Electronic Journal of Probability

Electron. J. Probab. **19** (2014), no. 104, 1–22.
ISSN: 1083-6489 DOI: 10.1214/EJP.v19-3388

Convergence in L^p and its exponential rate for a branching process in a random environment

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Abstract

We consider a supercritical branching process (Z_n) in a random environment ξ. Let W be the limit of the normalized population size $W_n = Z_n/\mathbb{E}[Z_n|\xi]$. We first show a necessary and sufficient condition for the quenched L^p ($p > 1$) convergence of (W_n), which completes the known result for the annealed L^p convergence. We then show that the convergence rate is exponential, and we find the maximal value of $\rho > 1$ such that $\rho^n(W - W_n) \to 0$ in L^p, in both quenched and annealed sense. Similar results are also shown for a branching process in a varying environment.

Keywords: branching process; varying environment; random environment; moments; exponential convergence rate; L^p convergence.
AMS MSC 2010: 60K37; 60J80.
Submitted to EJP on March 18, 2014, final version accepted on August 21, 2014.

1 Introduction and main results

Branching process in a random environment (BPRE), is an important extension of the Galton-Watson process in the aspect of random environments. This model was first introduced by Smith & Wilkinson [18] in the independent and identically distributed environment case, and then by Athreya & Karlin [3, 4] in the stationary and ergodic environment case. As it is a fundamental process for branching systems such as branching random walks, branching Markov processes in random environments, where the offspring distributions vary according to a random environment, the asymptotic properties of BPRE received many authors' attention recently, see for example [6, 7, 8, 12, 15]. Meanwhile, during our previous related works, we notice that many limit behavior such as large deviations of branching systems in random environments may rely on the convergence (especially the L^p convergence) and its rates of the martingale of the corresponding BPRE. For this reason, in this paper we focus on the L^p convergence and its exponential rates of the martingale for a supercritical BPRE. We study the sufficient

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conditions for the L^p convergence (at an exponential rate), and find the critical value of the rate, in both quenched and annealed sense. Our results complete the annealed convergence of Guivarc'h & Liu [11], and extend the corresponding ones of Liu [17] and Alsmeyer & Iksanov*et al.* [1] on the Galton-Watson process.

Let us give a description of the model – *a branching process in a stationary and ergodic random environment*. Let $\xi = (\xi_0, \xi_1, \xi_2, \cdots)$ be a stationary and ergodic process taking values in some measurable space $(\Theta, \mathcal{E})$. Without loss of generality we can suppose that ξ is defined on the product space $(\Theta^{\mathbb{N}}, \mathcal{E}^{\otimes \mathbb{N}}, \tau)$, with $\mathbb{N} = \{0, 1, 2, \cdots\}$ and τ the law of ξ. Each realization of ξ_n corresponds to a probability distribution on $\mathbb{N}$, denoted by $p(\xi_n) = (p_k(\xi_n))_{k\in\mathbb{N}}$, where

$$p_k(\xi_n) \geq 0, \quad \sum_k p_k(\xi_n) = 1 \quad \text{and} \quad \sum_k k p_k(\xi_n) \in (0, \infty).$$

The sequence $\xi = (\xi_n)$ will be called *random environment*. A branching process (Z_n) in the random environment ξ is a class of branching processes in varying environment indexed by ξ. By definition,

$$Z_0 = 1, \qquad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i} \quad (n \geq 0), \tag{1.1}$$

where $X_{n,i}(i = 1, 2, \cdots)$ denotes the number of offspring of the ith particle in the nth generation. Given ξ, $\{X_{n,i} : n \geq 0, i \geq 1\}$ is a family of (conditionally) independent random variables and each $X_{n,i}$ has distribution $p(\xi_n)$.

For each realization $\xi \in \Theta^{\mathbb{N}}$ of the environment sequence, let $(\Gamma, \mathcal{G}, \mathbb{P}_\xi)$ be the probability space under which the process is defined (when the environment ξ is fixed to be the given realization). The probability $\mathbb{P}_\xi$ is usually called *quenched law*. The total probability space can be formulated as the product space $(\Theta^{\mathbb{N}} \times \Gamma, \mathcal{E}^{\mathbb{N}} \otimes \mathcal{G}, \mathbb{P})$, where $\mathbb{P} = \mathbb{E}(\delta_\xi \otimes \mathbb{P}_\xi)$ with δ_ξ the Dirac measure at ξ and $\mathbb{E}$ the expectation with respect to the law of ξ, so that for all measurable and positive g defined on $\Theta^{\mathbb{N}} \times \Gamma$, we have

$$\int_{\Theta^{\mathbb{N}}\times\Gamma} g(x, y) d\mathbb{P}(x, y) = \mathbb{E} \int_\Gamma g(\xi, y) d\mathbb{P}_\xi(y).$$

The total probability $\mathbb{P}$ is usually called *annealed law*. The quenched law $\mathbb{P}_\xi$ may be considered to be the conditional probability of $\mathbb{P}$ given ξ. The expectation with respect to $\mathbb{P}$ will still be denoted by $\mathbb{E}$; there will be no confusion for reason of consistence. The expectation with respect to $\mathbb{P}_\xi$ will be denoted by $\mathbb{E}_\xi$.

Let $\mathcal{F}_0 = \sigma(\xi)$ and $\mathcal{F}_n = \sigma(\xi, (X_{l,i} : \ 0 \leq l \leq n-1, \ i = 1, 2, \cdots))$ be the σ-field generated by ξ and $X_{l,i}$ $(0 \leq l \leq n-1, \ i = 1, 2, \cdots)$. For $n \geq 0$ and $p \geq 1$, set

$$m_n(p) = m_n(p, \xi) = \sum_k k^p p_k(\xi_n), \quad m_n = m_n(1), \tag{1.2}$$

and

$$P_0 = 1, \qquad P_n = \prod_{i=0}^{n-1} m_i \ \ (n \geq 1). \tag{1.3}$$

So $m_n(p) = \mathbb{E}_\xi X_{n,i}^p$ and $P_n = \mathbb{E}_\xi Z_n$. It is well known that the normalized population size

$$W_n = \frac{Z_n}{P_n} \tag{1.4}$$

is a non-negative martingale with respect to $\mathcal{F}_n$ both under $\mathbb{P}_\xi$ for every ξ and under $\mathbb{P}$, hence the limit

$$W = \lim_{n\to\infty} W_n \tag{1.5}$$

exists almost surely (a.s.) with $\mathbb{E}W \leq 1$ by Fatou's lemma. Assume throughout the paper that the process is supercritical in the sense that $\mathbb{E}\log m_0$ is well defined with

$$\mathbb{E}\log m_0 > 0.$$

Here we are interested in the L^p convergence rate of W_n to W both in the quenched sense (under $\mathbb{P}_\xi$) and in the annealed sense (under $\mathbb{P}$).

We first show a criterion for the quenched L^p convergence of W_n.

Theorem 1.1 (Quenched L^p convergence)**.** *Let $p > 1$. Consider the following assertions:*

(i) $\mathbb{E}\log \mathbb{E}_\xi \left(\frac{Z_1}{m_0}\right)^p < \infty$; (ii) $\sup_n \mathbb{E}_\xi W_n^p < \infty$ $a.s.$;
(iii) $W_n \to W$ *in* L^p *under* $\mathbb{P}_\xi$ *for almost all* ξ; (iv) $0 < \mathbb{E}_\xi W^p < \infty$ $a.s.$.

Then the following implications hold: (i) ⇒ (ii) ⇔ (iii) ⇔ (iv). If additionally (ξ_n) are i.i.d. and $\mathbb{E}\log m_0 < \infty$, then all the four assertions are equivalent.

For the almost notion in (ii)-(iv), we mean that the concerned statement holds for almost every realization $\xi \in \Theta^{\mathbb{N}}$, that is, it holds for τ- almost every $\xi \in \Theta^N$ (recall that we use the same letter ξ to denote both the random variable and a realization), or equivalently, the statement holds for $\mathbb{P}$-almost every $(\theta,\gamma) \in (\Theta^N \times \Gamma, \mathbb{P})$ if we regard $\xi = \xi(\theta,\gamma)$ as a random variable defined on the total probability space $(\Theta^N \times \Gamma, \mathbb{P})$. The equivalence can be easily seen by the definition of $\mathbb{P}$.

The implications (ii) ⇔ (iii) ⇒ (iv) are direct consequences of the L^p convergence theorem for martingales. The non evident part is the sufficiency of the condition (i) for the quenched L^p convergence of W_n, which is also necessary in the independent environment case.

It can be easily seen that $\forall p > 0$, $\mathbb{E}\log \mathbb{E}_\xi (\frac{Z_1}{m_0})^p < \infty$ if and only if $\mathbb{E}\log^+ \mathbb{E}_\xi |\frac{Z_1}{m_0} - 1|^p < \infty$, where and hereafter we use the following usual notations:

$$\log^+ x = \max(\log x, 0), \quad \log^- x = \max(-\log x, 0).$$

Next we give a description of the quenched L^p convergence rate, with the notations

$$a \wedge b = \min(a,b), \quad a \vee b = \max(a,b).$$

Theorem 1.2 (Exponential rate of quenched L^p convergence)**.** *Let $p > 1$, $\rho > 1$ and $m = \exp(\mathbb{E}\log m_0) > 1$.*

(a) If $\mathbb{E}\log \mathbb{E}_\xi \left(\frac{Z_1}{m_0}\right)^p < \infty$, then

$$\lim_{n\to\infty} \rho^n (\mathbb{E}_\xi |W - W_n|^p)^{1/p} = 0 \quad a.s. \qquad \textit{for } \rho < \min\{m^{1-1/p}, m^{1/2}\}.$$

(b) If $\mathbb{E}\log^- \mathbb{E}_\xi \left|\frac{Z_1}{m_0} - 1\right|^{p\wedge 2} < \infty$ and $\mathbb{E}\log^+ \mathbb{E}_\xi \left|\frac{Z_1}{m_0} - 1\right|^{p\vee 2} < \infty$, then a.s.

$$\limsup_{n\to\infty} \rho^n (\mathbb{E}_\xi |W - W_n|^p)^{1/p} \begin{cases} = 0 & \textit{if } \rho < \bar{\rho}_c, \\ > 0 & \textit{if } \rho > \bar{\rho}_c, \end{cases}$$

where $\bar{\rho}_c = m^{1/2} = \exp(\frac{1}{2}\mathbb{E}\log m_0) > 1$.

We mention that the theorem is valid with evident interpretation even if $\mathbb{E}\log m_0 = \infty$ (so that $m = \infty$).

Theorem 1.2(a) shows that $W_n \to W$ in L^p under $\mathbb{P}_\xi$ at an exponential rate; Theorem 1.2(b) means that $\bar{\rho}_c$ is the critical value of $\rho > 1$ for which $\rho^n(W - W_n) \to 0$ in L^p under $\mathbb{P}_\xi$ for almost all ξ.

For the classical Galton-Watson process, Theorem 1.2(a) reduces to the result of Liu [17] that if $\mathbb{E}Z_1^p < \infty$, then $\rho^n(W - W_n) \to 0$ in L^p for $1 < \rho < \min\{m^{1-1/p}, m^{1/2}\}$, where $m = \mathbb{E}Z_1 \in (1, \infty)$; Theorem 1.2(b) can be obtained by a result of Alsmeyer & Iksanov *et al.* [1] on branching random walks.

Recall that for a Galton-Watson process with $m = \mathbb{E}Z_1 \in (1, \infty)$ and $\mathbb{P}(W > 0) > 0$, Asmussen [2] showed that for $p \in (1, 2)$, $W - W_n = o(m^{-n/q})$ a.s. if and only if $\mathbb{E}Z_1^p < \infty$, where $1/p + 1/q = 1$. As an application of Theorem 1.2, we immediately obtain the following similar result for a branching process in a random environment.

Corollary 1.3 (Exponential rate of a.s. convergence)**.** *Let* $p \in (1, 2)$ *and* $m = \exp(\mathbb{E}\log m_0) \in (0, \infty)$. *If* $\mathbb{E}\log \mathbb{E}_\xi(\frac{Z_1}{m_0})^p < \infty$, *then* $\forall \varepsilon > 0$,

$$W - W_n = o(m^{-\frac{n}{q+\varepsilon}}) \quad a.s., \tag{1.6}$$

where $1/p + 1/q = 1$.

In fact, to see the conclusion, let $\rho_1 = m^{\frac{1}{q+\varepsilon}}$ and take ρ satisfying $\rho_1 < \rho < m^{1/q}$. By Theorem 1.2(a), $\rho^n(\mathbb{E}_\xi|W - W_n|^p)^{1/p} \to 0$, so that

$$\begin{aligned} \mathbb{E}_\xi\left(\sum_n \rho_1^n |W - W_n|\right) &\leq \left(\mathbb{E}_\xi\left(\sum_n \rho_1^n |W - W_n|\right)^p\right)^{1/p} \\ &\leq \sum_n \left(\frac{\rho_1}{\rho}\right)^n \rho^n (\mathbb{E}_\xi|W - W_n|^p)^{1/p} < \infty \ \ a.s.. \end{aligned} \tag{1.7}$$

Therefore the series $\sum_n \rho_1^n|W - W_n|$ converges a.s., which implies (1.6).

Corollary 1.3 has recently been shown by Huang & Liu [13] by a truncating argument. The approach here is quite different.

We now turn to the annealed L^p convergence of W_n. When the environment is i.i.d., a necessary and sufficient condition was shown by Guivarc'h and Liu ([11], Theorem 3).

Proposition 1.4 (Annealed L^p convergence [11])**.** *Assume that* (ξ_n) *are i.i.d. and* $p > 1$. *Then the following assertions are equivalent:*

$$\begin{array}{ll} (i)\ \mathbb{E}\left(\frac{Z_1}{m_0}\right)^p < \infty \text{ and } \mathbb{E}m_0^{1-p} < 1; & (ii)\ \sup_n \mathbb{E}W_n^p < \infty; \\ (iii)\ W_n \to W \text{ in } L^p \text{ under } \mathbb{P}; & (iv)\ 0 < \mathbb{E}W^p < \infty. \end{array}$$

We shall prove the following theorem about the rate of convergence.

Theorem 1.5 (Exponential rate of annealed L^p convergence)**.** *Assume that* (ξ_n) *are i.i.d.. Let* $p > 1$ *and* $\rho > 1$.

(a) Assume that $\mathbb{E}\left(\frac{Z_1}{m_0}\right)^p < \infty$ *and* $\mathbb{E}m_0^{1-p} < 1$. *Then*

$$\lim_{n\to\infty} \rho^n(\mathbb{E}|W - W_n|^p)^{1/p} = 0 \quad \text{for } \rho < \rho_0,$$

where $\rho_0 > 1$ *is defined by*

$$\rho_0 = \begin{cases} (\mathbb{E}m_0^{1-p})^{-1/p} & \text{if } p \in (1, 2), \\ \min\{(\mathbb{E}m_0^{1-p})^{-1/p}, (\mathbb{E}m_0^{-p/2})^{-1/p}\} & \text{if } p \geq 2. \end{cases}$$

(b) Assume that $\mathbb{P}(W_1 = 1) < 1$ *and that either of the following conditions is satisfied:*

(i) $p \in (1, 2)$, $\mathbb{E}\left(\mathbb{E}_\xi\left(\frac{Z_1}{m_0}\right)^2\right)^{p/2} < \infty$, $\mathbb{E}m_0^{-p/2}\log m_0 > 0$ *and* $\mathbb{E}m_0^{-p/2-1}Z_1\log^+ Z_1 < \infty$;

(ii) $p \geq 2$ *and* $\mathbb{E}(\frac{Z_1}{m_0})^p < \infty$.

Set

$$\rho_c = \begin{cases} (\mathbb{E}m_0^{-p/2})^{-1/p} & \textit{if } p \in (1, 2), \\ \min\{(\mathbb{E}m_0^{1-p})^{-1/p}, (\mathbb{E}m_0^{-p/2})^{-1/p}\} & \textit{if } p \geq 2. \end{cases}$$

Then

$$\limsup_{n\to\infty} \rho^n(\mathbb{E}|W - W_n|^p)^{1/p} \begin{cases} = 0 & \textit{if } \rho < \rho_c, \\ > 0 & \textit{if } \rho > \rho_c. \end{cases}$$

Remark 1.6. By the convexity of the function $\mathbb{E}m_0^{-x}$, the condition $\mathbb{E}m_0^{-p/2}\log m_0 > 0$ implies that $\mathbb{E}m_0^{-x}$ is strictly decreasing on $(-\infty, \frac{p}{2}]$. Thus $\mathbb{E}m_0^{-p/2} < \mathbb{E}m_0^{1-p}$ for $p \in (1, 2)$, so that $\rho_0 \leq \rho_c$.

Theorem 1.5(a) implies that $W_n \to W$ in L^p under $\mathbb{P}$ (annealed) at an exponential rate. Theorem 1.5(b) shows that under certain moment conditions, ρ_c is the critical value of $\rho > 1$ for the annealed L^p convergence of $\rho^n(W - W_n)$ to 0, while Theorem 1.2(b) shows that $\bar{\rho}_c$ is the critical value for the quenched L^p convergence. Notice that by Jensen's inequality,

$$\mathbb{E}m_0^{-p/2} = \mathbb{E}\exp(-\frac{p}{2}\log m_0) \geq \exp(-\frac{p}{2}\mathbb{E}\log m_0),$$

so that $(\mathbb{E}m_0^{-p/2})^{-1/p} \leq \exp(\frac{1}{2}\mathbb{E}\log m_0)$. This shows that $\rho_c \leq \bar{\rho}_c$.

Finally, thanks to the exponential rates of W_n to W, we get the convergence of the series $\sum_n |W - W_n|$ by arguments similar to (1.7).

Corollary 1.7 (Convergence of the series). *Let* $p > 1$. *If* $\mathbb{E}\log\mathbb{E}_\xi\left(\frac{Z_1}{m_0}\right)^p < \infty$, *then the series* $\sum_n |W - W_n|$ *converges a.s. and in* L^p *under* $\mathbb{P}_\xi$ *for almost all* ξ. *If additionally* (ξ_n) *are i.i.d. and* $\mathbb{E}\left(\frac{Z_1}{m_0}\right)^p < \infty$, *the convergence also holds in* L^p *under* $\mathbb{P}$.

The rest of this paper is organized as follows. In Section 2, we consider the L^p convergence of the martingale W_n and its exponential rate for a branching process in a varying environment. In Sections 3 and 4, we study the random environment case, and give the proofs of the main results: in Section 3, we consider the quenched case and prove Theorems 1.1 and 1.2; in Section 4, we consider the annealed case and give the proof of Theorem 1.5.

2 Branching process in a varying environment

In this section, as preliminaries, we study the L^p convergence and its convergence rate for a branching process (Z_n) in a varying environment (BPVE). By definition,

$$Z_0 = 1, \qquad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i} \quad (n \geq 0), \tag{2.1}$$

where $X_{n,i}(i = 1, 2, \cdots)$ denotes the number of offspring of the ith particle in the nth generation, each $X_{n,i}$ has distribution $p(n) = (p_k(n))_{k\in\mathbb{N}}$ on $\mathbb{N} = \{0, 1, \cdots\}$, where

$$p_k(n) \geq 0, \quad \sum_k p_k(n) = 1 \quad \text{and} \quad \sum_k kp_k(n) \in (0, \infty);$$

all the random variables $X_{n,i}(n \geq 0, i \geq 1)$ are independent of each other. Let $(\Gamma, \mathbb{P})$ be the underlying probability space. For $n \geq 0$ and $p \geq 1$, set

$$m_n(p) = \mathbb{E}X_{n,i}^p = \sum_k k^p p_k(n), \qquad m_n = m_n(1), \tag{2.2}$$

and

$$\bar{m}_n(p) = \mathbb{E}\left|\frac{X_{n,i}}{m_n} - 1\right|^p = \sum_k \left|\frac{k - m_n}{m_n}\right|^p p_k(n). \tag{2.3}$$

Let $\mathcal{F}_0 = \{\emptyset, \Gamma\}$ and $\mathcal{F}_n = \sigma((X_{l,i} : \ 0 \leq l \leq n-1, \ i = 1, 2, \cdots))$ be the σ-field generated by $X_{l,i}$ $(0 \leq l \leq n-1, \ i = 1, 2, \cdots)$. Similarly to the case of BPRE, set

$$P_0 = 1, \qquad P_n = \prod_{i=0}^{n-1} m_i \ \ (n \geq 1). \tag{2.4}$$

Then the normalized population size $W_n = Z_n / P_n$ is a non-negative martingale with respect to the filtration $\mathcal{F}_n$, and $\lim_{n\to\infty} W_n = W$ a.s. for some non-negative random variable W with $\mathbb{E}W \leq 1$. It is well known that there is a non-negative but possibly infinite random variable Z_∞ such that $Z_n \to Z_\infty$ in distribution as $n \to \infty$. We are interested in the supercritical case where $\mathbb{P}(Z_\infty = 0) < 1$, so that by ([14], Corollary 3), either $\sum_{n=0}^{\infty}(1 - p_1(n)) < \infty$, or $\lim_{n\to\infty} P_n = \infty$. Here we assume that $\lim_{n\to\infty} P_n = \infty$.

We are interested in the L^p convergence of the martingale W_n and its convergence rate. We have the following theorem.

Theorem 2.1 (Exponential rate of L^p convergence of W_n for BPVE)**.** *Let (Z_n) be the BPVE defined in (2.1) and let $\rho \geq 1$.*

(i) Let $p \in (1, 2)$. If the series $\sum_n \rho^{pn} P_n^{p(1/r-1)} \bar{m}_n(r)^{p/r} < \infty$ for some $r \in [p, 2]$, then

$$(\mathbb{E}|W - W_n|^p)^{1/p} = o(\rho^{-n}). \tag{2.5}$$

Conversely, if $\liminf_{n\to\infty} \frac{\log P_n}{n} > 0$ and (2.5) holds, then we have for any $s > 0$, the series $\sum_n P_n^{-s-p/2} \bar{m}_n(p) < \infty$, and $\sum_n {\rho_1}^{pn} P_n^{-s-p/2} \bar{m}_n(p) < \infty$ for all $\rho_1 \in (1, \rho)$ if $\rho > 1$.

(ii) Let $p \geq 2$. If the series $\sum_n \rho^{2n} P_n^{-1} \bar{m}_n(p)^{2/p} < \infty$, then (2.5) holds. Conversely, if (2.5) holds, then we have for any $r \in [2, p]$, the series $\sum_n P_n^{p(1/r-1)} \bar{m}_n(r)^{p/r} < \infty$, and $\sum_n {\rho_1}^{pn} P_n^{p(1/r-1)} \bar{m}_n(r)^{p/r} < \infty$ for all $\rho_1 \in (1, \rho)$ if $\rho > 1$.

When $\rho = 1$, Theorem 2.1 actually show a criteria for the L^p convergence of W_n, or equivalently, $\sup_n \mathbb{E}W_n^p < \infty$. In particular, for $p = 2$, one can see that $\sup_n \mathbb{E}W_n^2 = 1 + \sum_{n=0}^{\infty} P_n^{-1} \bar{m}_n(2)$. So $\sup_n \mathbb{E}W_n^2 < \infty$ if and only if $\sum_n P_n^{-1} \bar{m}_n(2) < \infty$, as shown by Jagers ([14], Theorem 4).

2.1 The martingale $\{\hat{A}_n\}$

To estimate the exponential rate of L^p convergence of W_n, following [1], we consider the series

$$A(\rho) = \sum_{n=0}^{\infty} \rho^n (W - W_n) \quad (\rho > 1). \tag{2.6}$$

Here $A(\rho)$ denotes the series; it will also denote the sum of the series when the series converges. The convergence of the series $A(\rho)$ reflects the exponential rate of $W - W_n$. More precisely, if the series $A(\rho)$ converges a.s. (resp. in L^p, $p > 1$), then $\rho^n(W - W_n) \to 0$

a.s. (resp. in L^p). Conversely, if $\rho^n(W - W_n) \to 0$ a.s. (resp. in L^p), then for $\rho_1 \in (1, \rho)$, the series $A(\rho_1)$ converges a.s. (resp. in L^p). Moreover, according to Remark 2.2 below, we shall see that the L^p convergence of the series $A(\rho)$ implies its a.s. convergence.

As in [1], we introduce an associated martingale $\{\hat{A}_n\}$. Let $\rho \geq 1$, and define

$$\hat{A}_n = \hat{A}_n(\rho) = \sum_{k=0}^{n} \rho^k (W_{k+1} - W_k), \qquad \hat{A}(\rho) = \sum_{n=0}^{\infty} \rho^n (W_{n+1} - W_n). \tag{2.7}$$

As in the case of $A(\rho)$, here $\hat{A}(\rho)$ also denotes the series and it also denote the sum of the series when the series converges. It is easy to see that $\{(\hat{A}_n; \mathcal{F}_{n+1})\}$ forms a martingale. In particular, for $\rho = 1$, $\hat{A}_n = W_{n+1} - 1$. By the convergence theorems for martingales, $\sup_n \mathbb{E}|\hat{A}_n|^p < \infty$ implies that the series $\hat{A}(\rho)$ converges a.s. and in L^p. Therefore the L^p convergence of $\hat{A}(\rho)$ is equivalent to $\sup_n \mathbb{E}|\hat{A}_n|^p < \infty$. Moreover, if $\hat{A}(\rho)$ converges in L^p, then it also converges a.s..

It is known that the series $A(\rho)$ and $\hat{A}(\rho)$ have the following relations.

Lemma 2.2 ([1], Lemma 3.1)**.** *Let $p > 1$ and $\rho > 1$. The series $A(\rho)$ converges a.s. (resp. in L^p) if and only if the same is true for the series $\hat{A}(\rho)$.*

Remark 2.3. According to the relations between $\hat{A}_n$ and $\hat{A}(\rho)$ stated above, Lemma 2.2 in fact tells us that $A(\rho)$ converges in L^p if and only if $\sup_n \mathbb{E}|\hat{A}_n|^p < \infty$, and the L^p convergence of $A(\rho)$ implies its a.s. convergence.

We shall study the L^p convergence of $A(\rho)$ through the existence of the pth-moment of the martingale $\{\hat{A}_n\}$. The main tool is Burkholder's inequality for martingales.

Lemma 2.4 (Burkholder's inequality, see e.g. [9])**.** *Let $\{S_n\}$ be a L^1 martingale with $S_0 = 0$. Let $Q_n = (\sum_{k=1}^{n} (S_k - S_{k-1})^2)^{1/2}$ and $Q = (\sum_{n=1}^{\infty} (S_n - S_{n-1})^2)^{1/2}$. Then $\forall p > 1$,*

$$c_p \parallel Q_n \parallel_p \leq \parallel S_n \parallel_p \leq C_p \parallel Q_n \parallel_p,$$

$$c_p \parallel Q \parallel_p \leq \sup_n \parallel S_n \parallel_p \leq C_p \parallel Q \parallel_p,$$

where $c_p = (p-1)/18p^{3/2}$, $C_p = 18p^{3/2}/(p-1)^{1/2}$.

Put

$$a_p = \left[(p-1)/18p^{3/2}\right]^p \qquad \text{and} \qquad b_p = \left[18p^{3/2}/(p-1)^{1/2}\right]^p.$$

The following lemma gives the relations between $\sup_n \mathbb{E}|\hat{A}_n|^p$ and $\mathbb{E}|W_{n+1} - W_n|^p$ that we shall use later.

Lemma 2.5. *Let $p > 1$. Then:*

(i) For $p \in (1, 2)$ and $N \geq 1$,

$$a_p N^{p/2-1} \sum_{n=0}^{N-1} \rho^{pn} \mathbb{E}|W_{n+1} - W_n|^p \leq \sup_n \mathbb{E}|\hat{A}_n|^p \leq b_p \sum_{n=0}^{\infty} \rho^{pn} \mathbb{E}|W_{n+1} - W_n|^p. \tag{2.8}$$

(ii) For $p = 2$,

$$\sup_n \mathbb{E}|\hat{A}_n|^2 = \sum_{n=0}^{\infty} \rho^{2n} \mathbb{E}|W_{n+1} - W_n|^2. \tag{2.9}$$

(iii) For $p > 2$,

$$a_p \sum_{n=0}^{\infty} \rho^{pn} \mathbb{E}|W_{n+1} - W_n|^p \leq \sup_n \mathbb{E}|\hat{A}_n|^p \leq b_p \left(\sum_{n=0}^{\infty} \rho^{2n} (\mathbb{E}|W_{n+1} - W_n|^p)^{2/p} \right)^{p/2}. \tag{2.10}$$

Proof. Firstly, (2.9) is obvious by the orthogonality of martingale. The upper bound in (2.8) and (2.10) are directly from Burkholder's inequality. The lower bound in (2.8) can be obtained following similar arguments in [1] (p.25). □

Remark 2.6. For a BPRE, notice that $\{W_n\}$ is a martingale under both $\mathbb{P}_\xi$ (for every ξ) and $\mathbb{P}$, and the same is true for $\{\hat{A}_n\}$. Thus Lemmas 2.2 and 2.5 hold for both expectations $\mathbb{E}_\xi$ and $\mathbb{E}$.

Let

$$\bar{X}_{n,i} = \frac{X_{n,i}}{m_n}, \qquad \bar{X}_n = \bar{X}_{n,1}. \tag{2.11}$$

From the definitions of Z_n and W_n, we have

$$W_{n+1} - W_n = \frac{1}{P_n} \sum_{i=1}^{Z_n} (\bar{X}_{n,i} - 1). \tag{2.12}$$

This fact leads us to estimate $\mathbb{E}|W_{n+1} - W_n|^p$ through the moments of $\bar{X}_n - 1$.

Lemma 2.7. *Let $p > 1$, $n \geq 0$. Then:*

(i) For $p \in (1,2)$ and $r \in [p,2]$,

$$a_p P_n^{-p/2} \mathbb{E} W_n^{p/2} \mathbb{E}|\bar{X}_n - 1|^p \leq \mathbb{E}|W_{n+1} - W_n|^p \leq b_p P_n^{p(1/r-1)} (\mathbb{E}|\bar{X}_n - 1|^r)^{p/r}. \tag{2.13}$$

(ii) For $p = 2$,

$$\mathbb{E}|W_{n+1} - W_n|^2 = P_n^{-1} \mathbb{E}|\bar{X}_n - 1|^2. \tag{2.14}$$

(iii) For $p > 2$ and $r \in [2,p]$,

$$a_p P_n^{p(1/r-1)} (\mathbb{E}|\bar{X}_n - 1|^r)^{p/r} \leq \mathbb{E}|W_{n+1} - W_n|^p \leq b_p P_n^{-p/2} \mathbb{E} W_n^{p/2} \mathbb{E}|\bar{X}_n - 1|^p. \tag{2.15}$$

Proof. We first prove (ii). By (2.12),

$$\mathbb{E}|W_{n+1} - W_n|^2 = \frac{1}{P_n^2} \mathbb{E} \sum_{i=1}^{Z_n} (\bar{X}_{n,i} - 1)^2 = P_n^{-1} \mathbb{E}|\bar{X}_n - 1|^2.$$

We then prove (i) and (iii). Let $p > 1$. Fix $n \geq 0$ and let

$$S_0 = 0, \qquad S_k = P_n^{-1} \sum_{i=1}^{k} (\bar{X}_{n,i} - 1) \mathbf{1}_{\{Z_n \geq i\}}.$$

Let $\mathcal{G}_0 = \mathcal{F}_n$ and $\mathcal{G}_k = \sigma(\mathcal{F}_n, X_{n,i}, 1 \leq i \leq k)$. It is not difficult to verify that $\{S_k\}$ forms a martingale with respect to $\mathcal{G}_k$ and $\{S_k\}$ is uniformly integrable, so that $\sup_k \mathbb{E}|S_k|^p = \mathbb{E}|S|^p$, where $S = \lim_{k\to\infty} S_k = P_n^{-1} \sum_{i=1}^{Z_n} (\bar{X}_{n,i} - 1) = W_{n+1} - W_n$. By Burkholder's inequality,

$$a_p \mathbb{E} \left| \sum_{k=1}^{\infty} (S_k - S_{k-1})^2 \right|^{p/2} \leq \mathbb{E}|S|^p \leq b_p \mathbb{E} \left| \sum_{k=1}^{\infty} (S_k - S_{k-1})^2 \right|^{p/2},$$

which means that

$$a_p \mathbb{E} \left| \frac{1}{P_n^2} \sum_{i=1}^{Z_n} (\bar{X}_{n,i} - 1)^2 \right|^{p/2} \leq \mathbb{E}|W_{n+1} - W_n|^p \leq b_p \mathbb{E} \left| \frac{1}{P_n^2} \sum_{i=1}^{Z_n} (\bar{X}_{n,i} - 1)^2 \right|^{p/2}. \tag{2.16}$$

For $p \in (1,2)$ and $r \in [p,2]$, by the concavity of $x^{r/2}$, $x^{p/r}$ and $x^{p/2}$, we have

$$\mathbb{E}\left|\frac{1}{P_n^2}\sum_{i=1}^{Z_n}(\bar{X}_{n,i}-1)^2\right|^{p/2} \le P_n^{-p}(\mathbb{E}\sum_{i=1}^{Z_n}|\bar{X}_{n,i}-1|^r)^{p/r} \le P_n^{p(1/r-1)}(\mathbb{E}|\bar{X}_n-1|^r)^{p/r}, \quad (2.17)$$

and

$$\mathbb{E}\left|\frac{1}{P_n^2}\sum_{i=1}^{Z_n}(\bar{X}_{n,i}-1)^2\right|^{p/2} \ge P_n^{-p}\mathbb{E}Z_n^{p/2-1}\sum_{i=1}^{Z_n}|\bar{X}_{n,i}-1|^p = P_n^{-p/2}\mathbb{E}W_n^{p/2}\mathbb{E}|\bar{X}_n-1|^p. \quad (2.18)$$

Combing (2.17), (2.18) with (2.16), we obtain (2.13).

For $p>2$ and $r \in [2,p]$, since $x^{r/2}$, $x^{p/r}$ and $x^{p/2}$ are convex, (2.17) holds with " $\le$ " replaced by " $\ge$ ", while (2.18) holds with " $\ge$ " replaced by " $\le$ ". □

2.2 Moments of $\{\hat{A}_n\}$; Proof of Theorem 2.1

In this section, we study the pth-moment of $\{\hat{A}_n\}$, and prove Theorem 2.1.

Proposition 2.8 (Moments of $\hat{A}_n$ for BPVE)**.** *Let $\rho \ge 1$.*

(i) Let $p \in (1,2)$. If $\sum_n \rho^{pn} P_n^{p(1/r-1)} \bar{m}_n(r)^{p/r} < \infty$ for some $r \in [p,2]$, then

$$\sup_n \mathbb{E}|\hat{A}_n|^p < \infty. \quad (2.19)$$

Conversely, if $\liminf_{n\to\infty} \frac{\log P_n}{n} > 0$ and (2.19) holds, then $\sum_n \rho^{pn} P_n^{-s-p/2} \bar{m}_n(p) < \infty$ for any $s>0$.

(ii) Let $p \ge 2$. If $\sum_n \rho^{2n} P_n^{-1} \bar{m}_n(p)^{2/p} < \infty$, then (2.19) holds. Conversely, if (2.19) holds, then for any $r \in [2,p]$, $\sum_n \rho^{pn} P_n^{p(1/r-1)} \bar{m}_n(r)^{p/r} < \infty$.

Before the proof of Proposition 2.8, we give another lower bound of $\sup_n \mathbb{E}|\hat{A}_n|^p$ for $p \in (1,2)$, which is different from (2.8).

Lemma 2.9. *Let $p \in (1,2)$ and $s>0$. If $\eta = \eta(s) := \sum_n P_n^{-s} < \infty$, then*

$$\sup_n \mathbb{E}|\hat{A}_n|^p \ge a_p \eta^{p/2-1} \sum_{n=0}^{\infty} \rho^{pn} P_n^{s(p/2-1)} \mathbb{E}|W_{n+1}-W_n|^p. \quad (2.20)$$

Proof. Applying Burkholder's inequality and Jensen's inequality, we get

$$\begin{aligned}
\sup_n \mathbb{E}|\hat{A}_n|^p &\ge a_p \mathbb{E}\left|\sum_{n=0}^{\infty}\rho^{2n}(W_{n+1}-W_n)^2\right|^{p/2} \\
&= a_p \mathbb{E}\left(\sum_{n=0}^{\infty}\frac{1}{\eta P_n^s}(\eta P_n^s \rho^{2n}|W_{n+1}-W_n|^2)\right)^{p/2} \\
&\ge a_p \mathbb{E}\sum_{n=0}^{\infty}\frac{1}{\eta P_n^s}(\eta P_n^s \rho^{2n}|W_{n+1}-W_n|^2)^{p/2} \\
&= a_p \eta^{p/2-1}\sum_{n=0}^{\infty}\rho^{pn}P_n^{s(p/2-1)}\mathbb{E}|W_{n+1}-W_n|^p.
\end{aligned}$$

So (2.20) is proved. □

Proof of Proposition 2.8. (i) By Lemmas 2.5 and 2.7, for $r \in [p,2]$,

$$\sup_n \mathbb{E}|\hat{A}_n|^p \le C\sum_{n=0}^{\infty}\rho^{pn}\mathbb{E}|W_{n+1}-W_n|^p \le C\sum_{n=0}^{\infty}\rho^{pn}P_n^{p(1/r-1)}(\mathbb{E}|\bar{X}_n-1|^r)^{p/r}.$$

Here and throughout this paper C denotes a general positive constant (maybe different from line to line). Hence $\sup_n \mathbb{E}|\hat{A}_n|^p < \infty$, if $\sum_n \rho^{pn} P_n^{p(1/r-1)} \bar{m}_n(r)^{p/r} < \infty$ for some $r \in [p, 2]$. Conversely, assume that $\sup_n \mathbb{E}|\hat{A}_n|^p < \infty$. For any $s > 0$, let $s' = \frac{2s}{2-p} > 0$. Since $\eta = \eta(s') < \infty$, by (2.20) and Lemma 2.7,

$$\sup_n \mathbb{E}|\hat{A}_n|^p \geq C\eta^{p/2-1} \inf_n \mathbb{E}W_n^{p/2} \sum_{n=0}^{\infty} \rho^{pn} P_n^{-s-p/2} \bar{m}_n(p).$$

Thus $\sum_n \rho^{pn} P_n^{-s-p/2} \bar{m}_n(p) < \infty$, $\forall s > 0$.

(ii) For $p = 2$, by (2.9) and (2.14),

$$\sup_n \mathbb{E}|\hat{A}_n|^2 = \sum_{n=0}^{\infty} \rho^{2n} P_n^{-1} \mathbb{E}|\bar{X}_n - 1|^2 = \sum_{n=0}^{\infty} \rho^{2n} P_n^{-1} \bar{m}_n(2).$$

Thus $\sup_n \mathbb{E}|\hat{A}_n|^2 < \infty$ if and only if $\sum_n \rho^{2n} P_n^{-1} \bar{m}_n(2) < \infty$.

Let $p > 2$. We first assume that $\sum_n \rho^{2n} P_n^{-1} (\mathbb{E}|\bar{X}_n - 1|^p)^{2/p}$ $(= \sum_n \rho^{2n} P_n^{-1} \bar{m}_n(p)^{2/p})$ $< \infty$. By (2.10) and (2.15),

$$\begin{aligned} \sup_n \mathbb{E}|\hat{A}_n|^p &\leq C \left(\sum_{n=0}^{\infty} \rho^{2n} (\mathbb{E}|W_{n+1} - W_n|^p)^{2/p} \right)^{p/2} \\ &\leq C \left(\sum_{n=0}^{\infty} \rho^{2n} P_n^{-1} (\mathbb{E}W_n^{p/2})^{2/p} (\mathbb{E}|\bar{X}_n - 1|^p)^{2/p} \right)^{p/2} \\ &\leq C \sup_n \mathbb{E}W_n^{p/2} \left(\sum_{n=0}^{\infty} \rho^{2n} P_n^{-1} (\mathbb{E}|\bar{X}_n - 1|^p)^{2/p} \right)^{p/2} < \infty, \qquad (2.21) \end{aligned}$$

provided that $\sup_n \mathbb{E}W_n^{p/2} < \infty$, which holds obviously when $\sup_n \mathbb{E}W_n^p < \infty$. Therefore, it suffices to prove that for every integer $b \geq 1$,

$$\sup_n \mathbb{E}W_n^p < \infty \quad \text{if} \quad \sum_n P_n^{-1} (\mathbb{E}|\bar{X}_n - 1|^p)^{2/p} < \infty, \quad \forall p \in (2^b, 2^{b+1}]. \qquad (2.22)$$

We shall prove (2.22) by induction on b. For $b = 1$, we consider $p \in (2, 2^2]$, so that $p/2 \in (1, 2]$. By Hölder's inequality,

$$\sum_n P_n^{-1} \mathbb{E}|\bar{X}_n - 1|^2 \leq \sum_n P_n^{-1} (\mathbb{E}|\bar{X}_n - 1|^p)^{2/p} < \infty.$$

Hence $\sup_n \mathbb{E}W_n^2 < \infty$, so that $\sup_n \mathbb{E}W_n^{p/2} < \infty$. By (2.21) (with $\rho = 1$),

$$\sup_n \mathbb{E}|W_n - 1|^p \leq C \sup_n \mathbb{E}W_n^{p/2} \left(\sum_{n=0}^{\infty} P_n^{-1} (\mathbb{E}|\bar{X}_n - 1|^p)^{2/p} \right)^{p/2} < \infty. \qquad (2.23)$$

So (2.22) holds for $b = 1$. Now assume that (2.22) holds for $p \in (2^b, 2^{b+1}]$ for some integer $b \geq 1$. For $p \in (2^{b+1}, 2^{b+2}]$, we have $p/2 \in (2^b, 2^{b+1}]$. By Hölder's inequality,

$$\sum_n P_n^{-1} (\mathbb{E}|\bar{X}_n - 1|^{p/2})^{4/p} \leq \sum_n P_n^{-1} (\mathbb{E}|\bar{X}_n - 1|^p)^{2/p} < \infty.$$

Using (2.22) for $p/2$, we obtain $\sup_n \mathbb{E}W_n^{p/2} < \infty$, so that $\sup_n \mathbb{E}W_n^p < \infty$ from (2.23). Therefore (2.22) still holds for $p \in (2^{b+1}, 2^{b+2}]$, which implies that (2.22) holds for all integers $b \geq 1$.

Conversely, assume that $\sup_n \mathbb{E}|\hat{A}_n|^p < \infty$. Notice that by (2.10) and (2.15), $\forall r \in [2, p]$,

$$\sup_n \mathbb{E}|\hat{A}_n|^p \geq C \sum_{n=0}^{\infty} \rho^{pn} P_n^{p(1/r-1)} (\mathbb{E}|\bar{X}_n - 1|^r)^{p/r}.$$

This implies that $\sum_n \rho^{pn} P_n^{p(1/r-1)} \bar{m}_n(r)^{p/r} < \infty$, $\forall r \in [2, p]$. □

Now we give proof of Theorem 2.1.

Proof of Theorem 2.1. For $\rho = 1$, notice that $W_{n+1} - 1 = \hat{A}_n(1)$ and $W_n \to W$ in L^p is equivalent to $\sup_n \mathbb{E}W_n^p < \infty$. For $\rho > 1$, notice that the assertion $\sup_n \mathbb{E}|\hat{A}_n|^p < \infty$ is equivalent to the L^p convergence of $\hat{A}(\rho)$, which is also equivalent to the L^p convergence of $A(\rho)$ by Lemma 2.2. Applying Proposition 2.8 yields Theorem 2.1. □

3 Quenched moments and quenched L^p convergence rate for BPRE; Proofs of Theorems 1.1 and 1.2

Let us return to a BPRE (Z_n). Notice that for each fixed ξ, (Z_n) is a BPVE. So all the results for BPVE can be directly applied to BPRE by considering the quenched law $\mathbb{P}_\xi$ and the corresponding expectation $\mathbb{E}_\xi$. The following lemma will be used to prove our theorems for BPRE.

Lemma 3.1. *Let $(\alpha_n, \beta_n)_{n\geq 0}$ be a stationary and ergodic sequence of non-negative random variables. If $\mathbb{E}\log\alpha_0 < 0$ and $\mathbb{E}\log^+\beta_0 < \infty$, then*

$$\sum_{n=0}^{\infty} \alpha_0 \cdots \alpha_{n-1} \beta_n < \infty \quad a.s.. \tag{3.1}$$

Conversely, we have:

(a) if $(\alpha_n, \beta_n)_{n\geq 0}$ are $i.i.d.$ and $\mathbb{E}\log\alpha_0 \in (-\infty, 0)$, then (3.1) implies that $\mathbb{E}\log^+\beta_0 < \infty$;

(b) if $\mathbb{E}|\log\beta_0| < \infty$, then (3.1) implies that $\mathbb{E}\log\alpha_0 \leq 0$.

Proof. The sufficiency is a direct consequence of the ergodic theorem and Cauchy's test for the convergence of series, remarking that if $\mathbb{E}\log\alpha_0 < 0$ and $\mathbb{E}\log\max(\beta_0, 1) < \infty$, then

$$\limsup_{n\to\infty} \frac{1}{n} \log(\alpha_0 \cdots \alpha_{n-1} \max(\beta_n, 1)) < 0.$$

For the necessity, part (a) was shown in the proof of ([10], Theorem 4.1). For part (b), again by Cauchy's test, if (3.1) holds, then

$$\limsup_{n\to\infty} (\alpha_0 \cdots \alpha_{n-1}\beta_n)^{1/n} \leq 1 \quad a.s.,$$

which is equivalent to

$$\limsup_{n\to\infty} \left(\frac{1}{n} \sum_{i=0}^{n-1} \log\alpha_i + \frac{1}{n}\log\beta_n\right) \leq 0 \quad a.s..$$

By the ergodic theorem,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log\alpha_i = \mathbb{E}\log\alpha_0 \qquad a.s.,$$

and

$$\lim_{n\to\infty}\frac{1}{n}\log\beta_n=\lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=0}^{n}\log\beta_i-\frac{1}{n}\sum_{i=0}^{n-1}\log\beta_i\right)=\mathbb{E}\log\beta_0-\mathbb{E}\log\beta_0=0\quad a.s..$$

Hence $\mathbb{E}\log\alpha_0\le 0$. □

Applying Proposition 2.8 to the case in random environment, and thanks to Lemma 3.1, we obtain the following results for the quenched moments of $\hat{A}_n$.

Proposition 3.2 (Quenched moments of $\hat{A}_n$)**.** *Let* $\rho\ge 1$ *and* $m=\exp(\mathbb{E}\log m_0)>1$.

(i) Let $p\in(1,2)$. *If* $\mathbb{E}\log^+\mathbb{E}_\xi|\frac{Z_1}{m_0}-1|^r<\infty$ *and* $\rho<m^{1-1/r}$ *for some* $r\in[p,2]$, *then*

$$\sup_n\mathbb{E}_\xi|\hat{A}_n|^p<\infty\quad a.s.. \tag{3.2}$$

Conversely, if $\mathbb{E}\left|\log\mathbb{E}_\xi|\frac{Z_1}{m_0}-1|^p\right|<\infty$ *and (3.2) holds, then* $\rho\le m^{1/2}$.

(ii) Let $p\ge 2$. *If* $\mathbb{E}\log^+\mathbb{E}_\xi|\frac{Z_1}{m_0}-1|^p<\infty$ *and* $\rho<m^{1/2}$, *then (3.2) holds. Conversely, if* $\mathbb{E}\left|\log\mathbb{E}_\xi|\frac{Z_1}{m_0}-1|^r\right|<\infty$ *for some* $r\in[2,p]$ *and (3.2) holds, then* $\rho\le m^{1-1/r}$.

Proof. (i) Let $p\in(1,2)$. Suppose that $\mathbb{E}\log^+\mathbb{E}_\xi|\frac{Z_1}{m_0}-1|^r<\infty$ and $\rho<m^{1-1/r}$ for some $r\in[p,2]$. Then by Lemma 3.1, the series $\sum_n\rho^{pn}P_n^{p(1/r-1)}\bar{m}_n(r)^{p/r}<\infty$ a.s.. Thus $\sup_n\mathbb{E}_\xi|\hat{A}_n|^p<\infty$ a.s. by Proposition 2.8.

Conversely, suppose that $\mathbb{E}\left|\log\mathbb{E}_\xi|\frac{Z_1}{m_0}-1|^p\right|<\infty$ and $\sup_n\mathbb{E}_\xi|\hat{A}_n|^p<\infty$ a.s.. By Proposition 2.8, we have $\forall s>0$, $\sum_n\rho^{pn}P_n^{-s-p/2}\bar{m}_n(p)<\infty$ a.s.. Hence by Lemma 3.1, $\rho\le m^{1/2+s/p}$. Letting $s\to 0$, we get $\rho\le m^{1/2}$.

(ii) Let $p\ge 2$. Suppose that $\mathbb{E}\log^+\mathbb{E}_\xi|\frac{Z_1}{m_0}-1|^p<\infty$ and $\rho<m^{1/2}$. Then by Lemma 3.1, the series $\sum_n\rho^{2n}P_n^{-1}\bar{m}_n(p)^{2/p}<\infty$ *a.s.*, which implies that $\sup_n\mathbb{E}_\xi|\hat{A}_n|^p<\infty$ a.s. by Proposition 2.8.

Conversely, suppose that $\mathbb{E}\left|\log\mathbb{E}_\xi|\frac{Z_1}{m_0}-1|^r\right|<\infty$ for some $r\in[2,p]$ and $\sup_n\mathbb{E}_\xi|\hat{A}_n|^p<\infty$ a.s.. Proposition 2.8 shows that $\sum_n\rho^{pn}P_n^{p(1/r-1)}\bar{m}_n(r)^{p/r}<\infty$ a.s., which implies that $\rho\le m^{1-1/r}$ by Lemma 3.1. □

Proof of Theorem 1.1. The implications "(ii) ⇒ (iii) ⇒ (iv)" are evident. And the implication "(i) ⇒ (ii)" is directly from Proposition 3.2 (with ρ=1). We prove that (iv) implies (ii). Notice that for $n\ge 1$,

$$W=\frac{1}{P_n}\sum_{i=1}^{Z_n}W(n,i)\qquad a.s., \tag{3.3}$$

where under $\mathbb{P}_\xi$, $(W(n,i))_{i\ge 1}$ are independent of each other and independent of Z_n, with distribution $\mathbb{P}_\xi(W(n,i)\in\cdot)=\mathbb{P}_{T^n\xi}(W\in\cdot)$. Here the notation T denotes the shift operator such that $T^n\xi=(\xi_n,\xi_{n+1},\cdots)$ if $\xi=(\xi_0,\xi_1,\cdots)$. Taking conditional expectation at both sides of (3.3), we see that

$$\mathbb{E}_\xi W=\mathbb{E}_{T^n\xi}W\qquad a.s..$$

Therefore, by the ergodicity, $\mathbb{E}_\xi W=c$ a.s. for some constant $c\in[0,\infty]$. As $\mathbb{E}_\xi W^p>0$ a.s., we have $c>0$. Again by (3.3) and Jensen's inequality,

$$\mathbb{E}_\xi(W^p|\mathcal{F}_n)\ge\left(\mathbb{E}_\xi\left(\frac{1}{P_n}\sum_{i=1}^{Z_n}W(n,i)\middle|\mathcal{F}_n\right)\right)^p=c^pW_n^p\qquad a.s.,$$

so that

$$\mathbb{E}_\xi W_n^p \le c^{-p}\mathbb{E}_\xi W^p \qquad a.s., \quad \forall n \ge 1.$$

Therefore, $\sup_n \mathbb{E}_\xi W_n^p \le c^{-p}\mathbb{E}_\xi W^p < \infty$ a.s. (so that $c = 1$ as then $W_n \to W$ in L^p under $\mathbb{P}_\xi$).

We finally prove that (ii) implies (i) when the environment is i.i.d.. Assume that $(\xi_n)_{n\ge 0}$ are i.i.d, $\mathbb{E}\log m_0 < \infty$ and $\sup_n \mathbb{E}_\xi W_n^p < \infty$ a.s.. By Theorem 2.1, we have

$$\sum_n P_n^{-s-p/2}\bar{m}_n(p) < \infty \; a.s., \forall s > 0, \quad if \; p \in (1,2),$$

and

$$\sum_n P_n^{1-p}\bar{m}_n(p) < \infty \; a.s. \quad if \; p \ge 2.$$

As $(\xi_n)_{n\ge 0}$ are $i.i.d.$ and $\mathbb{E}\log m_0 \in (0,\infty)$, by Lemma 3.1, $\mathbb{E}\log^+ \mathbb{E}_\xi|\frac{Z_1}{m_0} - 1|^p < \infty$, so that $\mathbb{E}\log \mathbb{E}_\xi \left(\frac{Z_1}{m_0}\right)^p < \infty$. □

By the relations among $\hat{A}_n$, $A(\rho)$ and $\rho^n(W - W_n)$ (discussed at the beginning of Section 2.1), together with Proposition 3.2, we immediately obtain the following criteria for the quenched L^p convergence rate of W_n.

Theorem 3.3 (Exponential rate of quenched L^p convergence of W_n)**.** *Let $\rho > 1$ and $m = \exp(\mathbb{E}\log m_0) > 1$.*

(i) Let $p \in (1,2)$. If $\mathbb{E}\log \mathbb{E}_\xi \left(\frac{Z_1}{m_0}\right)^r < \infty$ and $\rho < m^{1-1/r}$ for some $r \in [p,2]$, then

$$(\mathbb{E}_\xi|W - W_n|^p)^{1/p} = o(\rho^{-n}) \qquad a.s.. \tag{3.4}$$

Conversely, if $\mathbb{E}\left|\log \mathbb{E}_\xi \left|\frac{Z_1}{m_0} - 1\right|^p\right| < \infty$ and (3.4) holds, then $\rho \le m^{1/2}$.

(ii) Let $p \ge 2$. If $\mathbb{E}\log \mathbb{E}_\xi \left(\frac{Z_1}{m_0}\right)^p < \infty$ and $\rho < m^{1/2}$, then (3.4) holds. Conversely, if $\mathbb{E}\left|\log \mathbb{E}_\xi \left|\frac{Z_1}{m_0} - 1\right|^r\right| < \infty$ for some $r \in [2,p]$ and (3.4) holds, then $\rho \le m^{1-1/r}$.

Proof of Theorem 1.2. The assertion (a) is a direct consequence of Theorem 3.3(i) with $r = p$ for $p \in (1,2)$ and Theorem 3.3(ii) for $p \ge 2$.

For the assertion (b), since the L^p norm $(\mathbb{E}_\xi|X|^p)^{1/p}$ is increasing in p and the function $\log^+ x$ is increasing in x, we have $\frac{1}{p_1}\log^+ \mathbb{E}_\xi|X|^{p_1} \le \frac{1}{p_2}\log^+ \mathbb{E}_\xi|X|^{p_2}$ if $1 \le p_1 \le p_2$. Thus the condition $\mathbb{E}\log^+ \mathbb{E}_\xi \left|\frac{Z_1}{m_0} - 1\right|^{p\vee 2} < \infty$ ensures that $\mathbb{E}\log^+ \mathbb{E}_\xi \left|\frac{Z_1}{m_0} - 1\right|^p < \infty$ and $\mathbb{E}\log^+ \mathbb{E}_\xi \left|\frac{Z_1}{m_0} - 1\right|^2 < \infty$. If $\rho < m^{1/2}$, applying Theorem 3.3(i) with $r = 2$ for $p \in (1,2)$ and Theorem 3.3(ii) for $p \ge 2$, we have

$$\lim_{n\to\infty} \rho^n(\mathbb{E}_\xi|W - W_n|^p)^{1/p} = 0 \qquad a.s..$$

Now consider the case where $\rho > m^{1/2}$. Denote

$$D = \{\xi : \lim_{n\to\infty} \rho^n(\mathbb{E}_\xi|W - W_n|^p)^{1/p} = 0\}.$$

First, we show that $\mathbb{P}(D) = 0$ or 1. By the ergodicity, it suffices to show that $T^{-1}D = D$ a.s.. By (3.3),

$$W = \frac{1}{m_0}\sum_{i=1}^{Z_1} W(1,i) \qquad a.s..$$

Similarly, we can write W_n as

$$W_n = \frac{1}{m_0} \sum_{i=0}^{Z_1} W_{n-1}(1,i) \qquad a.s., \tag{3.5}$$

where $W_n(k,i) = \frac{Z_n(k,i)}{m_k \cdots m_{k+n-1}}$ with $Z_n(k,i)$ denoting the branching process starting with the ith particle in the kth generation. Under $\mathbb{P}_\xi$, the sequence $(W_n(k,i))_{i\geq 1}$ are independent of each other and independent of Z_k, and have a common conditional distribution $\mathbb{P}_\xi(W_n(k,i) \in \cdot) = \mathbb{P}_{T^k\xi}(W_n \in \cdot)$. Therefore,

$$W - W_n = \frac{1}{m_0} \sum_{i=1}^{Z_1} (W(1,i) - W_{n-1}(1,i)) \qquad a.s.. \tag{3.6}$$

By (3.6) and the convexity of x^p, we have

$$\begin{aligned} \mathbb{E}_\xi |W - W_n|^p &\leq \frac{1}{m_0^p} \mathbb{E}_\xi \left(\sum_{i=1}^{Z_1} |W(1,i) - W_{n-1}(1,i)| \right)^p \\ &\leq \frac{1}{m_0^p} \mathbb{E}_\xi Z_1^{p-1} \sum_{i=1}^{Z_1} |W(1,i) - W_{n-1}(1,i)|^p \\ &= \mathbb{E}_\xi \left(\frac{Z_1}{m_0} \right)^p \mathbb{E}_{T\xi} |W - W_{n-1}|^p. \end{aligned} \tag{3.7}$$

Therefore for almost all ξ, if $T\xi \in D$, then $\xi \in D$. So we have proved that $T^{-1}D \subset D$ a.s.. On the other hand, notice that by Theorem 1.1, $\mathbb{E}_\xi W = 1$ a.s.. Using (3.6) and Burkholder's inequality, we get

$$\begin{aligned} \mathbb{E}_\xi |W - W_n|^p &\geq \frac{C}{m_0^p} \mathbb{E}_\xi \left(\sum_{i=1}^{Z_1} (W(1,i) - W_{n-1}(1,i))^2 \right)^{p/2} \\ &\geq \frac{C}{m_0^p} \mathbb{E}_\xi \mathbf{1}_{\{Z_1 \geq 1\}} \sum_{i=1}^{Z_1} |W(1,i) - W_{n-1}(1,i)|^p \\ &= C \frac{1 - p_0(\xi_0)}{m_0^p} \mathbb{E}_{T\xi} |W - W_{n-1}|^p \qquad a.s.. \end{aligned} \tag{3.8}$$

Notice that $p_0(\xi_0) < 1$ since $m_0 \in (0, \infty)$. It follows from (3.8) that for almost all ξ, if $\xi \in D$, then $T\xi \in D$. Hence $D \subset T^{-1}D$ a.s.. So we have proved that $T^{-1}D = D$ a.s..

For $\rho > m^{1/2}$, assume that $\mathbb{P}(D) = 1$, so that $\lim_{n\to\infty} \rho^n (\mathbb{E}_\xi |W - W_n|^p)^{1/p} = 0$ a.s.. Since the L^p norm is increasing in p and the function $\log^- x$ is decreasing in x, we have $\frac{1}{p_2} \log^- \mathbb{E}_\xi |X|^{p_2} \leq \frac{1}{p_1} \log^- \mathbb{E}_\xi |X|^{p_1}$ if $1 \leq p_1 \leq p_2$. Therefore the condition $\mathbb{E} \log^- \mathbb{E}_\xi \left| \frac{Z_1}{m_0} - 1 \right|^{p\wedge 2} < \infty$ ensures that $\mathbb{E} \log^- \mathbb{E}_\xi \left| \frac{Z_1}{m_0} - 1 \right|^p < \infty$ and $\mathbb{E} \log^- \mathbb{E}_\xi \left| \frac{Z_1}{m_0} - 1 \right|^2 < \infty$. So we have $\mathbb{E} \left| \log \mathbb{E}_\xi \left| \frac{Z_1}{m_0} - 1 \right|^p \right| < \infty$ and $\mathbb{E} \left| \log \mathbb{E}_\xi \left| \frac{Z_1}{m_0} - 1 \right|^2 \right| < \infty$. Applying Theorem 3.3(i) for $p \in (1,2)$ and Theorem 3.3(ii) with $r = 2$ for $p \geq 2$, we get $\rho \leq m^{1/2}$. This contradicts the condition that $\rho > m^{1/2}$. Thus $\mathbb{P}(D) = 0$, which implies that

$$\mathbb{P} \left(\limsup_{n\to\infty} \rho^n (\mathbb{E}_\xi |W - W_n|^p)^{1/p} > 0 \right) = \mathbb{P}(D^c) = 1.$$

So the proof is finished. □

4 Annealed moments and annealed L^p convergence rate for BPRE; Proof of Theorem 1.5

In this section, we consider a branching process in an $i.i.d.$ environment: we assume that $(\xi_n)_{n\geq 0}$ are $i.i.d.$. We also assume that

$$\mathbb{P}(W_1 = 1) < 1, \tag{4.1}$$

which avoids the trivial case where $W_n = 1$ a.s..

Let us study the annealed moments of $\hat{A}_n$ at first. We shall distinguish two cases: (i) $p \in (1, 2)$; (ii) $p \geq 2$. Our approach is inspired by ideas from [1] and [16], especially for the case where $p \geq 2$.

4.1 Annealed moments of $\hat{A}_n$: case $p \geq 2$

We first consider the case where $p \geq 2$.

Proposition 4.1 (Annealed moments of $\hat{A}_n$ for $p \geq 2$)**.** *Let $p \geq 2$ and $\rho \geq 1$. Then $\sup_n \mathbb{E}|\hat{A}_n|^p < \infty$ if and only if $\mathbb{E}\left(\frac{Z_1}{m_0}\right)^p < \infty$ and $\rho \max\{(\mathbb{E}m_0^{1-p})^{1/p}, (\mathbb{E}m_0^{-p/2})^{1/p}\} < 1$.*

To prove Proposition 4.1 for $p > 2$, we need two lemmas below. Denote

$$u_n(s, r) = \mathbb{E}P_n^{-s} W_n^r \quad (s \in \mathbb{R}, r > 1). \tag{4.2}$$

Lemma 4.2. *For $r > 2$, $u_n(s, r)$ satisfies the following recursive formula:*

$$u_n(s, r)^{\frac{1}{r-1}} \leq (\mathbb{E}m_0^{1-r-s})^{\frac{1}{r-1}} u_{n-1}(s, r)^{\frac{1}{r-1}} + (\mathbb{E}m_0^{-s} W_1^r)^{\frac{1}{r-1}} u_{n-1}(s, r-1)^{\frac{1}{r-1}}. \tag{4.3}$$

Proof. Denote $\varphi_n^{(k)}(t) = \mathbb{E}_{T^k\xi} e^{itW_n}$ and $\varphi_n(t) = \varphi_n^{(0)}(t) = \mathbb{E}_\xi e^{itW_n}$. By (3.5), we get the functional equation

$$\varphi_n(s) = \mathbb{E}_\xi \varphi_{n-1}^{(1)}(\frac{t}{m_0})^{Z_1} \qquad a.s..$$

By differentiations, this yields

$$\varphi_n'(t) = \mathbb{E}_\xi \frac{Z_1}{m_0} \left(\varphi_{n-1}^{(1)}(\frac{t}{m_0})\right)^{Z_1 - 1} \left(\varphi_{n-1}^{(1)}(\frac{t}{m_0})\right)' \qquad a.s.. \tag{4.4}$$

Recall that $\mathbb{E}_\xi W_n = 1$ for all ξ and all n. Therefore we can define a random variable V_n whose distribution is determined by

$$\mathbb{E}_\xi g(V_n) = \mathbb{E}_\xi W_n g(W_n)$$

for all bounded and measurable function g. For each $k \geq 1$, let $V_n^{(k)}$ be a random variable with law $\mathbb{P}_\xi(V_n^{(k)} \in \cdot) = \mathbb{P}_{T^k\xi}(V_n \in \cdot)$. Let M_n be a random variable independent of $V_{n-1}^{(1)}$ under $\mathbb{P}_\xi$, whose distribution is determined by

$$\mathbb{E}_\xi g(M_n) = \mathbb{E}_\xi \frac{Z_1}{m_0} g\left(\frac{1}{m_0} \sum_{i=0}^{Z_1 - 1} W_{n-1}(1, i)\right),$$

for all bounded and measurable function g. (The probability space $(\Gamma, \mathcal{G}, \mathbb{P}_\xi)$ can be taken large enough to define the random variables V_n, $V_n^{(k)}$ and M_n.) The Fourier transform of V_n is

$$\mathbb{E}_\xi e^{itV_n} = \mathbb{E}_\xi W_n e^{itW_n} = -i\varphi_n'(t).$$

So (4.4) implies that

$$\mathbb{E}_\xi e^{itV_n} = \mathbb{E}_\xi e^{it(\frac{1}{m_0} V_{n-1}^{(1)} + M_n)} \qquad a.s.,$$

which is equivalent to the distributional equation

$$V_n \stackrel{d}{=} \frac{1}{m_0} V_{n-1}^{(1)} + M_n$$

under $\mathbb{P}_\xi$. Therefore,

$$\begin{aligned} u_n(s,r) = \mathbb{E}P_n^{-s}W_n^r &= \mathbb{E}P_n^{-s}\mathbb{E}_\xi W_n^r = \mathbb{E}P_n^{-s}\mathbb{E}_\xi V_n^{r-1} \\ &= \mathbb{E}P_n^{-s}\mathbb{E}_\xi \left(\frac{1}{m_0}V_{n-1}^{(1)} + M_n\right)^{r-1} \\ &= \mathbb{E}\left(P_n^{-\frac{s}{r-1}} m_0^{-1} V_{n-1}^{(1)} + P_n^{-\frac{s}{r-1}} M_n\right)^{r-1}. \end{aligned}$$

By the triangular inequality in L^{r-1},

$$u_n(s,r)^{\frac{1}{r-1}} \le \left(\mathbb{E}P_n^{-s} m_0^{1-r} \left(V_{n-1}^{(1)}\right)^{r-1}\right)^{\frac{1}{r-1}} + \left(\mathbb{E}P_n^{-s} M_n^{r-1}\right)^{\frac{1}{r-1}}. \tag{4.5}$$

We now calculate the two expectations of the right hand side. We have

$$\begin{aligned} \mathbb{E}P_n^{-s} m_0^{1-r} \left(V_{n-1}^{(1)}\right)^{r-1} &= \mathbb{E}P_n^{-s} m_0^{1-r} \mathbb{E}_{T\xi} V_{n-1}^{r-1} \\ &= \mathbb{E}m_0^{1-r-s} \mathbb{E}P_n^{-s} V_{n-1}^{r-1} \\ &= \mathbb{E}m_0^{1-r-s} u_{n-1}(s,r), \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} \mathbb{E}P_n^{-s} M_n^{r-1} &= \mathbb{E}P_n^{-s}\mathbb{E}_\xi M_n^{r-1} \\ &= \mathbb{E}P_n^{-s}\mathbb{E}_\xi \frac{Z_1}{m_0}\left(\frac{1}{m_0}\sum_{i=0}^{Z_1-1} W_{n-1}(1,i)\right)^{r-1} \\ &\le \mathbb{E}P_n^{-s} m_0^{-r} \mathbb{E}_\xi Z_1^r \mathbb{E}_{T\xi} W_{n-1}^{r-1} \\ &= \mathbb{E}m_0^{-s}\left(\frac{Z_1}{m_0}\right)^r \mathbb{E}P_{n-1}^{-s} W_{n-1}^{r-1} \\ &= \mathbb{E}m_0^{-s}\left(\frac{Z_1}{m_0}\right)^r u_{n-1}(s,r-1). \end{aligned} \tag{4.7}$$

So (4.3) is a combination of (4.5), (4.6) and (4.7). □

Remark 4.3. In particular, $u_n(0,r) = \mathbb{E}W_n^r$. By Lemma 4.2, we can obtain the recursive formula for $\mathbb{E}W_n^r$:

$$(\mathbb{E}W_n^r)^{\frac{1}{r-1}} \le (\mathbb{E}m_0^{r-1})^{\frac{1}{r-1}} + (\mathbb{E}W_1^r)^{\frac{1}{r-1}} (\mathbb{E}W_{n-1}^{r-1})^{\frac{1}{r-1}} \quad (r > 2).$$

Lemma 4.4. *Let $s \in \mathbb{R}$ and $r \in (b, b+1]$, where $b \ge 1$ is an integer. If $\mathbb{E}m_0^{-s} < \infty$ and $\mathbb{E}m_0^{-s}\left(\frac{Z_1}{m_0}\right)^r < \infty$, then*

$$u_n(s,r) = O(n^{1+(b-1)r-(b-1)b/2}(\max\{\max_{1\le i\le b} \mathbb{E}m_0^{i-r-s},\ \mathbb{E}m_0^{-s}\})^n).$$

Proof. We shall prove this lemma by induction on b. For $b = 1$, let $r \in (1,2]$. By Burkholder's inequality,

$$\mathbb{E}_\xi W_n^r \quad \le \quad 1 + \sup_n \mathbb{E}_\xi |W_n - 1|^r \le 1 + C\sum_{k=0}^{n-1} P_k^{1-r} \mathbb{E}_\xi |\bar{X}_n - 1|^r \quad a.s..$$

Hence

$$\begin{aligned} u_n(s,r) &= \mathbb{E}P_n^{-s}\mathbb{E}_\xi W_n^r \\ &\le \mathbb{E}P_n^{-s}\left(1+C\sum_{k=0}^{n-1}P_k^{1-r}\mathbb{E}_\xi|\bar{X}_n-1|^r\right) \\ &= (\mathbb{E}m_0^{-s})^n + C\sum_{k=0}^{n-1}(\mathbb{E}m_0^{1-r-s})^k(\mathbb{E}m_0^{-s})^{n-k-1}\mathbb{E}m_0^{-s}|\bar{X}_0-1|^r \\ &\le (\mathbb{E}m_0^{-s})^n + Cn\max\{\mathbb{E}m_0^{1-r-s},\ \mathbb{E}m_0^{-s}\}^{n-1} \\ &= O(n(\max\{\mathbb{E}m_0^{1-r-s},\ \mathbb{E}m_0^{-s}\})^n). \end{aligned}$$

So the conclusion holds for $b=1$.

Now we assume that the conclusion is true for $r\in(b,b+1]$ for some integer $b\ge 1$. Then for $r\in(b+1,b+2]$, $r-1\in(b,b+1]$. By Hölder's inequality,

$$\mathbb{E}m_0^{-s}\left(\frac{Z_1}{m_0}\right)^{r-1} = \mathbb{E}m_0^{-s/r}m_0^{-s(r-1)/r}\left(\frac{Z_1}{m_0}\right)^{r-1} \le (\mathbb{E}m_0^{-s})^{1/r}\left(\mathbb{E}m_0^{-s}\left(\frac{Z_1}{m_0}\right)^r\right)^{(r-1)/r},$$

which implies that $\mathbb{E}m_0^{-s}(\frac{Z_1}{m_0})^{r-1}<\infty$, since $\mathbb{E}m_0^{-s}<\infty$ and $\mathbb{E}m_0^{-s}(\frac{Z_1}{m_0})^r<\infty$. By the induction assumption,

$$\begin{aligned} u_{n-1}(s,r-1) &= O((n-1)^{1+(b-1)(r-1)-(b-1)b/2}(\max\{\max_{1\le i\le b}\mathbb{E}m_0^{i+1-r-s},\ \mathbb{E}m_0^{-s}\})^{n-1}) \\ &= O(n^{1+(b-1)(r-1)-(b-1)b/2}(\max\{\max_{2\le i\le b+1}\mathbb{E}m_0^{i-r-s},\ \mathbb{E}m_0^{-s}\})^n). \end{aligned} \tag{4.8}$$

It is easy to verify that any solution to the recursive inequality

$$c_n \le \alpha c_{n-1} + O(n^\gamma\beta^n) \quad (\alpha,\beta,\gamma\ge 0) \tag{4.9}$$

satisfies $c_n = O(n^{\gamma+1}\max\{\alpha,\beta\}^n)$. Lemma 4.2 and (4.8) show that $u_n(s,r)^{\frac{1}{r-1}}$ is a solution of (4.9) with $\alpha=(\mathbb{E}m_0^{1-r-s})^{\frac{1}{r-1}}$, $\beta=\max\{\max_{2\le i\le b+1}(\mathbb{E}m_0^{i-r-s})^{\frac{1}{r-1}},(\mathbb{E}m_0^{-s})^{\frac{1}{r-1}}\}$ and $\gamma=\frac{1+(b-1)(r-1)-(b-1)b/2}{r-1}$. Thus

$$u_n(s,r)^{\frac{1}{r-1}} = O(n^{\gamma+1}\max\{\alpha,\beta\}^n). \tag{4.10}$$

Notice that $\gamma+1=\frac{1+br-b(b+1)/2}{r-1}$ and

$$\max\{\alpha,\beta\} = \max\{\max_{1\le i\le b+1}(\mathbb{E}m_0^{i-r-s})^{\frac{1}{r-1}},(\mathbb{E}m_0^{-s})^{\frac{1}{r-1}}\}.$$

Hence (4.10) becomes

$$u_n(s,r) = O(n^{1+br-b(b+1)/2}(\max\{\max_{1\le i\le b+1}\mathbb{E}m_0^{i-r-s},\ \mathbb{E}m_0^{-s}\})^n).$$

So the conclusion still holds for $r\in(b+1,b+2]$. This completes the proof. □

Remark 4.5. In Lemma 4.4, since $1-(b-1)b/2\le 0$ for $b\ge 2$, we in fact obtain

$$u_n(s,r) = O(n(\max\{\mathbb{E}m_0^{1-r-s},\ \mathbb{E}m_0^{-s}\})^n) \quad \text{for } r\in(1,2],$$

and for any integer $b\ge 1$,

$$u_n(s,r) = O(n^{br}(\max\{\max_{1\le i\le b+1}\mathbb{E}m_0^{i-r-s},\ \mathbb{E}m_0^{-s}\})^n) \quad \text{for } r\in(b+1,b+2].$$

Proof of Proposition 4.1. For $p=2$, by Lemmas 2.5 and 2.7,

$$\sup_n \mathbb{E}|\hat{A}_n|^2 = \sum_{n=0}^{\infty} \rho^{2n} \mathbb{E}(P_n^{-1}\mathbb{E}_\xi|\bar{X}_n - 1|^2) = \mathbb{E}|\bar{X}_0 - 1|^2 \sum_{n=0}^{\infty} \left(\rho^2 \mathbb{E} m_0^{-1}\right)^n . \tag{4.11}$$

Therefore, $\sup_n \mathbb{E}|\hat{A}_n|^2 < \infty$ if and only if $\mathbb{E}(\frac{Z_1}{m_0})^2 < \infty$ and $\rho(\mathbb{E}m_0^{-1})^{1/2} < 1$.

Now we consider the case where $p > 2$. Assume that $\mathbb{E}\left(\frac{Z_1}{m_0}\right)^p < \infty$ and $\rho \max\{(\mathbb{E}m_0^{1-p})^{1/p}, (\mathbb{E}m_0^{-p/2})^{1/p}\} < 1$. By Lemma 2.5,

$$\sup_n \mathbb{E}|\hat{A}_n|^p \le C \left(\sum_{n=0}^{\infty} \rho^{2n} (\mathbb{E}|W_{n+1} - W_n|^p)^{2/p} \right)^{p/2} .$$

To prove $\sup_n \mathbb{E}|\hat{A}_n|^p < \infty$, it suffices to show that

$$\sum_{n=0}^{\infty} \rho^{2n} (\mathbb{E}|W_{n+1} - W_n|^p)^{2/p} < \infty.$$

By Lemma 2.7,

$$\begin{aligned} \mathbb{E}|W_{n+1} - W_n|^p &\le C\mathbb{E}P_n^{-p/2}\mathbb{E}_\xi W_n^{p/2}\mathbb{E}_\xi|\bar{X}_n - 1|^p \\ &= C\mathbb{E}P_n^{-p/2}W_n^{p/2}\mathbb{E}|\bar{X}_0 - 1|^p \\ &= Cu_n(p/2, p/2). \end{aligned}$$

Notice that

$$\mathbb{E}m_0^{-p/2}\left(\frac{Z_1}{m_0}\right)^{p/2} = \mathbb{E}m_0^{-p}Z_1^{p/2}\mathbf{1}_{\{Z_1 \ge 1\}} \le \mathbb{E}m_0^{-p}Z_1^{p}\mathbf{1}_{\{Z_1 \ge 1\}} \le \mathbb{E}\left(\frac{Z_1}{m_0}\right)^p < \infty,$$

and $\mathbb{E}m_0^{-p/2} < 1 < \infty$. Remark 4.4 shows that

$$u_n(p/2, p/2) = O(n^\gamma (\max\{\max_{1\le i\le b+1} \mathbb{E}m_0^{i-p}, \mathbb{E}m_0^{-p/2}\})^n)$$

for $p/2 \in (b+1, b+2]$ with $\gamma = 1$ for $b = 0$ and $\gamma = bp/2$ for $b \ge 1$. Notice that $\mathbb{E}m_0^x$ is log convex. Therefore we have

$$\max\{\max_{1\le i\le b+1} \mathbb{E}m_0^{i-p},\ \mathbb{E}m_0^{-p/2}\} \le \sup_{1-p\le x\le -p/2} \{\mathbb{E}m_0^x\} = \max\{\mathbb{E}m_0^{1-p},\ \mathbb{E}m_0^{-p/2}\}.$$

Thus

$$\sum_{n=0}^{\infty} \rho^{2n}(\mathbb{E}|W_{n+1} - W_n|^p)^{2/p} \le C\sum_{n=0}^{\infty} \rho^{2n} n^{2\gamma/p} (\max\{(\mathbb{E}m_0^{1-p})^{2/p}, (\mathbb{E}m_0^{-p/2})^{2/p}\})^n.$$

The series in the right side of the above inequality is finite if and only if $\rho \max\{(\mathbb{E}m_0^{1-p})^{1/p}, (\mathbb{E}m_0^{-p/2})^{1/p}\} < 1$.

Conversely, assume that $\sup_n \mathbb{E}|\hat{A}_n|^p < \infty$. Obviously, $\mathbb{E}\left(\frac{Z_1}{m_0}\right)^p < \infty$, since $\mathbb{E}|\frac{Z_1}{m_0} - 1|^p = \mathbb{E}|\hat{A}_0|^p < \infty$. By Lemmas 2.5 and 2.7, we have $\forall r \in [2, p]$,

$$\begin{aligned} \sup_n \mathbb{E}|\hat{A}_n|^p &\ge C\sum_{n=0}^{\infty} \rho^{pn}\mathbb{E}|W_{n+1} - W_n|^p \\ &\ge C\sum_{n=0}^{\infty} \rho^{pn}\mathbb{E}P_n^{p(1/r-1)}(\mathbb{E}_\xi|\bar{X}_n - 1|^r)^{p/r} \\ &= C\sum_{n=0}^{\infty} \rho^{pn}(\mathbb{E}m_0^{p(1/r-1)})^n\mathbb{E}(\mathbb{E}_\xi|\bar{X}_0 - 1|^r)^{p/r}. \end{aligned}$$

Thus $\rho(\mathbb{E}m_0^{p(1/r-1)})^{1/p} < 1$ holds for all $r \in [2,p]$. Taking $r = p, 2$, we get $\rho \max\{(\mathbb{E}m_0^{1-p})^{1/p}, (\mathbb{E}m_0^{-p/2})^{1/p}\} < 1$. □

4.2 Annealed moments of $\hat{A}_n$: case $p \in (1,2)$

For the case where $p \in (1,2)$, we have the proposition below.

Proposition 4.6 (Annealed moments of $\hat{A}_n$ for $p \in (1,2)$). *Let $p \in (1,2)$ and $\rho \geq 1$. If $\mathbb{E}\left(\mathbb{E}_\xi\left(\frac{Z_1}{m_0}\right)^r\right)^{p/r} < \infty$ and $\rho(\mathbb{E}m_0^{p(1/r-1)})^{1/p} < 1$ for some $r \in [p,2]$, then*

$$\sup_n \mathbb{E}|\hat{A}_n|^p < \infty. \tag{4.12}$$

Conversely, if (4.12) holds, then $\mathbb{E}(\frac{Z_1}{m_0})^p < \infty$ and $\rho(\mathbb{E}m_0^s)^{-1/2s} < 1$ for all $s > 0$, so that $\rho \leq \exp(\frac{1}{2}\mathbb{E}\log m_0)$; if additionally $\mathbb{E}m_0^{-p/2}\log m_0 > 0$ and $\mathbb{E}m_0^{-p/2-1}Z_1\log^+ Z_1 < \infty$, then $\rho(\mathbb{E}m_0^{-p/2})^{1/p} < 1$.

Proof. Suppose that $\mathbb{E}\left(\mathbb{E}_\xi\left(\frac{Z_1}{m_0}\right)^r\right)^{p/r} < \infty$ and $\rho(\mathbb{E}m_0^{p(1/r-1)})^{1/p} < 1$ for some $r \in [p,2]$. By Lemma 2.7,

$$\mathbb{E}_\xi|W_{n+1} - W_n|^p \leq CP_n^{p(1/r-1)}(\mathbb{E}_\xi|\bar{X}_n - 1|^r)^{p/r}.$$

Taking expectation we obtain

$$\mathbb{E}|W_{n+1} - W_n|^p \leq C(\mathbb{E}m_0^{p(1/r-1)})^n\mathbb{E}(\mathbb{E}_\xi|\bar{X}_0 - 1|^r)^{p/r}. \tag{4.13}$$

Notice that

$$\mathbb{E}(\mathbb{E}_\xi|\bar{X}_0 - 1|^r)^{p/r} \leq C\left(\mathbb{E}\left(\mathbb{E}_\xi\left(\frac{Z_1}{m_0}\right)^r\right)^{p/r} + 1\right) < \infty.$$

By Lemma 2.5 and (4.13),

$$\begin{aligned}
\sup_n \mathbb{E}|\hat{A}_n|^p &\leq C\sum_{n=0}^{\infty}\rho^{pn}\mathbb{E}|W_{n+1} - W_n|^p \\
&\leq C\mathbb{E}\left(\mathbb{E}_\xi|\bar{X}_0 - 1|^r\right)^{p/r}\sum_{n=0}^{\infty}\rho^{pn}(\mathbb{E}m_0^{p(1/r-1)})^n < \infty.
\end{aligned}$$

Conversely, assume that $\sup_n \mathbb{E}|\hat{A}_n|^p < \infty$. It is obvious that $\mathbb{E}|\frac{Z_1}{m_0} - 1|^p = \mathbb{E}|\hat{A}_0|^p < \infty$. By Lemmas 2.5 and 2.7, we have $\forall N \geq 1$,

$$\begin{aligned}
\sup_n \mathbb{E}|\hat{A}_n|^p &\geq CN^{p/2-1}\sum_{n=0}^{N-1}\rho^{pn}\mathbb{E}|W_{n+1} - W_n|^p \\
&\geq CN^{p/2-1}\sum_{n=0}^{N-1}\rho^{pn}\mathbb{E}P_n^{-p/2}\mathbb{E}_\xi W_n^{p/2}\mathbb{E}_\xi|\bar{X}_n - 1|^p \\
&= CN^{p/2-1}\sum_{n=0}^{N-1}\rho^{pn}\mathbb{E}P_n^{-p/2}W_n^{p/2}\mathbb{E}|\bar{X}_0 - 1|^p. \qquad (4.14)
\end{aligned}$$

The assumption $\mathbb{P}(W_1 = 1) < 1$ ensures that $\mathbb{E}|\bar{X}_0 - 1|^p > 0$. For $\alpha > 0$, Hölder's inequality gives

$$\mathbb{E}W_n^\alpha = \mathbb{E}W_n^\alpha P_n^{-\alpha}P_n^\alpha \leq (\mathbb{E}W_n^{\alpha p_1}P_n^{-\alpha p_1})^{1/p_1}(\mathbb{E}P_n^{\alpha q_1})^{1/q_1}, \tag{4.15}$$

where $p_1, q_1 > 1$ and $1/p_1 + 1/q_1 = 1$. For $s > 0$, take $\alpha = \frac{sp}{p+2s}$, $p_1 = 1 + p/2s$ and $q_1 = 1 + 2s/p$. Then (4.15) becomes

$$(\mathbb{E}W_n^{\alpha})^{p_1} \le \mathbb{E}W_n^{p/2} P_n^{-p/2} (\mathbb{E}m_0^s)^{pn/2s}. \tag{4.16}$$

Combing (4.16) with (4.14), we get

$$\begin{aligned}\sup_n \mathbb{E}|\hat{A}_n|^p &\ge CN^{p/2-1} \sum_{n=0}^{N-1} \rho^{pn} (\mathbb{E}m_0^s)^{-pn/2s} (\mathbb{E}W_n^{\alpha})^{p_1} \\ &\ge C(\inf_n \mathbb{E}W_n^{\alpha})^{p_1} N^{p/2-1} \sum_{n=0}^{N-1} \left(\rho^p (\mathbb{E}m_0^s)^{-p/2s}\right)^n.\end{aligned}$$

Hence $\sup_n \mathbb{E}|\hat{A}_n|^p < \infty$ implies that $\rho(Em_0^s)^{-1/2s} < 1$ for all $s > 0$, so that $\log\rho < \frac{1}{2s}\log\mathbb{E}m_0^s$ for all $s > 0$. Notice that $(\mathbb{E}m_0^s)^{1/s}$ is increasing as s increases. We have

$$\log\rho \le \inf_{s>0} \frac{1}{2s}\log\mathbb{E}m_0^s = \frac{1}{2}\lim_{s\to 0^+}\frac{1}{s}\log(\mathbb{E}m_0^s) = \frac{1}{2}\mathbb{E}\log m_0,$$

so that $\rho \le \exp(\frac{1}{2}\mathbb{E}\log m_0)$.

If additionally $\mathbb{E}m_0^{-p/2}\log m_0 > 0$ and $\mathbb{E}m_0^{-p/2-1} Z_1 \log^+ Z_1 < \infty$, we introduce a new BPRE. Denote the distribution of ξ_0 by τ_0. Define a new distribution $\tilde{\tau}_0$ as

$$\tilde{\tau}_0(dx) = \frac{m(x)^{-p/2}\tau_0(dx)}{\mathbb{E}m_0^{-p/2}},$$

where $m(x) = \mathbb{E}[Z_1|\xi_0 = x] = \sum_{k=0}^{\infty} kp_k(x)$. Consider the new BPRE whose environment distribution is $\tilde{\tau} = \tilde{\tau}_0^{\otimes\mathbb{N}}$ instead of $\tau = \tau_0^{\otimes\mathbb{N}}$. The corresponding probability and expectation are denoted by $\tilde{\mathbb{P}}$ and $\tilde{\mathbb{E}}$, respectively. Then

$$\mathbb{E}P_n^{-p/2}W_n^{p/2} = \tilde{\mathbb{E}}W_n^{p/2}(\mathbb{E}m_0^{-p/2})^n. \tag{4.17}$$

Combing (4.17) with (4.14), we obtain

$$\sup_n \mathbb{E}|\hat{A}_n|^p \ge C\inf_n \tilde{\mathbb{E}}W_n^{p/2} N^{p/2-1} \sum_{n=0}^{N-1} \left(\rho^p \mathbb{E}m_0^{-p/2}\right)^n.$$

Notice that

$$\tilde{\mathbb{E}}\log m_0 = \mathbb{E}m_0^{-p/2}\log m_0 > 0,$$

and

$$\tilde{\mathbb{E}}\frac{Z_1}{m_0}\log^+ Z_1 = \mathbb{E}m_0^{-p/2-1} Z_1 \log^+ Z_1 < \infty.$$

Hence W is non-degenerate under $\tilde{\mathbb{P}}$, i.e. $\tilde{\mathbb{P}}(W > 0) > 0$ (cf. e.g. [5], [19]), so that $\inf_n \tilde{\mathbb{E}}W_n^{p/2} = \tilde{\mathbb{E}}W^{p/2} > 0$. Therefore, $\sup_n \mathbb{E}|\hat{A}_n|^p < \infty$ implies that $\rho(\mathbb{E}m_0^{-p/2})^{1/p} < 1$. □

4.3 Exponential rate of W_n

Again, by the relations of $\hat{A}_n$, $A(\rho)$ and $\rho^n(W - W_n)$, combined with Propositions 4.1 and 4.6, we obtain the following criteria for the annealed L^p convergence rate of W_n.

Theorem 4.7 (Exponential rate of annealed L^p convergence of W_n)**.** *Let* $\rho > 1$.

(i) Let $p \in (1,2)$. *If* $\mathbb{E}\left(\mathbb{E}_\xi\left(\frac{Z_1}{m_0}\right)^r\right)^{p/r} < \infty$ *and* $\rho(\mathbb{E}m_0^{p(1/r-1)})^{1/p} < 1$ *for some* $r \in [p,2]$*, then*

$$(\mathbb{E}|W - W_n|^p)^{1/p} = o(\rho^{-n}). \tag{4.18}$$

Conversely, if (4.18) holds, then $\rho \leq \exp(\frac{1}{2}\mathbb{E}\log m_0)$*; if additionally* $\mathbb{E}m_0^{-p/2}\log m_0 > 0$ *and* $\mathbb{E}m_0^{-p/2-1}Z_1\log^+ Z_1 < \infty$*, then* $\rho(\mathbb{E}m_0^{-p/2})^{1/p} \leq 1$.

(ii) Let $p \geq 2$. *If* $\mathbb{E}(\frac{Z_1}{m_0})^p < \infty$ *and* $\rho\max\{(\mathbb{E}m_0^{1-p})^{1/p}, (\mathbb{E}m_0^{-p/2})^{1/p}\} < 1$*, then (4.18) holds. Conversely, if (4.18) holds, then* $\rho\max\{(\mathbb{E}m_0^{1-p})^{1/p}, (\mathbb{E}m_0^{-p/2})^{1/p}\} \leq 1$.

Note that $\rho^{pn}\mathbb{E}|W - W_n|^p \to 0$ implies that $\forall \rho_1 \in (1,\rho)$, $\rho_1^{pn}\mathbb{E}_\xi|W - W_n|^p \to 0$ a.s. by Borel-Cantelli's lemma and Markov's inequality. So under the conditions of Theorem 4.7, we can also obtain (3.4). However, by Jensen's inequality, it can be seen that the conditions of Theorem 4.7 are stronger than those of Theorem 3.3.

The proof of Theorem 1.5 is now easy.

Proof of Theorem 1.5. Theorem 1.5 is a direct consequence of Theorem 4.7: taking $r = p$ in Theorem 4.7 gives (a), and taking $r = 2$ yields (b). □

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Acknowledgments. The authors would like to thank the anonymous referees for valuable comments and remarks. This work was partially supported by the Natural Scientific Research Innovation Foundation in Harbin Institute of Technology (HIT. NSRIF. 2015102), the National Natural Science Foundation of China (Nos. 11171044, 11101039), and by the Natural Science Foundation of Hunan Province (No. 11JJ2001).