \sup_{0 \le s \le 1} |B_s^i| \le \sqrt{\varepsilon/t} \right) = \left(\mathbf{P}\left(\max_{0 \le s \le 1} |b_s| \le \sqrt{\varepsilon/t} \right) \right)^k ,$$

where b stands for a one-dimensional standard Brownian motion. We now invoke again (1.3) in [35], which yields $\lim_{\varepsilon \to 0} \mathbf{P}(\sup_{0 \le s \le 1} |B_s| \le \sqrt{\frac{\varepsilon}{t}}) / \exp(-\frac{\pi^2}{8} \frac{t}{\varepsilon}) = 1$. Thus, when ε is sufficiently small,

$$\mathbf{P}\left(\sup_{0 \le s \le 1} |B_s| \le \sqrt{\frac{\varepsilon}{t}} \right) \ge \exp\left(-C \frac{t}{\varepsilon} \right) ,$$

for some positive constant C which does not depend on t. Hence, we end up with the following lower bound:

$$I_{t,k,\varepsilon} \ge \exp\left(C_1 k^2 t^{2-\beta} \frac{1}{\sqrt{\varepsilon}} - C_2 \frac{t}{\varepsilon} \right) .$$

As in the proof of Theorem 6.4, we optimize this expression by choosing $\varepsilon = \frac{4C_2^2}{C_1^2 k^3 t^{2-2\beta}}$, and we obtain that

$$I_{t,k,\varepsilon} \ge \exp(C_3 t^{3-2\beta} k^3) \qquad (6.10)$$

when t is sufficiently large, where the positive constant C_3 does not depend on t or k.

For the Stratonovich case, the lower bound is obvious and for the upper bound we use the Cauchy-Schwartz inequality and Lemma 2.2 in [11]. The estimate (6.8) is proved similarly, which completes the proof. $\qquad \square$

Remark 6.10. As a consequence of Theorems 6.4, 6.5 and 6.9, the solution u of both the Skorohod and Stratonivich equations is intermittent in the sense of condition (1.4).

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