From Sine kernel to Poisson statistics

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Abstract

We study the Sine$_\beta$ process introduced in [B. Valkó and B. Virág. Invent. math. **177** 463-508 (2009)] when the inverse temperature β tends to 0. This point process has been shown to be the scaling limit of the eigenvalues point process in the bulk of β-ensembles and its law is characterized in terms of the winding numbers of the Brownian carrousel at different angular speeds. After a careful analysis of this family of coupled diffusion processes, we prove that the Sine$_\beta$ point process converges weakly to a Poisson point process on $\mathbb{R}$. Thus, the Sine$_\beta$ point processes establish a smooth crossover between the rigid clock (or picket fence) process (corresponding to $\beta = \infty$) and the Poisson process.

1 Introduction and main result

Although random matrices were originally introduced by John Wishart [24] in 1928 as a tool to study population dynamics in biology through principal component analysis, they became very popular much later in 1951 when Wigner [23] postulated that the fluctuations in positions of the energy levels of heavy nuclei are well described (in terms of statistical properties) by the eigenvalues of a very large Hermitian random matrix. Random matrix theory (RMT) is now an active research area in mathematics and theoretical physics with applications in statistics, biology, financial mathematics, engineering and telecommunications, number theory etc. (see [6, 7, 18, 13, 1] for a state of the art).

The classical models of Hermitian random matrices are the Gaussian orthogonal, unitary and symplectic ensembles. It is well known that the joint law of the eigenvalues of the matrices in those Gaussian ensembles is the Boltzmann-Gibbs equilibrium measure of a one-dimensional repulsive Coulomb gas confined in a harmonic well. More precisely, this joint law has a probability density $\mathcal{P}_\beta$ on $\mathbb{R}^N$ (N is the dimension of the square matrices) given by

$$\mathcal{P}_\beta(\lambda_1, \cdots, \lambda_N) = \frac{1}{Z_N} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \exp\left(-\frac{N\beta}{4} \sum_{i=1}^N \lambda_i^2\right) \tag{1.1}$$

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where the inverse temperature $\beta = 1$ for the Gaussian orthogonal ensemble, respectively $\beta = 2, 4$ for the unitary and symplectic ensembles. The linear statistics of the point processes with joint probability density functions (jpdf) $\mathcal{P}_\beta, \beta = 1, 2, 4$ have been extensively studied in the literature with different methods [6, 18].

In 2002, Dumitriu and Edelman [11] came up with a new explicit ensemble of random tri-diagonal matrices whose eigenvalues are distributed according to the jpdf $\mathcal{P}_\beta$ for any $\beta > 0$ (see also [5, 2] where invariant ensembles associated to general β were constructed).

Those tri-diagonal matrices have been very useful in the last decade, leading to important progress on the understanding of the local eigenvalues statistics in the limit of large dimension N for general $\beta > 0$. At the edge of the spectrum, it was first proved [20] that the largest eigenvalues converge jointly (when zooming in the edge-scaling region of width $N^{-2/3}$ around 2) to the low lying eigenvalues of a random Schrödinger operator called the stochastic Airy operator (see also [12]). Similar results were proved for the bulk in [22] by Valkó and Virág. For λ belonging to the Wigner sea $(-2, 2)$, the authors of [22] consider the point process

$$\Lambda_N := (2\pi N \rho(\lambda) (\lambda_i - \lambda))_{i=1,\cdots,N} \tag{1.2}$$

where $(\lambda_1, \cdots, \lambda_N)$ is distributed according to $\mathcal{P}_\beta$ and $\rho(\lambda) = \frac{1}{2\pi}\sqrt{4 - \lambda^2}$ is the Wigner semi-circle density. Indeed, the mean level spacing around level λ for the points with law $\mathcal{P}_\beta$ is approximately $1/(N\rho(\lambda))$ when $N \gg 1$. The mean point spacing of Λ_N defined in (1.2) is therefore of order 2π and in this scaling, one can now investigate the limiting statistics of this point process when $N \to \infty$. The authors of [22] precisely answer this question proving that the point process Λ_N converges in law [1] to a point process Sine_β on $\mathbb{R}$ first introduced in [22] and characterized in terms of a family $(\alpha_\lambda)_{\lambda \in \mathbb{R}}$ of coupled one-dimensional diffusion processes, the *stochastic sine equations*. As expected for the eigenvalues statistics in the bulk, the point process Sine_β is translation-invariant in law. The family of diffusions $(\alpha_\lambda)_{\lambda \in \mathbb{R}}$ can be interpreted as the hyperbolic angle of the Brownian carousel with parameter λ and its law is characterized as follows: Given a (driving) complex Brownian motion $(Z_t)_{t \geq 0}$, the diffusions $\alpha_\lambda, \lambda \in \mathbb{R}$ satisfy

$$d\alpha_\lambda = \lambda \frac{\beta}{4} e^{-\frac{\beta}{4}t} \, dt + \mathrm{Re}((e^{-i\alpha_\lambda} - 1)dZ_t), \quad \alpha_\lambda(0) = 0 \,. \tag{1.3}$$

Note that all the diffusions $\alpha_\lambda, \lambda \in \mathbb{R}$ are adapted to the filtration of the Brownian motion (Z_t). This coupling induces a strong interaction between the diffusions which makes the joint law difficult to analyse, as we shall see. A key feature shared by the processes $\alpha_\lambda, \lambda \in \mathbb{R}$ is that they all converge almost surely as $t \to \infty$ to a limit $\alpha_\lambda(\infty)$ which is an integer multiple of 2π. The characterization of the law of the Sine_β point process [2] can now be enunciated as follows

$$(\mathrm{Sine}_\beta([\lambda, \lambda']))_{\lambda < \lambda'} \stackrel{(d)}{=} \left(\frac{\alpha_{\lambda'}(\infty) - \alpha_\lambda(\infty)}{2\pi} \right)_{\lambda < \lambda'}. \tag{1.4}$$

In this paper, we are interested in the limiting law of the Sine_β process when the inverse temperature β goes to 0. Theorem 1.1 is the main result of this paper and gives the convergence as $\beta \to 0$ of the Sine_β process towards a Poisson point process on $\mathbb{R}$. This convergence at the continuous ($N = \infty$) level seems rather natural since taking $\beta \to 0$ amounts to decreasing the electrostatic repulsion (and hence the correlation) at the

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[1] The convergence is with respect to vague topology for the counting measure of the point process.

[2] If A is a Borel set of $\mathbb{R}$, $\mathrm{Sine}_\beta(A)$ inside A denotes the number of points inside A. In other words, Sine_β is the counting measure of the point process.
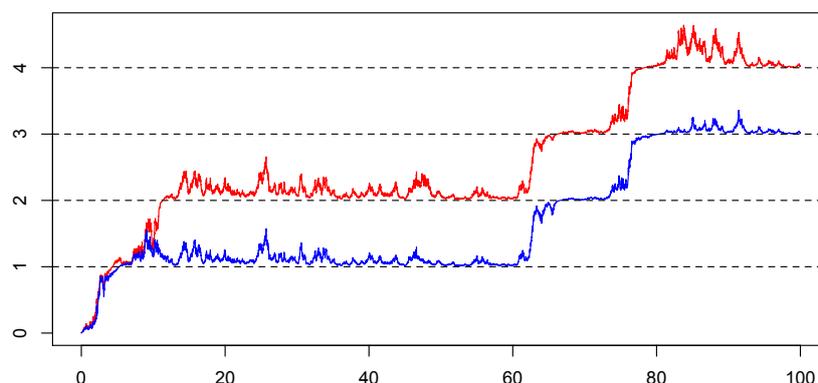
Figure 1: (Color online). Sample paths of two diffusions $\alpha_\lambda/(2\pi)$ (blue curve) and $\alpha_{\lambda'}/(2\pi)$ (red curve) for $\lambda < \lambda'$ and for a small value of β. We see that whenever $\lfloor \alpha_\lambda/(2\pi) \rfloor$ jumps, $\lfloor \alpha_{\lambda'}/(2\pi) \rfloor$ jumps at the same time in agreement with Lemma 3.4.

discrete level, i.e. between the N points distributed according to $\mathcal{P}_\beta$. If one takes $\beta = 0$ abruptly for fixed N, the probability density $\mathcal{P}_0$ corresponds (up to a rescaling depending on β) to the joint law of independent Gaussian variables and it is then straightforward to check the convergence (as $N \to \infty$) of the point process with law $\mathcal{P}_0$ towards a Poisson process. Theorem 1.1 exchanges the order of the limits $\beta \to 0$ and $N \to \infty$, describing the statistics when $N \to \infty$ *first* and then $\beta \to 0$. It also gives the precise rate of the convergence.

Theorem 1.1. *As $\beta \to 0$, the Sine$_\beta$ point process converges weakly in the space of Radon measure (equipped with the topology of vague convergence [15]) to a Poisson point process on $\mathbb{R}$ with intensity $\frac{d\lambda}{2\pi}$. In particular, we have, for any $k \in \mathbb{N}$ and $\lambda < \lambda'$,*

$$\mathbb{P}[\text{Sine}_\beta[\lambda, \lambda'] = k] \to_{\beta \to 0} \exp\left(\frac{\lambda - \lambda'}{2\pi}\right) \frac{(\lambda' - \lambda)^k}{(2\pi)^k k!},$$

and the numbers of points of Sine$_\beta$ inside two disjoint intervals are asymptotically independent.

Let us briefly discuss some implications of Theorem 1.1 and mention a few related questions on the spectral statistics of random matrices and random Schrödinger operators.

In [16], the authors have shown that the circular β-ensemble, which was later shown to be Sine$_\beta$ in [19], interpolates between Poisson and clock distributions on the circle (point process with rigid spacings like the numerals on a clock) by considering random CMV matrices. Theorem 1.1 provides a more precise description of this interpolating process on the Poisson process side.

In our study, we are led to examine a time homogeneous family of diffusions $(\theta_\lambda)_{\lambda \in \mathbb{R}}$ defined as

$$d\theta_\lambda = \lambda \frac{\beta}{4} dt + \text{Re}((e^{-i\theta_\lambda} - 1)dZ_t), \quad \theta_\lambda(0) = 0. \tag{1.5}$$

This family of coupled diffusions also appears in [17] to describe the law of the limiting point process of a certain critical random discrete Schrödinger operator. Our result can

be extended in this context to prove that this critical Schrödinger operator continuously interpolates between the extended (clock/picket fence) and localized (Poisson) regimes. More precisely, one could prove using our ideas that the random spectrum has Poissonian statistics in the limit of large temperature.

Let us also compare the results stated in Theorem 1.1 with those of a previous work [4] where we consider the stochastic Airy ensemble, Airy$_\beta$, obtained in the scaling limit of β-ensembles at the edge of the spectrum. In this context, we proved that the number of points Airy$_\beta(]-\infty, \lambda])$ inside the interval $]-\infty, \lambda]$ displays Poisson statistics in the small β limit. This permitted us to obtain the limiting distributions as $\beta \to 0$ of each of the lowest eigenvalues (individually) of the Airy$_\beta$ ensemble. In particular, we obtained the weak convergence of the $TW(\beta)$ distribution towards the Gumbel distribution. Although the Sine$_\beta$ and Airy$_\beta$ characterizations in law (in terms of a family of coupled diffusions) look very similar, the analysis of the limiting marginal statistics of the number of points inside a finite closed interval Airy$_\beta[\lambda, \lambda']$ for $\lambda < \lambda'$ and the asymptotic independence when $\beta \to 0$ of the respective numbers of points of the Airy$_\beta$ point process into two disjoint intervals remain open even after our study [4]. In this aspect, Theorem 1.1 gives a much more powerful and complete description of the Sine$_\beta$ process in the small β limit. In this case, we are able to prove the asymptotic independence between the number of points of the Sine$_\beta$ process in two disjoint intervals. This part of the proof requires new ideas in order to obtain estimates on the relative positions between two coupled diffusions α_λ and $\alpha_{\lambda'}$. The nice feature of the Sine$_\beta$ process is its translation invariance in law. This property makes the analysis of Sine$_\beta$ easier than the one of the Airy$_\beta$ process. The non-homogeneous intensity of the Airy$_\beta$ process is governed by the edge-scaling crossover spectral density of β-ensembles computed explicitly in [8, 14] for $\beta = 2$ (see also [4]).

Other spectral statistics of random matrices at high temperature, i.e. when $\beta \to 0$, have been investigated in [2, 3]. In [2], the authors study the empirical eigenvalue density in the limit of large dimension N for β-ensembles when β tends to 0 with N as $\beta = 2c/N$ where $c > 0$ is a constant. The authors compute the limiting spectral density $\rho_c(\lambda)$ explicitly in terms of parabolic cylinder function and establish a Gauss-Wigner crossover, in the sense that the family ρ_c interpolate between the Gaussian probability distribution ($c = 0$) and the Wigner semi-circle ($c \to +\infty$). The case of Gaussian Wishart matrices has also been studied in [3].

It would be interesting to have a description of the crossover statistics of the β-ensembles obtained in the double scaling limits when β tends to 0 with $N \to \infty$ (this question is briefly discussed in [4] for the statistics at the edge of the spectrum).

Organization of the paper. We start in section 2 by looking at the limiting marginal distributions of the random variables Sine$_\beta[\lambda, \lambda']$ for $\lambda < \lambda'$. We first study a classical problem on the exit time of a diffusion trapped in the well of a stationary potential (obtained by neglecting the slow evolution with time). Then, we prove that the jump process of $\lfloor \alpha_\lambda/(2\pi) \rfloor$ converges weakly to an (inhomogeneous) Poisson point process by first approximating α_λ with diffusions processes with piecewise constant drifts on a subdivision of small intervals and then by using the convergence of the exit time of the stationary well established previously. This requires estimates on the sample paths of a single diffusion, in a spirit similar to [4, 10]. In Section 3, we investigate the asymptotic independence of the numbers of points of Sine$_\beta$ in two disjoint intervals. We prove a crucial estimate regarding the typical relative positions of two diffusions α_λ and α'_λ for $\lambda < \lambda'$. Loosely speaking, the main point is to use this estimate to prove that, in the limit $\beta \to 0$, the jumps of the process $\lfloor \alpha_\lambda/(2\pi) \rfloor$ immediately follow those of $\lfloor \alpha_{\lambda'}/(2\pi) \rfloor$ (see Fig. 1) while the processes $\lfloor \alpha_\lambda/(2\pi) \rfloor$ and $\lfloor (\alpha_{\lambda'} - \alpha_\lambda)/(2\pi) \rfloor$ never jump at the same time (see Fig. 4). The asymptotic independence follows essentially from the fact that

two Poisson point processes adapted to the same filtration are independent if and only if they never jump simultaneously.

We gather in the next paragraph important properties of the family (α_λ) already established in [Section 2.2, [22]] that we will use throughout the paper.

First properties of the coupling of the diffusions α_λ:

(i) For all $\lambda < \lambda'$, $\alpha_{\lambda'} - \alpha_\lambda$ has the same distribution as $\alpha_{\lambda'-\lambda}$;

(ii) "Increasing property": $\alpha_\lambda(t)$ is increasing in λ;

(iii) $\lfloor \alpha_\lambda(t)/(2\pi) \rfloor$ is non-decreasing in t;

(iv) $\mathbb{E}[\alpha_\lambda(t)] = \lambda \frac{\beta}{4} \int_0^t e^{-\beta s/4} ds$;

(v) $\lim_{t\to\infty} \alpha_\lambda(t)/(2\pi)$ exists and is an integer a.s.

We will also use the following notation:

$$\{x\}_{2\pi} = x - 2\pi \left\lfloor \frac{x}{2\pi} \right\rfloor .$$

2 Limiting marginal distributions

We are first interested in the limiting law of the random variable $\text{Sine}_\beta[0, \lambda]$ when $\beta \to 0$ for a single fixed λ. In this case, we re-write the diffusion α_λ in a more convenient way:

$$d\alpha_\lambda = \lambda \frac{\beta}{4} e^{-\frac{\beta}{4}t} dt + 2\sin(\frac{\alpha_\lambda}{2}) dB_t , \tag{2.1}$$

where B is a real standard Brownian motion (which depends on λ). Let us introduce the change of variable $R_\lambda := \log(\tan(\alpha_\lambda/4))$. A straightforward computation (see [22]) shows that:

$$dR_\lambda = \frac{1}{2} \left(\lambda \frac{\beta}{4} e^{-\frac{\beta}{4}t} \cosh(R_\lambda) + \tanh(R_\lambda) \right) dt + dB_t , \quad R_\lambda(0) = -\infty . \tag{2.2}$$

2.1 Trapping in the stationary potential

In this subsection, we study an exit time problem for a Langevin diffusion S_λ evolving in a stationary potential V_β defined for $r \in \mathbb{R}$ as

$$V_\beta(r) := -\frac{1}{2} \left(\lambda \frac{\beta}{4} \sinh(r) + \log\cosh(r) \right) .$$

This problem is relevant to our study thanks to the slow variation with time of the non-stationary potential as $\beta \to 0$ in which the diffusion R_λ evolves. The diffusion S_λ satisfies the following stochastic differential equation

$$dS_\lambda = -V'_\beta(S_\lambda)\, dt + dB_t\,, \quad S_\lambda(0) = -\infty\,. \tag{2.3}$$

In the whole paper, the diffusions R_λ and S_λ defined respectively in (2.2) and (2.3) are coupled, driven by the same Brownian motion B. Under this coupling, we have almost surely

$$R_\lambda(t) \leq S_\lambda(t)$$

for all $t \leq \zeta$ where ζ is the first explosion (stopping) time

$$\zeta := \inf\{t \geqslant 0 : S_\lambda(t) = +\infty\}\,.$$
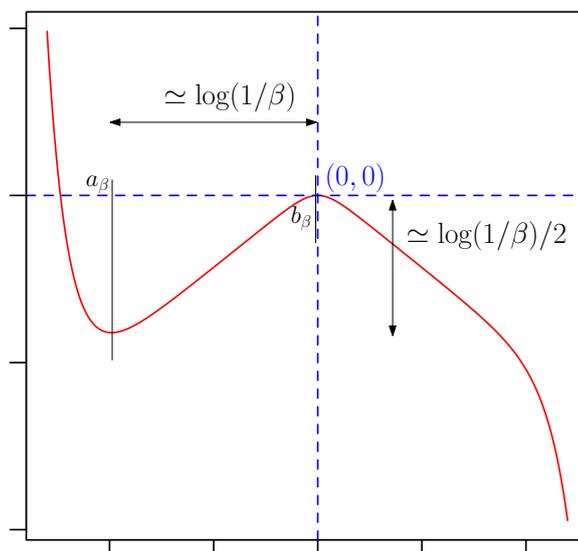
Figure 2: The potential $V_\beta(r)$ as a function of r.

In this paragraph we investigate the limiting law of the stopping time ζ through its Laplace transform defined for $\xi > 0$ as

$$g_\xi(r) := \mathbb{E}_r[e^{-\xi\zeta}]\,,$$

where r is the initial position of the diffusion S_λ.

We know from classical diffusion theory (see also [4]) that g_ξ satisfies

$$\frac{1}{2}g''_\xi(r) - V'_\beta(r)g'_\xi(r) = \xi g_\xi(r)\,, \text{ with } g_\xi(r) \to 1 \text{ as } r \to +\infty\,. \tag{2.4}$$

Let us first examine the expectation of the first explosion time of S_λ. Due to the strong barrier separating the local minimum and the local maximum (see Fig. 2), it is natural to expect the asymptotic of this mean exit time not to depend on the starting point of the diffusion, when $\beta \to 0$, as long as it is located in the well ("memory-loss property"). This is the purpose of Proposition 2.1. We then show that this first exit time properly rescaled by its mean value converges in law to an exponential distribution (Proposition 2.2) when the starting point is in the well.

Proposition 2.1. *Suppose that the diffusion S_λ starts from $r := r_\beta$ such that $r_\beta \to -\infty$. Then its expected exit time denoted $t_\beta(r) := \mathbb{E}_r[\zeta]$ has the following equivalent when $\beta \to 0$:*

$$t_\beta(r_\beta) \sim \frac{8\pi}{\beta\lambda}\,.$$

Proof of Proposition 2.1. From (2.4), it is easy to see that the expected exit time $t_\beta(r)$ satisfies the boundary value problem

$$\frac{1}{2}t_\beta''(r) - V_\beta'(r)t_\beta'(r) = -1\,, \quad \text{with } t_\beta(r) \to 0 \text{ as } r \to +\infty\,. \tag{2.5}$$

Solving (2.5) explicitly we obtain the integral form

$$t_\beta(r) = 2\int_r^{+\infty} dx \int_{-\infty}^{x} \exp\left(2\left[V_\beta(x) - V_\beta(y)\right]\right) dy\,. \tag{2.6}$$

By extracting carefully the asymptotic behavior of this integral in the limit $\beta \to 0$ (see Appendix A.1), we obtained the desired result. $\qquad\square$

The Laplace transform of the exit time satisfies the fixed point equation

$$g_\xi(r) = 1 - 2\xi\int_r^{+\infty} dx \int_{-\infty}^{x} \exp(2[V_\beta(x) - V_\beta(y)])g_\xi(y)dy.$$

With classical arguments similar to those of [4], we prove the following proposition in Appendix A.1.

Proposition 2.2. *Conditionally on $S_\lambda(0) = r := r_\beta$ such that $r_\beta \to -\infty$ when $\beta \to 0$, the first explosion time ζ of the diffusion $(S_\lambda(t))_{t \geqslant 0}$ converges weakly when rescaled by the expected exit time $8\pi/(\beta\lambda)$ towards an exponential distribution with mean 1.*

The process θ_λ such that $S_\lambda := \log\tan(\theta_\lambda/4)$ satisfies

$$d\theta_\lambda = \lambda\frac{\beta}{4}dt + 2\sin(\frac{\theta_\lambda}{2})dB_t\,.$$

Proposition (2.2) translates into a convergence of the stopping time $\zeta_{2\pi} := \inf\{t \geqslant 0 : \theta_\lambda(t) \geqslant 2\pi\}$.

Proposition 2.3. *Conditionally on $\theta_\lambda(0) = \theta_0 := \theta_0^\beta$ such that $\theta_0^\beta \to 0_+$ when $\beta \to 0$, the stopping time $\frac{\beta\lambda}{8\pi}\zeta_{2\pi}$ converges weakly when $\beta \to 0$ towards an exponential distribution with mean 1.*

2.2 Convergence of the jump process

We consider the diffusion α_λ defined in (1.5) or equivalently for a single λ, in (2.1). For $k \in \mathbb{N}$, let

$$\zeta_k := \inf\{t \geqslant 0 : \alpha_\lambda(t) \geqslant 2k\pi\}\,. \tag{2.7}$$

Note that those stopping times correspond to the jumps of the process $\lfloor \alpha_\lambda/(2\pi) \rfloor$. We denote by $\mathcal{F}_t$ the filtration associated to the diffusion process α_λ. In this paragraph, we

will sometimes omit the subscript λ in α_λ to simplify the notations. We consider the rescaled empirical measure μ_λ^β of the $\frac{\beta}{4}\zeta_k$ defined on $\mathbb{R}_+$

$$\mu_\lambda^\beta[0,t] = \sum_{k=1}^{+\infty} \delta_{\zeta_k}\left[0, \frac{8\pi t}{\beta}\right].$$ (2.8)

We divide the time interval $\mathbb{R}_+$ into random small intervals $I_k := [\frac{S_k}{n}\frac{8\pi}{\beta}, \frac{S_{k+1}}{n}\frac{8\pi}{\beta}], k \in \mathbb{N}$, *independent* of the diffusion α where $S_0 = 0$ and:

$$S_k := \sum_{i=1}^{k} \tau_i,$$

where the τ_i are i.i.d. random variables with mean 1 uniformly distributed on $[1/2, 3/2]$.

For each $k \in \mathbb{N}$, conditionally on $\mathcal{F}_{\frac{S_k}{n}\frac{8\pi}{\beta}}$, we define the two diffusion processes α_n^+ and α_n^- such that, for $t \in I_k$,

$$d\alpha_n^+ = \lambda\frac{\beta}{4}e^{-\frac{S_k}{n}2\pi}dt + 2\sin(\frac{\alpha_n^+}{2})dB_t, \quad \alpha_n^+(\frac{S_k}{n}\frac{8\pi}{\beta}) = \alpha(\frac{S_k}{n}\frac{8\pi}{\beta}),$$

$$d\alpha_n^- = \lambda\frac{\beta}{4}e^{-\frac{S_{k+1}}{n}2\pi}dt + 2\sin(\frac{\alpha_n^-}{2})dB_t, \quad \alpha_n^-(\frac{S_k}{n}\frac{8\pi}{\beta}) = \alpha(\frac{S_k}{n}\frac{8\pi}{\beta}).$$

By the increasing property, it follows that for any $k \in \mathbb{N}$ and $t \in I_k$, we have

$$\alpha_n^-(t) \leq \alpha(t) \leq \alpha_n^+(t).$$

Theorem 2.4. *As $\beta \to 0$, the empirical measure μ_λ^β converges weakly in the space of Radon measures (equipped with the topology of vague convergence [15]) to a Poisson point process on $\mathbb{R}_+$ with inhomogeneous intensity $\lambda e^{-2\pi t} dt$. In particular, we have, for any $k \in \mathbb{N}$,*

$$\mathbb{P}[\mu_\lambda^\beta[0,t] = k] \to_{\beta \to 0} \exp(-\frac{\lambda}{2\pi}(1 - e^{-2\pi t}))\frac{(\frac{\lambda}{2\pi})^k(1 - e^{-2\pi t})^k}{k!}.$$ (2.9)

The proof of Theorem 2.4 is done in the next paragraph 2.3. It uses a careful analysis of the behaviour of a single diffusion α_λ in the small β limit.

Remark 2.5. Theorem 2.4 establishes the convergence of the rescaled empirical measure of the jumps of the process $\lfloor \alpha_\lambda/2\pi \rfloor$ under the initial condition $\alpha_\lambda(0) = 0$. It is easy to generalize this result to other starting times s_0 and positions $\alpha_\lambda(s_0) \neq 0$ such that s_0 scales with $1/\beta$ and $\alpha_\lambda(s_0)$ is close enough to 0 modulo 2π.

More precisely, fix a time $s > 0$ and denote by $\mu_{\lambda,s,x}^\beta$ the rescaled counting measure of the jumps of $\lfloor \alpha_\lambda/2\pi \rfloor$ on the time interval $[8\pi s/\beta, +\infty)$ with the initial condition $\alpha_\lambda(8\pi s/\beta) = x$. Then, for any family of starting points (x_β) such that $\{x_\beta\}_{2\pi} \leqslant 4\arctan(\beta^{1/4})$, the measure $\mu_{\lambda,s,x_\beta}^\beta$ converges weakly in the space of Radon measures towards a Poisson point process on $[s, +\infty)$ with inhomogeneous intensity $\lambda e^{-2\pi t} dt$.

Thanks to the equality in law

$$\mu_\lambda^\beta(\mathbb{R}_+) \overset{(d)}{=} \frac{\alpha_\lambda(\infty)}{2\pi} \overset{(d)}{=} \text{Sine}_\beta[0, \lambda],$$

we easily deduce from Theorem 2.4 the convergence of the marginals of the Sine_β point process.

Corollary 2.6. *Let $\lambda < \lambda'$. The random variable $\text{Sine}_\beta[\lambda, \lambda']$ converges weakly as $\beta \to 0$ to a Poisson law with parameter $\frac{\lambda'-\lambda}{2\pi}$.*

2.3 Estimates for a single diffusion and proof of Theorem 2.4

We analyse in this paragraph the diffusion α_λ and the jumps of $\lfloor \alpha_\lambda/(2\pi) \rfloor$ in the limit $\beta \to 0$. The results we obtain are derived using the diffusion $R_\lambda := \log(\tan(\alpha_\lambda/4))$ satisfying the SDE (2.2). We defer the proof of those technical lemmas in Appendix A.2. In the following, for all $t > 0$ and $x \in \mathbb{R}_+$, the law of the diffusion process α_λ starting from x at time t is denoted $\mathbb{P}_{x,t}$. When $t = 0$, we use the short cuts $\mathbb{P}_x$ for $\mathbb{P}_{x,0}$ and and $\mathbb{P}$ for $\mathbb{P}_{0,0}$.

In the following Lemma, we first prove that if the diffusion α_λ starts just below 2π modulo 2π, then $\lfloor \alpha_\lambda/(2\pi) \rfloor$ will jump in a time of order $\log \frac{1}{\beta}$ much shorter than the typical time between two jumps (of order $1/\beta$) with probability going to 1.

Lemma 2.7. *Let $0 < \varepsilon < 1$ and $s > 0$. Denote $\hat{s} := 8\pi s/\beta$. We define the first reaching time*

$$\zeta_{2\pi}^\lambda := \inf\{t \geq \hat{s} : \alpha_\lambda(t) = 2\pi + 2\pi \lfloor \alpha_\lambda(\hat{s})/(2\pi) \rfloor\}. \tag{2.10}$$

Then, there exists a constant $c > 0$ (depending only on s, ε and λ) such that for all $\beta > 0$ small enough,

$$\mathbb{P}\left[\zeta_{2\pi}^\lambda < 9 \log \frac{1}{\beta} \,\middle|\, \{\alpha_\lambda(\hat{s})\}_{2\pi} = 2\pi - 4\arctan(\beta^\varepsilon)\right] \geq 1 - \beta^c.$$

Lemma 2.7 permits us to show that the time spent by α_λ away from 0 modulo 2π is negligible compared to the time scale $1/\beta$. This is the content of Lemma 2.8.

Lemma 2.8. *Let $t > s \geq 0$, $x \in \mathbb{R}_+$. Let us define*

$$\Xi_\beta(s,t,x) := \mathbb{E}_{x,8\pi s/\beta}\left[\int_{\frac{8\pi}{\beta}s}^{\frac{8\pi}{\beta}t} 1_{\{\{\alpha_\lambda(u)\}_{2\pi} \geq 4\arctan(\beta^{1/4})\}} du\right]. \tag{2.11}$$

Then, for all $t \geq s > 0$, there exists $C > 0$ (depending only on t and λ) such that for all $\beta > 0$ and $x \in \mathbb{R}_+$,

$$\Xi_\beta(s,t,x) \leq \frac{C}{\sqrt{\beta}} \log \frac{1}{\beta}.$$

We can now prove Theorem 2.4 using the previous estimates:

Proof of Theorem 2.4.

The proof follows the same lines as the proof of [Theorem 4.1 in [4]]. The idea is to approximate the number of jumps of the diffusion α by those of stationary diffusions and to use the increasing property. To this end, we will use the subdivision of $\mathbb{R}_+$ introduced above and the diffusions α_n^+ and α_n^-.

From Kallenberg's theorem [15], we just need to see that, for any finite union I of disjoint and bounded intervals, we have when $\beta \to 0$,

$$\mathbb{E}[\mu_\lambda^\beta(I)] \longrightarrow \lambda \int_I e^{-2\pi t}\, dt, \tag{2.12}$$

$$\mathbb{P}[\mu_\lambda^\beta(I) = 0] \longrightarrow \exp(-\lambda \int_I e^{-2\pi t}\, dt). \tag{2.13}$$

Denote by $[t_1; t_2]$ the right most interval of I, by J the union of disjoint and bounded intervals such that $I = J \cup [t_1; t_2]$ and by t_0 the supremum of J.

Let us use the random subdivision introduced above. It is crucial to control the position of the diffusion at the starting point of all the sub-intervals. As a consequence of

Lemma 2.8, with large probability, the diffusion α is close to 0 modulo 2π in the beginning of each of the sub-intervals overlapping $[0, t_2]$[3].

More precisely, denote by $\mathcal{C}_k := \{\{\alpha((8\pi/\beta)S_k/n)\}_{2\pi} \leqslant 4 \arctan(\beta^{1/4})\}$ and consider:

$$\mathcal{C} := \bigcap_{k=0}^{\lfloor 2nt_2 \rfloor + 1} \mathcal{C}_k.$$

Every sub-interval intersecting $[0, t_2]$ is taken care of in this event as by definition $S_k \in [k/2, 3k/2]$. Its probability is bounded from above by

$$\mathbb{P}[\mathcal{C}^c] \leqslant \sum_{k=0}^{\lfloor 2nt_2 \rfloor + 1} \mathbb{P}[\{\alpha((8\pi/\beta)S_k/n)\}_{2\pi} > 4 \arctan(\beta^{1/4})]$$

$$\leqslant 2 \mathbb{E}[\int_0^{3nt_2 + 3} 1_{\{\{\alpha((8\pi/\beta)u/n)\}_{2\pi} > 4 \arctan(\beta^{1/4})\}} du]$$

$$\leqslant \frac{n}{4\pi} \beta \, \Xi_\beta(0, 3t_2 + 1, 0) \to_{\beta \to 0} 0$$

where we used the notation and result of Lemma 2.8 for the last line.

We now turn to the proof of 2.12. Note that thanks to the linearity of the expectation, we simply need to prove (2.12) for intervals I of the form $I = [0, t_2]$. The upper bound simply follows from the SDE form:

$$\mathbb{E}[\alpha(t)] = \lambda \frac{\beta}{4} \int_0^t e^{-\beta s/4} ds \,.$$

We immediately derive:

$$\mathbb{E}[\mu_\beta[0, t_2]] = \mathbb{E}[\lfloor \alpha((8\pi/\beta)t_2)/(2\pi) \rfloor] \leqslant \lambda \int_0^{t_2} e^{-2\pi s} ds \,.$$

For the lower bound, denote by N_k^+, N_k^- and N_k the number of jumps of α_n^+, α_n^- and α in the sub-interval $[(8\pi/\beta)S_k/n, (8\pi/\beta)S_{k+1}/n]$.

$$\mathbb{E}[\mu_\beta[0, t_2]] \geqslant \sum_{k=0}^{\lfloor 2nt_2 \rfloor + 1} \mathbb{E}[N_k^- 1_{\{(S_k/n) < t_2\}} | \mathcal{C}_k] - \sum_{k=0}^{\lfloor 2nt_2 \rfloor + 1} \mathbb{E}[N_k^- 1_{\{(S_k/n) < t_2\}} | \mathcal{C}_k] \mathbb{P}[\mathcal{C}_k^c] \,.$$

The second sum of the RHS can be simply bounded from above by:

$$\mathbb{E}[\text{number of jumps of } \theta_\lambda \text{ in } [0, 3\, t_2(8\pi)/\beta]] \times \sup_{k \in \{0, \cdots, \lfloor 2nt_2 \rfloor + 1\}} \frac{\mathbb{P}[\mathcal{C}_k^c]}{\mathbb{P}[\mathcal{C}_k]} \,.$$

The first expectation is bounded from above by a number independent of β and the second term (bounded by $\mathbb{P}[\mathcal{C}^c]/(1 - \mathbb{P}[\mathcal{C}^c])$) tends to 0 as $\beta \to 0$. Moreover, thanks to Proposition 2.3, we have

$$\mathbb{E}\big[N_k^+ \, \big| \mathcal{C}_k \big] \to_{\beta \to 0} \mathbb{E}[\lambda(\tau_{k+1}/n) \, \exp(-2\pi S_k/n)] \,.$$

Using the convergence of the Riemann sum when $n \to \infty$, it gives the desired lower bound.

Let us now examine the convergence 2.13 for a single interval $[t_1, t_2]$ first. By the Markov property:

$$\mathbb{P}[\mu_\beta[t_1, t_2] = 0] \leqslant \mathbb{E}\Big[\prod_{k=0}^\infty \mathbb{P}\big[N_k^+ = 0 \,\big| \mathcal{C}_k, (\tau_i)_i \big] 1_{\{S_{k+1}/n \geqslant t_1, \, S_k/n \leqslant t_2\}} \Big] + \mathbb{P}[\mathcal{C}^c] \,.$$

---

[3]For the generalization of Remark 2.5, the initial position modulo 2π at time s_0 belongs to the interval $[0, 4\arctan(\beta^{1/4}))$ so that the event $\mathcal{C}_0$ occurs almost surely by definition.

Using the convergence in each of the sub-intervals (given by Proposition 2.3)

$$\mathbb{P}\big[N_k^+ = 0 \,\big|\, \mathcal{C}_k\big] \to_{\beta \to 0} \mathbb{E}[\exp(-\lambda(\tau_{k+1}/n) \, \exp(-2\pi S_k/n)))],$$

we obtain:

$$\limsup_{\beta \to 0} \mathbb{P}[\mu_\beta[t_1, t_2] = 0] \leqslant \mathbb{E}[\prod_{k=0}^{\infty} \exp(-\lambda(\tau_{k+1}/n) \, \exp(-2\pi S_k/n)))1_{\{S_{k+1}/n \,\geqslant\, t_1, \, S_k/n \,\leqslant\, t_2\}}].$$

Thanks to the convergence of the Riemann sum when $n \to \infty$, we deduce the upper bound. The lower bound can be done using the same techniques as above. To generalize the result to any finite union of intervals I, we use the simple Markov property which gives

$$\mathbb{P}[\mu_\lambda^\beta(I) = 0] = \mathbb{E}[1_{\{\mu_\beta(J)=0\}} \, \mathbb{P}[\mu_\beta([t_1, t_2]) = 0|\alpha(8\pi t_0/\beta)]].$$

To conclude, let us introduce an independent random time U sampled uniformly in $[t_0, t_1]$. With probability going to 1, $\alpha(8\pi U/\beta)$ belongs to an interval of the type $[2\pi, 2\pi + 4\arctan(\beta^{1/4}))$. We can then iterate the previous arguments to obtain the result. $\qquad\square$

3 Asymptotic spatial independence

3.1 Ordering of two diffusions

For $\lambda < \lambda'$, we now control the expected time spent by the process $\{\alpha_{\lambda'}\}_{2\pi}$ below the process $\{\alpha_\lambda\}_{2\pi}$ where α_λ and $\alpha_{\lambda'}$ are coupled according to (1.5). The following Lemma is a crucial step towards the proof of the asymptotic independence of the limiting point process on disjoint intervals and will be used for the proof of Lemmas 3.4 and 3.5.

Lemma 3.1. *Let $t > 0, \lambda < \lambda'$ and*

$$\Theta_\beta(t) := \mathbb{E}\left[\int_0^{\frac{8\pi}{\beta}t} 1_{\{\{\alpha_{\lambda'}(u)\}_{2\pi} \leq \{\alpha_\lambda(u)\}_{2\pi}\}} du\right].$$

Then, there exists two constants $c, C > 0$ (depending only on t and λ') such that for all $\beta > 0$,

$$\Theta_\beta(t) \leqslant \frac{C}{\beta^{1-c}}.$$

Proof. We set

$$\mathcal{E}_u := \{\{\alpha_{\lambda'}(u)\}_{2\pi} < \{\alpha_\lambda(u)\}_{2\pi}\}.$$

Before evaluating the probability of the event $\mathcal{E}_u$, we need to introduce for $u > 0$ the last jump time of the process $\lfloor \frac{\alpha_{\lambda'}}{2\pi} \rfloor$ before time u, i.e.

$$\zeta^{(u)} := \sup\{\zeta_k^{\lambda'} \; : \; k \in \mathbb{N}, \; \zeta_k^{\lambda'} \leq u\}.$$

The main idea is to prove that any time u such that $\mathcal{E}_u$ occurs is associated to a jump time of $\alpha_{\lambda'}$ right before u.

We set $u_0 := u - 9\log\frac{1}{\beta}$ and bound from above the probability of the event $\mathcal{E}_u$ by

$$\mathbb{P}\left[\mathcal{E}_u\right] \leq \mathbb{P}\left[\mathcal{E}_u \cap \left\{\zeta^{(u)} \in [u_0, u]\right\}\right] + \mathbb{P}\left[\mathcal{E}_u \cap \left\{\zeta^{(u)} < u_0\right\}\right]$$

$$\leq \mathbb{P}\left[\mathcal{E}_u \cap \left\{\zeta^{(u)} \in [u_0, u]\right\}\right] + \mathbb{P}\left[\bigcap_{s \in [u_0, u]} \mathcal{E}_s\right] \qquad (3.1)$$
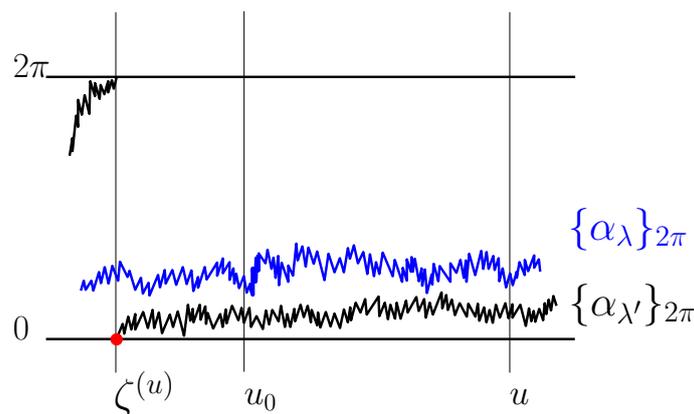
Figure 3: Representation of the event $\mathcal{E}_u \cap \{\zeta^{(u)} < u_0\}$.

where we have noticed the inclusion $\mathcal{E}_u \cap \{\zeta^{(u)} < u_0\} \subseteq \bigcap_{s \in [u_0, u]} \mathcal{E}_s$ for the second line (see Fig. 3). Indeed the definition of $\zeta^{(u)}$ implies that the process $\lfloor \frac{\alpha_{\lambda'}}{2\pi} \rfloor$ has not jumped during the time interval $[\zeta^{(u)}, u]$ so that the relative ordering (modulo 2π) $\{\alpha_{\lambda'}(t)\}_{2\pi} < \{\alpha_\lambda(t)\}_{2\pi}$ at time $t = u$ has to be preserved for all $t \in [\zeta^{(u)}, u)$.

We now tackle the second probability of (3.1) using the fact that α_λ is close to 0 modulo 2π with large probability:

$$\mathbb{P}\left[\bigcap_{s \in [u_0, u]} \mathcal{E}_s \right] \le \mathbb{P}\left[\bigcap_{s \in [u_0, u]} \mathcal{E}_s \cap \left\{ \{\alpha_\lambda(u_0)\}_{2\pi} \le 4 \arctan(\beta^{1/4}) \right\} \right]$$
$$+ \mathbb{P}\left[\{\alpha_\lambda(u_0)\}_{2\pi} \ge 4 \arctan(\beta^{1/4}) \right].$$

We introduce the stopping time

$$\zeta_>(u_0) := \inf\{t \ge u_0 : \{\alpha_{\lambda'}(t)\}_{2\pi} > \{\alpha_\lambda(t)\}_{2\pi}\}$$

and notice that, conditionally on the event $\mathcal{A}_{u_0} := \mathcal{E}_{u_0} \cap \left\{ \{\alpha_\lambda(u_0)\}_{2\pi} \le 4 \arctan(\beta^{1/4}) \right\}$, we have almost surely for all $t \in [u_0, \zeta_>(u_0)]$,

$$\{\alpha_{\lambda'}(t) - \alpha_\lambda(t)\}_{2\pi} = 2\pi - (\{\alpha_\lambda(t)\}_{2\pi} - \{\alpha_{\lambda'}(t)\}_{2\pi}).$$

We define a diffusion process $\widetilde{\alpha}_{\lambda' - \lambda}$ and its associated first reaching time of 2π such that

$$d\widetilde{\alpha}_{\lambda' - \lambda}(t) = (\lambda' - \lambda)\frac{\beta}{4} e^{-\frac{\beta}{4} t} dt + \mathrm{Re}((e^{-i\widetilde{\alpha}_{\lambda' - \lambda}(t)} - 1) dZ_t), \quad t \ge u_0$$
$$\text{with } \widetilde{\alpha}_{\lambda' - \lambda}(u_0) = 2\pi - (\{\alpha_\lambda(u_0)\}_{2\pi} - \{\alpha_{\lambda'}(u_0)\}_{2\pi}),$$
$$\text{and } \widetilde{\zeta}_{2\pi} := \inf\{t \ge u_0 : \widetilde{\alpha}_{\lambda' - \lambda}(t) = 2\pi\}.$$

Endowed with those definitions, we have the following upper-bound

$$\mathbb{P}\left[\bigcap_{s \in [u_0, u]} \mathcal{E}_s \cap \mathcal{A}_{u_0} \right] = \mathbb{P}\left[\{\zeta_>(u_0) \ge u\} \cap \mathcal{A}_{u_0} \right]$$
$$\le \mathbb{P}\left[\widetilde{\zeta}_{2\pi} \ge 9 \log \frac{1}{\beta} \right] \le \beta^c$$

where we have used the equality in law $(\widetilde{\alpha}_{\lambda' - \lambda}(t), \ t \ge u_0) = (\alpha_{\lambda'}(t) - \alpha_\lambda(t), \ t \ge u_0)$ for the second line and Lemma 2.7 (as well as the increasing property) to obtain the last

upper-bound. Gathering the above inequalities, we obtain

$$\mathbb{P}\left[\mathcal{E}_u\right] \leq \mathbb{P}\left[\zeta^{(u)} \in [u_0, u]\right] + \beta^c + \mathbb{P}\left[\{\alpha_\lambda(u_0)\}_{2\pi} \geq 4 \arctan(\beta^{1/4})\right].$$

Now, to conclude, we just have to integrate this latter inequality with respect to u. First notice that we have almost surely

$$\int_{9\log(1/\beta)}^{\frac{8\pi}{\beta}t} 1_{\left\{\zeta^{(u)} \in \left[u - 9\log\frac{1}{\beta}, u\right]\right\}} du \leq 9\log\frac{1}{\beta}\,\mu_{\lambda'}^\beta[0, \frac{8\pi}{\beta}t].$$

Integrating the other terms as well with respect to $u \in [0, \frac{8\pi}{\beta}t]$ and taking the expectation, we get

$$\Theta_\beta(t) \leq 9\log\frac{1}{\beta}\left(1 + \mathbb{E}\left[\mu_{\lambda'}^\beta[0, \frac{8\pi}{\beta}t]\right]\right) + \Xi_\beta(0, t, 0)$$

$$\leq 9\log\frac{1}{\beta}\left(1 + \frac{\lambda}{2\pi}\right) + 8\pi\,t\,\beta^{c-1} + \Xi_\beta(0, t, 0)$$

where $\Xi_\beta(0, t, 0)$ is defined in (2.11). The conclusion now follows from Lemma (2.8). $\quad\square$

3.2 Limiting coupled Poisson processes

Lemma 3.1 shall be an important tool to prove the asymptotic independence between $\alpha_\lambda(\infty)$ and $\alpha_{\lambda'}(\infty) - \alpha_\lambda(\infty)$ for $\lambda < \lambda'$.

Theorem 2.4 gives the weak convergence of the random measures μ_λ^β and $\mu_{\lambda'}^\beta$ in the space of measures on $\mathbb{R}_+$ equipped with the topology of vague convergence denoted $\mathcal{M}_+(\mathbb{R}_+)$. Due to the equality in law

$$\alpha_{\lambda'} - \alpha_\lambda \overset{(d)}{=} \alpha_{\lambda'-\lambda}, \tag{3.2}$$

Theorem 2.4 also implies the weak convergence of the (positive) random measure $\mu_{\lambda'-\lambda}^\beta$ such that for all $t \geq 0$,

$$\mu_{\lambda'-\lambda}^\beta[0, t] := \left\lfloor \frac{\alpha_{\lambda'}(t) - \alpha_\lambda(t)}{2\pi} \right\rfloor$$

towards a Poisson measure $\mathcal{P}_{\lambda'-\lambda}$ with intensity $(\lambda' - \lambda)\,e^{-2\pi t}dt$.

We now fix $\lambda < \lambda' < \lambda''$ and work with the three diffusions α_λ, $\alpha_{\lambda'}$ and $\alpha_{\lambda''}$ coupled according to (1.5). We are interested in the limiting joint distribution of the triplet of random measures $(\mu_\lambda^\beta, \mu_{\lambda'}^\beta, \mu_{\lambda''-\lambda'}^\beta)$ according to this coupling.

From the above convergences, it is straightforward to deduce the relative-compactness of the family of the triplets of (random) measures

$$\{(\mu_\lambda^\beta, \mu_{\lambda'}^\beta, \mu_{\lambda''-\lambda'}^\beta), \beta > 0\} \tag{3.3}$$

for the weak topology over $(\mathcal{M}_+(\mathbb{R}_+))^3$ equipped with the product topology of vague convergence.

Let us take a sequence $\beta_k \to 0$ when $k \to \infty$ such that the processes

$$\left(\mu_\lambda^{\beta_k}, \mu_{\lambda'}^{\beta_k}, \mu_{\lambda''-\lambda'}^{\beta_k}\right) \tag{3.4}$$

converge jointly weakly in the product space when $k \to \infty$ to a triplet

$$\left(\mathcal{P}_\lambda^{(\beta_k)}, \mathcal{P}_{\lambda'}^{(\beta_k)}, \mathcal{P}_{\lambda''-\lambda'}^{(\beta_k)}\right)$$

of point measures on $\mathbb{R}_+$ whose marginal distributions are given respectively by the law of the Poisson measures $\mathcal{P}_\lambda$ and $\mathcal{P}_{\lambda'}$ and $\mathcal{P}_{\lambda''-\lambda'}$ and whose joint law depends *a priori* on the chosen sub-sequence (β_k). In the following, we study this triplet and we drop the superscript (β_k) to ease the notations. We shall in fact see later that all the possible limit points have the same law. Therefore the law of the triplet does not depend on the subsequence (β_k) and the weak convergence of (3.3) holds (see Remark 3.6).

In the next Lemma, we regard the point measures $\mathcal{P}_\lambda, \mathcal{P}_{\lambda'}$ and $\mathcal{P}_{\lambda''-\lambda'}$ as Poisson point processes and prove that they are indeed jointly Poisson processes on a common filtration. This is an important step for our needs.

Lemma 3.2. *Let $\lambda < \lambda' < \lambda''$ and $\mathcal{F} := (\mathcal{F}_t)_{t\geq 0}$ be the natural filtration associated to the process $(\mathcal{P}_\lambda, \mathcal{P}_{\lambda'}, \mathcal{P}_{\lambda''-\lambda'})$ i.e. such that*

$$\mathcal{F}_t := \sigma(\mathcal{P}_\lambda(s), \mathcal{P}_{\lambda'}(s), \mathcal{P}_{\lambda''-\lambda'}(s), 0 \leq s \leq t).$$

Then, the marginal processes $\mathcal{P}_\lambda$ and $\mathcal{P}_{\lambda'}$ and $\mathcal{P}_{\lambda''-\lambda'}$ are $(\mathcal{F}_t)$-Poisson point processes with respective intensities $\lambda e^{-2\pi t}dt$ and $\lambda' e^{-2\pi t}dt$ and $(\lambda'' - \lambda')e^{-2\pi t}dt$.

The main point used to prove this Lemma is that all the diffusions $(\alpha_\lambda, \lambda \geq 0$ are measurable with respect to the same driving (complex) Brownian motion (Z_t) as they are strong solutions of the SDEs (1.5). A proof can be found in Appendix A.3.

Lemma 3.2 can easily be generalized to finitely many λ's:

Lemma 3.3. *Let us fix an integer $k \geq 3$ and $\lambda_0 < \lambda_1 < \cdots < \lambda_k$. For all $i, j \in \{1, \cdots, k\}$ such that $j > i$, define the point measures $\mathcal{P}_{\lambda_i}$ and $\mathcal{P}_{\lambda_j-\lambda_i}$ as above. Let $\mathcal{F} := (\mathcal{F}_t)_{t\geq 0}$ be the filtration:*

$$\mathcal{F}_t := \sigma(\mathcal{P}_{\lambda_i}(s), \mathcal{P}_{\lambda_j-\lambda_i}(s), 0 \leq s \leq t, \ i, j \in \{1, \cdots, k\} \text{ such that } j > i).$$

Then, for all $i, j \in \{1, \cdots, k\}$ with $j > i$, the marginal processes $\mathcal{P}_{\lambda_i}$ and $\mathcal{P}_{\lambda_j-\lambda_i}$ are $(\mathcal{F}_t)$-Poisson point processes with respective intensities $\lambda_i\, e^{-2\pi t}dt$ and $(\lambda_j - \lambda_i)\, e^{-2\pi t}dt$.

To fix notations, we will denote by $(\xi_i^\lambda)_{i\in\mathbb{N}}$ resp. $(\xi_i^{\lambda'})_{i\in\mathbb{N}}, (\xi_i^{\lambda'-\lambda})_{i\in\mathbb{N}}$ the points associated to the Poisson processes $\mathcal{P}_\lambda$ and $\mathcal{P}_{\lambda'}$ and $\mathcal{P}_{\lambda'-\lambda}$ such that

$$\mathcal{P}_\lambda[0,t] = \sum_i \delta_{\xi_i^\lambda}[0,t], \quad \mathcal{P}_{\lambda'}[0,t] = \sum_i \delta_{\xi_i^{\lambda'}}[0,t], \quad \mathcal{P}_{\lambda'-\lambda}[0,t] = \sum_i \delta_{\xi_i^{\lambda'-\lambda}}[0,t].$$

For $\beta > 0$, we also recall the notations

$$\mu_\lambda^\beta[0,t] = \sum_i \delta_{\zeta_i^\lambda}[0, \frac{8\pi t}{\beta}], \quad \mu_{\lambda'}^\beta[0,t] = \sum_i \delta_{\zeta_i^{\lambda'}}[0, \frac{8\pi t}{\beta}], \quad \mu_{\lambda'-\lambda}^\beta[0,t] = \sum_i \delta_{\zeta_i^{\lambda'-\lambda}}[0, \frac{8\pi t}{\beta}].$$

Lemma 3.4. *Let $\lambda < \lambda'$. Then, we have almost surely*

$$\mathcal{P}_\lambda \subseteq \mathcal{P}_{\lambda'}$$

i.e. for all $i \in \mathbb{N}$, there exists $j \geq i$ such that $\xi_i^\lambda = \xi_j^{\lambda'}$.

Proof. We have to prove that for any $t > 0$,

$$\mathbb{P}\left[\exists i : \xi_i^\lambda < t, \xi_i^\lambda \notin \mathcal{P}_{\lambda'}\right] = 0. \tag{3.5}$$

The probability (3.5) is the increasing limit of the sequence $(p_n)_{n\in\mathbb{N}}$ defined as

$$p_n := \mathbb{P}\left[\exists i : \xi_i^\lambda < t, \forall j \geq i, |\xi_i^\lambda - \xi_j^{\lambda'}| > \frac{1}{2n}\right]. \tag{3.6}$$

It suffices to prove that $p_n = 0$ for any n. Let us introduce the probability

$$p_n^\beta := \mathbb{P}\left[\exists i : \frac{\beta}{8\pi}\zeta_i^\lambda < t, \forall j \geq i, \frac{\beta}{8\pi}|\zeta_i^\lambda - \zeta_j^{\lambda'}| > \frac{1}{2n}\right].\tag{3.7}$$

Denote by $\mathcal{M}_p(\mathbb{R}_+)$ the space of point measures and recall that it is closed in the space $\mathcal{M}_+(\mathbb{R}_+)$ for the vague convergence topology. Notice that the set

$$\{(\mu, \nu) \in (\mathcal{M}_p(\mathbb{R}_+))^2 \ : \ \mu = \sum_{i=1}^\infty \delta_{x_i}, \nu = \sum_{i=1}^\infty \delta_{y_i}, \exists i \text{ such that } y_i < t \text{ and } \forall j, |y_i - x_j| > \frac{1}{2n}\}$$

is open in $(\mathcal{M}_p(\mathbb{R}_+))^2$ equipped with the product topology of the vague convergence on the space of point measures. It comes from the straightforward fact that if $\mu_k \in \mathcal{M}_p(\mathbb{R}_+)$ converges towards $\mu \in \mathcal{M}_p(\mathbb{R}_+)$ for the vague topology, the points of μ_k belonging to $[0, t]$ for all but finitely many k converge to points of μ in $[0, t]$.

If n is fixed, we therefore have $p_n \leq \liminf_{\beta \to 0} p_n^\beta$ using the joint convergence of $(\mu_\lambda^\beta, \mu_{\lambda'}^\beta)$ in the space $(\mathcal{M}_p(\mathbb{R}_+))^2$ (along the subsequence (β_k)) and the Portmanteau theorem. It suffices to check that

$$\limsup_{\beta \to 0} p_n^\beta = 0.$$

We need to work with a random subdivision of the interval $[0, 8\pi t/\beta]$. As before, we consider a sequence $(\tau_k)_{k\in\mathbb{N}}$ of i.i.d. positive random variables distributed uniformly in $[\frac{1}{2}, \frac{3}{2}]$ and form the sum $S_k = \sum_{i=1}^k \tau_i$.

Noting that for all x, y such that $|x - y| \leq 1/(2n)$, there exists $k \in \mathbb{N}$ such that $x, y \in [S_k/n, S_k/n + 2/n]$, we obtain

$$p_n^\beta \leq \mathbb{P}\left[\exists k \leq \lfloor 2nt \rfloor + 1 : \lfloor \frac{\alpha_\lambda}{2\pi} \rfloor \text{ jumps on the interval } \frac{8\pi}{\beta}\left[\frac{S_k}{n}, (\frac{S_k}{n} + \frac{2}{n})\right] \text{ but not } \lfloor \frac{\alpha_{\lambda'}}{2\pi} \rfloor\right].$$

Due to the increasing property, the event inside the probability can not happen if the process $\{\alpha_\lambda\}_{2\pi}$ starts below $\{\alpha_{\lambda'}\}_{2\pi}$ at the beginning of the interval. Therefore,

$$p_n^\beta \leq \sum_{k=1}^{\lfloor 2nt \rfloor+1} \mathbb{P}\left[\{\alpha_{\lambda'}(\frac{8\pi}{\beta}\frac{S_k}{n})\}_{2\pi} \leq \{\alpha_\lambda(\frac{8\pi}{\beta}\frac{S_k}{n})\}_{2\pi}\right].$$

which can in turn be upper-bounded as follows

$$\sum_{k=1}^{\lfloor 2nt \rfloor+1} \mathbb{P}\left[\{\alpha_{\lambda'}(\frac{8\pi}{\beta}\frac{S_k}{n})\}_{2\pi} \leq \{\alpha_\lambda(\frac{8\pi}{\beta}\frac{S_k}{n})\}_{2\pi}\right] \leq \frac{\beta n}{4\pi} \mathbb{E}\left[\int_0^{\frac{8\pi}{\beta}(3t+1)} 1_{\{\{\alpha_{\lambda'}(u)\}_{2\pi} \leq \{\alpha_\lambda(u)\}_{2\pi}\}} du\right]$$

$$\leq \frac{\beta n}{4\pi}\Theta_\beta(3t + 1) \leq \beta^c \tag{3.8}$$

where we have used Lemma 3.1 to obtain the last inequality which holds for β small enough (c is a constant which does not depend on β).

$\square$

Lemma 3.5. *Let $\lambda < \lambda'$. Then the $(\mathcal{F}_t)$-Poisson point processes $\mathcal{P}_\lambda$ and $\mathcal{P}_{\lambda'-\lambda}$ are independent.*

Proof of Lemma 3.5. From a classical result (see Proposition (1.7) Chapter XII, §1, p.473 in [21]) on Poisson processes, we know that it suffices to prove that the two $(\mathcal{F}_t)$-Poisson processes $\mathcal{P}_\lambda$ and $\mathcal{P}_{\lambda'-\lambda}$ do not jump simultaneously, i.e. that for $t > 0$,

$$\mathbb{P}\left[\exists i, j \in \mathbb{N} : \xi_i^\lambda < t, \xi_j^{\lambda'-\lambda} < t, \xi_i^\lambda = \xi_j^{\lambda'-\lambda}\right] = 0.\tag{3.9}$$
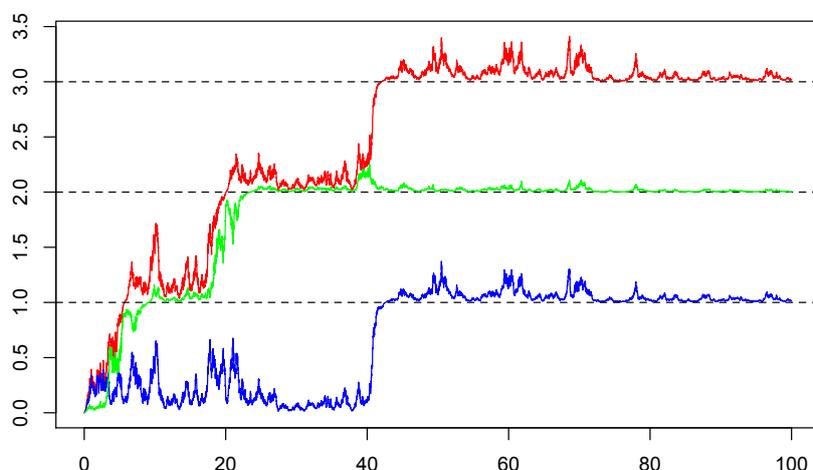
Figure 4: (Color online). Sample paths of the diffusions $\alpha_\lambda/(2\pi)$ (blue curve) and $\alpha_{\lambda'}/(2\pi)$ (red curve) together with $(\alpha_{\lambda'} - \alpha_\lambda)/(2\pi)$ (green curve) for $\lambda < \lambda'$ and for a small value of β. Again we see that $\lfloor \alpha_\lambda/(2\pi) \rfloor$ and $\lfloor (\alpha_{\lambda'} - \alpha_\lambda)/(2\pi) \rfloor$ never jump simultaneously while $\lfloor \alpha_{\lambda'}/(2\pi) \rfloor$ and $\lfloor (\alpha_{\lambda'} - \alpha_\lambda)/(2\pi) \rfloor$ always jump at the same times in agreement with Lemma 3.5, Lemma 3.4 and Remark 3.6.

For $n \in \mathbb{N}$, we consider the probability

$$p_n^\beta := \mathbb{P}\left[\exists i, j \in \mathbb{N} : \frac{\beta}{8\pi}\zeta_i^\lambda < t, \ \frac{\beta}{8\pi}\zeta_j^{\lambda'-\lambda} < t, \ \frac{\beta}{8\pi}|\zeta_i^\lambda - \zeta_j^{\lambda'-\lambda}| < \frac{1}{2n} \right]. \qquad (3.10)$$

If n is fixed, then we have the convergence (the studied set is open as in the proof Lemma 3.4)

$$\liminf_{\beta \to 0} p_n^\beta \geq p_n := \mathbb{P}\left[\exists i, j \in \mathbb{N} : \xi_i^\lambda < t, \ \xi_j^{\lambda'-\lambda} < t, \ |\xi_i^\lambda - \xi_j^{\lambda'-\lambda}| < \frac{1}{2n} \right].$$

To prove (3.9), it therefore suffices to prove that

$$\limsup_{n \to \infty} \limsup_{\beta \to 0} p_n^\beta = 0. \qquad (3.11)$$

For this, we need to work with a random subdivision of the interval $[0, 8\pi t/\beta]$. As before, we consider a sequence $(\tau_k, \ k \in \mathbb{N})$ of i.i.d. positive random variables uniformly distributed in $[1/2, 3/2]$ and independent of the processes and form the sum $S_k = \sum_{i=1}^k \tau_i$ ($S_0 := 0$).
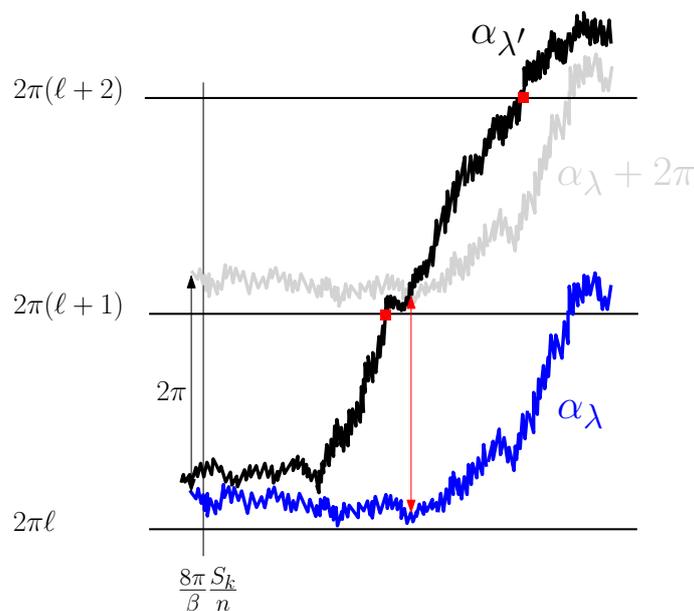
Figure 5: Illustration for inclusion (3.15).

The probability (3.10) can be upper-bounded as follows

$$\mathbb{P}\left[\exists i, j \in \mathbb{N}, k \leq \lfloor 2nt \rfloor + 1 : \frac{8\pi}{\beta}\frac{S_k}{n} \leq \zeta_i^{\lambda}, \zeta_j^{\lambda'-\lambda} \leq \frac{8\pi}{\beta}(\frac{S_k}{n} + \frac{2}{n})]\right]$$

$$= \mathbb{P}\left[\exists k \leq \lfloor 2nt \rfloor + 1 : \lfloor \frac{\alpha_{\lambda'} - \alpha_{\lambda}}{2\pi} \rfloor \text{ and } \lfloor \frac{\alpha_{\lambda}}{2\pi} \rfloor \text{ both jump on the interval } \frac{8\pi}{\beta}\left[\frac{S_k}{n}, (\frac{S_k}{n} + \frac{2}{n})\right]\right]$$

$$\leq \sum_{k=1}^{\lfloor 2nt \rfloor + 1} \mathbb{P}\left[\{\alpha_{\lambda'}(\frac{8\pi}{\beta}\frac{S_k}{n})\}_{2\pi} \leq \{\alpha_{\lambda}(\frac{8\pi}{\beta}\frac{S_k}{n})\}_{2\pi}\right] \tag{3.12}$$

$$+ \sum_{k=1}^{\lfloor 2nt \rfloor + 1} \mathbb{P}\left[\lfloor \frac{\alpha_{\lambda'}}{2\pi} \rfloor \text{ jumps two times during the interval } \frac{8\pi}{\beta}\left[\frac{S_k}{n}, (\frac{S_k}{n} + \frac{2}{n})\right]\right]. \tag{3.13}$$

For this bound, we have used the fact that, conditionally on

$$\{\alpha_{\lambda}(\frac{8\pi}{\beta}\frac{S_k}{n})\}_{2\pi} \leq \{\alpha_{\lambda'}(\frac{8\pi}{\beta}\frac{S_k}{n})\}_{2\pi}, \tag{3.14}$$

the increasing property and the equality in law (3.2) impose

$$\left\{\lfloor \frac{\alpha_{\lambda'} - \alpha_{\lambda}}{2\pi} \rfloor \text{ and } \lfloor \frac{\alpha_{\lambda}}{2\pi} \rfloor \text{ both jump on the interval } \frac{8\pi}{\beta}\left[\frac{S_k}{n}, (\frac{S_k}{n} + \frac{2}{n})\right]\right\} \tag{3.15}$$

$$\subseteq \left\{\lfloor \frac{\alpha_{\lambda'}}{2\pi} \rfloor \text{ jumps two times during the interval } \frac{8\pi}{\beta}\left[\frac{S_k}{n}, (\frac{S_k}{n} + \frac{2}{n})\right]\right\}.$$

Indeed, the equality in law (3.2) implies that $\lfloor \frac{\alpha_{\lambda'}(t) - \alpha_{\lambda}(t)}{2\pi} \rfloor$ is increasing with respect to t (once the difference $\alpha_{\lambda'} - \alpha_{\lambda}$ has reached the value $2k\pi$ where $k \in \mathbb{N}$, it remains forever above this value). This fact and the increasing property imply that under the event (3.15), the process $\lfloor \frac{\alpha_{\lambda'}}{2\pi} \rfloor$ has to jump two times on the interval $\frac{8\pi}{\beta}[\frac{S_k}{n}, (\frac{S_k}{n} + \frac{2}{n})]$ (see Fig. 5). We now estimate the two sums in (3.12). The first one was already tackled in (3.8).

For the other sum in (3.12), we write

$$\sum_{k=1}^{\lfloor 2nt \rfloor+1} \mathbb{P}\left[\lfloor \frac{\alpha_{\lambda'}}{2\pi} \rfloor \text{ jumps two times during the interval } \frac{8\pi}{\beta}\left[\frac{S_k}{n}, (\frac{S_k}{n} + \frac{2}{n})\right]\right]$$

$$= \sum_{k=1}^{\lfloor 2nt \rfloor+1} \mathbb{P}\left[\mu_{\lambda'}^{\beta}\left[\frac{8\pi}{\beta}\left[\frac{S_k}{n}, (\frac{S_k}{n} + \frac{2}{n})\right]\right] = 2\right].$$

If n is fixed and $\beta \to 0$, Theorem 2.4 gives the following convergence

$$\sum_{k=1}^{\lfloor 2nt \rfloor+1} \mathbb{P}\left[\mu_{\lambda'}^{\beta}[\frac{8\pi}{\beta}\frac{S_k}{n}[1, 1 + \frac{2}{n}]] = 2\right]$$

$$\to_{\beta \to 0} \frac{\lambda^2}{16\pi^2} \mathbb{E}\left[\sum_{k=1}^{\lfloor 2nt \rfloor+1} \exp\left(-\frac{\lambda}{2\pi} e^{-2\pi \frac{S_k}{n}}(1 - e^{-2\pi \frac{2}{n}})\right) e^{-4\pi \frac{S_k}{n}}(1 - e^{-2\pi \frac{2}{n}})^2\right]$$

$$= O(n^{-1}). \tag{3.16}$$

The convergence (3.11) now follows from the two estimates (3.8) and (3.16) and the independence between the two Poisson processes $\mathcal{P}_\lambda$ and $\mathcal{P}_{\lambda'-\lambda}$ is proved. $\qquad\square$

Remark 3.6. Using similar arguments as in the proof of Lemma 3.4, one could also show that for all $\lambda < \lambda'$, $\mathcal{P}_{\lambda'-\lambda} \subseteq \mathcal{P}_{\lambda'}$. Using then Lemma 3.5, it implies the almost-sure equality $\mathcal{P}_{\lambda'} = \mathcal{P}_\lambda + \mathcal{P}_{\lambda'-\lambda}$[4]. It therefore determines the law of the triplet $(\mathcal{P}_\lambda, \mathcal{P}_{\lambda'}, \mathcal{P}_{\lambda''-\lambda'})$ and the weak convergence of the family (3.3) when $\beta \to 0$ holds.

Nevertheless, we do not need this stronger result as we will only use the previous equality for the total number of points in $\mathbb{R}_+$ $\alpha_{\lambda'}(\infty)/(2\pi) = \alpha_\lambda(\infty)/(2\pi) + (\alpha_{\lambda'}(\infty) - \alpha_\lambda(\infty))/(2\pi)$ which holds trivially.

As an immediate consequence of Lemma 3.4 and Lemma 3.5, we obtain the following result using the characterization of independent Poisson processes mentioned above.

Lemma 3.7. Let $\lambda_0 < \lambda_1 < \lambda_2$. Then, the $(\mathcal{F}_t)$-Poisson processes $\mathcal{P}_{\lambda_0}$ and $\mathcal{P}_{\lambda_2-\lambda_1}$ are independent.

3.3   Proof of Theorem 1.1

From Kallenberg's theorem [15], we simply need to check that, for any finite union I of disjoint and bounded intervals, we have when $\beta \to 0$,

$$\mathbb{E}[\text{Sine}_\beta(I)] \longrightarrow \frac{1}{2\pi} \int_I d\lambda, \tag{3.17}$$

$$\mathbb{P}[\text{Sine}_\beta(I) = 0] \longrightarrow \exp(-\frac{1}{2\pi} \int_I d\lambda). \tag{3.18}$$

The convergence (3.17) follows from Corollary 2.6.

Towards (3.18), we consider first a union of two disjoint interval of the form

$$I := [\lambda_0, \lambda_1] \cup [\lambda_2, \lambda_3],$$

where $\lambda_0 < \lambda_1 < \lambda_2 < \lambda_3$. By translation invariance of the Sine_β point process, we can suppose without loss of generality that $\lambda_0 = 0$. By definition of Sine_β, we have the following equality in law

$$(\text{Sine}_\beta[0, \lambda_1], \text{Sine}_\beta[\lambda_2, \lambda_3],) \stackrel{(d)}{=} \left(\frac{\alpha_{\lambda_1}(\infty)}{2\pi}, \frac{\alpha_{\lambda_3}(\infty) - \alpha_{\lambda_2}(\infty)}{2\pi}\right).$$

---

[4]Note that this equality is obviously wrong when $\beta > 0$ i.e we don't have the equality between $\lfloor \alpha_{\lambda'}(t)/(2\pi) \rfloor$ and $\lfloor \alpha_{\lambda'}(t)/(2\pi) \rfloor + \lfloor (\alpha_{\lambda'}(t) - \alpha_\lambda(t))/(2\pi) \rfloor$.

Using Lemma 3.7, we deduce that

$$\mathbb{P}\left[\mathrm{Sine}_\beta[0, \lambda_1] = 0, \mathrm{Sine}_\beta[\lambda_2, \lambda_3] = 0\right]$$

$$\rightarrow_{\beta \to 0} \mathbb{P}\left[\mathcal{P}_{\lambda_1}(\infty) = 0\right] \mathbb{P}\left[\mathcal{P}_{\lambda_3 - \lambda_2}(\infty) = 0\right] = \exp\left(-\frac{\lambda_1}{2\pi}\right) \exp\left(-\frac{\lambda_3 - \lambda_2}{2\pi}\right).$$

Now, if $k \geq 3$ is an odd integer and $I := [\lambda_0, \lambda_1] \cup [\lambda_2, \lambda_3] \cup \cdots \cup [\lambda_{k-1}, \lambda_k]$ where $\lambda_0 \leq \cdots \leq \lambda_k$, the limiting Poisson point processes $(\mathcal{P}_{\lambda_i - \lambda_{i-1}})_{i=1, \cdots, k}$ defined in Lemma 3.3 are adapted to a common filtration and never jump simultaneously. Using again the characterization of independent Poisson processes, we know that they are independent processes (see Proposition (1.7) Chapter XII, §1, p.473 in [21]).

Theorem 1.1 is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

A Proof of auxiliary results.

A.1 Asymptotic of integrals and Laplace transform

Asymptotic of the integral (2.6). We denote by a_β (respectively b_β) the local minimum (resp. maximum) of the potential $V_\beta(r)$ and derive asymptotic expansions for a_β and b_β in the limit $\beta \to 0$ using

$$V_\beta'(r) = 0 \Leftrightarrow \frac{\lambda\beta}{8} = \frac{e^{-r} - e^r}{(e^r + e^{-r})^2}.$$

We deduce that

$$a_\beta = \log\beta + \log(\lambda) - 3\log 2 + O(\beta^2),$$

$$b_\beta = -\frac{\lambda\beta}{4} + o(\beta^2),$$

$$V_\beta(a_\beta) = \frac{1}{2}\log\beta + \frac{1}{2}\log\lambda - \log 2 + \frac{1}{2} + O(\beta^2),$$

$$V_\beta(b_\beta) = \frac{\lambda^2}{64}\beta^2 + O(\beta^2) = O(\beta^2).$$

More generally, we have for $x, y \in \mathbb{R}$

$$V_\beta(a_\beta + y) = \frac{1}{2}\log\frac{\beta\lambda}{4} + \frac{y + e^{-y}}{2} - \frac{1}{2}(\frac{\lambda\beta}{8})^2 e^y - \frac{1}{2}\log(1 + (\frac{\lambda\beta}{8})^2 e^{2y}) + O(\beta^2),$$

$$V_\beta(b_\beta + x) = -\frac{1}{2}\log\frac{e^x + e^{-x}}{2}(1 - \frac{\lambda\beta}{8} + O(\beta^2)) - \frac{\lambda\beta}{8}\sinh(x) + \frac{\lambda^2\beta^2}{64}\cosh(x) + O(\beta^2).$$

Those computations permit us to find the asymptotic behaviour of $t_\beta(r_\beta)$ (using also the bounded convergence theorem),

$$t_\beta(r_\beta) \sim \frac{8}{\beta\lambda}\int_{r_\beta - b_\beta}^{+\infty}\frac{dx}{\cosh(x)}\int_{-\infty}^{x + a_\beta} e^{-y}e^{-e^{-y}}dy \sim \frac{8}{\beta\lambda}\int_{r_\beta - b_\beta}^{+\infty}\frac{dx}{\cosh(x)} \sim \frac{8\pi}{\beta\lambda}. \qquad \text{(A.1)}$$

$$\square$$

Proof of Proposition 2.2.

We consider the Laplace transform $g_{\beta\lambda/(8\pi)\xi}(r)$ of the rescaled exit time $\beta\lambda/(8\pi)\zeta$ of the diffusion starting from r. If suffices to prove that, if $(r_\beta)_{\beta>0}$ is such that $r_\beta \to -\infty$ as $\beta \to 0$, we have for any $0 < \xi < 1$, $g_{\beta\lambda/(8\pi)\xi}(r_\beta) \to 1/(1 + \xi)$ (Laplace transform of an exponential distribution with parameter 1). To simplify notations, we will just write $g_\beta(r)$ for $g_{\beta\lambda/(8\pi)\xi}(r)$.

Let us recall the fixed point equation

$$g_\beta(r) = 1 - 2\frac{\beta\lambda}{8\pi}\xi \int_r^{+\infty} dx \int_{-\infty}^x \exp\left(2\left[V_\beta(x) - V_\beta(y)\right]\right) g_\beta(y)dy \,. \tag{A.2}$$

Noting that $g_\beta(r) \leq 1$, we obtain the lower bound

$$1 - \xi\frac{\beta\lambda}{8\pi}t_\beta(r) \leq g_\beta(r) \,. \tag{A.3}$$

Using the asymptotic (A.1), we obtain a lower bound

$$1 - \xi \leq \liminf_{\beta\to 0} g_\beta(r_\beta) \,.$$

Plugging (A.3) into (A.2), we get an upper bound

$$g_\beta(r) \leq 1 - \xi\frac{\beta\lambda}{8\pi}t_\beta(r) + 2\xi^2(\frac{\beta\lambda}{8\pi})^2 \int_r^{+\infty} dx \int_{-\infty}^x \exp\left(2\left[V_\beta(x) - V_\beta(y)\right]\right) t_\beta(y)dy \,.$$

After the derivation of the asymptotic quadruple integral (similar to the one of (A.1)), we obtain

$$\limsup_{\beta\to 0} g_\beta(r_\beta) \leq 1 - \xi + \xi^2 \,.$$

Iterating this argument (the multiple integrals are always such that the integration ranges permit to catch the maximum value of $V_\beta(x) - V_\beta(y)$ inside the exponential as in (A.1)), we get the result. □

A.2 Proof of the estimates for a single diffusion

Recall the definition of $R_\lambda := \log(\tan(\alpha_\lambda/4))$ and its differential equation (2.2):

$$dR_\lambda = \frac{1}{2}\left(\lambda\frac{\beta}{4}e^{-\frac{\beta}{4}t}\cosh(R_\lambda) + \tanh(R_\lambda)\right) dt + dB_t \,, \quad R_\lambda(0) = -\infty \,,$$

and denote by $T_r(t)$ the first passage time to $r \in \mathbb{R} \cup \{+\infty\}$ after time t i.e.

$$T_r(t) := \inf\{s \geqslant t : R_\lambda(s) = r\} \,.$$

In this sub-section, we first study the diffusion R_λ and then translate the estimates to the diffusion α_λ. We will denote by $\mathbb{P}_{r_0,t}$ the law of the diffusion R_λ starting from position r_0 at time t. To simplify notations, we omit the subscript t in $\mathbb{P}_{r_0,t}$ if $t = 0$ and write T_r instead of $T_r(t)$ if it appears in the probability space $\mathbb{P}_{r_0,t}$. We will also denote by P_x the law of a Brownian motion starting from x.

Our first lemma A.1 shows that if the diffusion R_λ is outside the well, then the probability that it explodes in a short time (compared to $1/\beta$) tends to 1.

Lemma A.1. *Let $0 < \varepsilon < 1$ and $s > 0$. Then, there exists a constant $c > 0$ (depending only on λ, s and ε) such that for all $\beta > 0$ small enough and $u \leq 8\pi s/\beta$,*

$$\mathbb{P}_{\varepsilon\log\frac{1}{\beta},u}\left[T_{+\infty} < 9\log\frac{1}{\beta}\right] \geqslant 1 - \beta^c \,.$$

Note that it immediately gives the analogous result Lemma 2.7 for α_λ.

Proof of Lemma A.1. We prove the desired inequality in the case $u = 0$ as the general one follows easily by changing λ in equation (2.2). We have

$$\mathbb{P}_{\varepsilon\log\frac{1}{\beta}}\left[T_{+\infty} < 9\log\frac{1}{\beta}\right] \geq \mathbb{P}_{\varepsilon\log\frac{1}{\beta}}\left[T_{2\log\frac{1}{\beta}} < 8\log\frac{1}{\beta} \wedge T_{\frac{\varepsilon}{2}\log\frac{1}{\beta}}\right] \mathbb{P}_{2\log\frac{1}{\beta},8\log\frac{1}{\beta}}\left[T_{+\infty} < \log\frac{1}{\beta}\right] \,.$$

If $\frac{\varepsilon}{2}\ln\frac{1}{\beta} < r < 2\log\frac{1}{\beta}$, the drift term $-V'_\beta(r) \geq \frac{1}{2}\frac{e^r - e^{-r}}{e^r + e^{-r}} \geq \frac{1}{4}$ for β small enough. Thus, for $t < T_{\frac{\varepsilon}{2}\log\frac{1}{\beta}}$, we have $R_\lambda(t) \geq B_t + t/4$. Therefore,

$$\mathbb{P}_{\varepsilon\log\frac{1}{\beta}}\left[T_{2\log\frac{1}{\beta}} < 8\log\frac{1}{\beta} \wedge T_{\frac{\varepsilon}{2}\log\frac{1}{\beta}}\right] \geq P_{\varepsilon\log\frac{1}{\beta}}\left[\inf_{0\leq t\leq 8\log\frac{1}{\beta}} B_t > \frac{\varepsilon}{2}\log\frac{1}{\beta}\right].$$

This latter probability is easily computed with the reflection principle for Brownian motion.

$$P_{\varepsilon\log\frac{1}{\beta}}\left[\inf_{0\leq t\leq 8\log\frac{1}{\beta}} B_t > \frac{\varepsilon}{2}\log\frac{1}{\beta}\right] = P_0\left[\sup_{0\leq t\leq 8\log\frac{1}{\beta}} B_t \leq \frac{\varepsilon}{2}\log\frac{1}{\beta}\right]$$

$$= P_0\left[|B(1)| \leq \frac{\varepsilon}{4\sqrt{2}}\sqrt{\log\frac{1}{\beta}}\right] = 1 - O(\exp(-c\log\frac{1}{\beta})).$$

We now consider the probability

$$\mathbb{P}_{2\log\frac{1}{\beta}, 8\log\frac{1}{\beta}}\left[T_{+\infty} < \log\frac{1}{\beta}\right]$$

We define $G_\lambda(t) := R_\lambda(t) - B_t$ such that $G_\lambda(0) = 2\log\frac{1}{\beta}$. The function G_λ satisfies the following random *ordinary* differential equation

$$G'_\lambda(t) = \frac{1}{2}\left(\frac{\lambda\beta}{8}e^{-\frac{\beta}{4}t}\frac{e^{G_\lambda(t)+B_t} + e^{-G_\lambda(t)-B_t}}{2} + \frac{e^{G_\lambda(t)+B_t} - e^{-G_\lambda(t)-B_t}}{e^{G_\lambda(t)+B_t} + e^{-G_\lambda(t)-B_t}}\right).$$

On the event

$$\mathcal{E}_\beta := \left\{\sup_{0\leq t\leq\log\frac{1}{\beta}} |B_t| \leq \frac{1}{2}\log\frac{1}{\beta}\right\},$$

which occurs with probability bigger than $1 - \exp(-c\log(\frac{1}{\beta}))$ where $c > 0$ is a constant independent of β, we can easily check that, for $t \in [0, \log\frac{1}{\beta}]$ and $\beta > 0$ small enough,

$$G'_\lambda(t) \geq \frac{\lambda\beta^{3/2}}{64}e^{G_\lambda(t)} - \frac{1}{2}.$$

This leads us to study the Cauchy problem

$$H'_\lambda(t) = \frac{\lambda\beta^{3/2}}{64}e^{H_\lambda(t)} - \frac{1}{2}, \quad H_\lambda(0) = 2\log\frac{1}{\beta}.$$

This ordinary differential equation can be solved explicitly. We find

$$e^{-H_\lambda(t)}e^{-\frac{t}{2}} = \beta^2 - \frac{\lambda\beta^{3/2}}{32}(1 - e^{-\frac{t}{2}}).$$

It is easy to see that the exploding time of H_λ is of order $\sqrt{\beta}$ as $\beta \to 0$. On the event $\mathcal{E}_\beta$, we have $H_\lambda(t) \leq G_\lambda(t)$ for all $t \geq 0$.

We can conclude that

$$\mathbb{P}_{2\log\frac{1}{\beta}, 8\log\frac{1}{\beta}}\left[T_{+\infty} < \log\frac{1}{\beta}\right] \geq P_0[\mathcal{E}_\beta] = 1 - O(\exp(-c\log(\frac{1}{\beta}))).$$

$\square$

Lemma A.2. *Let us fix $s > 0$. There exists $c' > 0$ (depending only on λ and s) such that for all $\beta > 0$ small enough, and for all $u \leq 8\pi s/\beta$, we have*

$$\mathbb{P}_{-\frac{1}{4}\log\frac{1}{\beta}, u}\left[T_{+\infty} < 10\log\frac{1}{\beta}\right] \geqslant c'\sqrt{\beta}. \tag{A.4}$$

Proof of Lemma A.2. As before, we prove (A.4) in the case $u = 0$ as the general case follows easily by changing λ in equation (2.2). The probability under consideration can be bounded below by

$$\mathbb{P}_{-\frac{1}{4}\log\frac{1}{\beta}}\left[T_{\frac{1}{4}\log\frac{1}{\beta}} < \log\frac{1}{\beta}\right]\mathbb{P}_{\frac{1}{4}\log\frac{1}{\beta},\log\frac{1}{\beta}}\left[T_{+\infty} < 9\log\frac{1}{\beta}\right].$$

For the first probability, we note that $R_\lambda(t) \geqslant -\frac{t}{2} + B_t$ and thus

$$\mathbb{P}_{-\frac{1}{4}\log\frac{1}{\beta}}\left[T_{\frac{1}{4}\log\frac{1}{\beta}} < \log\frac{1}{\beta}\right] \geqslant P_0\left[B(\log\frac{1}{\beta}) - \frac{1}{2}\log\frac{1}{\beta} \geq \frac{1}{2}\log\frac{1}{\beta}\right]$$

$$= P_0\left[B_1 \geqslant \sqrt{\log\frac{1}{\beta}}\right] \sim \frac{\sqrt{\beta}}{\sqrt{2\pi\log\frac{1}{\beta}}}.$$

From Lemma A.1, the second probability is bigger than $1 - \beta^c$ for some positive constant $c > 0$. $\qquad\square$

On the large scale-time of the order $1/\beta$, we prove that the time spent by R_λ away from the well of the potential V (away from 0 modulo 2π for α_λ) is negligible, i.e. small compared to the typical time $1/\beta$ between two jumps. This is Lemma A.3:

Lemma A.3. *Let $t > s \geq 0$ and $x \in \mathbb{R} \cup \{-\infty\}$. Consider*

$$\Xi_\beta(s,t,x) := \mathbb{E}_{x,8\pi s/\beta}\left[\int_{\frac{8\pi}{\beta}s}^{\frac{8\pi}{\beta}t} 1_{\{R_\lambda(u) \geqslant -\frac{1}{4}\log\frac{1}{\beta}\}}du\right].$$

Then, for all $t > s \geq 0$, there exists $C > 0$ (depending only on λ and t) such that for all $x \in \mathbb{R} \cup \{-\infty\}$ and $\beta > 0$, we have:

$$\Xi_\beta(s,t,x) \leqslant \frac{C}{\sqrt{\beta}}\log\frac{1}{\beta}.$$

Lemma A.3 can be translated for α_λ, it gives Lemma 2.8.

Proof of Lemma A.3. The key estimate is the lower-bound (A.4) given by Lemma A.2 on the probability for the diffusion R_λ starting from the position $-\frac{1}{4}\log\frac{1}{\beta}$ to blow up in a short time. The idea is then to relate the time spent by the diffusion R_λ above the level $-\frac{1}{4}\log\frac{1}{\beta}$ with the number of explosions in the interval $[8\pi s/\beta, 8\pi t/\beta]$ which is of order $O(1)$ (the typical time between two explosions is of order $1/\beta$ from Lemma 2.2). To ease the notations, denote by $\hat{s} := 8\pi s/\beta$. For any $u \geq \hat{s}$, recall that $T_{+\infty}(u)$ is the first explosion time after time u. We have

$$\mathbb{P}_{x,\hat{s}}\left[R_\lambda(u) \geqslant -\frac{1}{4}\log\frac{1}{\beta}\right]$$

$$\leqslant \mathbb{P}_{x,\hat{s}}\left[R_\lambda(u) \geqslant -\frac{1}{4}\log\frac{1}{\beta}, T_{+\infty}(u) < 10\log\frac{1}{\beta}\right] + \mathbb{P}_{x,\hat{s}}\left[R_\lambda(u) \geqslant -\frac{1}{4}\log\frac{1}{\beta}, T_{+\infty}(u) \geq 10\log\frac{1}{\beta}\right]$$

$$\leqslant \mathbb{P}_{x,\hat{s}}\left[R_\lambda(u) \geqslant -\frac{1}{4}\log\frac{1}{\beta}, T_{+\infty}(u) < 10\log\frac{1}{\beta}\right] + \left(1 - c'\sqrt{\beta}\right)\mathbb{P}_{x,\hat{s}}\left[R_\lambda(u) \geqslant -\frac{1}{4}\log\frac{1}{\beta}\right],$$

where we have used the simple Markov property at time u, the inequality (A.4) as well as the increasing property in the third line. We deduce that

$$\mathbb{P}_{x,\hat{s}}\left[R_\lambda(u) \geqslant -\frac{1}{4}\log\frac{1}{\beta}\right] \leqslant \frac{1}{c'\sqrt{\beta}}\,\mathbb{P}_{x,\hat{s}}\left[R_\lambda(u) \geqslant -\frac{1}{4}\log\frac{1}{\beta}, T_{+\infty}(u) < 10\log\frac{1}{\beta}\right]$$

$$\leqslant \frac{1}{c'\sqrt{\beta}}\,\mathbb{P}_{x,\hat{s}}\left[\text{The interval } [u, u+10\log\tfrac{1}{\beta}] \text{ contains at least one explosion}\right]. \quad \text{(A.5)}$$

Denote by k the total number of explosions of the diffusion R_λ starting at $R_\lambda(\hat{s}) = x$ in the interval $[8\pi s/\beta, 8\pi t/\beta]$. Let $8\pi s/\beta < \zeta_1 < \cdots < \zeta_k < 8\pi t/\beta$ be the explosion times, we easily see that almost surely

$$\int_{\frac{8\pi s}{\beta}}^{\frac{8\pi t}{\beta}} 1_{\{\exists i: \zeta_i \in [u, u+10\log\frac{1}{\beta}]\}}\,du \leq \left(10\log\frac{1}{\beta}\right) \times k\,.$$

Using Property (iv) of the diffusion α_λ, we immediatly derive that $\mathbb{E}_{x,\hat{s}}(k)$ is smaller than $1 + \lambda/(2\pi)$. Using this inequality and integrating (A.5) with respect to u, we finally obtain

$$\Xi_\beta(s,t,x) \leqslant \left(1 + \frac{\lambda}{2\pi}\right)\frac{10}{c'\sqrt{\beta}}\log\frac{1}{\beta}\,.$$

$\square$

A.3 Proof of Lemma 3.2

Denote $\mathcal{C}_0(\mathbb{R}_+)$ the space of continuous functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(t) \to 0$ as $t \to +\infty$. To prove that $\mathcal{P}_\lambda$ is indeed a $(\mathcal{F}_t)$-Poisson process with the correct intensity, it suffices to check that its Laplace functional satisfies

$$\mathbb{E}\left[\exp(-\int_s^t f(u)\mathcal{P}_\lambda(du))\big|\,\mathcal{F}_s\right] = \exp\left(-\int_s^t (1 - e^{-f(u)})\lambda e^{-2\pi u}du\right)\,. \quad \text{(A.6)}$$

We have to check that the filtration $(\mathcal{F}_t)$ does not contain too much information compared to the natural filtration denoted $(\mathcal{F}_t^\lambda)$ associated to the process $\mathcal{P}_\lambda$ only, for (A.6) to remain valid.

We denote by $(\mathcal{G}_t)$ the filtration associated to the complex Brownian motion (Z_t) which drives the processes $\alpha_\lambda, \alpha_{\lambda'}$ and $\alpha_{\lambda''}$ according to (1.5).

The conclusion of Theorem 2.4 is equivalent to the convergence of the Laplace functional of the point process μ_λ^β to the Laplace functional of the Poisson process $\mathcal{P}_\lambda$ (see Proposition 11.1.VIII (ii) of [9]). Let us fix $\varepsilon > 0$. Let us prove first that for any $\varphi \in \mathcal{C}_0(\mathbb{R}_+)$ and any $0 < s < t$, the following convergence holds in probability:

$$\mathbb{E}\left[\exp(-\int_{s+\varepsilon}^t \varphi(u)\mu_\lambda^\beta(du))\big|\,\mathcal{G}_{\frac{8\pi}{\beta}s}\right] \to_{\beta\to 0} \exp\left(-\int_{s+\varepsilon}^t \left(1 - e^{-\varphi(u)}\right)\lambda e^{-2\pi u}du\right)\,. \quad \text{(A.7)}$$

To show (A.7), the main point is that the sigma field $\mathcal{G}_{\frac{8\pi}{\beta}s}$ already contains the information on the two diffusions $(\alpha_\lambda(\frac{8\pi}{\beta}u))_{0\leq u\leq s}, (\alpha_{\lambda'}(\frac{8\pi}{\beta}u))_{0\leq u\leq s}$ and $(\alpha_{\lambda''}(\frac{8\pi}{\beta}u))_{0\leq u\leq s}$ up to time s as they are strong solutions to the stochastic differential equation system (1.5). Note that we have introduced a small gap of macroscopic size $\varepsilon > 0$ after time t as (A.7) is not true for $\varepsilon = 0$ because of the conditioning: it is indeed possible that the position of $\alpha_\lambda((\frac{8\pi}{\beta})s)$ (if it is very close to 2π modulo 2π) induces a jump quickly after time s.

This issue is circumvented by introducing an independent random time U distributed uniformly over the interval $[s, s+\varepsilon]$. The position of $\alpha_\lambda(\frac{8\pi}{\beta}U)$ belongs to an interval of

the type $[2k\pi, 2k\pi + 4\arctan(\beta^{1/4})]$ with probability going to 1 when $\beta \to 0$. Indeed, we have

$$\mathbb{P}[\{\alpha_\lambda(\frac{8\pi U}{\beta})\}_{2\pi} \geq 4\arctan(\beta^{1/4})|\,\mathcal{G}_{\frac{8\pi}{\beta}s}] = \frac{\beta}{8\pi}\mathbb{E}[\int_{\frac{8\pi s}{\beta}}^{\frac{8\pi(s+\varepsilon)}{\beta}} 1_{\{\{\alpha_\lambda(v)\}_{2\pi} \geq 4\arctan(\beta^{1/4})\}} dv|\,\mathcal{G}_{\frac{8\pi}{\beta}s}]$$

$$\leqslant \sup_{x\in\mathbb{R}_+} \frac{\beta}{8\pi}\mathbb{E}_{x,8\pi s/\beta}[\int_{\frac{8\pi s}{\beta}}^{\frac{8\pi(s+\varepsilon)}{\beta}} 1_{\{\{\alpha_\lambda(v)\}_{2\pi} \geq 4\arctan(\beta^{1/4})\}} dv]$$

$$= \frac{\beta}{8\pi} \sup_{x\in\mathbb{R}_+} \Xi_\beta(s, s+\varepsilon, x) \xrightarrow{}_{\beta\to 0} 0,$$

using for the second line the notation and result of Lemma 2.8.

Remark 2.5 after Theorem 2.4 then shows that the conditional law of the measure μ_λ^β over $[U, \infty)$ when α_λ modulo 2π starts below $4\arctan(\beta^{1/4})$ at time $8\pi U/\beta$ converges to the Poisson measure of intensity $\lambda \exp(-2\pi t)dt$ and (A.7) holds.

From the joint convergence of the triplet (3.4) along the sequence (β_k), we can deduce for $\varphi, f, g, h \in \mathcal{C}_0(\mathbb{R}_+)$,

$$\lim_{k\to\infty} \mathbb{E}\left[\exp\left(-\int_{s+\varepsilon}^t \varphi(u)\mu_\lambda^{\beta_k}(du)\right)\exp\left(-\int_0^s (f(u)\mu_\lambda^{\beta_k}(du) + g(u)\mu_{\lambda'}^{\beta_k}(du) + h(u)\mu_{\lambda''-\lambda'}^{\beta_k}(du))\right)\right]$$

$$= \mathbb{E}\left[\exp\left(-\int_{s+\varepsilon}^t \varphi(u)\mathcal{P}_\lambda(du)\right)\exp\left(-\int_0^s (f(u)\mathcal{P}_\lambda(du) + g(u)\mathcal{P}_{\lambda'}(du) + h(u)\mathcal{P}_{\lambda''-\lambda'}(du))\right)\right].$$
$$\tag{A.8}$$

On the other hand, using in addition (A.7), we can check that

$$\lim_{k\to\infty} \mathbb{E}\left[\exp\left(-\int_{s+\varepsilon}^t \varphi(u)\mu_\lambda^{\beta_k}(du)\right)\exp\left(-\int_0^s (f(u)\mu_\lambda^{\beta_k}(du) + g(u)\mu_{\lambda'}^{\beta_k}(du) + h(u)\mu_{\lambda''-\lambda'}^{\beta_k}(du))\right)\right]$$

$$= \exp\left(-\int_{s+\varepsilon}^t \left(1 - e^{-\varphi(u)}\right)\lambda e^{-2\pi u}du\right)\mathbb{E}\left[\exp\left(-\int_0^s (f(u)\mathcal{P}_\lambda(du) + g(u)\mathcal{P}_{\lambda'}(du) + h(u)\mathcal{P}_{\lambda''-\lambda'}(du))\right)\right].$$
$$\tag{A.9}$$

Gathering (A.8) and (A.9) and taking the limit $\varepsilon \to 0$ (as a.s. $\mathcal{P}_\lambda$ does not jump on s), we obtain (A.6).

$\square$

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