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Maximum principle for an optimal control problem associated to a stochastic variational inequality with delay*

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Abstract

We deal with a stochastic control problem subject to a stochastic variational inequality with delay. By deriving the adjoint equation as an anticipated backward stochastic differential equation, we are able to establish necessary conditions of optimality under the form of a Pontryagin-Bensoussan stochastic maximum principle. This is achieved first for càdlàg controls, by explicitly writing the coefficients of the adjoint equation in terms of the local time of the state process. The general result is then obtained by approximating the optimal control with continuous controls and applying Ekeland's variational principle to the approximating sequence.

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1 Introduction

In this paper we establish necessary conditions for the existence of an optimal control u^* minimizing the cost functional

$$J(u) := \mathbb{E}\left[\int_0^T g(t, R(X)_t, u_t)dt + h(X_T^u)\right]$$

subject to the one-dimensional stochastic variational inequality (SVI) with delay

$$\begin{cases} dX_t + \partial\varphi(X_t)dt \ni b(t, R(X)_t, u_t)dt + \langle \sigma(t, R(X)_t, u_t), dW_t\rangle \,,\, t \in [0,T]; \\ X_t = \eta(t),\ t \in [-\delta, 0]. \end{cases} \tag{1.1}$$

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where $\partial\varphi$ is the subdifferential of a lower semi-continuous (l.s.c.) convex function φ and

$$R(X)_t := \int_{-\delta}^{0} X_{t+r}\lambda(dr),\ t\in[0,T]$$

is a delay term applied to the dynamics of the system. In order to reach this goal we will employ one of the essential approaches in solving optimal control problems, the maximum principle.

The maximum principle approach has been introduced by Pontryagin and his group in the 1950's to establish necessary conditions of optimality for deterministic controlled systems. Since then, the number of papers on the subject sharply increased and a lot of work has been done on different type of systems. One major difficulty that arises in the extension to the stochastic controlled systems is that the adjoint equation becomes an SDE with terminal conditions, called backward SDEs (BSDEs). Pioneering work in this direction was achieved by Kushner [14], Bismut [6] or Haussman [12]. The results therein concern the case where the diffusion does not depend on control. Peng removed this restriction in [21], by establishing a maximum principle containing two adjoint equations, both in the form of linear BSDEs, because one needs to take into account both the first-order and second-order terms in the Taylor expansion. There is also another possibility of treating the case where the diffusion is controlled: if the action space of controls is convex, it is possible to derive the maximum principle in a local form. This is accomplished by using a convex perturbation of the control instead of a spike variation. Important results in this direction have been obtained by Bensoussan [3] or [4].

Variational inequalities, on the other hand, form an important class of problems appearing in applications, ranging from electrostatics to optimization and game theory. In the stochastic case, variational inequalities given by subdifferential operators were introduced by Răşcanu [24]. General variational inequalities on non-convex domains have been considered in [8]. Concerning the control of such systems, Barbu [2] initiated systematic studies on controlled variational inequalities in the deterministic case. On stochastic control, results have been obtained in the following directions: existence of an optimal control ([25]) and the study of associated Hamilton-Jacobi-Bellman equation ([9], [26]).

For delayed stochastic controlled systems, the delay responses bring more difficulties in solving control problems. One of the first results in this topic can be found in [13]. In general the problem is by its nature infinite dimensional but nevertheless, it happens that the delayed systems can be reduced to finite dimensional systems under certain conditions. We refer to [11], [15] and [16] for contributions in this direction, mainly by the dynamic programming principle approach. Concerning the maximum principle, a general result was considered in [19], where the state system is an SDE driven by a Wiener-Poisson process, with delay of the form $R(X)_t := \left(X_{t-\delta}, \int_{-\delta}^{0} e^{\rho r} X_{t+r}dr\right)$. The authors establish sufficient and necessary stochastic maximum principle, where the associated adjoint equation is a time-advanced backward stochastic differential equation.

The main difficulty we have to deal to in this paper is the lack of smoothness for the subdifferential operator $\partial\varphi$; the only regularity which it possess is maximal monotonicity (generalizing monotonicity and continuity for single-valued functions). In general, in order to derive the maximum principle for the optimal control, one needs C^1-differentiability of the coefficients (when the control space is convex, otherwise we need C^2-differentiability). We are able to overcome this difficulty by considering the second derivative of φ in a generalized form and write "$\partial^2\varphi(X_t)$" in terms of the local time of X. In order to find the solution of the variation equation, corresponding heuristically to "dX/du", we adapt the standard approach for controlled SDEs (developed in [3], for example) to solutions of a SDE with delay which approximate the solution of the variational inequality (1.1) *via* Moreau-Yosida regularization of φ. Then, using an indirect method we obtain the weak derivative of the dynamics of the system with respect to the control by passing to the limit in the approximated variation equation. However, this works only for càdlàg controls, due to the extensive use of weak convergence of measures on the real line. Also, the methods employed force us to restrict to the case where the domain of φ is the whole real line and the diffusion is non-degenerate. We are able then to show that optimal controls which are càdlàg satisfy a maximum principle, obtained by the duality with the adjoint equation, following the main ideas encountered in [19]. In order to pass to the case where the optimal control is not necessarily càdlàg, we use Ekeland's variational principle, by approximating the optimal control with continuous controls and then passing to the limit. The difference to the càdlàg case is that now an unknown parameter k appears in the adjoint equation.

This paper is organized as follows. In section 2, we introduce some notations and recall some preliminary results concerning the well-posedness of stochastic variational inequalities. Section 3 is devoted to the optimal control problem and is divided in three parts: derivation of the variation equation, the maximum principle for near-optimal controls and finally, the necessary conditions of optimality. The Appendix is concerned with the proof of Proposition 3.8, which is the core result on the variation equation.

2 Preliminaries

Throughout this paper we fix a time horizon $T > 0$, a delay constant $\delta \in [0, T]$ and a vector of m finite positive finite scalar measures on $\mathcal{B}([-\delta, 0])$, $\lambda = (\lambda^1, \dots, \lambda^m)^\top$. We will denote by $|\lambda|$ the measure $\lambda^1 + \cdots + \lambda^m$. The space of controls is a convex closed set $U \subseteq \mathbb{R}^l$. For the sake of simplicity, we will suppose that U is also bounded. This assumption is quite natural, since in the literature it is often assumed that the control space is compact, especially for existence purposes.

The Euclidean norm and the scalar product in an Euclidean space are denoted $|\cdot|$, respectively $\langle \cdot, \cdot \rangle$. For a closed set E of an Euclidean space, $-\delta \le s \le t < +\infty$ and a finite measure ν on $[s, t]$, we will use the following (standard) notations:

- For $p \ge 1$, $L^p_\nu([s,t]; E)$ denotes the space (of equivalence classes) of E-valued, p-integrable functions on $[s, t]$, endowed with the p-*norm*
$$\|x\|_{L^p_\nu([s,t];E)} := \left[\int_s^t |x(r)|^p \, \nu(dr) \right]^{1/p}.$$
- $C([s,t]; E)$ is the space of E-valued continuous functions on $[s, t]$, endowed with the $\sup$-*norm*[1]:
$$\|x\|_{C([s,t];E)} := \sup_{r \in [s,t]} |x(r)|.$$
In order to shorten the formulae, we also denote $\|\cdot\|_{s,t}$, $\|\cdot\|_t$ instead of $\|\cdot\|_{C([s,t];E)}$, respectively $\|\cdot\|_{C([0,t];E)}$. Sometimes we extend this notation to càdlàg functions on $[s, t]$.
- By $BV([s,t]; E)$ we denote the space of E-valued, bounded variation functions x on $[s, t]$. The *total variation* of a function $x \in BV([s,t]; E)$ is
$$\|x\|_{BV([s,t];E)} := \sup \sum_{i=0}^{k-1} |x(t_{i+1}) - x(t_i)|,$$
the supremum being taken on all $k \in \mathbb{N}^*$ and $s \le t_0 < t_1 < \cdots < t_k \le t$.

[1] the p-norm and sup-norm are not really norms unless E is a linear space. However, these "norms" determine complete metrics on $L^p_\nu([s,t]; E)$, respectively $C([s,t]; E)$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, W a d-dimensional standard Brownian motion and $\mathbb{F} := \{\mathcal{F}_t\}_{t\geq 0}$ the filtration generated by W augmented by the null-sets of $\mathcal{F}$. We prolong the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ on $[-\delta, 0)$ by setting $\mathcal{F}_t := \mathcal{F}_0$ for $t \in [-\delta, 0)$ (and we still denote by $\mathbb{F}$ the filtration $\{\mathcal{F}_t\}_{t\geq -\delta}$).

Sometimes it is interesting to restrict the information available to the controller and consider a subfiltration $\mathbb{G} := \{\mathcal{G}_t\}_{t\geq 0}$ of $\mathbb{F}$, instead of $\mathbb{F}$.

For $\mathbb{G}$, E, s, t, p and ν like above and $q \geq 1$, $L^q_{\mathbb{G}}(\Omega; \mathbf{E})$ is the space (of equivalence classes) of $\mathbb{G}$-progressively measurable E-valued processes Z on $[s,t]$ such that $\mathbb{E}\, \|Z\|^p_{\mathbf{E}} < +\infty$, for $\mathbf{E}$ denoting $L^p_\nu([s,t]; E)$, $C([s,t]; E)$ or $BV([s,t]; E)$. In the case that E is an Euclidean space and $\mathbf{E}$ is $L^p_\nu([s,t]; E)$ or $C([s,t]; E)$, $L^q_{\mathbb{G}}(\Omega; \mathbf{E})$ is a Banach space with respect to the norm $\left(\mathbb{E}\, \|Z\|^p_{\mathbf{E}}\right)^{1/p}$. We often denote $L^p_{\mathbb{G},\nu}(\Omega \times [s,t]\,; E)$ instead of $L^p_{\mathbb{G}}(\Omega; L^p_\nu([s,t]; E))$.

Since much of the work is done on the real line, in the case $E = \mathbb{R}$ we will simplify the notations by denoting $L^p_\nu[s,t]$, $C[s,t]$, $BV[s,t]$ instead of $L^p_\nu([s,t]; \mathbb{R})$, $C([s,t]; \mathbb{R})$, respectively $BV([s,t]; \mathbb{R})$. We can equally drop the subscript ν when ν is the Lebesgue measure.

We say that a multivalued operator $A : \mathbb{R} \to 2^{\mathbb{R}}$ is a *maximal monotone* operator if it is *monotone*, *i.e.*

$$(x^* - y^*)\,(x - y) \geq 0,\ \forall x, y \in \mathbb{R},\ \forall x^* \in A(x),\ \forall y^* \in A(y)$$

and is *maximal* with respect to monotonicity, *i.e.* if $x, x^* \in \mathbb{R}$ satisfy

$$(x^* - y^*)\,(x - y) \geq 0,\ \forall y \in \mathbb{R},\ \forall y^* \in A(y),$$

then $x^* \in A(x)$.

By [7], if $\varphi : \mathbb{R} \to (-\infty, +\infty]$ is a l.s.c. convex function with $\operatorname{Dom}\varphi \neq \emptyset$[2], then its *subdifferential*, defined by

$$\partial\varphi(x) := \{x^* \in \mathbb{R} \mid x^*(y - x) + \varphi(x) \leq \varphi(y),\ \forall y \in \mathbb{R}\}$$

is a maximal monotone operator. The converse is also true: every maximal monotone operator on $\mathbb{R}$ can be written as a subdifferential as above.

We consider the following SVI with delay

$$\begin{cases} dX_t + \partial\varphi(X_t)dt \ni b(t, R(X)_t, u_t)dt + \langle \sigma(t, R(X)_t, u_t), dW_t \rangle\,,\ t \in [0,T]; \\ X_t = \eta(t),\ t \in [-\delta, 0]. \end{cases} \tag{2.1}$$

where:

- u is an *admissible control, i.e.* u is an U-valued, progressively measurable process with respect to $\mathbb{G}$;
- R is the delay term defined by $R(x)(t) := \displaystyle\int_{-\delta}^{0} x(t + r)\lambda(dr)$ for $x \in C[-\delta, T]$ and $t \in [0,T]$;
- the measurable functions $b : [0,T] \times \mathbb{R}^m \times U \to \mathbb{R}$, $\sigma : [0,T] \times \mathbb{R}^m \times U \to \mathbb{R}^d$ are the coefficients of the equation;
- $\varphi : \mathbb{R} \to (-\infty, +\infty]$ is a l.s.c. convex function with $\operatorname{int}\operatorname{Dom}\varphi \neq \emptyset$;
- η represents the starting deterministic process, satisfying the following condition:

  (A$_0$) $\eta \in C[-\delta, 0]$ and $\eta(0) \in \overline{\operatorname{Dom}\varphi}$.

[2] the *domain* of φ is defined as $\operatorname{Dom}\varphi := \{x \in \mathbb{R} \mid \varphi(x) < +\infty\}$.

As a convention, we regard σ, u, λ (and hence $R(\cdot)$) and W as column vectors.

We mention that coefficients depending also on the present state of the solution X_t can be envisaged by replacing λ with $\lambda' := (\lambda, \boldsymbol{\delta}_0)$, where $\boldsymbol{\delta}_0$ is the Dirac measure on $[-\delta, 0]$ concentrated in 0.

Definition 2.1. *A pair of one-dimensional, continuous $\mathbb{F}$-adapted processes (X, K) is called a solution of (2.1) if the following hold $\mathbb{P}$-a.s.:*

(i) $||K||_{BV[0,T]} < \infty$; $K_t = 0,\ \forall t \in [-\delta, 0]$;

(ii) $X_t = \eta(t),\ \forall t \in [-\delta, 0]$;

(iii) $X_t + K_t = \eta(0) + \int_0^t b(s, R(X)_s, u_s)ds + \int_0^t \langle \sigma(s, R(X)_s, u_s), dW_s \rangle\,,\ \forall\, t \in [0, T]$;

(iv) $\int_0^T (y(r) - X_r)dK_r + \int_0^T \varphi(X_r)dr \leq \int_0^T \varphi(y(r))dr, \forall y \in C[0, T]$.

Remark 2.2. In general, one cannot expect to show that K is an absolutely continuous process such that $\frac{dK_r}{dr} \in \partial\varphi(X_r)$, dr-a.e.; the last condition (iv) is introduced as a natural weakening and can be understood as the rigorous translation of the expression "$dK_r \in \partial\varphi(X_r)dr$". The reader can find equivalent conditions to it in [1], for example.

As a consequence of (iv) and of the continuity of X, we have that $X_t \in \overline{\mathrm{Dom}\,\varphi},\ \forall t \in [0, T]\,$, $\mathbb{P}$-a.s.

In order to have existence and uniqueness of strong solutions for equation (2.1), we impose the following conditions on the coefficients:

(A$_1$) there exists a constant $L > 0$ such that for every $t \in [0, T]$, $y, \tilde{y} \in \mathbb{R}^m$ and $u \in U$:

- (i) $|b(t, y, u) - b(t, \tilde{y}, u)| \leq L\,|y - \tilde{y}|$;
- (ii) $|\sigma(t, y, u) - \sigma(t, \tilde{y}, u)| \leq L\,|y - \tilde{y}|$.

Theorem 2.3. *Let $p \geq 2$. Under assumptions (A$_0$) and (A$_1$), for each control u, equation (2.1) has a unique solution (X^u, K^u) in the space $L^p_{\mathbb{F}}(\Omega; C[-\delta, T])) \times \big(L^p_{\mathbb{F}}(\Omega; C[-\delta, T])) \cap L^{p/2}_{\mathbb{F}}(\Omega; BV[-\delta, T]))\big)$.*

This result is the generalization to the delay case of Theorem 2.1. in [1] and its proof follows essentially the same lines. It was stated in a more general setting in [25] (Theorem 2.3) and was proved in detail in [9] by the penalization method *via* the Moreau-Yosida regularization of φ, though in the particular case where the delay has the form $R(X)_t = \Big(X_{t-\delta}, \int_{-\delta}^0 e^{\rho r} X_{t+r}\, dr\Big)$. For the above reasons, we skip the proof.

In order to have continuous dependence on controls, we impose the supplementary Lipschitz condition:

(A$_2$) there exists a constant $\tilde{L} > 0$ such that for every $t \in [0, T]$, $y \in \mathbb{R}^m$ and $u, v \in U$:

- (i) $|b(t, y, u) - b(t, y, v)| \leq \tilde{L}\,|u - v|$;
- (ii) $|\sigma(t, y, u) - \sigma(t, y, v)| \leq \tilde{L}\,|u - v|$.

Proposition 2.4. *Under assumptions (A$_0$)-(A$_2$), for every $p \geq 2$, there exists a constant $C_p > 0$ such that*

$$\mathbb{E}\,\|X^u - X^v\|^p_T + \mathbb{E}\,\|K^u - K^v\|^p_T \leq C\mathbb{E}\int_0^T |u_t - v_t|^p\, dt,$$

for all admissible controls u, v.

Proof. Since the proof involves only standard calculus, we give just a sketch, leaving the details to the reader. Applying Itô's formula to $|X_t^u - X_t^v|^2$, we get, for every $t \in [0,T]$:

$$\begin{aligned}
|X_t^u - X_t^v|^2 + 2\int_0^t (X_s^u - X_s^v) d(K_s^u - K_s^v) \\
=& 2\int_0^t (X_s^u - X_s^v)\left[b(s, R(X^u)_s, u_s) - b(s, R(X^v)_s, v_s)\right] ds \\
&+ 2\int_0^t \langle (X_s^u - X_s^v)\left[\sigma(s, R(X^u)_s, u_s) - \sigma(s, R(X^v)_s, v_s)\right], dW_s\rangle \\
&+ \int_0^t |\sigma(s, R(X^u)_s, u_s) - \sigma(s, R(X^v)_s, v_s)|^2 ds.
\end{aligned}$$

The monotonicity of $\partial\varphi$ implies that $2\int_0^t (X_s^u - X_s^v) d(K_s^u - K_s^v) \geq 0,\ \forall t \geq 0$. Hence, using ($A_1$), ($A_2$) and Burkholder-Davis-Gundy's inequality, we obtain

$$\begin{aligned}
\mathbb{E}||X^u - X^v||_t^p \leq & C\mathbb{E}\left(\int_0^t \left(||X^u - X^v||_s^2 + |u_s - v_s|^2\right) ds\right)^{p/2} \\
&+ C\mathbb{E}\left[||X^u - X^v||_t^2 \int_0^t \left(||X^u - X^v||_s^2 + |u_s - v_s|^2\right) ds\right]^{p/4} \\
\leq & \tfrac{1}{2}\mathbb{E}||X^u - X^v||_t^p + \left(C + \tfrac{C^2}{2}\right)\mathbb{E}\left(\int_0^t \left(||X^u - X^v||_s^2 + |u_s - v_s|^2\right) ds\right)^{p/2} \\
\leq & \tfrac{1}{2}\mathbb{E}||X^u - X^v||_t^p + \left(C + \tfrac{C^2}{2}\right)(2T)^{\frac{p}{2}-1}\,\mathbb{E}\int_0^t \left(||X^u - X^v||_s^p + |u_s - v_s|^p\right) ds,
\end{aligned}$$

where C is a constant depending only on p, L and $\tilde{L}$. We now use Gronwall's inequality in order to get the desired estimate for $X^u - X^v$. The one for $K^u - K^v$ is obtained directly from equation (2.1). $\square$

3 Necessary conditions of optimality

The purpose of this section is to give necessary conditions of optimality under the form of a maximum principle for the optimal control. We recall that the problem is to minimize the cost functional

$$J(u) := \mathbb{E}\left[\int_0^T g(t, R(X)_t, u_t) dt + h(X_T^u)\right] \tag{3.1}$$

subject to the SVI with delay (2.1):

$$\begin{cases} dX_t + \partial\varphi(X_t)dt \ni b(t, R(X)_t, u_t)dt + \langle\sigma(t, R(X)_t, u_t), dW_t\rangle,\ t \in [0,T]; \\ X_t = \eta(t),\ t \in [-\delta, 0]. \end{cases}$$

From now on we assume that $\operatorname{Dom}\varphi = \mathbb{R}$. This implies that for every $a \in \mathbb{R}$, there exist the left-hand side and the right-hand side derivatives of φ in a, denoted $\varphi'_-(a)$, respectively $\varphi'_+(a)$. It is clear that

$$\varphi'_-(a) \leq \varphi'_+(a),\ \forall a \in \mathbb{R}$$

and

$$\partial\varphi(a) = \left[\varphi'_-(a), \varphi'_+(a)\right],\ \forall a \in \mathbb{R}.$$

Moreover, by the monotonicity of $\partial\varphi$, $\varphi'_+(a) \leq \varphi'_-(a')$, if $a < a'$.

Let us define the *second-order generalized derivative* of φ as the unique σ-finite positive measure μ on $\mathcal{B}(\mathbb{R})$ such that

$$\mu([a,a']) = \varphi'_+(a') - \varphi'_-(a),\ \text{if}\ a \le a'.$$

The name of μ is justified by the following fact: if φ is second-order differentiable, then μ has φ'' as density.

On the coefficients of the state equation and cost functional we impose the following conditions:

(H_0) b, g, σ are continuous in $t \in [0,T]$.
(H_1) b, g, σ and h are C^1 in $(y,u) \in \mathbb{R}^m \times U$ with bounded derivatives.

By Theorem 2.3, for every control u we have, under conditions (A_0) and (H_1), the existence of a unique solution $(X_t^u, K_t^u)_{t\in[0,T]}$ in the space $L^2_{\mathbb{F}}(\Omega; C[0,T]) \times (L^2_{\mathbb{F}}(\Omega; C[0,T]) \cap L^1_{\mathbb{F}}(\Omega; BV[0,T]))$ for equation (2.1).

Also, as a straightforward consequence of Proposition 2.4 and the Lipschitz properties of g and h, which are derived from condition (H_1), we have the following result:

Proposition 3.1. *As a mapping from $L^2_{\mathbb{G}}(\Omega \times [0,T]; U)$ to $\mathbb{R}$, the cost functional J is continuous.*

On σ we impose also the non-degeneracy condition:

(H_2) $\sigma(t,y,u) \neq 0,\ \forall (t,y,u) \in [0,T] \times \mathbb{R}^m \times U.$

From now on, (A_0), (H_0)–(H_2) are the standing assumptions.

For an admissible control u we introduce the *local time* of the process X^u by

$$L_t^{a,u} := |X_t^u - a| - |X_0^u - a| - \int_0^t \operatorname{sgn}(X_s^u - a) dX_s^u.$$

By [23, p. 213] we always can (and will) choose a version which is measurable in $(a,t,\omega) \in \mathbb{R} \times [0,T] \times \Omega$, continuous and increasing in $t \ge 0$, càdlàg in $a \in \mathbb{R}$. We recall here some properties of the local time:

Proposition 3.2. *Let u be an admissible control. Then:*

1. *for every bounded, Borel function γ,*

$$\int_{\mathbb{R}} L_t^{a,u} \gamma(a) da = \int_0^t \gamma(X_s^u) \left|\sigma(s, R(X^u)_s, u_s)\right|^2 ds,\ \forall t \in [0,T]\,,\ \text{a.s.};$$

2. *for every $t \in [0,T]$ and $a \in \mathbb{R}$,*

$$(X_t^u - a)^+ - (X_0^u - a)^+ = \int_0^t \mathbf{1}_{\{X_s^u > a\}} dX_s^u + \frac{1}{2} L_t^{a,u},\ \text{a.s.};$$

3. *for every $t \in [0,T]$ and $a \in \mathbb{R}$,*

$$L_t^{a,u} - L_t^{a-,u} = 2 \int_0^t \mathbf{1}_{\{X_s^u = a\}} \left[b(s, R(X^u)_s, u_s) ds - dK_s^u\right],\ \text{a.s.}$$

Formulas 1 and 2 are called *occupation time density formula*, respectively *Tanaka formula*.

A consequence of (H_2) is the absolute continuity of the bounded variation process K^u:

Proposition 3.3. *For any admissible control u, the process K^u is absolutely continuous. Moreover, $\mathbb{P}$-a.s.,*

$$\frac{dK_t^u}{dt} = \varphi'_-(X_t^u) = \varphi'_+(X_t^u),\ dt\text{-a.e.} \tag{3.2}$$

and $a \mapsto L_t^{a,u}$ is continuous for every $t \in [0,T]$.

Proof. The formula of occupation time density gives us

$$\int_0^T \mathbf{1}_{\{X_t^u = x\}} |\sigma(t, R(X^u)_t, u_t)|^2 dt = \int_{\mathbb{R}} \mathbf{1}_{\{a=x\}} L_T^{a,u} da = 0 \text{ a.s.,}$$

which yields

$$\int_0^T \mathbf{1}_{\{X_t^u = x\}} dt = 0 \text{ a.s., } \forall x \in \mathbb{R}. \tag{3.3}$$

Let us set $\Lambda := \left\{ x \in \mathbb{R} \mid \varphi'_+(x) > \varphi'_-(x) \right\}$. Since Λ is at most countable, we obtain $\int_0^T \mathbf{1}_{\{X_t^u \in \Lambda\}} dt = 0$ a.s., hence

$$\int_0^t \varphi'_-(X_s^u) ds = \int_0^t \varphi'_+(X_s^u) ds \text{ a.s.} \tag{3.4}$$

On the other hand, according to [1, Proposition 1.2], a condition equivalent to (iv) of the definition of the solution is that $\mathbb{P}$-a.s., for every $0 \le s < t \le T$ and every $y \in C[0,T]$,

$$\int_s^t (y(r) - X_r^u) dK_r^u + \int_s^t \varphi(X_r^u) dr \le \int_s^t \varphi(y(r)) dr.$$

Choosing $y(r) := X_r + \varepsilon$ with arbitrary $\varepsilon \in (-1,1)$, we get

$$\varepsilon K_t^u \le \int_0^t \left[\varphi(X_r^u + \varepsilon) - \varphi(X_r^u) \right] dr,\ \forall t \in [0,T],\ \text{a.s.}$$

Since

$$\varphi'_-\Big(\min_{s\in[0,t]} X_s - 1 \Big) \le \frac{\varphi(X_r^u + \varepsilon) - \varphi(X_r^u)}{\varepsilon} \le \varphi'_+\Big(\max_{s\in[0,t]} X_s + 1 \Big),\ \forall \varepsilon \in (-1,1) \setminus \{0\},$$

we can apply Lebesgue's dominated convergence theorem for ε converging to 0, both from the left and right, in order to obtain

$$\int_0^t \varphi'_-(X_s^u) ds \le K_t^u \le \int_0^t \varphi'_+(X_s^u) ds \text{ a.s.}$$

Combining this inequality with relation (3.4) we deduce that K^u is absolutely continuous with its derivative given by formula (3.2). From Proposition 3.2-(3.), this property implies the continuity of $L_t^{\cdot,u}$. ☐

Remark 3.4. This result may seem in contradiction to the remark after the definition of the solution, in which we claimed that the process K^u is in general with bounded variation. However, in this particular case, the absolute continuity of K^u is due to the non-degeneracy of σ (condition (H_2)) and of the fact that $\operatorname{Dom}\varphi = \mathbb{R}$.

In the sequel we will need the following generalization of the occupation time density formula (still called as such):

Lemma 3.5. *Let u be an admissible control. Then, for every bounded (or positive) Borel function $\gamma : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$,*

$$\int_{\mathbb{R}} \int_0^t \gamma(s,a) L^{a,u}(ds) da = \int_0^t \gamma(s, X_s^u) \left| \sigma(s, R(X^u)_s, u_s) \right|^2 ds,\ \forall t \in [0,T],\ \text{a.s.} \tag{3.5}$$

Proof. It is clear that the equality is true for functions γ of the form $\gamma(\omega,s,a):=\mathbf{1}_{[t_0,t_1]}(s)\mathbf{1}_A(\omega)\tilde{\gamma}(a)$, where $0\le t_0\le t_1\le T$, $A\in\mathcal{F}$ and $\tilde{\gamma}:\mathbb{R}\to\mathbb{R}$ is a bounded Borel function. By linearity, every linear combination of bounded Borel functions satisfying (3.5) also satisfies this relation. Moreover, if $\gamma_n:\Omega\times[0,T]\times\mathbb{R}\to\mathbb{R}_+$ form an increasing sequence of Borel measurable functions satisfying (3.5) and converging to some function γ, then γ also satisfies (3.5), by the monotone convergence theorem. Since $\mathcal{F}\times(\{\emptyset\}\cup\{[t_0,t_1]\mid 0\le t_0\le t_1\le T\})\times\mathcal{B}(\mathbb{R})$ is a π-system generating $\mathcal{F}\otimes\mathcal{B}([0,T])\otimes\mathcal{B}(\mathbb{R})$, by the monotone class argument, it follows that every bounded Borel function satisfies (3.5). The extension to positive Borel functions (or for which the integrals make sense) is obvious. $\square$

Now, if φ were C^2, then by the above Lemma we would have

$$\int_0^t\varphi''(X_s^u)ds=\int_{\mathbb{R}}\int_0^t\frac{L^{a,u}(ds)}{|\sigma(s,R(X^u)_s,u_s)|^2}\varphi''(a)da=\int_{\mathbb{R}}\int_0^t\frac{L^{a,u}(ds)}{|\sigma(s,R(X^u)_s,u_s)|^2}\mu(da).$$

This serves as a motivation for introducing the increasing process:

$$A_t^u:=\int_{\mathbb{R}}\int_0^t\frac{L^{a,u}(ds)}{|\sigma(s,R(X^u)_s,u_s)|^2}\mu(da),\ t\in[0,T].$$

Since the function $s\mapsto\frac{1}{|\sigma(s,R(X^u)_s,u_s)|^2}$ is bounded and $L_t^{a,u}=0$ if $\|X^u\|_t\le|a|$, it follows that A^u is also finite and continuous.

3.1 Variation equation

In order to approximate φ and $\partial\varphi$ with smoother functions, we consider the Moreau-Yosida regularization of φ, given by:

$$\varphi_\varepsilon(x):=\inf\left\{\tfrac{1}{2\varepsilon}|z-x|^2+\varphi(z)\mid z\in\mathbb{R}\right\},\ x\in\mathbb{R},$$

for every $\varepsilon>0$. We list below some useful properties of φ_ε, which can be found, for instance, in [7, Chap. II]:

- φ_ε is a convex, C^1-function;
- $\varphi_\varepsilon(x)\nearrow\varphi(x)$ as $\varepsilon\to0$, $\forall x\in\mathbb{R}$;
- $|\varphi_\varepsilon'(x)-\varphi_\varepsilon'(y)|\le\frac{1}{\varepsilon}|x-y|$, $\forall x,y\in\mathbb{R}$;
- $\varphi_\varepsilon'(x)\to(\partial\varphi)^0(x)$ as $\varepsilon\to0$, $\forall x\in\mathbb{R}$, where $(\partial\varphi)^0(x)$ is the projection of 0 on $\partial\varphi(x)$;
- $\varepsilon\to|\varphi_\varepsilon'(x)|$ is a decreasing function on $(0,+\infty)$ for every $x\in\mathbb{R}$;
- $\varphi_\varepsilon'(x)\in\partial\varphi(x-\varepsilon\varphi_\varepsilon'(x))$, $\forall x\in\mathbb{R}$, $\forall\varepsilon>0$.

Since φ_ε is not necessarily of class C^2, we continue on approximating φ, by applying a mollification procedure on φ_ε. Let the function $\beta_\varepsilon:\mathbb{R}\to\mathbb{R}$ be defined by

$$\beta_\varepsilon(x):=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\varphi_\varepsilon'(x-\varepsilon^2y)\,\mathrm{e}^{-\frac{y^2}{2}}\,dy,\ x\in\mathbb{R}.$$

Lemma 3.6. *For every $\varepsilon>0$, β_ε is an increasing C^∞-function, with β_ε', β_ε'' bounded. We also have*

$$|\beta_\varepsilon(x)-\varphi_\varepsilon'(x)|<\varepsilon,\ \forall x\in\mathbb{R}. \tag{3.6}$$

Moreover, if φ is affine outside a compact interval, then $(\beta_\varepsilon)_{\varepsilon\in(0,1]}$ is uniformly bounded and there exists another compact interval I such that

$$\beta_\varepsilon'(x)\le\varepsilon,\ \forall x\in I^{\mathrm{c}},\ \forall\varepsilon\in(0,1]. \tag{3.7}$$

Proof. The first part is obvious, since φ'_ε is increasing and Lipschitz with constant $1/\varepsilon$. Given that $\varphi'_\varepsilon(x) \to (\partial\varphi)^0(x)$ as $\varepsilon \searrow 0$ and that the mapping $\varepsilon \to |\varphi'_\varepsilon(x)|$ is decreasing for every $x \in \mathbb{R}$, we have that

$$|\beta_\varepsilon(x)| \le |\varphi'_\varepsilon(x)| + \sqrt{\tfrac{2}{\pi}}\varepsilon \le |(\partial\varphi)^0(x)| + \varepsilon, \forall x \in \mathbb{R}, \varepsilon \in (0,1].$$

If φ is affine outside a compact interval, then $(\partial\varphi)^0$ is bounded, hence $(\beta_\varepsilon)_{\varepsilon\in(0,1]}$ is uniformly bounded. We also remark that φ'_ε is constant beyond a compact interval $[-a_0, a_0]$ for $\varepsilon \in (0,1]$, because $(\varphi'_\varepsilon)_{\varepsilon>0}$ is bounded, φ is affine outside a compact interval and $\varphi'_\varepsilon(x) \in \partial\varphi(x - \varepsilon\varphi'_\varepsilon(x))$ for $\varepsilon > 0$. Since φ'_ε is Lipschitz (with constant $1/\varepsilon$), φ''_ε exists a.e. and is bounded by $1/\varepsilon$, so we have

$$\beta'_\varepsilon(x) = \tfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi''_\varepsilon\left(x - \varepsilon^2 y\right) \mathrm{e}^{-\frac{y^2}{2}}\, dy.$$

Therefore, for $x > a_0 + 1$,

$$\beta'_\varepsilon(x) \le \tfrac{1}{\varepsilon\sqrt{2\pi}} \int_{1/\varepsilon^2}^{+\infty} \mathrm{e}^{-\frac{y^2}{2}}\, dy = \tfrac{1}{\varepsilon\sqrt{2\pi}} \int_0^{\varepsilon^2} \tfrac{1}{y^2}\, \mathrm{e}^{-\frac{y^2}{2}}\, dy \le \sqrt{\tfrac{2}{\pi}}\varepsilon.$$

A similar inequality holds for $x < -a_0 - 1$. □

For the moment, let us fix two controls u^0 and u^1; let us set, for $\theta \in (0,1)$,

$$u_t^\theta := u_t^0 + \theta(u_t^1 - u_t^0),\ t \in [0,T].$$

In order to simplify the notations, we write X^θ, K^θ, $L^{a,\theta}$, A^θ instead X^{u^θ}, K^{u^θ}, L^{a,u^θ}, respectively A^{u^θ}. The reason for studying the behavior of X^θ as $\theta \to 0$ is that $\theta \mapsto J(u^\theta)$ has a minimum in $\theta = 0$ if u^0 is an optimal control, hence we can derive necessary optimality conditions by calculating its derivative in 0. For that we need to study the derivative of $\theta \mapsto X^\theta$.

Let $X^{\varepsilon,\theta}$ be the solution of the penalized equation

$$dX_t^{\varepsilon,\theta} + \beta_\varepsilon(X_t^{\varepsilon,\theta})dt = b(t, R(X^{\varepsilon,\theta})_t, u_t^\theta)dt + \left\langle \sigma(t, R(X^{\varepsilon,\theta})_t, u_t^\theta), dW_t \right\rangle,\ t \in [0,T]; \tag{3.8}$$

with initial condition $X_t^{\varepsilon,\theta} = \eta(t)$ on $[-\delta, 0]$. We set $K_t^{\varepsilon,\theta} := \int_0^t \beta_\varepsilon(X_s^{\varepsilon,\theta})ds$, $t \in [0,T]$; $K_t^{\varepsilon,\theta} := 0,\ t \in [-\delta, 0)$.

From the proof of [1, Theorem 2.1.] and relation (3.6), $X^{\varepsilon,\theta}$ and $K^{\varepsilon,\theta}$ converge as $\varepsilon \searrow 0$ in $L^2_{\mathbb{F}}(\Omega; C[0,T])$ to X^θ, respectively K^θ, uniformly with respect to θ.

We also consider, for $\varepsilon > 0$ and $\theta \in [0,1]$, the solution $Y^{\varepsilon,\theta}$ of the delay equation[3,4]

$$\begin{aligned} dY_t^{\varepsilon,\theta} + \beta'_\varepsilon(X_t^{\varepsilon,\theta})Y_t^{\varepsilon,\theta}dt &= \left[(\partial_y b_t^{\varepsilon,\theta})R(Y^{\varepsilon,\theta})_t + (\partial_u b_t^{\varepsilon,\theta})(u_t^1 - u_t^0)\right]dt \\ &\quad + \left\langle (\partial_y \sigma_t^{\varepsilon,\theta})R(Y^{\varepsilon,\theta})_t + (\partial_u \sigma_t^{\varepsilon,\theta})(u_t^1 - u_t^0), dW_t \right\rangle,\ t \in [0,T], \end{aligned} \tag{3.9}$$

with initial condition $Y_t^{\varepsilon,\theta} = 0,\ t \in [-\delta, 0]$. We observe that the boundedness of β'_ε as well as condition (H_1) imply existence and uniqueness of the solution in $L^2_{\mathbb{F}}(\Omega; C[0,T])$.

By formally differentiating with respect to θ in (3.8), we obtain an equation of the form (3.9), suggesting that $\frac{d}{d\theta}X_t^{\varepsilon,\theta} = Y_t^{\varepsilon,\theta}$. This can be done rigorously by using a standard argument, developed in [3] for example, which gives that the differentiation takes place in $L^2_{\mathbb{F}}(\Omega; C[0,T])$:

[3] We denote $\partial_z b_t^{\varepsilon,\theta}$, $\partial_z \sigma_t^{\varepsilon,\theta}$ for $\frac{\partial b}{\partial z}(t, R(X^{\varepsilon,\theta})_t, u_t^\theta)$, respectively $\frac{\partial \sigma}{\partial z}(t, R(X^{\varepsilon,\theta})_t, u_t^\theta)$, where z stands for y or u.

[4] We regard $\frac{\partial b}{\partial y}$, $\frac{\partial b}{\partial u}$, $\frac{\partial g}{\partial y}$, $\frac{\partial g}{\partial u}$ as row vectors and $\frac{\partial \sigma}{\partial y}$, $\frac{\partial \sigma}{\partial u}$ as matrices of size $d \times m$, respectively $d \times l$.

$$\lim_{\theta\searrow\theta_0}\mathbb{E}\sup_{t\in[0,T]}\left[\left|\frac{X_t^{\varepsilon,\theta}-X_t^{\varepsilon,\theta_0}}{\theta-\theta_0}-Y_t^{\varepsilon,\theta_0}\right|^2+\left|\frac{K_t^{\varepsilon,\theta}-K_t^{\varepsilon,\theta_0}}{\theta-\theta_0}-\int_0^t\beta'_\varepsilon(X_s^{\varepsilon,\theta_0})Y_s^{\varepsilon,\theta_0}ds\right|^2\right]=0; \tag{3.10}$$

for every $\theta_0\in[0,T)$.

Our first task is to find an analogous derivative formula for X^θ and K^θ. For that, suppose for one moment that φ is C^2. Then, as before, we would have $\frac{d}{d\theta}X_t^\theta=Y_t^\theta$, where Y^θ is the solution of the equation[5]

$$\begin{aligned}dY_t^\theta+Y_t^\theta\varphi''(X_t^\theta)dt=&\left[(\partial_y b_t^\theta)R(Y^\theta)_t+(\partial_u b_t^\theta)(u_t^1-u_t^0)\right]dt\\&+\left\langle(\partial_y\sigma_t^\theta)R(Y^\theta)_t+(\partial_u\sigma_t^\theta)(u_t^1-u_t^0),dW_t\right\rangle,\ t\in[0,T].\end{aligned}$$

with $Y_t^\theta=0,\ t\in[-\delta,0]$. But as already explained at page 9, in this case $\int_0^t\varphi''(X_s^\theta)ds$ equals A_t^θ, so we can write the above equation as

$$\begin{aligned}dY_t^\theta+Y_t^\theta dA_t^\theta=&\left[(\partial_y b_t^\theta)R(Y^\theta)_t+(\partial_u b_t^\theta)(u_t^1-u_t^0)\right]dt\\&+\left\langle(\partial_y\sigma_t^\theta)R(Y^\theta)_t+(\partial_u\sigma_t^\theta)(u_t^1-u_t^0),dW_t\right\rangle,\ t\in[0,T],\end{aligned} \tag{3.11}$$

with initial condition $Y_t^\theta=0,\ t\in[-\delta,0]$. This makes sense even when φ is no longer C^2, which leads us to believe that equation (3.11) will deliver the derivative of X^θ also for our general standing assumptions. In fact, we will show that the derivation formula $\frac{d}{d\theta}X_t^\theta=Y_t^\theta$ is still valid, however in an weaker sense. First we have to prove the existence of a solution to equation (3.11).

Proposition 3.7. *Equation (3.11) has a unique solution* $Y^\theta\in L^2_{\mathbb{F}}(\Omega;C[-\delta,T])$.

Proof. Uniqueness follows by the monotonicity of $t\mapsto A_t^\theta$. Indeed, if we have two solutions $Y,\ Y'\in L^2_{\mathbb{F}}(\Omega;C[0,T])$ of equation (3.11), then

$$\begin{aligned}&|Y_t-Y_t'|^2+2\int_0^t|Y_s-Y_s'|^2dA_s^\theta\\&=2\int_0^t(\partial_y b_s^\theta)R(Y-Y')_s(Y_s-Y_s')+(\partial_u b_s^\theta)(u_s^1-u_s^0)(Y_s-Y_s')ds\\&\quad+\int_0^t\left[(\partial_y\sigma_s^\theta)R(Y-Y')_s+(\partial_u\sigma_s^\theta)(u_s^1-u_s^0)\right]^2ds\\&\quad+2\int_0^t\left\langle(\partial_y\sigma_s^\theta)R(Y-Y')_s(Y_s-Y_s')+(\partial_u\sigma_s^\theta)(u_s^1-u_s^0)(Y_s-Y_s'),dW_s\right\rangle.\end{aligned}$$

Standard estimates and Gronwall's inequality allow us to conclude that $Y=Y'$. In order to prove existence, we let $\tau_n^\theta:=\inf\left\{t\in[0,T]\mid A_t^\theta>n\right\}\wedge T$ and $A_t^{n,\theta}:=A_{t\wedge\tau_n}^\theta$. Then the equation

$$\begin{aligned}d\bar{Y}_t^{n,\theta}=&\left[(\partial_y b_t^\theta)\bar{R}^n(\bar{Y}^{n,\theta})_t+\mathrm{e}^{A_t^{n,\theta}}(\partial_u b_t^\theta)(u_t^1-u_t^0)\right]dt\\&+\left\langle(\partial_y\sigma_t^\theta)\bar{R}^n(\bar{Y}^{n,\theta})_t+\mathrm{e}^{A_t^{n,\theta}}(\partial_u\sigma_t^\theta)(u_t^1-u_t^0),dW_t\right\rangle,\ t\in[0,T],\end{aligned} \tag{3.12}$$

where

$$\bar{R}^n(x)_t:=\int_{-\delta}^0\mathrm{e}^{A_t^{n,\theta}-A_{t+r}^{n,\theta}}x(t+r)\lambda(dr),\ t\in[0,T]$$

[5] Here again, $\partial_z b_t^\theta$, $\partial_z\sigma_t^\theta$ stand for $\frac{\partial b}{\partial z}(t,R(X^\theta)_t,u_t^\theta)$, respectively $\frac{\partial\sigma}{\partial z}(t,R(X^\theta)_t,u_t^\theta)$.

has a unique solution $\bar{Y}^{n,\theta} \in L^2_{\mathbb{F}}(\Omega; C[-\delta, T])$ with initial condition $\bar{Y}^{n,\theta}_t = 0,\ t \in [-\delta, 0]$. The transformation $Y^{n,\theta}_t := \mathrm{e}^{-A^{n,\theta}_t}\,\bar{Y}^{n,\theta}_t$ gives us a solution of the equation

$$\begin{aligned} dY^{n,\theta}_t + Y^{n,\theta}_t dA^{n,\theta}_t = &\left[(\partial_y b^\theta_t)R(Y^{n,\theta})_t + (\partial_u b^\theta_t)(u^1_t - u^0_t)\right] dt \\ &+ \left\langle (\partial_y \sigma^\theta_t)R(Y^{n,\theta})_t + (\partial_u \sigma^\theta_t)(u^1_t - u^0_t), dW_t \right\rangle,\ t \in [0, T], \end{aligned} \tag{3.13}$$

with initial condition $Y^{n,\theta}_t = 0,\ t \in [-\delta, 0]$. By uniqueness, it is clear that $Y^{n,\theta}_t = Y^{m,\theta}_t$ for $t \in \left[0, \tau^\theta_n\right]$ if $n \le m$. Since $\mathbb{P}(\tau^\theta_n = T) \nearrow 1$ as $n \to \infty$, we can define $Y^\theta_t := Y^{n,\theta}_t$ for $t \in \left[0, \tau^\theta_n\right]$. Obviously, Y^θ is a solution of equation (3.11). It remains only to show that $Y^\theta \in L^2_{\mathbb{F}}(\Omega; C[-\delta, T])$, which is done by applying Itô's formula to $\left|Y^\theta_t\right|^2$. □

Since the convergence in formula (3.10) is not necessarily uniform in $\varepsilon > 0$ (the speed of the convergence depends on the Lipschitz constants of the coefficients), we cannot derive a similar relation for X^θ, K^θ and Y^θ directly from that. In this regard, we will adapt an idea from [17] concerning the Malliavin derivatives for processes without control and we will define the derivative of $\theta \to X^\theta$ in a Sobolev space.

Proposition 3.8. *If u^0 and u^1 are càdlàg, the following derivation formula holds:*

$$\lim_{\theta \to 0} \mathbb{E}\left[\int_0^T \left|\frac{X^\theta_t - X^0_t}{\theta} - Y^0_t\right|^2 dt + \left|\frac{X^\theta_T - X^0_T}{\theta} - Y^0_T\right|^2\right] = 0. \tag{3.14}$$

The proof of this result is postponed to the Appendix, given its length and technicality.

3.2 Maximum principle for near optimal controls

As in the case of SDEs, the adjoint equation associated with our optimal control problem is a linear BSDE. We define the Hamiltonian of the system $H : [0, T] \times \mathbb{R}^m \times U \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ by

$$H(t, y, u, \zeta, \vartheta) = g(t, y, u) + \zeta b(t, y, u) + \vartheta\sigma(t, y, u)$$

For every control u, we consider the following anticipated BSDE[6]:

$$\begin{cases} -dP_t + P_t dA^u_t = \mathbb{E}^{\mathcal{F}_t}[f(t, R(X^u), u, P, Q)]dt - Q_t dW_t,\ t \in [0, T]; \\ P_T = h'(X^u_T), \end{cases} \tag{3.15}$$

where[7]

$$f(t, y, u, \zeta, \vartheta) := \int_t^{(t+\delta)\wedge T} \frac{\partial H}{\partial y}(s, y(s), u(s), \zeta(s), \vartheta(s))\lambda(t - ds)$$

for $(t, y, u, \zeta, \vartheta) \in [0, T] \times C([0, T]; \mathbb{R}^m) \times L^2([0, T]; U) \times L^2\,[0, T] \times L^2([0, T]; \mathbb{R}^d)$.

The main problem in proving the well-posedness of the above equation is given by the fact that the increasing process A^u could be unbounded. However, it stands on the "right side" in the equation, thus simplifying the estimates. Our strategy is to consider first the case where A^u is bounded and then to search for a solution as a limit of solutions of approximating equations driven by bounded increasing processes.

It turns out that, using the mapping $(P, Q) \mapsto (e^{-A^u}P, e^{-A^u}Q)$, our equation can be transformed into one of the following form:

$$\begin{cases} -dP_t = \mathbb{E}^{\mathcal{F}_t}[F(t, (P_{t+s})_{s\in[0,\delta]}, (Q_{t+s})_{s\in[0,\delta]})]dt - Q_t dW_t,\ t \in [0, T]\,; \\ P_t = G(t),\ t \in [T, T + \delta]\,; \\ Q_t = \tilde{G}(t),\ t \in [T, T + \delta]\,,\ \text{a.e.}, \end{cases} \tag{3.16}$$

[6] $\mathbb{E}^{\mathcal{G}}\xi$ denotes the conditional expectation of a random variable ξ with respect to a subalgebra $\mathcal{G}$ of $\mathcal{F}$.

[7] We regard Q, $\frac{\partial H}{\partial y}$, $\frac{\partial H}{\partial u}$ as row vectors.

where

$$F(t,\zeta,\vartheta) := \int_t^{(t+\delta)\wedge T} \mathrm{e}^{A_s^u - A_t^u} \frac{\partial \tilde{H}}{\partial y}(s, A_s^u, R(X^u)_s, u_s, \zeta(s-t), \vartheta(s-t))\lambda(t-ds) \quad (3.17)$$

with

$$\tilde{H}(t,a,y,u,\zeta,\vartheta) = \mathrm{e}^{-a}\, g(t,y,u) + \zeta b(t,y,u) + \vartheta\sigma(t,y,u)$$

and

$$G(t) := e^{-A_T^u} h'(X_T^u),\ \tilde{G}(t) := 0. \quad (3.18)$$

We are now going to show the existence and uniqueness of the solution of equation (3.16) for general driver $F : \Omega \times [0,T] \times C[0,\delta] \times L^2_\nu([0,\delta];\mathbb{R}^d) \to \mathbb{R}$ and final states $(G,\tilde{G}) : \Omega \times [T,T+\delta] \to \mathbb{R} \times \mathbb{R}^d$ satisfying, for a finite measure ν on $[0,\delta]$ and $p \geq 2$, the following conditions:

(B$_1$) F is measurable, a.s. continuous and there exists a constant $L > 0$ such that for every $(\omega,t) \in \Omega \times [0,T]$ and every $(\zeta,\vartheta),(\tilde{\zeta},\tilde{\vartheta}) \in C[0,\delta] \times L^2_\nu([0,\delta];\mathbb{R}^d)$:

$$(i)\ |F(\omega,t,\zeta,\vartheta) - F(\omega,t,\tilde{\zeta},\tilde{\vartheta})| \leq L\left(\|\zeta - \tilde{\zeta}\|_\delta + \|\vartheta - \tilde{\vartheta}\|_{L^2_\nu([0,\delta];\mathbb{R}^d)}\right);$$
$$(ii)\ |F(\omega,t,0,0)| \leq L.$$

(B$_2$) G, $\tilde{G}$ are $\mathcal{B}([T,T+\delta]) \otimes \mathcal{F}_T$-measurable, G is a.s. continuous and

$$\mathbb{E}\left[\sup_{t\in[T,T+\delta]} |G(t)|^p + \int_T^{T+\delta} |\tilde{G}(t)|^2 dt\right] < +\infty.$$

Remark 3.9. If A^u is bounded, then the driver F introduced by (3.17) satisfies condition (B$_1$) with $\nu(ds) := |\lambda(-ds)|$. Indeed, this is easily seen if we write F as

$$F(t,\zeta,\vartheta) = \int_0^\delta \mathrm{e}^{A_{s+t}^u - A_t^u} \frac{\partial \tilde{H}}{\partial y}(s+t, A_{s+t}^u, R(X^u)_{s+t}, u_{s+t}, \zeta(s), \vartheta(s))\lambda(-ds),$$

where $\frac{\partial \tilde{H}}{\partial y}(\cdot,a,y,u,\zeta,\vartheta)$, A^u, $R(X^u)$ and u are prolonged to $(T,T+\delta]$ by 0. Also, in this case, if G and $\tilde{G}$ are defined by (3.18), then they satisfy (B$_2$) with any $p \geq 2$.

Equation (3.16) was already studied in [19], but we cannot use the existence result stated there because ν is not so general: it is the sum of the Lebesgue measure and Dirac measures concentrated on 0 and δ, respectively. Nevertheless, the proof of Theorem 5.3 in [19] can be easily adapted to our case in order to show the well-posedness of this equation in $\mathcal{S}_p := L^p_{\mathbb{F}}(\Omega; C[0,T+\delta]) \times L^2_{\mathbb{F}}(\Omega \times [0,T+\delta]\,;\mathbb{R}^d)$.

Remark 3.10. For $(P,Q) \in \mathcal{S}_2$ and $t \in [0,T]$, it may be possible that the random variable $F(t,(P_{t+s})_{s\in[0,\delta]},(Q_{t+s})_{s\in[0,\delta]})$ is not defined (since for fixed $\omega \in \Omega$, $(Q_{t+s})_{s\in[0,\delta]}(\omega)$ does not necessarily belong to $L^2_\nu([0,\delta];\mathbb{R}^d)$, but to $L^2([0,\delta];\mathbb{R}^d)$), and even if it is, $\mathbb{E}|F(t,(P_{t+s})_{s\in[0,\delta]},(Q_{t+s})_{s\in[0,\delta]})|$ could be infinite. However, due to (B$_1$) and Fubini's theorem,

$$\begin{aligned}\int_0^T \mathbb{E}\big|F(t,(P_{t+s})_{s\in[0,\delta]},(Q_{t+s})_{s\in[0,\delta]})\big|^2 dt &\leq 2L\int_0^T \left(\mathbb{E}\,\|P\|^2_{t,t+\delta} + \mathbb{E}\int_0^\delta |Q_{t+s}|^2\,\nu(ds)\right)dt \\ &\leq 2LT\mathbb{E}\,\|P\|^2_{T+\delta} + 2L\nu([0,\delta])\mathbb{E}\int_0^{T+\delta} |Q_t|^2\,dt \\ &< +\infty;\end{aligned}$$

therefore $\mathbb{E}^{\mathcal{F}_t}[F(t,(P_{t+s})_{s\in[0,\delta]},(Q_{t+s})_{s\in[0,\delta]})]$ is defined dt-a.e. (the second inequality from above also implies that $(Q_{t+s})_{s\in[0,\delta]} \in L^2_\nu([0,\delta];\mathbb{R}^d)$, $dtd\mathbb{P}$-a.e.). Moreover, it can be seen as an operator mapping $\mathcal{S}_2$ into $L^2_{\mathbb{F}}(\Omega\times[0,T])$, meaning that there is no risk in identifying processes P and Q with their equivalence classes.

We emphasize also that a final condition for Q is necessary for the uniqueness (indeed, supposing that F does not depend on Q, one can modify Q on $[T,T+\delta]$ and still get a solution).

Proposition 3.11. *If $p>2$, under conditions (B_1) and (B_2), equation (3.16) has a unique solution $(P,Q)\in\mathcal{S}_p$.*

Proof. We will prove this result by applying twice the contraction principle, first in the argument P and second in the argument Q.

Step I. Let us first suppose that F does not depend on P. Let $\mathcal{H}$ be the space of processes $Z\in L^2_{\mathbb{F}}(\Omega\times[0,T+\delta]\,;\mathbb{R}^d)$ such that $Z_t=\tilde G(t),\ dt$-a.e. on $[T,T+\delta]$. If Z is an arbitrary element of $\mathcal{H}$, it follows from the martingale representation theorem (or the classical theory of BSDEs, see [20]) that the following equation:

$$\begin{cases} -dP_t=\mathbb{E}^{\mathcal{F}_t}[F(t,(Z_{t+s})_{s\in[0,\delta]})]dt-Q_tdW_t,\ t\in[0,T]\,;\\ P_t=G(t),\ t\in[T,T+\delta]\,;\\ Q_t=\tilde G(t),\ t\in[T,T+\delta]\,,\ \text{a.e.}\end{cases}\tag{3.19}$$

has a unique solution $(P,Q)\in\mathcal{S}_p$. Let us prove that the mapping $\Phi:\mathcal{H}\to\mathcal{H}$, defined by $\Phi(Z):=Q$, is a contraction under an appropriate norm.

Let $Z,\tilde Z\in\mathcal{H}$ and (P,Q), $(\tilde P,\tilde Q)$ be the solutions of equation (3.19) corresponding to Z, respectively $\tilde Z$. Then, applying Itô's formula to $e^{\gamma t}\,|P_t-\tilde P_t|^2$ for an arbitrary $\gamma>0$, we get

$$\begin{aligned}&e^{\gamma t}\,|P_t-\tilde P_t|^2+\gamma\int_t^T e^{\gamma s}\,|P_s-\tilde P_s|^2ds+\int_t^T e^{\gamma s}\,|Q_s-\tilde Q_s|^2ds\\ &\qquad=2\int_t^T e^{\gamma s}(P_s-\tilde P_s)\mathbb{E}^{\mathcal{F}_s}[F(s,(Z_{s+r})_{r\in[0,\delta]})-F(s,(\tilde Z_{s+r})_{r\in[0,\delta]})]ds\\ &\qquad\qquad-2\int_t^T e^{\gamma s}(P_s-\tilde P_s)(Q_s-\tilde Q_s)dW_s.\end{aligned}$$

Since $e^{\gamma\cdot}(P-\tilde P)(Q-\tilde Q)\in L^1_{\mathbb{F}}\left(\Omega;L^2([0,T];\mathbb{R}^d)\right)$, it follows that $\int_0^{\cdot} e^{\gamma s}(P_s-\tilde P_s)(Q_s-\tilde Q_s)dW_s$ is a martingale. Taking the expectance and using (B_1), we obtain

$$\begin{aligned}&e^{\gamma t}\,\mathbb{E}|P_t-\tilde P_t|^2+\gamma\int_t^T e^{\gamma s}\,\mathbb{E}|P_s-\tilde P_s|^2ds+\mathbb{E}\int_t^T e^{\gamma s}\,|Q_s-\tilde Q_s|^2ds\\ &\leq 2L\mathbb{E}\int_t^T e^{\gamma s}\,|P_s-\tilde P_s|\bigg(\int_0^\delta|Z_{s+r}-\tilde Z_{s+r}|^2\nu(dr)\bigg)^{1/2}ds\\ &\leq\frac{L}{\alpha}\int_t^T e^{\gamma s}\,\mathbb{E}|P_s-\tilde P_s|^2ds+L\alpha\mathbb{E}\int_t^T e^{\gamma s}\int_0^\delta|Z_{s+r}-\tilde Z_{s+r}|^2\nu(dr)ds,\end{aligned}\tag{3.20}$$

where $\alpha>0$ is a constant chosen such that $L\alpha\nu([0,\delta])\leq\frac12$. Now, since Z and $\tilde Z$ agree dt-a.e. on $[T,T+\delta]$,

$$\begin{aligned}\mathbb{E}\int_t^T e^{\gamma s}\int_0^\delta|Z_{s+r}-\tilde Z_{s+r}|^2\nu(dr)ds&=\mathbb{E}\int_0^\delta\int_t^T e^{\gamma s}\,|Z_{s+r}-\tilde Z_{s+r}|^2ds\nu(dr)\\ &\leq\mathbb{E}\int_0^\delta\int_t^T e^{\gamma s}\,|Z_s-\tilde Z_s|^2ds\nu(dr)\\ &=\nu([0,\delta])\mathbb{E}\int_t^T e^{\gamma s}\,|Z_s-\tilde Z_s|^2ds.\end{aligned}$$

From (3.20) we deduce that, for γ large enough ($\gamma \geq \frac{L}{\alpha}$), we have

$$\mathbb{E}\int_0^T \mathrm{e}^{\gamma s}\,|Q_s - \tilde{Q}_s|^2 ds \leq \frac{1}{2}\mathbb{E}\int_0^T \mathrm{e}^{\gamma s}\,|Z_s - \tilde{Z}_s|^2 ds;$$

therefore, Φ is a contraction on the Banach space $\mathcal{H}$ endowed with the norm

$$Z \mapsto \left[\mathbb{E}\int_0^{T+\delta} \mathrm{e}^{\gamma s}\,|Z_s|^2\, ds\right]^{1/2}.$$

By Banach fixed-point theorem, the equation $\Phi(Q) = Q$ has a unique solution in $\mathcal{H}$.

Step II. We pass now to the general case and consider the Banach space $\tilde{\mathcal{H}}$ consisting in those processes $Z \in L^p_{\mathbb{F}}(\Omega; C[0, T+\delta])$ such that $Z_t = G(t)$, $\forall t \in [T, T+\delta]$. For arbitrary $Z \in \tilde{\mathcal{H}}$, we consider the equation

$$\begin{cases} -dP_t = \mathbb{E}^{\mathcal{F}_t}[F(t, (Z_{t+s})_{s\in[0,\delta]}\,, (Q_{t+s})_{s\in[0,\delta]})]dt - Q_t dW_t,\ t \in [0,T]\,; \\ P_t = G(t),\ t \in [T, T+\delta]\,; \\ Q_t = \tilde{G}(t),\ t \in [T, T+\delta]\,,\ \text{a.e.} \end{cases}$$

According to the previous step, it has a unique solution $(P, Q) \in \mathcal{S}_p$ and we denote $\tilde{\Phi}(Z) := P$.

Again, similarly to step I, we take two processes $Z, \tilde{Z} \in \tilde{\mathcal{H}}$ and the corresponding solutions, (P, Q), respectively $(\tilde{P}, \tilde{Q})$. Then, applying Itô's formula to $\mathrm{e}^{\gamma t}\,|P_t - \tilde{P}_t|^2$, we get

$$\begin{aligned} &\mathrm{e}^{\gamma t}\,|P_t - \tilde{P}_t|^2 + \gamma\int_t^T \mathrm{e}^{\gamma s}\,|P_s - \tilde{P}_s|^2 ds + \int_t^T \mathrm{e}^{\gamma s}\,|Q_s - \tilde{Q}_s|^2 ds \\ &= 2\int_t^T \mathrm{e}^{\gamma s}(P_s - \tilde{P}_s)\mathbb{E}^{\mathcal{F}_s}[F(s, (Z_{s+r})_{r\in[0,\delta]}, (Q_{s+r})_{r\in[0,\delta]}) - F(s, (\tilde{Z}_{s+r})_{r\in[0,\delta]}, (\tilde{Q}_{s+r})_{r\in[0,\delta]})]ds \\ &\qquad - 2\int_t^T \mathrm{e}^{\gamma s}(P_s - \tilde{P}_s)(Q_s - \tilde{Q}_s)dW_s. \end{aligned}$$

Taking the conditional expectance with respect to $\mathcal{F}_t$ and using (B_1), we obtain

$$\begin{aligned} &\mathrm{e}^{\gamma t}|P_t - \tilde{P}_t|^2 + \gamma\mathbb{E}^{\mathcal{F}_t}\int_t^T \mathrm{e}^{\gamma s}\,|P_s - \tilde{P}_s|^2 ds + \mathbb{E}^{\mathcal{F}_t}\int_t^T \mathrm{e}^{\gamma s}\,|Q_s - \tilde{Q}_s|^2 ds \\ &\leq 2L\mathbb{E}^{\mathcal{F}_t}\int_t^T \mathrm{e}^{\gamma s}\,|P_s - \tilde{P}_s|\mathbb{E}^{\mathcal{F}_s}\left[\|Z - \tilde{Z}\|_{s,s+\delta} + \left(\int_0^\delta |Q_{s+r} - \tilde{Q}_{s+r}|^2\nu(dr)\right)^{1/2}\right]ds \\ &\leq L\mathbb{E}^{\mathcal{F}_t}\int_t^T \left[\frac{1}{\alpha}\,\mathrm{e}^{\gamma s}\,|P_s - \tilde{P}_s|^2 + 2\alpha\,\mathrm{e}^{\gamma s}\,\mathbb{E}^{\mathcal{F}_s}\|Z - \tilde{Z}\|^2_{s,s+\delta}\right]ds \\ &\quad + 2L\alpha\mathbb{E}^{\mathcal{F}_t}\int_t^T\int_0^\delta \mathrm{e}^{\gamma s}\,|Q_{s+r} - \tilde{Q}_{s+r}|^2\nu(dr)ds \\ &\leq \frac{L}{\alpha}\mathbb{E}^{\mathcal{F}_t}\int_t^T \mathrm{e}^{\gamma s}\,|P_s - \tilde{P}_s|^2 ds + 2LT\alpha\mathbb{E}^{\mathcal{F}_t}\big\|\,\mathrm{e}^{\frac{\gamma}{2}\cdot}(Z - \tilde{Z})\big\|^2_T \\ &\quad + 2L\alpha\mathbb{E}^{\mathcal{F}_t}\int_t^T \mathrm{e}^{\gamma s}\int_0^\delta |Q_{s+r} - \tilde{Q}_{s+r}|^2\nu(dr)ds \\ &\leq \frac{L}{\alpha}\mathbb{E}^{\mathcal{F}_t}\int_t^T \mathrm{e}^{\gamma s}\,|P_s - \tilde{P}_s|^2 ds + 2LT\alpha\mathbb{E}^{\mathcal{F}_t}\big\|\,\mathrm{e}^{\frac{\gamma}{2}\cdot}(Z - \tilde{Z})\big\|^2_T + \frac{1}{2}\mathbb{E}^{\mathcal{F}_t}\int_t^T \mathrm{e}^{\gamma s}\,|Q_s - \tilde{Q}_s|^2 ds \end{aligned}$$

by a similar argument as before, where $\alpha > 0$ is chosen such that $L\alpha\nu([0,\delta]) \le \frac{1}{4}$ and $\frac{2LT\alpha p}{p-2} < 1$. This implies that, for any $t \in [0,T]$,

$$\mathrm{e}^{\gamma t}\,|P_t - \tilde{P}_t|^2 + \left(\gamma - \tfrac{L}{\alpha}\right)\mathbb{E}^{\mathcal{F}_t}\int_t^T \mathrm{e}^{\gamma s}\,|P_s - \tilde{P}_s|^2 ds + \frac{1}{2}\mathbb{E}^{\mathcal{F}_t}\int_t^T \mathrm{e}^{\gamma s}\,|Q_s - \tilde{Q}_s|^2 ds \\ \le 2LT\alpha\mathbb{E}^{\mathcal{F}_t}\big\|\,\mathrm{e}^{\frac{\gamma}{2}\cdot}(Z - \tilde{Z})\big\|_T^2.$$

By taking $\gamma \ge \frac{L}{\alpha}$, we obtain

$$\big\|\,\mathrm{e}^{\frac{\gamma}{2}\cdot}(P - \tilde{P})\big\|_T^2 \le 2LT\alpha \sup_{t\in[0,T]} \mathbb{E}^{\mathcal{F}_t}\big\|\,\mathrm{e}^{\frac{\gamma}{2}\cdot}(Z - \tilde{Z})\big\|_T^2.$$

Since

$$\mathbb{E}^{\mathcal{F}_t}\big\|\,\mathrm{e}^{\gamma\cdot}(Z - \tilde{Z})^2\big\|_T,\ t \in [0,T]$$

is a martingale, it follows, by Doob's maximal inequality, that

$$\begin{aligned}\mathbb{E}\big\|\,\mathrm{e}^{\frac{\gamma}{2}\cdot}(P - \tilde{P})\big\|_T^p &\le (2LT\alpha)^{p/2}\,\mathbb{E}\sup_{t\in[0,T]}\left(\mathbb{E}^{\mathcal{F}_t}\big\|\,\mathrm{e}^{\frac{\gamma}{2}\cdot}(Z - \tilde{Z})\big\|_T^2\right)^{p/2} \\ &\le \left(\tfrac{2LT\alpha p}{p-2}\right)^{p/2}\mathbb{E}\big\|\,\mathrm{e}^{\frac{\gamma}{2}\cdot}(Z - \tilde{Z})\big\|_T^p.\end{aligned}$$

Since $\left(\frac{2LT\alpha p}{p-2}\right)^{p/2} < 1$, it follows that $\tilde{\Phi}$ is a contraction on $\tilde{\mathcal{H}}$, which is a Banach space when endowed with the norm

$$Z \longmapsto \left[\mathbb{E}\big\|\,\mathrm{e}^{\frac{\gamma}{2}\cdot}(Z - \tilde{Z})\big\|_{T+\delta}^p\right]^{1/p}.$$

By the contraction principle, $\tilde{\Phi}$ has a unique fixed point in $\tilde{\mathcal{H}}$, which amounts to say that equation (3.16) has a unique solution in $\mathcal{S}_p$. □

Let us now return to equation (3.15); we look for a solution in $\mathcal{S} := L^2_{\mathbb{F}}(\Omega; C[0,T]) \times L^2_{\mathbb{F}}(\Omega \times [0,T]; \mathbb{R}^d)$.

Theorem 3.12. *Equation (3.15) has a unique solution $(P,Q) \in \mathcal{S}$.*

Proof. Uniqueness is straightforward, by applying Itô's formula to $|P - P'|^2$, where (P,Q) and (P',Q') are two solutions in $\mathcal{S}$. In order to prove existence, suppose first that A^u is bounded and let F, G and $\tilde{G}$ be defined by (3.17) and (3.18), respectively. Then the equation

$$\begin{cases} -d\tilde{P}_t = \mathbb{E}^{\mathcal{F}_t}[F(t,\tilde{P},\tilde{Q})]dt - \tilde{Q}_t dW_t,\ t \in [0,T]; \\ \tilde{P}_T = e^{-A^u_T} h'(X^u_T) \end{cases} \tag{3.21}$$

has a unique solution $(\tilde{P},\tilde{Q})$ with $(\tilde{P},\tilde{Q}) \in \mathcal{S}_p$ for every $p \ge 2$, according to Proposition 3.11. Applying Itô's formula to the process $e^{A^u_t}\tilde{P}_t$, it is clear that $(P,Q) := (e^{A^u}\tilde{P}, e^{A^u}\tilde{Q})$ satisfies equation (3.15) and $(P,Q) \in \mathcal{S}_p$ for every $p \ge 2$.

Suppose now that A^u is bounded no more. Let, for $n \in \mathbb{N}^*$, $A^n_\cdot := A^u_\cdot \wedge n$. Since A^n is bounded, there exists a solution $(P^n, Q^n) \in \mathcal{S}$ of the equation

$$\begin{cases} -dP^n_t + P^n_t dA^n_t = \mathbb{E}^{\mathcal{F}_t}[f(t,R(X^u),u,P^n,Q^n)]dt - Q^n_t dW_t; \\ P^n_T = h'(X^u_T). \end{cases} \tag{3.22}$$

with $\mathbb{E}\|P^n\|_T^p < +\infty$, $\forall p \geq 2$. Let us first prove that the sequence (P^n, Q^n) is bounded. By applying Itô's formula to $|P_t^n|^2$ we obtain

$$\begin{aligned}|P_t^n|^2 &+ 2\int_t^T |P_s^n|^2 \, dA_s^n + \int_t^T |Q_s^n|^2 \, ds \\ &= |h'(X_T^u)|^2 + 2\int_t^T P_s^n \mathbb{E}^{\mathcal{F}_s}[f(s, R(X^u), u, P^n, Q^n)]ds - 2\int_t^T P_s^n Q_s^n dW_s.\end{aligned}$$

Since the term $\int_t^T |P_s^n|^2 \, dA_s^n$ is positive, $\frac{\partial g}{\partial y}$, $\frac{\partial b}{\partial y}$, $\frac{\partial \sigma}{\partial y}$ are bounded and the stochastic integral is a martingale, taking the conditional expectation with respect to $\mathcal{F}_t$ yields

$$\begin{aligned}&|P_t^n|^2 + \mathbb{E}^{\mathcal{F}_t}\int_t^T |Q_s^n|^2 \, ds \\ &\leq c_0 + c_1 \mathbb{E}^{\mathcal{F}_t}\int_t^T |P_s^n| \left(1 + \mathbb{E}^{\mathcal{F}_s}\int_s^{(s+\delta)\wedge T} (|P_r^n| + |Q_r^n|) \, |\lambda|(s - dr)\right)ds \\ &\leq c_0 + \frac{c_1 T}{2} + \left(\frac{1}{2} + |\lambda|([-\delta, 0]) + \frac{c_1}{4\alpha}\right) c_1 \mathbb{E}^{\mathcal{F}_t}\left[\int_t^T \|P^n\|_{s,T}^2 ds\right] \\ &\quad + \alpha \mathbb{E}^{\mathcal{F}_t}\int_t^T \left(\int_{-\delta\vee(s-T)}^0 \left|Q_{s-r}^n\right| |\lambda|(dr)\right)^2 ds,\end{aligned}$$

where c_0, c_1 are positive constants not depending on n and $\alpha > 0$ is chosen such that $\alpha\big[|\lambda|([-\delta, 0])\big]^2 \leq \frac{1}{2}$. For the last term of the right-hand side of this inequality we have the following estimate:

$$\begin{aligned}\mathbb{E}^{\mathcal{F}_t}\int_t^T \left(\int_{-\delta\vee(s-T)}^0 \left|Q_{s-r}^n\right| |\lambda|(dr)\right)^2 ds &\leq |\lambda|([-\delta, 0])\mathbb{E}^{\mathcal{F}_t}\int_t^T\int_{-\delta\vee(s-T)}^0 \left|Q_{s-r}^n\right|^2 |\lambda|(dr)ds \\ &= |\lambda|([-\delta, 0])\mathbb{E}^{\mathcal{F}_t}\int_{-\delta\vee(t-T)}^0\int_t^{T+r} \left|Q_{s-r}^n\right|^2 ds|\lambda|(dr) \\ &\leq |\lambda|([-\delta, 0])\mathbb{E}^{\mathcal{F}_t}\int_{-\delta}^0\int_t^T |Q_s^n|^2 \, ds|\lambda|(dr) \\ &= \big[|\lambda|([-\delta, 0])\big]^2 \mathbb{E}^{\mathcal{F}_t}\int_t^T |Q_s^n|^2 \, ds.\end{aligned}$$

This implies that, for $\tilde{c}_0 := c_0 + \frac{c_1 T}{2}$, $\tilde{c}_1 := \left(\frac{1}{2} + |\lambda|([-\delta, 0]) + \frac{c_1}{4\alpha}\right) c_1$,

$$|P_t^n|^2 + \frac{1}{2}\mathbb{E}^{\mathcal{F}_t}\int_t^T |Q_s^n|^2 \, ds \leq \tilde{c}_0 + \tilde{c}_1 \mathbb{E}^{\mathcal{F}_t}\left[\int_t^T \|P^n\|_{s,T}^2 ds\right], \tag{3.23}$$

hence

$$\|P^n\|_{s,T}^2 \leq \tilde{c}_0 + \tilde{c}_1 \sup_{s\in[t,T]} \mathbb{E}^{\mathcal{F}_s}\left[\int_t^T \|P^n\|_{r,T}^2 dr\right].$$

Let now $p > 1$. Since

$$\mathbb{E}^{\mathcal{F}_s}\left[\int_t^T \|P^n\|_{r,T}^2 dr\right], \ s \in [0, T]$$

is a martingale, it follows, by Doob's maximal inequality, that

$$\begin{aligned}\mathbb{E}\sup_{s\in[t,T]}\left(\mathbb{E}^{\mathcal{F}_s}\left[\int_t^T \|P^n\|_{r,T}^2 dr\right]\right)^p &\leq \left(\tfrac{p}{p-1}\right)^p \mathbb{E}\left[\int_t^T \|P^n\|_{r,T}^2 dr\right]^p \\ &\leq \left(\tfrac{p}{p-1}\right)^p T^{p/2}\mathbb{E}\int_t^T \|P^n\|_{r,T}^{2p} dr.\end{aligned}$$

Therefore

$$\mathbb{E}\|P^n\|_{t,T}^{2p} \le c_{p,T}\left(1 + \int_t^T \mathbb{E}\|P^n\|_{r,T}^{2p} dr\right),$$

where $c_{p,T}$ is a constant independent of n. By Gronwall lemma (for $t \mapsto \mathbb{E}\|P^n\|_{T-t,T}^{2p}$), we get

$$\sup_{n\ge 1} \mathbb{E}\|P^n\|_T^{2p} < +\infty.$$

Inserting this into (3.23) with $t = 0$, we obtain

$$\sup_{n\ge 1} \mathbb{E}\int_0^T |Q_t^n|^2 \, dt < +\infty.$$

Having these estimates, we would like to pass to the limit in (3.22). Let us consider the finite measure ρ on $\Omega \times [0,T]$, defined by

$$\rho(d\omega dt) = \left[dt + \frac{dA_t^u(\omega)}{A_T^u(\omega)}\right] \mathbb{P}(d\omega)$$

(if $A_T^u(\omega) = 0$ then $dA_t^u(\omega) \equiv 0$; we use the convention $\frac{0}{0} = 0$ in this case) and let $L^2_{\mathbb{F},\rho}(\Omega \times [0,T])$ be the linear space of square-integrable processes on $[0,T]$ with respect to the measure ρ which are progressively measurable. Since (P^n, Q^n) is bounded in $L^2_{\mathbb{F}}(\Omega; C[0,T]) \times L^2_{\mathbb{F}}(\Omega \times [0,T]\,; \mathbb{R}^d)$, it follows that (P^n, Q^n) is also bounded in the Hilbert space $\mathcal{S}' := L^2_{\mathbb{F},\rho}(\Omega \times [0,T]) \times L^2_{\mathbb{F}}(\Omega \times [0,T]\,; \mathbb{R}^d)$. Hence, there exists a sub-sequence, still denoted by (P^n, Q^n) and converging weakly to an element $(\bar{P}, Q)$ of $\mathcal{S}'$. This has as consequence the existence, by Mazur's lemma, of a convex combination

$$(\bar{P}^n, \bar{Q}^n) = \textstyle\sum_{i=n}^{N_n} \alpha_i^n (P^i, Q^i),\ \alpha_n^n + ... + \alpha_{N_n}^n = 1,\ \alpha_i^n \ge 0,\ \forall n \le i \le N_n$$

converging strongly to $(\bar{P}, Q)$. Without restricting the generality, we can suppose that $\bar{P}^n$ converges to $\bar{P}$, ρ-a.e. and $\bar{Q}^n$ converges to Q, $dtd\mathbb{P}$-a.e. We have that

$$\left|\sum_{i=n}^{N_n} \alpha_i^n \int_0^t P_s^i dA_s^i - \int_0^t \bar{P}_s dA_s^u\right| \le \int_0^T \left|\bar{P}_s^n - \bar{P}_s\right| dA_s^u + \sum_{i=n}^{N_n} \alpha_i^n \|P^i\|_T (A_T^u - A_T^i),$$

It is clear that the sequence $\left(\sum_{i=n}^{N_n} \alpha_i^n \|P^i\|_T (A_T^u - A_T^i)\right)$ converges to 0 a.s., since $A_T^u - A_T^i = 0$ on $\{A_T^u \le i\}$ for every $i \ge 1$. On the other hand,

$$\mathbb{E}\frac{1}{A_T^u}\int_0^T \left|\bar{P}_s^n - \bar{P}_s\right| dA_s^u \le \int_{\Omega\times[0,T]} \left|\bar{P}_s^n - \bar{P}_s\right| d\rho \to 0,$$

so we can extract a subsequence, still labelled $(\bar{P}^n)$, such that $\frac{1}{A_T^u}\int_0^T \left|\bar{P}_s^n - \bar{P}_s\right| dA_s^u$ converges a.s. to 0. Hence $\int_0^T \left|\bar{P}_s^n - \bar{P}_s\right| dA_s^u \to 0$. The two convergences we have obtained imply that $\sum_{i=n}^{N_n} \alpha_i^n \int_0^t P_s^i dA_s^i$ converges to $\int_0^t \bar{P}_s dA_s^u$, $\forall t \in [0,T]$, a.s. Consequently, since equation (3.22) is linear in (P^n, Q^n) and $\int_0^\cdot P_s^n dA_s^n$, passing to the limit in the relation

$$\bar{P}_t^n + \textstyle\sum_{i=n}^{N_n} \alpha_i^n \displaystyle\int_0^t P_s^i dA_s^i = h'(X_T^u) + \int_t^T \mathbb{E}^{\mathcal{F}_s}[f(s, R(X^u), u, \bar{P}^n, \bar{Q}^n)]ds - \int_t^T \bar{Q}_s^n dW_s,$$

leads to

$$\bar{P}_t + \int_t^T \bar{P}_s dA_s^u = h'(X_T^u) + \int_t^T \mathbb{E}^{\mathcal{F}_s}[f(s, R(X^u), u, \bar{P}, Q)]ds - \int_t^T Q_s dW_s,$$

$dtd\mathbb{P}$-a.e. By denoting, for $t \in [0,T]$,

$$P_t := h'(X_T^u) + \int_t^T \mathbb{E}^{\mathcal{F}_s}[f(s, R(X^u), u, \bar{P}, Q)]ds - \int_t^T Q_s dW_s - \int_t^T \bar{P}_s dA_s^u,$$

it is clear that P is a continuous process and $P_t = \bar{P}_t$ a.s., dt-a.e. A first consequence is that P_t is $\mathcal{F}_t$-measurable dt-a.e., therefore P is adapted (by the properties of the filtration $\mathbb{F}$). The second one is that (P,Q) satisfies the equation

$$P_t + \int_t^T P_s dA_s^u = h'(X_T^u) + \int_t^T \mathbb{E}^{\mathcal{F}_s}[f(s, R(X^u), u, P, Q)]ds - \int_t^T Q_s dW_s,$$

for every $t \in [0,T]$, a.s. We will prove now that $P \in L^2_{\mathbb{F}}(\Omega; C[0,T])$, *i.e.* $(P,Q) \in \mathcal{S}$. By applying Itô's formula to $|P_t|^2$ we obtain

$$\begin{aligned}|P_t|^2 + 2\int_t^T |P_s|^2 \, dA_s^u + \int_t^T |Q_s|^2 \, ds = |h'(X_T^u)|^2 + 2\int_t^T P_s \mathbb{E}^{\mathcal{F}_s}[f(s, R(X^u), u, P, Q)]ds \\ + 2\int_t^T P_s Q_s dW_s.\end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}\|P\|_T^2 \le & c_0 + c_1 \mathbb{E}\int_0^T |P_s| \left(1 + \mathbb{E}^{\mathcal{F}_s}\int_s^{(s+\delta)\wedge T} (|P_r| + |Q_r|)\,|\lambda|(s-dr)\right)ds \\ & + 2\mathbb{E}\sup_{t\in[0,T]}\left|\int_t^T P_s Q_s dW_s\right| \\ \le & c_0 + \frac{c_1 T}{2} + c_1 \mathbb{E}\int_0^T |P_s|^2 \, ds \\ & + c_1 |\lambda|\left([-\delta, 0]\right)\mathbb{E}\int_0^T \int_s^{(s+\delta)\wedge T} \left(|P_r|^2 + |Q_r|^2\right)|\lambda|(s-dr)ds \\ & + 4\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t P_s Q_s dW_s\right|.\end{aligned}$$

We have that

$$\begin{aligned}\mathbb{E}\int_0^T \int_s^{(s+\delta)\wedge T} & \left(|P_r|^2 + |Q_r|^2\right)|\lambda|(s-dr)ds \\ & = \mathbb{E}\int_0^T \int_{-\delta\vee(s-T)}^0 \left(|P_{s-r}|^2 + |Q_{s-r}|^2\right)|\lambda|(dr)ds \\ & = \mathbb{E}\int_{-\delta}^0 \int_0^{T+r} \left(|P_{s-r}|^2 + |Q_{s-r}|^2\right)ds|\lambda|(dr) \\ & \le |\lambda|\left([-\delta, 0]\right)\mathbb{E}\int_0^T \left(|P_s|^2 + |Q_s|^2\right)ds.\end{aligned}$$

Also, by Burkholder-Davis-Gundy inequality,

$$\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t P_s Q_s dW_s\right| \le c_2 \mathbb{E}\left[\int_0^T |P_s Q_s|^2 \, ds\right]^{1/2},$$

for a constant $c_2 > 0$. Combining the two inequalities, we obtain

$$\mathbb{E}\|P\|_T^2 \le c_0 + \frac{c_1 T}{2} + c_1\left(1 + \left[|\lambda|([-\delta, 0])\right]^2\right)\mathbb{E}\int_0^T \left(|P_s|^2 + |Q_s|^2\right)ds + 4c_2 \mathbb{E}\left[\int_0^T |P_s Q_s|^2 \, ds\right]^{\frac{1}{2}}.$$

We already know that $(P,Q) \in L^2_{\mathbb{F}}(\Omega\times[0,T])\times L^2_{\mathbb{F}}(\Omega\times[0,T];\mathbb{R}^d)$, so it remains to show that $PQ \in L^1_{\mathbb{F}}\left(\Omega; L^2([0,T];\mathbb{R}^d)\right)$. Actually we will prove more: we will show that $PQ \in L^p_{\mathbb{F}}\left(\Omega; L^2([0,T];\mathbb{R}^d)\right)$ for every $p\in(1,2)$. We recall that the sequence of processes $(\bar P^n,\bar Q^n)$ converges $dtd\mathbb{P}$-a.e. to (P,Q). We also have that $(\bar P^n,\bar Q^n)$ is bounded in $L^{\frac{2p}{2-p}}_{\mathbb{F}}(\Omega;C[0,T])\times L^2_{\mathbb{F}}(\Omega\times[0,T];\mathbb{R}^d)$; since, by Hölder inequality,

$$\begin{aligned}\mathbb{E}\Big[\int_0^T\left|\bar P^n_s\bar Q^n_s\right|^2ds\Big]^{\frac p2} &\le \mathbb{E}\left[\sup_{s\in[0,T]}\left|\bar P^n_s\right|^p\left(\int_0^T\left|\bar Q^n_s\right|^2ds\right)^{\frac p2}\right]\\ &\le\left[\mathbb{E}\sup_{s\in[0,T]}\left|\bar P^n_s\right|^{\frac{2p}{2-p}}\right]^{\frac{2-p}{2}}\mathbb{E}\Big[\int_0^T\left|\bar Q^n_s\right|^2ds\Big]^{\frac p2},\end{aligned}$$

it follows that $(\bar P^n\bar Q^n)$ is bounded in $L^p_{\mathbb{F}}\left(\Omega;L^2([0,T];\mathbb{R}^d)\right)$, which is a reflexive Banach space. Consequently, there exists a weak limit point $Z\in L^p_{\mathbb{F}}\left(\Omega;L^2([0,T];\mathbb{R}^d)\right)$ of the sequence $(\bar P^n\bar Q^n)$. But $\bar P^n\bar Q^n$ converges $dtd\mathbb{P}$-a.e. to PQ, so $Z=PQ$, $dtd\mathbb{P}$-a.e. This implies, of course, that $PQ\in L^p_{\mathbb{F}}\left(\Omega;L^2([0,T];\mathbb{R}^d)\right)$. □

Every control u can be approximated, in $L^2_{\mathbb{G}}(\Omega\times[0,T];U)$ by continuous controls u^ε: for instance, we may take

$$u^\varepsilon_t := \frac1\varepsilon\int_{t-\varepsilon}^t u_s ds,\ t\in[0,T].$$

(we prolong u by 0 on the negative axis). Hence, if u^* is an optimal control, since $J: L^2_{\mathbb{G}}(\Omega\times[0,T];U)\to\mathbb{R}$ is continuous by Proposition 3.1, we can find continuous controls $\bar u^n$ with $\bar u^n\to u^*$ in $L^2_{\mathbb{G}}(\Omega\times[0,T];U)$ and $J(\bar u^n)\le J(u^*)+n^{-1}$.

Let us recall Ekeland's variational principle (see [10]):

Theorem 3.13. *Let V be a complete metric space, and $F:V\to\mathbb{R}\cup\{+\infty\}$ a proper, l.s.c. function, bounded from below. For every $\varepsilon,\lambda>0$ and every point $u\in V$ such that $F(u)\le\inf_{w\in V}F(w)+\varepsilon$, there exists some point $v\in V$ such that:*

1. *$F(v)\le F(u)$;*
2. *$d(u,v)\le\lambda$;*
3. *$F(v)<F(w)+\frac{\varepsilon}{\lambda}d(v,w),\ \forall w\ne v.$*

We apply the above result with $F=J$, $V=L^2_{\mathbb{G}}(\Omega;C([0,T];U))$ endowed with the metric

$$d(u,v):=\left(\mathbb{E}\left\|u-v\right\|_T^2\right)^{1/2},$$

$\varepsilon=n^{-1}$, $\lambda=n^{-1/2}$ and $u=\bar u^n$. Therefore, for every $n\in\mathbb{N}^*$, there exist $u^n\in L^2_{\mathbb{G}}(\Omega;C([0,T];U))$ such that

$$\mathbb{E}\left\|u^n-\bar u^n\right\|_T^2\le n^{-1}$$

and

$$J(u^n)\le J^n(u):=J(u)+n^{-1/2}\left(\mathbb{E}\left\|u^n-u\right\|_T^2\right)^{1/2},\ \forall u\in L^2_{\mathbb{G}}(\Omega;C([0,T];U)),$$

meaning that u^n is an optimal control corresponding to the perturbed cost functional J^n. Of course, since $\bar u^n$ approximates u^*, it is clear that $u^n\to u^*$ in $L^2_{\mathbb{G}}(\Omega\times[0,T];U)$.

We now formulate the maximum principle for the near optimal controls u^n. Let $X^n:=X^{u^n}$, $A^n:=A^{u^n}$ and (P^n,Q^n) be the solution of equation (3.15) with parameter u^n.

Proposition 3.14. *For every admissible control v we have*

$$\mathbb{E}\int_0^T\frac{\partial H}{\partial u}(t,R(X^n)_t,u^n_t,P^n_t,Q^n_t)(v_t-u^n_t)dt\ge-\sqrt{\frac1n\mathbb{E}\sup_{t\in[0,T]}\left|v_t-u^n_t\right|^2}. \tag{3.24}$$

Proof. Suppose first that v is a continuous control and let $Y^n \in L^2_{\mathbb{F}}(\Omega; C[0,T])$ be the unique solution of equation (3.11) with $u^0 = u^n$, $u^1 = v$ and $\theta = 0$.

Applying Itô's formula to the process $(P^n_t Y^n_t)_{t\in[0,T]}$ yields[8]

$$\begin{aligned}P^n_T Y^n_T = &\int_0^T \left[(P^n_t(\partial_u b^n_t) + Q^n_t(\partial_u \sigma^n_t))(v_t - u^n_t) + (P^n_t(\partial_y b^n_t) + Q^n_t(\partial_y \sigma^n_t))R(Y^n)_t\right] dt \\ &- \int_0^T \mathbb{E}^{\mathcal{F}_t}[f(t, R(X^n), u^n, P^n, Q^n)]Y^n_t dt \\ &+ \int_0^T \langle P^n_t(\partial_y \sigma^n)R(Y^n)_t + (\partial_u \sigma^n_t)(v_t - u^n_t), dW_t\rangle + \int_0^T Q^n_t Y^n_t dW_t.\end{aligned}$$

Consequently, since $(P^n, Q^n) \in \mathcal{S}$ and $Y^n \in L^2_{\mathbb{F}}(\Omega; C[0,T])$, the expectation of stochastic integrals in the above relation is 0; therefore

$$\begin{aligned}\mathbb{E}P^n_T Y^n_T =&\mathbb{E}\int_0^T \left[(P^n_t(\partial_u b^n_t) + Q^n_t(\partial_u \sigma^n_t))(v_t - u^n_t)\right] dt \\ &+ \mathbb{E}\int_0^T \left[(P^n_t(\partial_y b^n_t) + Q^n_t(\partial_y \sigma^n_t))R(Y^n)_t\right] dt \\ &- \mathbb{E}\int_0^T f(t, R(X^n), u^n, P^n, Q^n)Y^n_t dt.\end{aligned} \tag{3.25}$$

Let us analyze the last term of the right-hand side of the above equality:

$$\begin{aligned}&\mathbb{E}\int_0^T f(t, R(X^n), u^n, P^n, Q^n)Y^n_t dt \\ &= \mathbb{E}\int_0^T \int_t^{(t+\delta)\wedge T} \left[\partial_y g^n_s + P^n_s(\partial_y b^n_t) + Q^n_s(\partial_y \sigma^n_s)\right] \lambda(t - ds)Y^n_t dt \\ &= \mathbb{E}\int_0^T \int_{-\delta\vee(t-T)}^0 \left[\partial_y g^n_{t-s} + P^n_{t-s}(\partial_y b^n_{t-s}) + Q^n_{t-s}(\partial_y \sigma^n_{t-s})\right] \lambda(ds)Y^n_t dt \\ &= \mathbb{E}\int_{-\delta}^0 \int_{-s}^T \left[\partial_y g^n_t + P^n_t(\partial_y b^n_t) + Q^n_t(\partial_y \sigma^n_t)\right] Y^n_{t+s} dt\lambda(ds).\end{aligned}$$

Since $Y^n_s = 0$ for $s \in [-\delta, 0]$ we have

$$\begin{aligned}\mathbb{E}\int_0^T f(t, R(X^n), u^n, P^n, Q^n)Y^n_t dt &= \mathbb{E}\int_{-\delta}^0 \int_0^T \left[\partial_y g^n_t + P^n_t(\partial_y b^n_t) + Q^n_t(\partial_y \sigma^n_t)\right] Y^n_{t+s} dt\lambda(ds) \\ &= \mathbb{E}\int_0^T \int_{-\delta}^0 \left[\partial_y g^n_t + P^n_t(\partial_y b^n_t) + Q^n_t(\partial_y \sigma^n_t)\right] R(Y^n)_t dt.\end{aligned}$$

Inserting this relation into (3.25) we obtain

$$\mathbb{E}P^n_T Y^n_T = \mathbb{E}\int_0^T \left[(P^n_t(\partial_u b^n_t) + Q^n_t(\partial_u \sigma^n_t))(v_t - u^n_t) - (\partial_y g^n_t)R(Y^n)_t\right] dt. \tag{3.26}$$

On the other hand, since $J^n(u^\theta) \geq J(u^n)$, we have (as before, u^θ denotes $u^n + \theta(v - u^n)$)

$$\begin{aligned}0 \leq \mathbb{E}\int_0^T &\Big[g\big(t, R\big(X^{u^\theta}\big)_t, u^\theta_t\big) - g(t, R(X^n)_t, u^n_t)\Big] dt + \mathbb{E}\Big[h\big(X^{u^\theta}_T\big) - h(X^n_T)\Big] \\ &+ \frac{1}{\sqrt{n}}\big(\mathbb{E}\left\|u^\theta - u^n\right\|^2_T\big)^{1/2}\end{aligned}$$

[8] As before, $\partial_z b^n_t$, $\partial_z \sigma^n_x$, $\partial_z g^n_t$ denote $\dfrac{\partial b}{\partial z}(t, R(X^n)_t, u^n_t)$, $\dfrac{\partial \sigma}{\partial z}(t, R(X^n)_t, u^n_t)$, $\dfrac{\partial g}{\partial z}(t, R(X^n)_t, u^n_t)$, respectively, where z stands for y or u.

Taking into account (3.14), we divide the right-hand side of this inequality by θ and take the limit with θ going to 0. This gives

$$0 \le \mathbb{E}\int_0^T [(\partial_y g_t^n)R(Y^n)_t + (\partial_u g_t^n)(v_t - u_t^n)]\,dt + \mathbb{E}h'\big(X_T^n\big)Y_T^n + \frac{1}{\sqrt{n}}\big(\mathbb{E}\,\|v-u^n\|_T^2\big)^{1/2}.$$

By (3.26), the previous inequality takes the form

$$-\frac{1}{\sqrt{n}}\big(\mathbb{E}\,\|v-u^n\|_T^2\big)^{1/2} \le \mathbb{E}\int_0^T [P_t^n(\partial_u b_t^n) + Q_t^n(\partial_u \sigma_t^n) + \partial_u g_t^n](v_t - u_t^n)dt. \tag{3.27}$$

Now we want to show that this inequality takes place for every control $v \in L^2_{\mathbb{G}}(\Omega \times [0,T];U)$. Indeed, as before, we can approximate v by continuous controls

$$v_t^\varepsilon := u_t^n + \frac{1}{\varepsilon}\int_{t-\varepsilon}^t (v_s - u_s^n)ds$$

(we prolong v and u^n by 0 on the negative axis). We then have $v_t^\varepsilon \to v_t$ $dtd\mathbb{P}$-a.e. when ε goes to 0, and

$$\sup_{t\in[0,T]} |v_t^\varepsilon - u_t^n| \le \sup_{t\in[0,T]} |v_t - u_t^n|\,.$$

Since the process v^ε verifies (3.27) by the previous step, the inequality will be also verified for v, by passing to the limit. □

3.3 Maximum principle

We are able to retrieve the necessary conditions of optimality for u^* by passing to the limit in inequality (3.24). Let X^* denote the state of the system corresponding to the optimal control u^*.

Theorem 3.15 (maximum principle)**.** *If u^* is an optimal control, then there exists a càdlàg process $k \in L^2_{\mathbb{F}}(\Omega; BV[0,T])$ such that, $dtd\mathbb{P}$-a.e.,*

$$\mathbb{E}^{\mathcal{G}_t}\left[P_t^*\frac{\partial b}{\partial u}(t,R(X^*)_t,u_t^*) + Q_t^*\frac{\partial \sigma}{\partial u}(t,R(X^*)_t,u_t^*) + \frac{\partial g}{\partial u}(t,R(X^*)_t,u_t^*)\right](v-u_t^*) \ge 0,\ \forall v \in U,$$

where $(P^*,Q^*) \in L^2_{\mathbb{F}}(\Omega\times[0,T]) \times L^2_{\mathbb{F}}(\Omega\times[0,T];\mathbb{R}^d)$ is a solution of the equation

$$\begin{cases} -dP_t^* = -dk_t + \mathbb{E}^{\mathcal{F}_t}\left[f(t,R(X^*),u^*,P^*,Q^*)\right]dt - Q_t^* dW_t,\ t\in[0,T]\,; \\ P_T^* = h'(X_T^*). \end{cases} \tag{3.28}$$

Proof. First we would like to pass to the limit in the equation

$$\begin{cases} -dP_t^n + P_t^n dA_t^n = \mathbb{E}^{\mathcal{F}_t}\left[f(t,R(X^n),u^n,P^n,Q^n)\right]dt - Q_t^n dW_t,\ t\in[0,T]; \\ P_T^n = h'(X_T^n), \end{cases} \tag{3.29}$$

whose solution exists and is unique thanks to Theorem 3.12. Exactly as in the proof of this result, by applying Itô's formula to $|P_t^n|^2$, one can prove that $(P^n,Q^n)_{n\ge1}$ is bounded in $L^2_{\mathbb{F}}(\Omega;C\,[0,T]\,;\mathbb{R}^d) \times L^2_{\mathbb{F}}(\Omega\times[0,T]\,;\mathbb{R}^d)$. In order to show that $\big(\int_0^\cdot |P_s^n|\,dA_s^n\big)_{n\ge1}$ is bounded in $L^2_{\mathbb{F}}(\Omega\times[0,T])$, we introduce the local time of P^n at 0:

$$L_t^{P^n,0} := |P_t^n| - |P_0^n| - \int_0^t |P_s^n|\,dA_s^n + \int_0^t (\operatorname{sgn} P_s^n)\mathbb{E}^{\mathcal{F}_s}\left[f(s,R(X^n),u^n,P^n,Q^n)\right]ds \\ - \int_0^t (\operatorname{sgn} P_s^n)Q_s^n dW_s.$$

We know (see [23]) that $L_t^{P^n,0}$ is nonnegative, therefore, by condition (H$_1$) and the boundedness of (P^n, Q^n),

$$\begin{aligned}
&\mathbb{E}\Big(\int_0^T |P_s^n|\, dA_s^n\Big)^2 \\
&\le C\left(\mathbb{E}\,|P_T^n|^2 + \mathbb{E}\,|P_0^n|^2 + \mathbb{E}\bigg[\int_0^T \mathbb{E}^{\mathcal{F}_s}\left[f(s, R(X^n), u^n, P^n, Q^n)\right] ds\bigg]^2 + \mathbb{E}\int_0^T |Q_s^n|^2\, ds\right) \\
&\le C + C\mathbb{E}\bigg[\int_0^T \mathbb{E}^{\mathcal{F}_s}\int_s^{(s+\delta)\wedge T} (1 + |P_r^n| + |Q_r^n|)\,|\lambda|(s - dr)ds\bigg]^2 \\
&\le C + C\mathbb{E}\int_0^T \int_{-\delta\vee(s-T)}^0 \left(1 + |P_{s-r}^n|^2 + |Q_{s-r}^n|^2\right)|\lambda|(dr)ds \\
&= C + C\mathbb{E}\int_{-\delta\vee(t-T)}^0 \int_t^{T+r} \left(1 + |P_{s-r}^n|^2 + |Q_{s-r}^n|^2\right)|\lambda|(dr)ds \\
&\le C + C\mathbb{E}\int_0^T \left(1 + |P_s^n|^2 + |Q_s^n|^2\right) ds \le C.
\end{aligned}$$

Here and below C denotes a positive constant not depending on n, possibly changing value from one occurrence to another. By the boundedness of the sequence

$$\left(P^n, Q^n, \int_0^{\cdot}(P_s^n)^+ dA_s^n, \int_0^{\cdot}(P_s^n)^- dA_s^n\right)_{n\ge1}$$

in $\mathcal{S}'' := L^2_{\mathbb{F}}(\Omega\times[0,T]) \times L^2_{\mathbb{F}}(\Omega\times[0,T];\mathbb{R}^d) \times L^2_{\mathbb{F}}(\Omega\times[0,T])^2$, it has a weak limit point $(\bar P, Q^*, \bar k, \underline{k}) \in \mathcal{S}''$. As in the proof of Theorem 3.12, we can suppose that there exists a convex combination $\sum_{i=\underline{n}}^{N_n}\alpha_i^n\big(P^i, Q^i, \int_0^{\cdot}(P_s^i)^+ dA_s^i, \int_0^{\cdot}(P_s^i)^- dA_s^i\big)$ converging strongly in $\mathcal{S}''$ and $dtd\mathbb{P}$-a.e. to $(\bar P, Q^*, \bar k, \underline{k})$. It is easy to show that $\bar k$ and $\underline{k}$ admit càdlàg, increasing, adapted modifications, so we can suppose that they are càdlàg and increasing. Let $k := \bar k - \underline{k}$; obviously k is càdlàg and $k \in L^2_{\mathbb{F}}(\Omega; BV[0,T])$. We want first to show that $(\bar P, Q^*, k)$ satisfies equation (3.28) $dtd\mathbb{P}$-a.e. The main difference to the proof of Theorem 3.12 is that the integral with respect to Lebesgue measure in (3.28) is no longer stable to convex combinations. However, we will use the fact that $u^n \to u^*$ in $L^2_{\mathbb{G}}(\Omega\times[0,T];U)$ and $X^n \to X^*$ in $L^2_{\mathbb{F}}(\Omega; C[0,T])$, due to Proposition 2.4; we may assume, without restricting the generality, that these convergences hold $dtd\mathbb{P}$-a.e. We claim that the sequence $\Big(\sum_{i=n}^{N_n}\alpha_i^n\int_t^T \mathbb{E}^{\mathcal{F}s}\left[f(s, R(X^i), u^i, P^i, Q^i)\right] ds\Big)_{n\ge1}$ converges to $\int_t^T \mathbb{E}^{\mathcal{F}s}\left[f(s, R(X^*), u^*, \bar P, Q^*)\right] ds$ for every $t \in [0,T]$. Indeed,

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\left|\sum_{i=n}^{N_n}\alpha_i^n\int_t^T \mathbb{E}^{\mathcal{F}s}\left[f(s, R(X^i), u^i, P^i, Q^i)\right] ds - \int_t^T \mathbb{E}^{\mathcal{F}s}\left[f(s, R(X^*), u^*, \bar P, Q^*)\right] ds\right| \\
&\qquad\le \sum_{i=n}^{N_n}\alpha_i^n\mathbb{E}\int_0^T \left|f(s, R(X^i), u^i, P^i, Q^i) - f(s, R(X^*), u^*, P^i, Q^i)\right| ds \\
&\qquad\quad + \mathbb{E}\int_0^T \left|\sum_{i=n}^{N_n}\alpha_i^n f(s, R(X^*), u^*, P^i, Q^i) - f(s, R(X^*), u^*, \bar P, Q^*)\right| ds.
\end{aligned}$$

For the first term of the right-hand side of the inequality, we have the estimate:

$$\begin{aligned}
&\mathbb{E}\int_0^T \left|f(s,R(X^i),u^i,P^i,Q^i)-f(s,R(X^*),u^*,P^i,Q^i)\right|ds\\
&\le \mathbb{E}\int_0^T\int_s^{(s+\delta)\wedge T}\left|\frac{\partial H}{\partial y}(r,R(X^i)_r,u^i_r,P^i_r,Q^i_r)-\frac{\partial H}{\partial y}(r,R(X^*)_r,u^*_r,P^i_r,Q^i_r)\right||\lambda|(s-dr)ds\\
&=|\lambda|([-\delta,0])\mathbb{E}\int_0^T\left|\frac{\partial H}{\partial y}(s,R(X^i)_s,u^i_s,P^i_s,Q^i_r)-\frac{\partial H}{\partial y}(s,R(X^*)_s,u^*_s,P^i_s,Q^i_s)\right|ds\\
&\le |\lambda|([-\delta,0])\mathbb{E}\int_0^T\Big[\left|\partial_y g^i_s-\partial_y g^i_s\right|+\left|\partial_y b^i_s-\partial_y b^*_s\right|\left|P^i_s\right|+\left|\partial_y\sigma^i_s-\partial_y\sigma^*_s\right|\left|Q^i_s\right|\Big]ds\\
&\le C\mathbb{E}\int_0^T\Big[\left|\partial_y g^i_s-\partial_y g^*_s\right|+\left|\partial_y b^i_s-\partial_y b^*_s\right|^2+\left|\partial_y\sigma^i_s-\partial_y\sigma^*_s\right|^2\Big]ds,
\end{aligned}$$

where $\partial_y g^i_s$ stands for $\frac{\partial g}{\partial y}(s,R(X^i)_s,u^i_s)$, etc. For the second term, we use the linearity of f:

$$\begin{aligned}
\mathbb{E}\int_0^T&\left|\textstyle\sum_{i=n}^{N_n}\alpha^n_i f(s,R(X^*),u^*,P^i,Q^i)-f(s,R(X^*),u^*,\bar P,Q^*)\right|ds\\
&\le \mathbb{E}\int_0^T\Big[\left|\partial_y b^*_s\right|\left|\textstyle\sum_{i=n}^{N_n}\alpha^n_i P^i_s-\bar P_s\right|+\left|\partial_y\sigma^*_s\right|\left|\textstyle\sum_{i=n}^{N_n}\alpha^n_i Q^i_s-Q^*_s\right|\Big]ds\\
&\le C\mathbb{E}\int_0^T\Big[\left|\textstyle\sum_{i=n}^{N_n}\alpha^n_i P^i_s-\bar P_s\right|^2+\left|\textstyle\sum_{i=n}^{N_n}\alpha^n_i Q^i_s-Q^*_s\right|^2\Big]ds.
\end{aligned}$$

Summing these inequalities and passing to the limit, we obtain the thesis. Since the sequence $(h'(X^n_T))_{n\ge1}$ is also converging a.s. to $h'(X^*_T)$, we get from (3.29):

$$\bar P_t+k_t-k_T=h'(X^*_T)+\int_t^T\mathbb{E}^{\mathcal{F}_s}\left[f(s,R(X^*),u^*,\bar P,Q^*)\right]ds-\int_t^T Q^*_s dW_s,\ dtd\mathbb{P}\text{-a.e.}$$

Let now, for $t\in[0,T]$,

$$P^*_t:=k_T-k_t+h'(X^*_T)+\int_t^T\mathbb{E}^{\mathcal{F}_s}\left[f(s,R(X^*),u^*,\bar P,Q^*)\right]ds-\int_t^T Q^*_s dW_s.$$

It is clear that P^* is càdlàg and $P^*_t=\bar P_t$, $dtd\mathbb{P}$-a.e. As in the proof of Theorem 3.12, this implies that P^* is adapted and (P^*,Q^*,k) satisfies equation (3.28) for every $t\in[0,T]$, a.s. Now, let us prove the maximum principle for (P^*,Q^*). By Proposition 3.14 and the boundedness of U,

$$\mathbb{E}\int_0^T\frac{\partial H}{\partial u}(t,R(X^n)_t,u^n_t,P^n_t,Q^n_t)(v_t-u^n_t)dt\ge-\frac{C}{\sqrt{n}},\tag{3.30}$$

for every admissible control v. Since $\frac{\partial H}{\partial u}$ is linear in (P^n,Q^n) and continuous in the other terms, we can pass to the limit in relation (3.30) as we did for equation (3.29), in order to obtain

$$\mathbb{E}\int_0^T\frac{\partial H}{\partial u}(t,R(X^*)_t,u^*_t,P^*_t,Q^*_t)(v_t-u^*_t)dt\ge0$$

for every $v\in L^2_{\mathbb{G}}(\Omega\times[0,T];U)$. It is now clear that for an arbitrary $v\in U$ we must have

$$\mathbb{E}^{\mathcal{G}_t}\left[\frac{\partial H}{\partial u}(t,R(X^*)_t,u^*_t,P^*_t,Q^*_t)\right](v-u^*_t)\ge0,\ dtd\mathbb{P}\text{-a.e.}$$

Since U is separable, this implies

$$\mathbb{E}^{\mathcal{G}_t}\left[\frac{\partial H}{\partial u}(t,R(X^*)_t,u^*_t,P^*_t,Q^*_t)\right](v-u^*_t)\ge0,\ \forall v\in U,\ dtd\mathbb{P}\text{-a.e.}$$

□

4 Appendix

This section is dedicated to the proof of Proposition 3.8.

For the moment we impose some restrictive assumptions:

(S_1) φ is affine outside a compact interval;

(S_2) there exists $\alpha > 0$ such that $|\sigma(t,y,u)| \geq \alpha$, for every $(t,y,u) \in [0,T] \times \mathbb{R}^m \times U$.

Condition (S_1) implies that μ is a finite measure having compact support.

First, we need a stability result for A^u with respect to the control u:

Proposition 4.1. *Let $(u^n)_{n\geq 0}$ be a sequence of controls such that $\sup_{t\in[0,T]} \left|u_t^n - u_t^0\right|^2 \to 0$ in $L^\infty(\Omega)$. Suppose that conditions (S_1)-(S_2) hold and u^0 is càdlàg. Then*

$$\mathbb{E}\left|A_t^{u^n} - A_t^{u^0}\right|^4 \to 0,\ \forall t \in [0,T]\,.$$

Proof. Let for simplicity X^n, K^n, $L^{a,n}$, A^n denote X^{u^n}, K^{u^n}, L_t^{a,u^n}, respectively A^{u^n}, for $n \in \mathbb{N}$. By Theorem 2.3 and Proposition 2.4, the sequence (X^n, K^n) converges to (X^0, K^0) in $L^4_{\mathbb{F}}(\Omega; C[-\delta,T])) \times L^4_{\mathbb{F}}(\Omega; C[-\delta,T]))$. We will assume, without loss of generality, that $\|X^n - X^0\|_T \to 0$, a.s.

In order to establish the boundedness of $(L^{a,n})$, we apply Tanaka's formula:

$$\begin{aligned}\frac{1}{2}L_t^{a,n} &= (X_t^n - a)^+ - (\eta(0) - a)^+ + \int_0^t \mathbf{1}_{\{X_s^n > a\}} dK_s^n \\ &\quad - \int_0^t \mathbf{1}_{\{X_s^n > a\}} b(s, R(X^n)_s, u_s^n) ds - \int_0^t \left\langle \mathbf{1}_{\{X_s^n > a\}} \sigma(s, R(X^n)_s, u_s^n), dW_s \right\rangle. \end{aligned} \tag{4.1}$$

As a consequence of the boundedness of (X^n, K^n) and condition (H_1), we obtain

$$\sup_{a\in\mathbb{R},\, n\geq 0} \mathbb{E}|L_T^{a,n}|^4 < +\infty, \tag{4.2}$$

which gives, by (S_1),

$$\sup_{n\geq 0} \mathbb{E}|A_T^n|^4 < +\infty.$$

By the uniform convergence of X^n to X^0, $\mathbf{1}_{\{X_t^n > a\}}$ converges to $\mathbf{1}_{\{X_t^0 > a\}}$ $dtd\mathbb{P}$-a.e., because $\{t \in [0,T] \mid X_t^n = a\}$ are negligible sets with respect to Lebesgue measure (see formula (3.3)). Consequently, passing to the limit in (4.1), $L^{a,n}$ converges to $L^{a,0}$ in $L^4_{\mathbb{F}}(\Omega; C\,[0,T])$. Indeed, by Proposition 3.3,

$$\begin{aligned}&\sup_{t\in[0,T]} \left| \int_0^t \mathbf{1}_{\{X_s^n > a\}} dK_s^n - \int_0^t \mathbf{1}_{\{X_s^0 > a\}} dK_s^0 \right| \\ &\qquad \leq \int_0^T \varphi'_-(X_s^0) \left| \mathbf{1}_{\{X_s^n > a\}} - \mathbf{1}_{\{X_s^0 > a\}} \right| ds + \int_0^T \mathbf{1}_{\{X_s^0 > a\}} \left| \varphi'_-(X_s^n) - \varphi'_-(X_s^0) \right| ds.\end{aligned}$$

Since φ'_- is increasing, the set of its discontinuity points is countable, so $\varphi'_-(X_s^n)$ converges to $\varphi'_-(X_s^0)$, ds-a.e. (again, by (3.3)). By Lebesgue's dominated convergence theorem,

$$\mathbb{E} \sup_{t\in[0,T]} \left| \int_0^t \mathbf{1}_{\{X_s^n > a\}} dK_s^n - \int_0^t \mathbf{1}_{\{X_s^0 > a\}} dK_s^0 \right|^4 \to 0.$$

For the convergence of the other terms in (4.1) we use similar arguments (and the inequality of Burkholder-Davis-Gundy for the stochastic integral).

For $k \in \mathbb{N}^*$, $a \in \mathbb{R}$ and $n \in \mathbb{N}$ we introduce the processes

$$L_t^{a,k,n} := L_t^{a,n} \wedge k,\; t \in [0,T]$$

and

$$A_t^{k,n} := \int_{\mathbb{R}} \int_0^t \frac{L^{a,k,n}(ds)}{|\sigma(s, R(X^n)_s, u_s^n)|^2} \mu(da),\; t \in [0,T]\,.$$

Clearly, by conditions (S$_2$) and (H$_1$),

$$\begin{aligned} \left|A_t^{k,n} - A_t^{k,0}\right| &\leq \int_{\mathbb{R}} \left| \int_0^t \frac{L^{a,k,n}(ds)}{|\sigma(s, R(X^n)_s, u_s^n)|^2} - \int_0^t \frac{L^{a,k,0}(ds)}{|\sigma(s, R(X^0)_s, u_s^0)|^2} \right| \mu(da) \\ &\leq \frac{2\sup\left[\left|\frac{\partial \sigma}{\partial y}\right| + \left|\frac{\partial \sigma}{\partial u}\right|\right]\left[\sup_{t\in[0,T]}\left|u_t^n - u_t^0\right| + \|X^n - X^0\|_T\right]}{\alpha^3} \int_{\mathbb{R}} L_t^{a,k,n} \mu(da) \\ &\qquad + \int_{\mathbb{R}} \left| \int_0^t \frac{(L^{a,k,n} - L^{a,k,0})(ds)}{|\sigma(s, R(X^0)_s, u_s^0)|^2} \right| \mu(da). \end{aligned} \tag{4.3}$$

Since $\|L^{a,k,n} - L^{a,k,0}\|_T \to 0$ in probability, by Helly-Bray's Theorem, $L^{a,k,n}$ converges weakly to $L^{a,k,0}$ in probability (meaning that $\rho(L^{a,k,n}, L^{a,k,0}) \to 0$ in probability, where ρ is the Prohorov's metric of the weak convergence). Furthermore, the set of discontinuities of $\frac{1}{|\sigma(s,R(X^0)_s,u_s^0)|^2}$ is countable (u^0 is càdlàg, σ continuous), therefore $L^{a,k,0}(ds)$-negligible, $\mu(da)d\mathbb{P}$-a.e., so it follows (see, for instance, [5, Theorem 2.7]) that the sequence $\left(\int_0^t \frac{L^{a,k,n}(ds)}{|\sigma(s,R(X^0)_s,u_s^0)|^2}\right)_{n\geq 1}$ converges to $\int_0^t \frac{L^{a,k,0}(ds)}{|\sigma(s,R(X^0)_s,u_s^0)|^2}$ in probability, for all $a \in \mathbb{R}$ and $t \in [0,T]$. Consequently, $\int_0^t \frac{(L^{a,k,n} - L^{a,k,0})(ds)}{|\sigma(s,R(X^0)_s,u_s^0)|^2}$ converges to 0 in measure with respect to $\mu \otimes \mathbb{P}$. By Lebesgue's dominated convergence theorem we infer from (4.3) that

$$\lim_{n\to\infty} \mathbb{E}\left|A_t^{k,n} - A_t^{k,0}\right|^4 = 0,\; \forall t \in [0,T]\,. \tag{4.4}$$

On the other hand, $A_t^{k,n} \leq A_t^n$ and

$$A_t^n - A_t^{k,n} = \int_{\mathbb{R}} \left[\int_0^t \frac{(L^{a,n} - L^{a,k,n})(ds)}{|\sigma(s,R(X^n)_s,u_s^n)|^2}\right] \mu(da) \leq \frac{1}{\alpha^2} \int_{\mathbb{R}} (L_t^{a,n} - k)^+ \mu(da),$$

which implies that

$$\mathbb{E}\left|A_t^n - A_t^{k,n}\right|^4 \leq \tfrac{\mu(\mathbb{R})^3}{\alpha^8} \int_{\mathbb{R}} \mathbb{E}\left[(L_t^{a,n} - k)^+\right]^4 \mu(da). \tag{4.5}$$

By combining this relation with (4.4), from Fatou's Lemma and relation (4.2) we derive for every $k \in \mathbb{N}^*$ and $t \in [0,T]$:

$$\begin{aligned} \limsup_{n\to\infty} \mathbb{E}\left|A_t^n - A_t^0\right|^4 &\leq \frac{8\mu(\mathbb{R})^3}{\alpha^8} \int_{\mathbb{R}} \left(\mathbb{E}\left[(L_t^{a,0} - k)^+\right]^4 + \limsup_{n\to\infty} \mathbb{E}\left[(L_t^{a,n} - k)^+\right]^4\right) \mu(da) \\ &= \frac{8\mu(\mathbb{R})^3}{\alpha^8} \int_{\mathbb{R}} \mathbb{E}\left[(L_t^{a,0} - k)^+\right]^4 \mu(da). \end{aligned}$$

Letting $k \to \infty$, this yields that

$$\lim_{n\to\infty} \mathbb{E}\left|A_t^n - A_t^0\right|^4 = 0,\; \forall t \in [0,T]\,.$$

□

Let $\mathcal{H} := L^2_{\mathbb{F},\nu_T}(\Omega \times [-\delta, T])$, where the measure $\nu_T(dt) := dt + \boldsymbol{\delta}_T(dt)$ is the sum of the Lebesgue measure on $[-\delta, T]$ and Dirac measure concentrated in T. We introduce the space $W^{1,2}([0,1]\,;\mathcal{H})$ of absolutely continuous functions (hence a.e. differentiable) defined on $[0,1]$ and $\mathcal{H}$-valued. Endowed with the scalar product $\langle \cdot, \cdot \rangle_{W^{1,2}([0,1];\mathcal{H})} := \langle \cdot, \cdot \rangle_{L^2([0,1];\mathcal{H})} + \langle \nabla \cdot, \nabla \cdot \rangle_{L^2([0,1];\mathcal{H})}$, the vector space $W^{1,2}([0,1]\,;\mathcal{H})$ is Hilbert; the reader is referred to [7] for properties of this space.

If X is an element of $W^{1,2}([0,1]\,;\mathcal{H})$, we will equally write $\nabla_\theta X$ instead of $\frac{dX}{d\theta}$. A result which we will use frequently in the sequel is:

Lemma 4.2 ([7])**.** *If the sequence $(X^n)_{n\geq 1} \subseteq W^{1,2}([0,1]\,;\mathcal{H})$ is bounded and converges in $L^2([0,1]\,;\mathcal{H})$ to some $X \in L^2([0,1]\,;\mathcal{H})$, then $X \in W^{1,2}([0,1]\,;\mathcal{H})$ and ∇X_n converges weakly to ∇X in $L^2([0,1]\,;\mathcal{H})$.*

Relation (3.10) shows that $X^\varepsilon := (X^{\varepsilon,\theta})_{\theta\in[0,1]}$ and $K^\varepsilon := (K^{\varepsilon,\theta})_{\theta\in[0,1]}$ are elements of $W^{1,2}([0,1]\,;\mathcal{H})$; moreover, $\nabla_\theta X^\varepsilon = Y^{\varepsilon,\theta}$ and

$$\nabla_\theta K^\varepsilon = \int_0^{\cdot} \beta'_\varepsilon(X_s^{\varepsilon,\theta})(\nabla_\theta X^\varepsilon)_s ds,\ \theta \in [0,1]\,. \tag{4.6}$$

Also, X^ε and K^ε converge in $L^2([0,1]\,;\mathcal{H})$ to $X := (X^\theta)_{\theta\in[0,1]}$ and $K := (K^\theta)_{\theta\in[0,1]}$, respectively. By Lemma 4.2, it follows that ∇X^ε and ∇K^ε converge weakly in $L^2([0,1]\,;\mathcal{H})$ to ∇X, respectively ∇K. Of course ∇X and ∇K are null on $[-\delta, 0]$.

By passing to the limit in equation (3.9) and using some *a priori* estimates, we have the following preliminary result:

Lemma 4.3. *We have $\nu_T(dt)d\mathbb{P}d\theta$-a.e.*

$$\begin{aligned}(\nabla_\theta X)_t + (\nabla_\theta K)_t = &\int_0^t \left[(\partial_y b_s^\theta)R(\nabla_\theta X)_s + (\partial_u b_s^\theta)(u_t^1 - u_t^0)\right] ds \\ &+ \int_0^t \left\langle (\partial_y \sigma_s^\theta)R(\nabla_\theta X)_s + (\partial_u \sigma_s^\theta)(u_t^1 - u_t^0), dW_s \right\rangle .\end{aligned} \tag{4.7}$$

Moreover, under conditions (S$_1$)-(S$_2$), $\nabla_\theta X$ can be chosen to be càdlàg, $\nabla_\theta K$ with bounded variation and càdlàg, satisfying

$$\underset{\theta\in[0,1]}{\operatorname{esssup}} \mathbb{E}\left[\|\nabla_\theta X\|_{0,T}^4 + \|\nabla_\theta K\|_{BV[0,T]}^4\right] < +\infty.$$

Proof. Relation (4.7) can be easily derived by passing to the limit in equation (3.9), thanks to the continuity and the linearity of deterministic and stochastic integrals on $L^2_{\mathbb{F}}(\Omega \times [-\delta, T])$. Let us first give some estimates on the limit processes. In this regard, we remark first that, slightly modifying the proof of Theorem 2.1 in [1] we obtain

$$\mathbb{E}\left[\|X^{\varepsilon,\theta}\|_t^8 + \|K^{\varepsilon,\theta}\|_t^8\right] \leq C, \tag{4.8}$$

for every $\varepsilon > 0$, $\theta \in [0,1]$. Here and below C denotes a positive constant, independent of ε and θ, which is allowed to change from line to line. By the positiveness of β'_ε and by applying Itô's formula to $|Y_t^{\varepsilon,\theta}|^2$, we obtain by the previous inequality that

$$\mathbb{E}\left[\|Y^{\varepsilon,\theta}\|_T^4 + \left(\int_0^T \beta'_\varepsilon(X_t^{\varepsilon,\theta})|Y_t^{\varepsilon,\theta}|^2 dt\right)^4\right] \leq C, \tag{4.9}$$

for every $\varepsilon > 0$, $\theta \in [0,1]$. Let B_ε be an anti-derivative of β_ε. Then, by Itô's formula applied to $B_\varepsilon(X_t^{\varepsilon,\theta})$, we get

$$\begin{aligned}\frac{1}{2}\int_0^T \beta_\varepsilon'(X_s^{\varepsilon,\theta})\left|\sigma(s,R(X^{\varepsilon,\theta})_s,u_s^\theta)\right|^2 ds &= \int_0^T \beta_\varepsilon(X_s^{\varepsilon,\theta})\left[\beta_\varepsilon(X_s^{\varepsilon,\theta}) - b(s,R(X^{\varepsilon,\theta})_s,u_s^\theta)\right] ds \\ &\quad - \int_0^T \left\langle \beta_\varepsilon(X_s^{\varepsilon,\theta})\sigma(s,R(X^{\varepsilon,\theta})_s,u_s^\theta), dW_s\right\rangle \\ &\quad + B_\varepsilon(X_T^{\varepsilon,\theta}) - B_\varepsilon(\eta(0)).\end{aligned}$$

Since (S_1) holds and (β_ε') is uniformly bounded by Lemma 3.6, by relation (4.8) and (S_2), we now obtain

$$\mathbb{E}\bigg(\int_0^T \beta_\varepsilon'(X_s^{\varepsilon,\theta})ds\bigg)^4 \le C.$$

Hence, by (4.9),

$$\mathbb{E}\bigg(\int_0^T \beta_\varepsilon'(X_t^{\varepsilon,\theta})|Y_t^{\varepsilon,\theta}|dt\bigg)^4 \le C, \tag{4.10}$$

for every $\varepsilon > 0$, $\theta \in [0,1]$. Since the convergence of equation (3.9) to relation (4.7) is obtained only almost everywhere, we need to look at the regularity properties of $\nabla_\theta X$ and $\nabla_\theta K$. For this purpose, we introduce the auxiliary processes

$$\begin{aligned}\bar{H}_t^{\varepsilon,\theta} &:= \int_0^t \beta_\varepsilon'(X_s^{\varepsilon,\theta})(Y_s^{\varepsilon,\theta})^+ ds,\ t\in[0,T]\,; \\ \underline{H}_t^{\varepsilon,\theta} &:= \int_0^t \beta_\varepsilon'(X_s^{\varepsilon,\theta})(Y_s^{\varepsilon,\theta})^- ds,\ t\in[0,T]\,.\end{aligned}$$

We observe that $\bar{H}^\varepsilon := (\bar{H}^{\varepsilon,\theta})_{\theta\in[0,1]}$ and $\underline{H}^\varepsilon := (\underline{H}^{\varepsilon,\theta})_{\theta\in[0,1]}$ are bounded in $L^2([0,1]\,;\mathcal{H})$ with respect to $\varepsilon > 0$ (see (4.10)). Hence, there exists a sequence $\varepsilon_n \searrow 0$ such that $(\bar{H}^{\varepsilon_n})_{n\ge1}$ and $(\underline{H}^{\varepsilon_n})_{n\ge1}$ converge weakly in $L^2([0,1]\,;\mathcal{H})$ to some elements $\bar{H}$, respectively $\underline{H}$. Since $\nabla K^{\varepsilon_n} = \bar{H}^{\varepsilon_n} - \underline{H}^{\varepsilon_n}$, it follows that $\nabla K = \bar{H} - \underline{H}$. The processes $\bar{H}^{\varepsilon,\theta}$ and $\underline{H}^{\varepsilon,\theta}$, being increasing and positive for every $\varepsilon > 0$ and $\theta\in[0,1]$, $\bar{H}(\theta)$ and $\underline{H}(\theta)$ admit càdlàg, progressively measurable, increasing and positive modifications for almost $\theta \in [0,1]$. Therefore we can suppose that $\underline{H}(\theta)$ and $\bar{H}(\theta)$ are positive, increasing and càdlàg. Consequently, $\nabla_\theta K$ is a càdlàg process with bounded variation for almost all $\theta\in[0,1]$. This implies the existence of a càdlàg modification of $\nabla_\theta X$ satisfying relation (4.7). By (4.10) we have

$$\mathbb{E}(\bar{H}_T^{\varepsilon,\theta})^4 + \mathbb{E}(\underline{H}_T^{\varepsilon,\theta})^4 \le C,\ \forall\theta\in[0,1]\,.$$

From Mazur's Lemma, we can choose a convex combination of elements of a subsequence $(\underline{H}^{\varepsilon_n})_{n\ge1}$ (with $\varepsilon_n \searrow 0$),

$$\textstyle\sum_{k=n}^{N_n} \alpha_k^n \underline{H}^{\varepsilon_k},\ \alpha_1^n + ... + \alpha_{N_n}^n = 1,\ \alpha_k^n \ge 0,\ \forall n\le k\le N_n,$$

converging in $L^2([0,1]\,;\mathcal{H})$ to $\underline{H}$. Therefore (at least on a subsequence), $\sum_{k=n}^{N_n}\alpha_k^n \underline{H}_T^{\varepsilon_k,\theta}$ converges $d\mathbb{P}d\theta$-a.e. to $\underline{H}_T^\theta$. By Fatou's lemma, $d\theta$-a.e.,

$$\mathbb{E}(\underline{H}_T^\theta)^4 \le \liminf_{n\to+\infty}\mathbb{E}(\textstyle\sum_{k=n}^{N_n}\alpha_k^n \underline{H}_T^{\varepsilon_k,\theta})^4 \le \liminf_{n\to+\infty}\sum_{k=n}^{N_n}\alpha_k^n\mathbb{E}(\underline{H}_T^{\varepsilon_k,\theta})^4 \le C. \tag{4.11}$$

In a similar manner we can prove that

$$\operatorname*{ess\,sup}_{\theta\in[0,1]}\mathbb{E}(\bar{H}_T^\theta)^4 < +\infty;$$

so finally we get

$$\mathbb{E}\left\|\nabla_\theta K\right\|_{BV[0,T]}^4 \le C,\ d\theta\text{-a.e.}$$

By (4.7) we have:

$$\|\nabla_\theta X\|_t \leq \underline{H}_t^\theta + C\int_0^t \left[|(\nabla_\theta X)_s| + (1+|X_s^\theta|^2)\left|u_s^1 - u_s^0\right|\right] ds \\ + \sup_{s\in[0,t]}\left|\int_0^s \left\langle(\partial_y\sigma_r^\theta)R(\nabla_\theta X)_r + (\partial_u\sigma_r^\theta)(u_r^1-u_r^0), dW_r\right\rangle\right|.$$

Let, for $n \in \mathbb{N}^*$,

$$\tau_n := \inf\left\{t\in[0,T] \mid |(\nabla_\theta X)_t| > n\right\}.$$

By estimate (4.11) and Burkholder-Davis-Gundy inequality we obtain

$$\mathbb{E}\,\|\nabla_\theta X\|_{\tau_n\wedge t}^4 \leq C + C\int_0^t \mathbb{E}\,\|\nabla_\theta X\|_{\tau_n\wedge s}^4\, ds$$

for every $t\in[0,T]$ and $n\in\mathbb{N}^*$. Therefore (since $\mathbb{E}\,\|\nabla_\theta X\|_{\tau_n}^4$ is finite), applying Gronwall's inequality and letting $n\to\infty$, we get

$$\mathbb{E}\,\|\nabla_\theta X\|_T^4 \leq C,\ d\theta\text{-a.e.} \tag{4.12}$$

□

We wish to transform relation (4.7) into a SDE by identifying $\nabla_\theta K$; therefore we should be able to pass to the limit in relation (4.6). Since the weak convergence of $\beta_\varepsilon(a)da$ to $\mu(da)$ is involved, it is sufficient to impose that $t\mapsto\sigma(t,R(X^\theta)_t,u_t^\theta)$ has at most countable many points of discontinuities, which is of course ensured under our current assumption that u^0 and u^1 are càdlàg processes.

Lemma 4.4. *Let*

$$\tilde{K}_t^{\varepsilon,\theta} := \int_0^t \beta_\varepsilon(X_s^\theta)ds,\ t\in[0,T].$$

Suppose that conditions (S$_1$)-(S$_2$) hold. Then, $\tilde{K}^\varepsilon \in W^{1,2}([0,1];\mathcal{H})$,

$$\nabla_\theta\tilde{K}^\varepsilon = \int_0^\cdot \beta'_\varepsilon(X_s^\theta)(\nabla_\theta X)_s ds \tag{4.13}$$

and $\tilde{K}^\varepsilon$ *converges weakly to* K *in* $W^{1,2}([0,1];\mathcal{H})$.

Proof. For $\varepsilon,\ \delta>0$, let $\tilde{K}^{\varepsilon,\delta,\theta} := \int_0^\cdot \beta_\varepsilon(X_s^{\delta,\theta})ds$. Then $\tilde{K}^{\varepsilon,\delta}\in W^{1,2}([0,1];\mathcal{H})$ and $\nabla_\theta\tilde{K}^{\varepsilon,\delta} = \int_0^\cdot \beta'_\varepsilon(X_s^{\delta,\theta})Y_s^{\delta,\theta}ds$. Moreover, for every $\varepsilon>0$, $\lim_{\delta\to0}\tilde{K}^{\varepsilon,\delta} = \tilde{K}^\varepsilon$ in $L^2([0,1];\mathcal{H})$ and $(\nabla\tilde{K}^{\varepsilon,\delta})_{\delta>0}$ is bounded in $L^2([0,1];\mathcal{H})$. The application of Lemma 4.2 concludes the proof of the first part. Since $\beta_\varepsilon(x)\to(\partial\varphi)^0(x) := \inf\{|y| \mid y\in\partial\varphi(x)\}$ for every $x\in\mathbb{R}$ (see [7], for example), it follows that

$$\tilde{K}_t^{\varepsilon,\theta} \to K_t^\theta = \int_0^t (\partial\varphi)^0(X_s^\theta)ds,\ \forall t\in[0,T],\ d\mathbb{P}d\theta\text{-a.e.}$$

Consequently, by Lebesgue's dominated convergence theorem, $\tilde{K}^\varepsilon$ converges to K in $L^2([0,1];\mathcal{H})$. In order to prove the weak convergence of $\tilde{K}^\varepsilon$ to K in $W^{1,2}([0,1];\mathcal{H})$, it is sufficient to show that $(\nabla\tilde{K}^\varepsilon)_{\varepsilon>0}$ is bounded in $L^2([0,1];\mathcal{H})$. By Itô's formula applied to $B_\varepsilon(X^\theta)$ (B_ε was defined in the proof of Lemma 4.3 as the anti-derivative of β_ε), we obtain

$$\frac{1}{2}\int_0^T \beta'_\varepsilon(X_s^\theta)\left|\sigma(s,R(X^\theta)_s,u_s^\theta)\right|^2 ds = \int_0^T \beta_\varepsilon(X_s^\theta)\left[(\partial\varphi)^0(X_s^\theta) - b(s,R(X^\theta)_s,u_s^\theta)\right]ds \\ - \int_0^T \left\langle\beta_\varepsilon(X_s^\theta)\sigma(s,R(X^\theta)_s,u_s^\theta), dW_s\right\rangle \\ + B_\varepsilon(X_T^\theta) - B_\varepsilon(\eta(0)),$$

from which we get the uniform boundedness of $\left(\int_0^T \beta'_\varepsilon(X_s^\theta)ds\right)_{\varepsilon>0}$ in $L^4(\Omega)$ with respect to θ. Relation (4.12), used together with (4.13), shows that $(\nabla\tilde{K}^\varepsilon)_{\varepsilon>0}$ is bounded in $L^2([0,1]\,;\mathcal{H})$. Applying once again Lemma 4.2, we obtain the weak convergence of $\nabla\tilde{K}^\varepsilon$ to ∇K in $L^2([0,1]\,;\mathcal{H})$ for $\varepsilon\to 0$. $\square$

Lemma 4.5. *Suppose that u^0 and u^1 are càdlàg. Under (S_1)-(S_2), the derivative of the mapping $\theta\longmapsto K^\theta$ in $W^{1,2}([0,1]\,;\mathcal{H})$ is given by:*

$$(\nabla_\theta K)_t = \int_0^t (\nabla_\theta X)_s dA_s^\theta,\ \nu_T(dt)d\mathbb{P}d\theta\text{-a.e.} \tag{4.14}$$

Proof. Let μ_ε be the measure on $\mathbb{R}$ defined by its density $\beta'_\varepsilon(a)$. Then $(\mu_\varepsilon)_{\varepsilon>0}$ converges weakly to μ. The continuity of $L_t^{a,\theta}$ in $a\in\mathbb{R}$ implies the weak continuity of measures $L^{a,\theta}(ds)$ in a. Since $s\longmapsto\sigma(s,R(X^\theta)_s,u_s^\theta)$ has a countable number of discontinuities, we obtain the continuity in a of $\int_0^t \frac{L^{a,\theta}(ds)}{|\sigma(s,R(X^\theta)_s,u_s^\theta)|^2}$, for every $t\in[0,T]$, $d\mathbb{P}d\theta$-a.e. We can find a compact interval I such that:

- $\mu_{\varepsilon|I}$ converges weakly to μ_I (see [5]);
- $\beta'_\varepsilon\le\varepsilon$ on I^{c};
- the support of μ is included in I.

It follows that $\int_I\int_0^t \frac{L^{a,\theta}(ds)}{|\sigma(s,R(X^\theta)_s,u_s^\theta)|^2}\mu_\varepsilon(da)$ converges to A_t^θ. Let

$$A_t^{\varepsilon,\theta} := \int_0^t \beta'_\varepsilon(X_s^\theta)ds,\ t\in[0,T]\,.$$

By the occupation time density formula (Lemma 3.5) we derive the equality

$$A_t^{\varepsilon,\theta} = \int_{\mathbb{R}}\int_0^t \frac{L^{a,\theta}(ds)}{\left|\sigma(s,R(X^\theta)_s,u_s^\theta)\right|^2}\beta'_\varepsilon(a)da.$$

Since

$$\int_{I^{\mathrm{c}}}\int_0^t \frac{L^{a,\theta}(ds)}{\left|\sigma(s,R(X^\theta)_s,u_s^\theta)\right|^2}\beta'_\varepsilon(a)da \le \frac{\varepsilon}{\alpha^2}\int_{I^{\mathrm{c}}} L_t^{a,\theta}da \le \frac{\varepsilon}{\alpha^2}\int_0^t \left|\sigma(s,R(X^\theta)_s,u_s^\theta)\right|^2 ds,$$

we obtain

$$\lim_{\varepsilon\to 0} A_t^{\varepsilon,\theta} = A_t^\theta, \forall t\in[0,T]\,,\ \text{a.s.},\ \forall\theta\in[0,1]\,.$$

Consequently, from (4.13),

$$(\nabla_\theta\tilde{K}^\varepsilon)_t = \int_0^t (\nabla_\theta X)_s dA_s^{\varepsilon,\theta}.$$

Since the set of discontinuity points of the function $t\mapsto(\nabla_\theta X)_t$ is at most countable (recall that $\nabla_\theta X$ is càdlàg), it follows that

$$\lim_{\varepsilon\to 0}(\nabla_\theta\tilde{K}^\varepsilon)_t = \int_0^t (\nabla_\theta X)_s dA_s^\theta,\ \forall t\in[0,T]\,,\ d\mathbb{P}d\theta\text{-a.e.}$$

On the other hand, $(\nabla_\theta\tilde{K}^\varepsilon)_{\varepsilon>0}$ converges weakly to $\nabla_\theta K$ in $L^2([0,1]\,;\mathcal{H})$. Therefore relation (4.14) is satisfied. $\square$

Relations (4.7) and (4.14), combined with the uniqueness of the solution of equation (3.11), give

$$(\nabla_\theta X)_t = Y_t^\theta,\ \forall t \in [0,T]\,,\ d\mathbb{P}d\theta\text{-a.e.} \tag{4.15}$$

Let us finally pass to the proof of Proposition 3.8.

Step I. Let us first suppose that (S_1)-(S_2) hold. By Itô's formula applied to $|Y_t^\theta|^2$ we obtain $Y^\theta \in L^4_{\mathbb{F}}(\Omega; C\,[0,T])$.

Since we already have obtained (4.15), in order to prove that X is everywhere derivable with respect to θ, it is sufficient to show that the function $[0,1] \ni \theta \mapsto Y^\theta \in \mathcal{H}$ is continuous.

For this, let, as in the proof of Proposition 3.7, $\tau_n^\theta := \inf\left\{t \in [0,T] \mid \left|A_t^\theta\right| > n\right\}$, $A_t^{n,\theta} := A_{t\wedge\tau_n}$ and $Y^{n,\theta} \in L^2_{\mathbb{F}}(\Omega; C\,[-\delta,T])$ solving the modified equation (3.13). Then $Y_t^{n,\theta} = Y_t^\theta$ on $\left[0,\tau_n^\theta\right]$.

By applying Itô's formula to $|Y_t^{n,\theta}|^2$, respectively to $|Y_t^\theta|^2$, we obtain the estimate

$$\mathbb{E}\|Y^{n,\theta}\|_T^4 + \mathbb{E}\|Y^\theta\|_T^4 \le C,$$

where C is a constant independent of n and θ.

Let $\theta_0 \in [0,1]$. Since, by Proposition 4.1,

$$\lim_{\theta\to\theta_0} \mathbb{E}|A_t^\theta - A_t^{\theta_0}|^4 = 0,\ \forall t \in [0,T]$$

and $A_t^{n,\theta}$ is bounded by a constant, standard estimates allow to show that

$$\lim_{\theta\to\theta_0} \mathbb{E}|Y_t^{n,\theta} - Y_t^{n,\theta_0}|^2 = 0,\ \forall t \in [0,T]\,. \tag{4.16}$$

Indeed, returning to the proof of Proposition 3.7, we see that $\bar{Y}^{n,\theta} := \mathrm{e}^{A^{n,\theta}_\cdot}\, Y^{n,\theta}$ is a solution of the transformed equation (3.12); since A^θ and A^{θ_0} appear only as integrands, one can establish without difficulty that

$$\lim_{\theta\to\theta_0} \mathbb{E}|\bar{Y}_t^{n,\theta} - \bar{Y}_t^{n,\theta_0}|^2 = 0,\ \forall t \in [0,T]\,,$$

by applying Itô's formula to $|\bar{Y}_t^{n,\theta} - \bar{Y}_t^{n,\theta_0}|^2$. Relation (4.16) then follows easily.

On the other hand,

$$\begin{aligned}\mathbb{E}|Y_t^{n,\theta} - Y_t^\theta|^2 &= \mathbb{E}|Y_t^{n,\theta} - Y_t^\theta|^2 \mathbf{1}_{\{\tau_n^\theta > t\}} \le C^{1/2}\left[\mathbb{P}(\tau_n^\theta > t)\right]^{1/2} \\ &\le C^{1/2}\left[\mathbb{P}(A^{n,\theta} > n)\right]^{1/2} \le \frac{C^{1/2}}{n}(\mathbb{E}|A_t^\theta|^2)^{1/2}.\end{aligned}$$

Hence

$$\mathbb{E}|Y_t^\theta - Y_t^{\theta_0}|^2 \le \frac{3C^{1/2}}{n}\left[(\mathbb{E}|A_t^\theta|^2)^{1/2} + (\mathbb{E}|A_t^{\theta_0}|^2)^{1/2}\right] + 3\mathbb{E}|Y_t^{n,\theta} - Y_t^{n,\theta_0}|^2.$$

Passing to the limit as $\theta \to \theta_0$ and $n \to \infty$, we get

$$\lim_{\theta\to\theta_0} \mathbb{E}|Y_t^\theta - Y_t^{\theta_0}|^2 = 0,\ \forall t \in [0,T]\,.$$

This proves that $Y \in C([0,1]\,;\mathcal{H})$.

Since $Y = \nabla X$, it follows that X is differentiable on $[0,1]$, *i.e.*

$$\lim_{\theta\to\theta_0} \mathbb{E}\left[\int_0^T \left|\frac{X_t^\theta - X_t^{\theta_0}}{\theta-\theta_0} - Y_t^{\theta_0}\right|^2 dt + \left|\frac{X_T^\theta - X_T^{\theta_0}}{\theta-\theta_0} - Y_T^{\theta_0}\right|^2\right] = 0,\ \forall\theta_0 \in [0,1].$$

In particular, for $\theta_0 = 0$ we obtain (3.14), which ends the proof under the conditions imposed on φ and σ.

We pass now to the general case.

Step II. We set, for $\gamma > 0$,

$$\varphi_\gamma(x) := \begin{cases} \varphi(-\gamma) + \varphi'_-(-\gamma)(x+\gamma), & x < -\gamma; \\ \varphi(x), & x \in [-\gamma, \gamma]\,; \\ \varphi(\gamma) + \varphi'_+(\gamma)(x-\gamma), & x > \gamma \end{cases}$$

and we choose σ_γ satisfying (S_2), (H_0) and (H_1) (with the same constants) such that $\sigma_\gamma(t,y,u) = \sigma(t,y,u)$ for every $t \in [0,T]$, $u \in U$ and $|y| \le \gamma|\lambda|([-\delta,0])$.

Let $(X^{\gamma,\theta}, K^{\gamma,\theta})$ be the solution of the equation

$$\begin{cases} dX_t^{\gamma,\theta} + \partial\varphi_\gamma(X_t^{\gamma,\theta})dt = b(t, R(X^{\gamma,\theta})_t, u_t^\theta)dt \\ \qquad\qquad\qquad + \left\langle \sigma_\gamma(t, R(X^{\gamma,\theta})_t, u_t^\theta), dW_t \right\rangle, \; t \in [0,T]\,; \\ X_t^{\gamma,\theta} = \eta(t), \; t \in [-\delta, 0]\,. \end{cases} \tag{4.17}$$

We define, for every $\gamma \ge \|\eta\|_{-\delta,0}$, the stopping time (with convention $\inf\emptyset = +\infty$)

$$\tau_\gamma^\theta := \inf\left\{t \in [0,T] \mid |X_t^\theta| \ge \gamma\right\}.$$

By the uniqueness of the solution of equation (2.1) we obtain

$$X_t^{\gamma,\theta} = X_t^\theta \text{ and } K_t^{\gamma,\theta} = K_t^\theta, \, \forall t \in \left[0, \tau_\gamma^\theta\right], \text{ a.s.}$$

Moreover, with $L_t^{a,\theta,\gamma}$ denoting the local time of $X^{\gamma,\theta}$, the application of Proposition 3.2 gives

$$\begin{aligned} &L_t^{a,\theta} = L_t^{a,\gamma,\theta}, \, \forall t \in \left[0, \tau_\gamma^\theta\right], \text{ a.s. and} \\ &L_t^{a,\theta} = 0, \, \forall t \in \left[0, \tau_\gamma^\theta\right], \text{ a.s., if } a \notin [-\gamma, \gamma]\,. \end{aligned}$$

We introduce now the process

$$A_t^{\gamma,\theta} := \int_{\mathbb{R}} \int_0^t \frac{L^{a,\gamma,\theta}(ds)}{|\sigma_\gamma(s, R(X^\theta)_s, u_s^\theta)|^2} \mu_\gamma(da), \, t \in [0,T]\,,$$

where $\mu_\gamma := \mu(\cdot \cap [-\gamma, \gamma])$ is the second order generalized derivative of φ_γ. Clearly,

$$A_t^\theta = A_t^{\gamma,\theta}, \, \forall t \in \left[0, \tau_\gamma^\theta\right], \text{ a.s.}$$

By the first step of the proof, the equation

$$\begin{cases} dY_t^\gamma + Y_t^\gamma dA_t^{\gamma,0} = \left[(\partial_y b_t^{\gamma,0})(R(Y^\gamma)_t) + (\partial_u b_t^{\gamma,0})(u_t^1 - u_t^0)\right]dt \\ \qquad\qquad + \left\langle (\partial_y \sigma_t^{\gamma,0}) R(Y^\gamma)_t + \langle (\partial_u \sigma_t^{\gamma,0})(u_t^1 - u_t^0), dW_t \right\rangle, \, t \in [0,T]\,; \\ Y_t^\gamma = 0, \; t \in [-\delta, 0] \end{cases} \tag{4.18}$$

has a unique solution $Y^\gamma \in L^2_{\mathbb{F}}(\Omega; C\,[-\delta, T])$ and

$$\lim_{\theta \to 0} \mathbb{E}\left[\int_0^T \left|\frac{X_t^{\gamma,\theta} - X_t^{\gamma,0}}{\theta} - Y_t^\gamma\right|^2 dt + \left|\frac{X_T^{\gamma,\theta} - X_T^{\gamma,0}}{\theta} - Y_T^\gamma\right|^2\right] = 0. \tag{4.19}$$

If $\gamma' > \gamma$, then $\tau_{\gamma'}^\theta \ge \tau_\gamma^\theta$ and we can apply Itô's formula to $|Y^\gamma_{t\wedge\tau_\gamma^\theta} - Y^{\gamma'}_{t\wedge\tau_\gamma^\theta}|^2$ in order to obtain

$$Y_t^\gamma = Y_t^{\gamma'}, \, \forall t \in \left[0, \tau_\gamma^\theta\right], \text{ a.s.}$$

Therefore, we can uniquely define an adapted continuous process satisfying

$$Y_t = Y_t^\gamma,\ \forall t \in [0, \tau_\gamma^0].$$

Since

$$\sup_{\gamma>0} \mathbb{E}\, \|Y^\gamma\|_T^2 < +\infty$$

(see the previous step), the process Y belongs to $L^2_{\mathbb{F}}(\Omega; C\,[0,T])$. Moreover, Y is a solution of the following equation:

$$\begin{cases} dY_t + Y_t dA_t^0 = \left[(\partial_y b_t^0) R(Y)_t + (\partial_u b_t^0)(u_t^1 - u_t^0)\right] dt \\ \qquad\qquad\qquad\qquad\qquad + \left\langle (\partial_y \sigma_t^0) R(Y)_t + (\partial_u \sigma_t^0)(u_t^1 - u_t^0), dW_t \right\rangle,\ t \in [0,T]\,; \\ \\ Y_t = 0,\ t \in [-\delta, 0]\,. \end{cases}$$

By the uniqueness of the solution of this equation, $Y = Y^0$. Then, for $t \in [0, \tau_\gamma^0]$:

$$\begin{aligned} \left|\tfrac{1}{\theta}(X_t^\theta - X_t^0) - Y_t\right| \leq& \left|\tfrac{1}{\theta}(X_t^{\gamma,\theta} - X_t^{\gamma,0}) - Y_t^\gamma\right| + \left|\tfrac{1}{\theta}(X_t^{\gamma,\theta} - X_t^\theta)\right| \\ =& \left|\tfrac{1}{\theta}(X_t^{\gamma,\theta} - X_t^{\gamma,0}) - Y_t^\gamma\right| + \left|\tfrac{1}{\theta}(X_t^{\gamma,\theta} - X_t^\theta)\right| \mathbf{1}_{\{t>\tau_\gamma^\theta\}} \\ \leq& \left|\tfrac{1}{\theta}(X_t^{\gamma,\theta} - X_t^{\gamma,0}) - Y_t^\gamma\right| + \tfrac{1}{\theta}(|X_t^{\gamma,\theta} - X_t^{\gamma,0}| \\ & + |X_t^\theta - X_t^0|)\mathbf{1}_{\{t>\tau_\gamma^\theta\}}. \end{aligned} \tag{4.20}$$

Since $\mathbb{E}\|X^\theta - X^0\|_T^2$ converges to 0 (by Proposition 2.4), we have

$$\lim_{\theta\to 0} \mathbb{P}(\tau_\gamma^0 > t > \tau_\gamma^\theta) = 0,\ \forall t \in [0,T]. \tag{4.21}$$

Indeed,

$$\begin{aligned} \mathbb{P}\left(\tau_\gamma^0 > t > \tau_\gamma^\theta\right) \leq& \mathbb{P}\left(\|X^0\|_t < \gamma \leq \|X^\theta\|_t\right) \leq \mathbb{P}\left(\|X^0\|_t + \tfrac{1}{n} < \gamma \leq \|X^\theta\|_t\right) \\ & + \mathbb{P}\left(\gamma - \tfrac{1}{n} \leq \|X^0\|_t < \gamma\right) \\ \leq& \mathbb{P}\left(\tfrac{1}{n} \leq \|X^\theta - X^0\|_t\right) + \mathbb{P}\left(\gamma - \tfrac{1}{n} \leq \|X^0\|_t < \gamma\right) \\ \leq& n^2 \mathbb{E}\|X^\theta - X^0\|_T^2 + \mathbb{P}\left(\gamma - \tfrac{1}{n} \leq \|X^0\|_t < \gamma\right), \end{aligned}$$

for every $\theta \in [0,1]$ and $n \in \mathbb{N}^*$. Hence

$$\limsup_{\theta\to 0} \mathbb{P}\left(\tau_\gamma^0 > t > \tau_\gamma^\theta\right) \leq \mathbb{P}\left(\gamma - \tfrac{1}{n} \leq \|X^0\|_t < \gamma\right),\ \forall n \in \mathbb{N}^*,$$

from which we get relation (4.21).

Proposition 2.4 shows that

$$\mathbb{E}\left[\left\|\frac{X^{\gamma,\theta} - X^{\gamma,0}}{\theta}\right\|_T^2 + \left\|\frac{X^\theta - X^0}{\theta}\right\|_T^2\right]$$

is uniformly bounded with respect to $\theta \in [0,1]$ (the Lipschitz constant of σ_γ is the same for all γ). Hence, by (4.19), (4.20) and (4.21) we derive, for each $\gamma > 0$,

$$\begin{aligned} &\limsup_{\theta\to 0} \mathbb{E}\left[\int_0^T \left|\frac{X_t^\theta - X_t^0}{\theta} - Y_t\right|^2 dt + \left|\frac{X_T^\theta - X_T^0}{\theta} - Y_T\right|^2\right] \mathbf{1}_{\{\tau_\gamma^0 > T\}} \\ &\leq 2 \limsup_{\theta\to 0} \mathbb{E}\left[\int_0^T \left|\frac{X_t^{\gamma,\theta} - X_t^{\gamma,0}}{\theta} - Y_t^\gamma\right|^2 dt + \left|\frac{X_T^{\gamma,\theta} - X_T^{\gamma,0}}{\theta} - Y_T\right|^2\right] \mathbf{1}_{\{\tau_\gamma^0 > T\}} \\ &\quad + 4 \limsup_{\theta\to 0} \mathbb{E} \int_0^T \frac{|X_t^{\gamma,\theta} - X_t^{\gamma,0}|^2 + |X_t^\theta - X_t^0|^2}{\theta^2} \mathbf{1}_{\{\tau_\gamma^0 > t > \tau_\gamma^\theta\}} dt \\ &\quad + 4 \limsup_{\theta\to 0} \mathbb{E} \int_0^T \frac{|X_T^{\gamma,\theta} - X_T^{\gamma,0}| + |X_T^\theta - X_T^0|}{\theta^2} \mathbf{1}_{\{\tau_\gamma^0 > T > \tau_\gamma^\theta\}} dt \\ &= 0. \end{aligned}$$

Consequently, since $\lim_{\gamma\to+\infty}\mathbb{P}(\tau_\gamma^0 \le T)=0$, we obtain

$$\lim_{\theta\to 0}\mathbb{E}\left[\int_0^T\left|\frac{X_t^\theta - X_t^0}{\theta}-Y_t\right|^2 dt+\left|\frac{X_T^\theta - X_T^0}{\theta}-Y_T\right|^2\right]=0, \tag{4.22}$$

which ends the proof of this result.

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