

Ergodic properties for α -CIR models and a class of generalized Fleming-Viot processes

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Dedicated to Professor Ken-iti Sato on the occasion of his 80th birthday

Abstract

We discuss a Markov jump process regarded as a variant of the CIR (Cox-Ingersoll-Ross) model and its infinite-dimensional extension. These models belong to a class of measure-valued branching processes with immigration, whose jump mechanisms are governed by certain stable laws. The main result gives a lower spectral gap estimate for the generator. As an application, a certain ergodic property is shown for the generalized Fleming-Viot process obtained as the time-changed ratio process.

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1 Introduction

The study of ergodic behaviors of a Markov process is of quite interest for various reasons. For instance, it is typical that the analysis of such behaviors depends heavily on the mathematical structure of the model, so that resulting properties are expected to yield deep understanding for it. In this paper, we discuss two specific classes of measure-valued Markov jump processes. The one consists of what we will call measure-valued α -CIR models, each of which is thought of as an infinite-dimensional extension for a jump-type version of the CIR model, and the other generalizes naturally a class of Fleming-Viot processes with parent-independent mutation. As for the measure-valued α -CIR model, identification of a stationary distribution is easy thanks to its nice structure as a measure-valued branching process with immigration (henceforth MBI-process). For the latter class of models, stationary distributions are identified recently in [4]. A key idea there is to exploit a special relationship with measure-valued α -CIR models, which enabled us to give an expression for stationary distributions of our generalized Fleming-Viot processes in terms of those of the measure-valued α -CIR models. It should be mentioned that such links have been discussed in another context in [1] and

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[2]. Our attempt here is to rely still on that relationship to explore ergodic properties for the generalized Fleming-Viot process.

It is worth illustrating by taking up a one-dimensional model which is regarded as a ‘prototype’ of the above mentioned MBI-process. Consider the well-known CIR model governed by generator

$$L_1 = z \frac{d^2}{dz^2} + (-bz + c) \frac{d}{dz}, \quad z \in \mathbf{R}_+ := [0, \infty), \tag{1.1}$$

where $b \in \mathbf{R}$ and $c > 0$ are constants. Rather than its importance in the context of mathematical finance, we emphasize that this model belongs to the class of continuous state branching processes with immigration (CBI-processes in short). (See [5] for fundamental results regarding this class.) Let $0 < \alpha < 1$ be arbitrary. As a natural non-local version of (1.1) within the class of generators of conservative CBI-processes (cf. Theorem 1.2 in [5]), we will be concerned with

$$\begin{aligned} L_\alpha F(z) &= \frac{\alpha + 1}{\Gamma(1 - \alpha)} z \int_0^\infty [F(z + y) - F(z) - yF'(z)] \frac{dy}{y^{\alpha+2}} \\ &\quad - \frac{b}{\alpha} z F'(z) + c \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty [F(z + y) - F(z)] \frac{dy}{y^{\alpha+1}}, \end{aligned} \tag{1.2}$$

where $\Gamma(\cdot)$ is the gamma function. The operator L_α with $b = 0$ is found in Example 1.1 of [5]. Observing that, as $\alpha \uparrow 1$, $L_\alpha F(z) \rightarrow L_1 F(z)$ for any $z > 0$ and ‘nice’ functions F on \mathbf{R}_+ , we call a Markov process associated with aL_α for some constant $a > 0$ an α -CIR model.

Although this class of models would be of interest in its own right especially in the mathematical finance context, our main motivation to study it is the analysis of ergodicity for a jump-type version of a Wright-Fisher diffusion model with mutation, which is obtained through normalization and random time-change from two independent processes with generators of the form (1.2), say L'_α and L''_α , with common α and b . On the level of generators such a link can be reformulated as the identity

$$(L'_\alpha F(\cdot, z_2))(z_1) + (L''_\alpha F(z_1, \cdot))(z_2) = C(z_1 + z_2)^{-\alpha} A_\alpha G\left(\frac{z_1}{z_1 + z_2}\right), \quad z_1, z_2 > 0, \tag{1.3}$$

where G is any smooth function on $[0, 1]$, F is defined by $F(z_1, z_2) = G(z_1/(z_1 + z_2))$, C is a positive constant independent of G and A_α is the generator of a jump-type version of the Wright-Fisher diffusion model. (See (1.3) in [4] for a concrete expression for A_α or (5.1) below for its generalization.) A significant consequence of (1.3) is that Dirichlet form associated with A_α is, up to some multiplicative constant, a restriction of Dirichlet form associated with the two independent α -CIR models. Therefore, ergodic properties of α -CIR models would be expected to help us obtain the same kind of results for the process associated with A_α .

Such an idea can extend naturally to the measure-valued α -CIR model, which is regarded roughly as ‘continuum direct sum’ of α -CIR models with coefficients depending on a spatial parameter. More precisely, it is a MBI-process on a type space E , say, with zero mutation, branching mechanism

$$\begin{aligned} E \times \mathbf{R}_+ \ni (r, \lambda) &\mapsto a(r) \frac{\alpha + 1}{\Gamma(1 - \alpha)} \int_0^\infty [e^{-\lambda y} - 1 + \lambda y] \frac{dy}{y^{\alpha+2}} - \frac{b(r)}{\alpha} \lambda \\ &= -\frac{1}{\alpha} (a(r) \lambda^{\alpha+1} + b(r) \lambda) \end{aligned} \tag{1.4}$$

and nonlocal immigration mechanism

$$f(\cdot) \mapsto \frac{\alpha}{\Gamma(1 - \alpha)} \int_E m(dr) \int_0^\infty [1 - e^{-f(r)y}] \frac{dy}{y^{\alpha+2}} = \int_E m(dr) f(r)^\alpha, \tag{1.5}$$

where $a(r) > 0, b(r) \in \mathbf{R}$ and m is a finite non-null measure on E . An important feature of such processes is that the transition semigroups (for fixed t) and stationary distributions form a convolution semigroup with respect to m . (See (2.15) and (2.17) below.) Because of this structure studying ergodic properties of the extended model would be reduced to the one-dimensional case at least under the assumption of uniform bounds for the coefficients. In addition, as observed in [4], the relation (1.3) admits a generalization in the setting of measure-valued processes. (See also (5.2) below.) For this reason the above mentioned extension of the α -CIR model is considered to play an important role in studying the generalized Fleming-Viot process obtained as the time-changed ratio process.

The organization of this paper is as follows. In Section 2, we introduce the measure-valued α -CIR model, and it is shown in Section 3 that a lower spectral gap estimate for the generator can reduce to the one-dimensional case in a suitable sense. In Section 4, we prove such an estimate for L_α , establishing exponential convergence to equilibrium for the measure-valued α -CIR model. The latter result will be applied to a class of generalized Fleming-Viot processes in Section 5.

2 The measure-valued α -CIR models

To discuss in the setting of measure-valued processes, we need the following notation. Let E be a compact metric space and $C(E)$ (resp. $B_+(E)$) the set of continuous (resp. nonnegative, bounded Borel) functions on E . Also, denote by $C_{++}(E)$ the set of functions in $C(E)$ which are uniformly positive. Define $\mathcal{M}(E)$ to be the totality of finite Borel measures on E , and we equip $\mathcal{M}(E)$ with the weak topology. Denote by $\mathcal{M}(E)^\circ$ the set of non-null elements of $\mathcal{M}(E)$. The set $\mathcal{M}_1(E)$ of Borel probability measures on E is regarded as a subspace of $\mathcal{M}(E)$. We also use notation $\langle \eta, f \rangle := \int_E f(r)\eta(dr)$. For each $r \in E$, let δ_r denote the delta distribution at r . Given a probability measure Q , we write also $E^Q[\cdot]$ for the expectation with respect to Q .

Suppose that $0 < \alpha < 1$, $a \in C_{++}(E)$, $b \in C(E)$ and $m \in \mathcal{M}(E)^\circ$ are given. As a natural generalization of the α -CIR model generated by (1.2), we shall discuss in this section the Markov process on $\mathcal{M}(E)$ associated with

$$\begin{aligned} \mathcal{L}_\alpha \Psi(\eta) &= \mathcal{L}_\alpha^{(1)} \Psi(\eta) + \mathcal{L}_\alpha^{(2)} \Psi(\eta) + \mathcal{L}_\alpha^{(3)} \Psi(\eta) \\ &:= \frac{\alpha + 1}{\Gamma(1 - \alpha)} \int_0^\infty \frac{dz}{z^{2+\alpha}} \int_E \eta(dr) a(r) \left[\Psi(\eta + z\delta_r) - \Psi(\eta) - z \frac{\delta \Psi}{\delta \eta}(r) \right] - \frac{1}{\alpha} \langle \eta, b \frac{\delta \Psi}{\delta \eta} \rangle \\ &\quad + \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty \frac{dz}{z^{1+\alpha}} \int_E m(dr) [\Psi(\eta + z\delta_r) - \Psi(\eta)], \quad \eta \in \mathcal{M}(E), \end{aligned} \tag{2.1}$$

where $\frac{\delta \Psi}{\delta \eta}(r) = \frac{d}{d\epsilon} \Psi(\eta + \epsilon \delta_r) \Big|_{\epsilon=0}$. The operator $\mathcal{L}_\alpha^{(3)}$ describes the mechanism of immigration. (See (9.25) in [6] for a general form of generators of MBI-processes. In our model, there is no ‘motion process’, whose generator is thus considered to be $A \equiv 0$.) Set $\Psi_f(\eta) = e^{-\langle \eta, f \rangle}$ for $f \in B_+(E)$ and define \mathcal{D} to be the linear span of functions Ψ_f with $f \in C_{++}(E)$. It is immediate to see from (1.4) and (1.5) that for any $f \in B_+(E)$

$$\mathcal{L}_\alpha \Psi_f(\eta) = \Psi_f(\eta) \frac{1}{\alpha} \langle \eta, a f^{\alpha+1} + b f \rangle - \Psi_f(\eta) \langle m, f^\alpha \rangle. \tag{2.2}$$

\mathcal{L}_α is well-defined also on the class \mathcal{F} of functions Ψ of the form

$$\Psi(\eta) = \varphi(\langle \eta, f_1 \rangle, \dots, \langle \eta, f_n \rangle) \tag{2.3}$$

for some $\varphi \in C_0^2(\mathbf{R}_+^n)$, $f_i \in C_{++}(E)$ and a positive integer n . Our first result below not only verifies this but also gives bounds for each $\mathcal{L}_\alpha^{(k)} \Psi$ ($k \in \{1, 2, 3\}$) for a more general

class of functions Ψ . In what follows $\|\cdot\|_\infty$ denotes the sup norm. Let $\tilde{\mathcal{F}}$ be the totality of functions Ψ of the form (2.3) with $\varphi \in C^2(\mathbf{R}_+^n)$ and $\mathbf{f} := (f_1, \dots, f_n) \in C_{++}(E)^n$ satisfying the following conditions; there exist nonnegative constants $C_j^{(i)} (1 \leq i, j \leq n)$, $C_k^{(ij)} (1 \leq i, j, k \leq n)$ and $\epsilon > 0$ such that for each $i, j \in \{1, \dots, n\}$

$$|\varphi_i(x_1, \dots, x_n)| \leq \sum_{k=1}^n \frac{C_k^{(i)}}{x_k + \epsilon} \quad \text{for any } (x_1, \dots, x_n) \in \mathbf{R}_f^n \tag{2.4}$$

and

$$|\varphi_{ij}(x_1, \dots, x_n)| \leq \sum_{k=1}^n \frac{C_k^{(ij)}}{(x_k + \epsilon)^2} \quad \text{for any } (x_1, \dots, x_n) \in \mathbf{R}_f^n, \tag{2.5}$$

where $\varphi_i = \frac{\partial \varphi}{\partial x_i}$, $\varphi_{ij} = \frac{\partial^2 \varphi}{\partial x_i \partial x_j}$ and \mathbf{R}_f^n is defined to be

$$\left\{ (x_1, \dots, x_n) \in (0, \infty)^n : \frac{\inf_{x \in E} f_i(x)}{\|f_j\|_\infty} \leq \frac{x_i}{x_j} \leq \frac{\|f_i\|_\infty}{\inf_{x \in E} f_j(x)} \quad (1 \leq i, j \leq n) \right\}.$$

Note that $\langle \eta, \mathbf{f} \rangle := (\langle \eta, f_1 \rangle, \dots, \langle \eta, f_n \rangle) \in \mathbf{R}_f^n$ for any $\eta \in \mathcal{M}(E)^\circ$. Intuitively, these conditions enable one to control the effect of long-range jumps governed by stable laws, and are inspired by the calculations in the proof of Proposition 3.4 in [4].

It will turn out in Section 5 that an important example of functions in $\tilde{\mathcal{F}} \setminus \mathcal{F}$ is

$$\Psi(\eta) = \langle \eta, f_1 \rangle \cdots \langle \eta, f_n \rangle (\langle \eta, f_{n+1} \rangle + \epsilon)^{-n},$$

where $f_i \in C_{++}(E)$, $\epsilon > 0$ and n is a positive integer. This function corresponds to $\varphi(x_1, \dots, x_{n+1}) = x_1 \cdots x_n (x_{n+1} + \epsilon)^{-n}$, for which the following are verified to hold:

$$\varphi_i(x_1, \dots, x_{n+1}) = \begin{cases} x_1 \cdots x_n (x_{n+1} + \epsilon)^{-n} & (i \in \{1, \dots, n\}) \\ -n x_1 \cdots x_n (x_{n+1} + \epsilon)^{-(n+1)} & (i = n + 1) \end{cases}$$

and

$$\varphi_{ij}(x_1, \dots, x_{n+1}) = \begin{cases} 0 & (i = j \in \{1, \dots, n\}) \\ x_1 \cdots x_n (x_{n+1} + \epsilon)^{-n} & (i, j \in \{1, \dots, n\}, i \neq j) \\ -n x_1 \cdots x_n (x_{n+1} + \epsilon)^{-(n+1)} & (i \in \{1, \dots, n\}, j = n + 1) \\ n(n + 1) x_1 \cdots x_n (x_{n+1} + \epsilon)^{-(n+2)} & (i = j = n + 1). \end{cases}$$

Here, $\overset{i}{\cdots}$ (resp. $\overset{i,j}{\cdots}$) indicates deletion of the i th (resp. i th and j th) factor(s). These equalities are sufficient to show inequalities of the form (2.4) and (2.5). We can take in particular $C_k^{(i)} = 0 = C_k^{(ij)}$ for any $i, j \in \{1, \dots, n + 1\}$ and $k \in \{1, \dots, n\}$.

Lemma 2.1. (i) It holds that $\mathcal{F} \subset \tilde{\mathcal{F}}$.

(ii) Let $\Psi \in \tilde{\mathcal{F}}$ be expressed as (2.3) with φ satisfying (2.4) and (2.5) and $f_i \in C_{++}(E)$. Then for any $\eta \in \mathcal{M}(E)$

$$\mathcal{L}_\alpha^{(1)} \Psi(\eta) = \frac{1}{\alpha \Gamma(1 - \alpha)} \int_E \eta(dr) a(r) \int_0^\infty u^{-\alpha} du \sum_{i,j=1}^n f_i(r) f_j(r) \varphi_{ij}(\langle \eta + u \delta_r, \mathbf{f} \rangle), \tag{2.6}$$

$$\mathcal{L}_\alpha^{(2)} \Psi(\eta) = -\frac{1}{\alpha} \sum_{i=1}^n \langle \eta, b f_i \rangle \varphi_i(\langle \eta, \mathbf{f} \rangle) \tag{2.7}$$

and

$$\mathcal{L}_\alpha^{(3)}\Psi(\eta) = \frac{1}{\Gamma(1-\alpha)} \int_E m(dr) \int_0^\infty w^{-\alpha} dw \sum_{i=1}^n f_i(r) \varphi_i(\langle \eta + w\delta_r, \mathbf{f} \rangle). \quad (2.8)$$

Also, we have the bounds

$$\begin{aligned} |\mathcal{L}_\alpha^{(1)}\Psi(\eta)| &\leq \Gamma(\alpha) \sum_{i,j,k=1}^n C_k^{(ij)} \frac{\langle \eta, a f_i f_j f_k^{\alpha-1} \rangle}{(\langle \eta, f_k \rangle + \epsilon)^{\alpha+1}} \\ &\leq \Gamma(\alpha) \sum_{i,j,k=1}^n C_k^{(ij)} \frac{\|a f_i f_j f_k^{\alpha-1}\|_\infty}{\inf_{x \in E} f_k(x)} (\langle \eta, f_k \rangle + \epsilon)^{-\alpha}, \end{aligned} \quad (2.9)$$

$$|\mathcal{L}_\alpha^{(2)}\Psi(\eta)| \leq \frac{1}{\alpha} \sum_{i,j=1}^n C_j^{(i)} \frac{\langle \eta, |b| f_i \rangle}{\langle \eta, f_j \rangle + \epsilon} \leq \frac{1}{\alpha} \sum_{i,j=1}^n C_j^{(i)} \frac{\|b f_i\|_\infty}{\inf_{x \in E} f_j(x)} \quad (2.10)$$

and

$$|\mathcal{L}_\alpha^{(3)}\Psi(\eta)| \leq \Gamma(\alpha) \sum_{i,j=1}^n C_j^{(i)} \frac{\langle m, f_i f_j^{\alpha-1} \rangle}{(\langle \eta, f_j \rangle + \epsilon)^\alpha}. \quad (2.11)$$

In particular, $\mathcal{L}_\alpha^{(1)}\Psi$, $\mathcal{L}_\alpha^{(2)}\Psi$ and $\mathcal{L}_\alpha^{(3)}\Psi$ are bounded.

Proof. (i) Let $\varphi \in C_0^2(\mathbf{R}_+^n)$ be given and take $R_1, \dots, R_n > 0$ large enough so that

$$\varphi(x_1, \dots, x_n) = 0 \quad \text{whenever} \quad \max\{x_1/R_1, \dots, x_n/R_n\} > 1.$$

Let $\epsilon > 0$ be arbitrary. Then it is easy to see that for all $(x_1, \dots, x_n) \in \mathbf{R}_+^n$

$$|\varphi_i(x_1, \dots, x_n)| \leq \frac{1}{n} \sum_{k=1}^n \frac{R_k + \epsilon}{x_k + \epsilon} \|\varphi_i\|_\infty$$

and

$$|\varphi_{ij}(x_1, \dots, x_n)| \leq \frac{1}{n} \sum_{k=1}^n \frac{(R_k + \epsilon)^2}{(x_k + \epsilon)^2} \|\varphi_{ij}\|_\infty.$$

In view of (2.4) and (2.5), what we have just seen suffice to imply that $\mathcal{F} \subset \tilde{\mathcal{F}}$.

(ii) First, we consider $\mathcal{L}_\alpha^{(2)}\Psi(\eta)$, assuming that $\eta \in \mathcal{M}(E)^\circ$. (If η is the null measure, (2.10) is trivial.) Observe that

$$\frac{\delta\Psi}{\delta\eta}(r) = \sum_{i=1}^n f_i(r) \varphi_i(\langle \eta, \mathbf{f} \rangle), \quad (2.12)$$

from which (2.7) follows. Also, (2.10) is immediate from (2.4).

The next task is to prove the assertions for $\mathcal{L}_\alpha^{(3)}\Psi(\eta)$. Since $\frac{d}{dz}\Psi(\eta + z\delta_r) = \frac{\delta\Psi}{\delta(\eta + z\delta_r)}(r)$, we have by Fubini's theorem

$$\begin{aligned} \int_0^\infty \frac{dz}{z^{1+\alpha}} [\Psi(\eta + z\delta_r) - \Psi(\eta)] &= \int_0^\infty \frac{dz}{z^{1+\alpha}} \int_0^z dw \frac{\delta\Psi}{\delta(\eta + w\delta_r)}(r) \\ &= \frac{1}{\alpha} \int_0^\infty w^{-\alpha} dw \frac{\delta\Psi}{\delta(\eta + w\delta_r)}(r). \end{aligned} \quad (2.13)$$

So (2.8) is deduced from (2.12). Noting that $\eta + w\delta_r \in \mathcal{M}(E)^\circ$ for $w > 0$, apply (2.4) to get

$$\begin{aligned} |\mathcal{L}_\alpha^{(3)}\Psi(\eta)| &\leq \frac{1}{\Gamma(1-\alpha)} \int_E m(dr) \int_0^\infty w^{-\alpha} dw \sum_{i,j=1}^n \frac{f_i(r) C_j^{(i)}}{\langle \eta, f_j \rangle + w f_j(r) + \epsilon} \\ &= \Gamma(\alpha) \sum_{i,j=1}^n C_j^{(i)} \langle m, f_i f_j^{\alpha-1} \rangle (\langle \eta, f_j \rangle + \epsilon)^{-\alpha}, \end{aligned}$$

which proves (2.11). In the above equality we have used

$$\int_0^\infty w^{-\alpha} \frac{dw}{sw+t} = \Gamma(\alpha)\Gamma(1-\alpha)s^{\alpha-1}t^{-\alpha}, \quad s, t > 0. \tag{2.14}$$

It remains to prove (2.6) and (2.9). Similarly to (2.13)

$$\begin{aligned} \mathcal{L}_\alpha^{(1)}\Psi(\eta) &= \frac{\alpha+1}{\Gamma(1-\alpha)} \int_E \eta(dr)a(r) \int_0^\infty \frac{dz}{z^{2+\alpha}} \int_0^z dw \left[\frac{\delta\Psi}{\delta(\eta+w\delta_r)}(r) - \frac{\delta\Psi}{\delta\eta}(r) \right] \\ &= \frac{1}{\Gamma(1-\alpha)} \int_E \eta(dr)a(r) \int_0^\infty \frac{dw}{w^{1+\alpha}} \left[\frac{\delta\Psi}{\delta(\eta+w\delta_r)}(r) - \frac{\delta\Psi}{\delta\eta}(r) \right] \end{aligned}$$

and by (2.12)

$$\frac{\delta\Psi}{\delta(\eta+w\delta_r)}(r) - \frac{\delta\Psi}{\delta\eta}(r) = \int_0^w du \sum_{i,j=1}^n f_i(r)f_j(r)\varphi_{ij}(\langle \eta+u\delta_r, \mathbf{f} \rangle).$$

Hence (2.6) is derived by Fubini’s theorem. (2.6) and (2.5) together yield

$$\begin{aligned} |\mathcal{L}_\alpha^{(1)}\Psi(\eta)| &\leq \frac{1}{\alpha\Gamma(1-\alpha)} \int_E \eta(dr)a(r) \int_0^\infty u^{-\alpha} du \sum_{i,j,k=1}^n \frac{f_i(r)f_j(r)C_k^{(ij)}}{(\langle \eta, f_k \rangle + u f_k(r) + \epsilon)^2} \\ &= \Gamma(\alpha) \sum_{i,j,k=1}^n C_k^{(ij)} \frac{\langle \eta, a f_i f_j f_k^{\alpha-1} \rangle}{(\langle \eta, f_k \rangle + \epsilon)^{\alpha+1}}. \end{aligned}$$

Here, the last equality is deduced from

$$\int_0^\infty u^{-\alpha} \frac{du}{(su+t)^2} = \alpha\Gamma(\alpha)\Gamma(1-\alpha)s^{\alpha-1}t^{-(\alpha+1)}, \quad s, t > 0,$$

which is verified by differentiating (2.14) in t . □

Following [10], we consider the operator $(\mathcal{L}_\alpha, \mathcal{F})$ as an operator on $C_\infty(\mathcal{M}(E))$, the set of continuous functions on $\mathcal{M}(E)$ vanishing at infinity. In the theorem below we collect basic properties of \mathcal{L}_α and the associated transition semigroup.

Theorem 2.2. (i) $(\mathcal{L}_\alpha, \mathcal{F})$ is closable in $C_\infty(\mathcal{M}(E))$ and the closure $(\overline{\mathcal{L}_\alpha}, D(\overline{\mathcal{L}_\alpha}))$ generates a C_0 -semigroup $(T(t))_{t \geq 0}$. Moreover, \mathcal{D} is a core for $\overline{\mathcal{L}_\alpha}$, and for each $f \in B_+(E)$ and $\eta \in \mathcal{M}(E)$

$$T(t)\Psi_f(\eta) = \exp \left[-\langle \eta, V_t f \rangle - \int_0^t \langle m, (V_s f)^\alpha \rangle ds \right], \quad t \geq 0, \tag{2.15}$$

where

$$V_t f(r) = \frac{e^{-b(r)t/\alpha} f(r)}{\left[1 + a(r)f(r)^\alpha \int_0^t e^{-b(r)s} ds \right]^{1/\alpha}}. \tag{2.16}$$

(ii) If $b \in C_{++}(E)$, then Markov process with transition semigroup $(T(t))_{t \geq 0}$ is ergodic in the sense that for every initial state $\eta \in \mathcal{M}(E)$, the law of the process at time t converges to a unique stationary distribution, say Q_α , as $t \rightarrow \infty$. Moreover, the Laplace functional of Q_α is given by

$$\int_{\mathcal{M}(E)^\circ} Q_\alpha(d\eta)\Psi_f(\eta) = \exp \left[-\langle m, a^{-1} \log(1 + ab^{-1} f^\alpha) \rangle \right], \quad f \in B_+(E). \tag{2.17}$$

Proof. (i) If m were the null measure, the assertions except (2.16) follow from more general Theorem 1.1 in [10], and also (2.16) is deduced from the proof of it. Indeed, $V_t f(r)$ was given there implicitly by

$$\frac{\partial}{\partial t} V_t f(r) = -\frac{a(r)}{\alpha} V_t f(r)^{1+\alpha} - \frac{b(r)}{\alpha} V_t f(r), \quad V_0 f(r) = f(r), \quad (2.18)$$

from which (2.16) is obtained. (See Example 3.1 in [6].)

Based on these facts, the proof of the assertions for $m \in \mathcal{M}(E)^\circ$ can be done by modifying suitably the proof of Corollary 1.3 in [10], which deals with the immigration mechanism described by the operator $\Psi \mapsto \langle m, \frac{\delta \Psi}{\delta \eta} \rangle$. A (possibly unique) non-trivial modification would be the step to construct, for each $\eta \in \mathcal{M}(E)$ and $t \geq 0$, $q_t(\eta, \cdot) \in \mathcal{M}_1(\mathcal{M}(E))$ with Laplace transform given by the right side of (2.15). By the observation made in the last paragraph, we have $p_t(\eta, \cdot) \in \mathcal{M}_1(\mathcal{M}(E))$ such that

$$\int_{\mathcal{M}(E)} p_t(\eta, d\eta') \Psi_f(\eta') = \exp[-\langle \eta, V_t f \rangle], \quad f \in B_+(E).$$

Additionally, for every $\eta \in \mathcal{M}(E)$, let $s_\alpha(\eta, \cdot)$ be the law of an α -stable random measure with parameter measure η , i.e.,

$$\int_{\mathcal{M}(E)} s_\alpha(\eta, d\eta') \Psi_f(\eta') = \exp[-\langle \eta, f^\alpha \rangle], \quad f \in B_+(E)$$

and define $p_{t,\alpha}(\eta, \cdot) \in \mathcal{M}_1(\mathcal{M}(E))$ to be the mixture

$$p_{t,\alpha}(\eta, \cdot) = \int_{\mathcal{M}(E)} s_\alpha(\eta, d\eta') p_t(\eta', \cdot).$$

It then follows that

$$\int_{\mathcal{M}(E)} p_{t,\alpha}(\eta, d\eta') \Psi_f(\eta') = \exp[-\langle \eta, (V_t f)^\alpha \rangle], \quad f \in B_+(E).$$

Therefore, for each $N = 1, 2, \dots$, the convolution

$$q_t^{(N)}(\eta, \cdot) := p_t(\eta, \cdot) * \left(\bigstar_{k=1}^N p_{t/N, \alpha} \left(\frac{t}{N} m, \cdot \right) \right)$$

has Laplace transform

$$\int_{\mathcal{M}(E)} q_t^{(N)}(\eta, d\eta') \Psi_f(\eta') = \exp \left[-\langle \eta, V_t f \rangle - \sum_{k=1}^N \frac{t}{N} \langle m, (V_{t/N} f)^\alpha \rangle \right],$$

which converges to the right side of (2.15) as $N \rightarrow \infty$. Thus, the weak limit of $q_t^{(N)}(\eta, \cdot)$ as $N \rightarrow \infty$ is identified with the desired probability measure $q_t(\eta, \cdot)$ on $\mathcal{M}(E)$. Hence the semigroup $(T(t))_{t \geq 0}$ defined by

$$T(t)\Psi(\eta) = \int_{\mathcal{M}(E)} q_t(\eta, d\eta') \Psi(\eta'), \quad \Psi \in B(\mathcal{M}(E))$$

satisfies (2.15). The identity $\frac{d}{dt} T(t)\Psi_f|_{t=0} = \mathcal{L}_\alpha \Psi_f$ for $f \in C_{++}(E)$ is verified by combining (2.2) with (2.18). Once (2.15) is in hand, the assertion that \mathcal{D} is a core for $\overline{\mathcal{L}_\alpha}$ follows as a direct consequence of Lemma 2.2 in [13].

(ii) As $t \rightarrow \infty$ the right side of (2.15) converges to

$$\exp \left[-\int_0^\infty \langle m, (V_t f)^\alpha \rangle dt \right] = \exp \left[-\langle m, a^{-1} \log(1 + ab^{-1} f^\alpha) \rangle \right]$$

since by (2.18)

$$\frac{d}{dt} \log [1 + a(r)b(r)^{-1}(V_t f(r))^\alpha] = -a(r)(V_t f(r))^\alpha.$$

This proves the required ergodicity and that the unique stationary distribution Q_α has the Laplace functional given by the right side of (2.17). The fact that Q_α is supported on $\mathcal{M}(E)^\circ$ follows by observing that the right side of (2.17) with $f \equiv \beta > 0$ tends to 0 as $\beta \rightarrow \infty$. \square

We call the Markov process on $\mathcal{M}(E)$ associated with (2.1) in the sense of Theorem 2.2 the measure-valued α -CIR model with triplet (a, b, m) . It is said to be ergodic if $b \in C_{++}(E)$.

Remark 2.3. (i) A random measure with law Q_α in Theorem 2.2 (ii) is an infinite-dimensional analogue of the random variable with law sometimes referred to as a (non-symmetric) Linnik distribution, whose Laplace exponent is of the form $\lambda \mapsto c \log(1 + d\lambda^\alpha)$ for some $c, d > 0$. Observe from (2.17) that, as $\alpha \uparrow 1$, Q_α converges to Q_1 , the law of a generalized gamma process such that

$$\int_{\mathcal{M}(E)^\circ} Q_1(d\eta) \Psi_f(\eta) = \exp [-\langle m, a^{-1} \log(1 + ab^{-1}f) \rangle], \quad f \in B_+(E).$$

In addition, one can see that

$$\lim_{\alpha \uparrow 1} \mathcal{L}_\alpha \Psi(\eta) = \langle \eta, a \frac{\delta^2 \Psi}{\delta \eta^2} \rangle - \langle \eta, b \frac{\delta \Psi}{\delta \eta} \rangle + \langle m, \frac{\delta \Psi}{\delta \eta} \rangle =: \mathcal{L}_1 \Psi(\eta)$$

for ‘nice’ functions Ψ , where $\frac{\delta^2 \Psi}{\delta \eta^2}(r) = \frac{d^2}{d\epsilon^2} \Psi(\eta + \epsilon \delta_r) \Big|_{\epsilon=0}$. (For instance, this is immediate for $\Psi = \Psi_f$ from (2.2).) \mathcal{L}_1 is the generator of an MBI-process discussed in Section 4 of [11] and in Section 3 of [10], where Q_1 was shown to be a reversible stationary distribution of the process associated with \mathcal{L}_1 .

(ii) In contrast, Q_α ($0 < \alpha < 1$) is not a reversible stationary distribution of the measure-valued α -CIR model. See Theorem 2.3 in [3] for an assertion of this type regarding CBI-processes. Essentially the same proof works at least in the case of ergodic measure-valued α -CIR models. Namely, one can show, by a proof by contradiction, that the formal symmetry $E^{Q_\alpha} [(-\mathcal{L}_\alpha) \Psi_f \cdot \Psi_g] = E^{Q_\alpha} [(-\mathcal{L}_\alpha) \Psi_g \cdot \Psi_f]$ fails for some $f, g \in C_{++}(E)$. For this purpose, an expression for the Dirichlet form $E^{Q_\alpha} [(-\mathcal{L}_\alpha) \Psi_f \cdot \Psi_g]$ given after Lemma 3.1 below is helpful.

3 Associated Dirichlet forms

From now on, we suppose additionally that $b \in C_{++}(E)$. Thus, only ergodic measure-valued α -CIR models will be discussed. To study the speed of convergence to equilibrium in the L^2 -sense, we consider the symmetric part of Dirichlet form associated with \mathcal{L}_α in (2.1). It is a bilinear form on $\mathcal{F} \times \mathcal{F}$ defined by $\tilde{\mathcal{E}}(\Psi, \Psi') := E^{Q_\alpha} [\Gamma(\Psi, \Psi')]$ with $\Gamma(\cdot, \cdot)$ being the ‘carré du champ’:

$$\begin{aligned} \Gamma(\Psi, \Psi')(\eta) &:= \frac{1}{2} [-\Psi(\eta) \mathcal{L}_\alpha \Psi'(\eta) - \Psi'(\eta) \mathcal{L}_\alpha \Psi(\eta) + \mathcal{L}_\alpha(\Psi \Psi')(\eta)] \\ &= \frac{1}{2} \int_0^\infty n_B(dz) \int_E \eta(dr) a(r) [\Psi(\eta + z\delta_r) - \Psi(\eta)] [\Psi'(\eta + z\delta_r) - \Psi'(\eta)] \\ &\quad + \frac{1}{2} \int_0^\infty n_I(dz) \int_E m(dr) [\Psi(\eta + z\delta_r) - \Psi(\eta)] [\Psi'(\eta + z\delta_r) - \Psi'(\eta)], \end{aligned}$$

where $n_B(dz) = (\alpha + 1)z^{-\alpha-2} dz / \Gamma(1 - \alpha)$ and $n_I(dz) = \alpha z^{-\alpha-1} dz / \Gamma(1 - \alpha)$ govern the jump mechanisms associated with branching and immigration, respectively. The same

argument as in the proof of Proposition 1.6 in [10] shows that $(\mathcal{L}_\alpha, \mathcal{F})$ is closable in $L^2(Q_\alpha)$ and that the closure $(\overline{\mathcal{L}_\alpha}^{(2)}, D(\overline{\mathcal{L}_\alpha}^{(2)}))$ generates a C_0 -semigroup $(T^2(t))_{t \geq 0}$ on $L^2(Q_\alpha)$ which coincides with $(T(t))_{t \geq 0}$ when restricted to $C_\infty(\mathcal{M}(E))$. We set $\mathcal{E}(\Psi) = E^{Q_\alpha} [(-\overline{\mathcal{L}_\alpha}^{(2)})\Psi \cdot \Psi]$ for any $\Psi \in D(\overline{\mathcal{L}_\alpha}^{(2)})$, remarking that $\mathcal{E}(\Psi) = \tilde{\mathcal{E}}(\Psi, \Psi)$ for $\Psi \in \mathcal{F}$.

Let $\text{var}(\Psi)$ stand for the variance of $\Psi \in L^2(Q_\alpha)$ with respect to Q_α , namely,

$$\text{var}(\Psi) = E^{Q_\alpha} [(\Psi - E^{Q_\alpha}[\Psi])^2].$$

It is known that the largest $\kappa \geq 0$ such that

$$\text{var}(T^2(t)\Psi) \leq e^{-\kappa t} \text{var}(\Psi) \quad \text{for all } \Psi \in L^2(Q_\alpha) \text{ and } t > 0$$

is identified with

$$\begin{aligned} \text{gap}(\overline{\mathcal{L}_\alpha}^{(2)}) &:= \inf \left\{ \mathcal{E}(\Psi) : \text{var}(\Psi) = 1, \Psi \in D(\overline{\mathcal{L}_\alpha}^{(2)}) \right\} \\ &= \sup \left\{ \kappa \geq 0 : \kappa \cdot \text{var}(\Psi) \leq \mathcal{E}(\Psi) \text{ for all } \Psi \in D(\overline{\mathcal{L}_\alpha}^{(2)}) \right\}. \end{aligned}$$

We refer the reader to e.g. Theorem 2.3 in [7] for the proof of this fact in a general setting. Besides, an estimate of the form $\text{gap}(\overline{\mathcal{L}_\alpha}^{(2)}) \geq \kappa$ implies that $\overline{\mathcal{L}_\alpha}^{(2)}$ has a spectral gap below 0 of size larger than or equal to κ . (See Remark 1.13 in [10].) In calculating Dirichlet form and the variance functional with respect to Q_α , we will make an essential use of the following expression for the ‘log-Laplace functional’ in (2.17):

$$\begin{aligned} \psi(f) &:= \langle m, a^{-1} \log(1 + ab^{-1}f^\alpha) \rangle \\ &= \int_E m_a(dr) \int_0^\infty \Lambda(dz) \left(1 - e^{-f^*(r)z}\right), \end{aligned} \tag{3.1}$$

where $m_a(dr) = a(r)^{-1}m(dr)$, Λ is the Lévy measure of the infinite divisible distribution on $(0, \infty)$ with Laplace exponent $\lambda \mapsto \log(1 + \lambda^\alpha)$ and $f^* = (a/b)^{1/\alpha}f$. In what follows the domain of integration is understood to be $(0, \infty)$ when suppressed. Define nonnegative functions K_B and K_I on \mathbf{R}_+^2 by

$$K_B(s, t) := \int n_B(dy)(1 - e^{-sy})(1 - e^{-ty}) = \alpha^{-1} [(s + t)^{\alpha+1} - s^{\alpha+1} - t^{\alpha+1}] \tag{3.2}$$

and

$$K_I(s, t) := \int n_I(dy)(1 - e^{-sy})(1 - e^{-ty}) = s^\alpha + t^\alpha - (s + t)^\alpha, \tag{3.3}$$

respectively. The above identities are verified easily by differentiating in s and t .

Lemma 3.1. For any $f, g \in B_+(E)$

$$\begin{aligned} \tilde{\mathcal{E}}(\Psi_f, \Psi_g) &= \frac{1}{2} e^{-\psi(f+g)} \int_E m(dr) \frac{a(r)^{1/\alpha}}{b(r)^{1/\alpha}} \int n_B(dy)(1 - e^{-f(r)y})(1 - e^{-g(r)y}) \\ &\quad \int \Lambda(dz) z e^{-(f^*(r)+g^*(r))z} \\ &\quad + \frac{1}{2} e^{-\psi(f+g)} \int_E m(dr) \int n_I(dy)(1 - e^{-f(r)y})(1 - e^{-g(r)y}) \\ &= \frac{1}{2} e^{-\psi(f+g)} \left(\langle m, \frac{\alpha(f+g)^{\alpha-1} a K_B(f, g)}{b + a(f+g)^\alpha} \rangle + \langle m, K_I(f, g) \rangle \right), \end{aligned}$$

which is finite.

Proof. It follows that

$$\begin{aligned} \Gamma(\Psi_f, \Psi_g) &= \frac{1}{2} e^{-\langle \eta, f+g \rangle} \int_E \eta(dr) a(r) \int n_B(dy) (1 - e^{-f(r)y}) (1 - e^{-g(r)y}) \\ &\quad + \frac{1}{2} e^{-\langle \eta, f+g \rangle} \int_E m(dr) \int n_I(dy) (1 - e^{-f(r)y}) (1 - e^{-g(r)y}) \\ &= \frac{1}{2} e^{-\langle \eta, f+g \rangle} (\langle \eta, aK_B(f, g) \rangle + \langle m, K_I(f, g) \rangle). \end{aligned}$$

Note that the function $r \mapsto K_I(f(r), g(r))$ is an element of $B_+(E)$. Defining $h \in B_+(E)$ by $h(r) = a(r)K_B(f(r), g(r))$ and recalling that $\tilde{\mathcal{E}}(\Psi_f, \Psi_g) = E^{Q_\alpha}[\Gamma(\Psi_f, \Psi_g)]$, we need only to show that

$$\begin{aligned} I(f + g; h) &:= E^{Q_\alpha} \left[e^{-\langle \eta, f+g \rangle} \langle \eta, h \rangle \right] \\ &= e^{-\psi(f+g)} \int_E m_a(dr) \frac{a(r)^{1/\alpha}}{b(r)^{1/\alpha}} h(r) \int \Lambda(dz) z e^{-(f^*(r)+g^*(r))z} \\ &= e^{-\psi(f+g)} \left\langle m, \frac{\alpha(f+g)^{\alpha-1} h}{b+a(f+g)^\alpha} \right\rangle \end{aligned} \tag{3.4}$$

and that this is finite. The second equality can be verified to hold by (2.17) and (3.1) together:

$$\begin{aligned} I(f + g; h) &= - \frac{d}{d\epsilon} E^{Q_\alpha} \left[e^{-\langle \eta, f+g+\epsilon h \rangle} \right] \Big|_{\epsilon=0} = - \frac{d}{d\epsilon} e^{-\psi(f+g+\epsilon h)} \Big|_{\epsilon=0} \\ &= e^{-\psi(f+g)} \int_E m_a(dr) h^*(r) \int \Lambda(dz) z e^{-(f^*(r)+g^*(r))z} \\ &= e^{-\psi(f+g)} \int_E m_a(dr) \frac{a(r)^{1/\alpha}}{b(r)^{1/\alpha}} h(r) \int \Lambda(dz) z e^{-(f^*(r)+g^*(r))z}. \end{aligned}$$

For the proof of the last equality in (3.4), we make use of another expression for $I(f + g; h)$ deduced from (2.17) only:

$$\begin{aligned} I(f + g; h) &= - \frac{d}{d\epsilon} \exp \left[-\langle m, a^{-1} \log(1 + ab^{-1}(f + g + \epsilon h)^\alpha) \rangle \right] \Big|_{\epsilon=0} \\ &= e^{-\psi(f+g)} \left\langle m, \frac{\alpha(f+g)^{\alpha-1} h}{b+a(f+g)^\alpha} \right\rangle. \end{aligned}$$

Here, by (3.2)

$$0 \leq \alpha(f + g)^{\alpha-1} h \leq (f + g)^{\alpha-1} a(f + g)^{\alpha+1} = a(f + g)^{2\alpha}$$

and so $I(f + g; h)$ is finite. □

Noting that (3.4) is clearly valid for every $h \in B_+(E)$ and combining (2.2) with (3.4), we get for any $f, g \in B_+(E)$

$$\begin{aligned} E^{Q_\alpha} [(-\mathcal{L}_\alpha)\Psi_f \cdot \Psi_g] &= -E^{Q_\alpha} \left[\Psi_{f+g}(\eta) \cdot \frac{1}{\alpha} \langle \eta, a f^{\alpha+1} + b f \rangle - \Psi_{f+g}(\eta) \langle m, f^\alpha \rangle \right] \\ &= -e^{-\psi(f+g)} \left(\left\langle m, \frac{(f+g)^{\alpha-1} (a f^{\alpha+1} + b f)}{b + a(f+g)^\alpha} \right\rangle - \langle m, f^\alpha \rangle \right), \end{aligned}$$

from which the last expression in Lemma 3.1 for the symmetric part

$$\tilde{\mathcal{E}}(\Psi_f, \Psi_g) = \frac{1}{2} (E^{Q_\alpha} [(-\mathcal{L}_\alpha)\Psi_f \cdot \Psi_g] + E^{Q_\alpha} [(-\mathcal{L}_\alpha)\Psi_g \cdot \Psi_f])$$

can be recovered.

Our objective is to show the positivity of $\text{gap}(\overline{\mathcal{L}}_\alpha^{(2)})$. The contribution here in this direction is the reduction to a certain estimate regarding the one-dimensional model. For a measurable function f on E , the essential supremum (resp. the essential infimum) of f with respect to m is denoted by $\text{ess sup}_{(E,m)} f$ (resp. $\text{ess inf}_{(E,m)} f$). Let D be the linear span of functions on \mathbf{R}_+ of the form $F_\lambda(z) := e^{-\lambda z}$ for some $\lambda > 0$.

Theorem 3.2. *Suppose that $b \in C_{++}(E)$. Let $\gamma > 0$ be a constant. If for every $F \in D$*

$$\int \Lambda(dz)(F(z) - F(0))^2 \leq \frac{\gamma}{2} \left[\int \Lambda(dz)z \int n_B(dy)(F(z+y) - F(z))^2 + \int n_I(dy)(F(y) - F(0))^2 \right], \tag{3.5}$$

then for any $\Psi \in \mathcal{D}$

$$\text{var}(\Psi) \leq \gamma \text{ess sup}_{(E,m)}(b^{-1})\mathcal{E}(\Psi) \tag{3.6}$$

and it holds that $\text{gap}(\overline{\mathcal{L}}_\alpha^{(2)}) \geq \gamma^{-1} \text{ess inf}_{(E,m)} b$.

This kind of reduction was discovered by Stannat [12] (Theorem 1.2) for a lower estimate for the quadratic form of gradient type. In particular, for the process associated with \mathcal{L}_1 in Remark 2.3 (i), the condition corresponding to (3.5) reads

$$\int \Lambda_1(dz)(F(z) - F(0))^2 \leq \gamma \int \Lambda_1(dz)z(F'(z))^2, \tag{3.7}$$

where $\Lambda_1(dz) = z^{-1}e^{-z}dz$ is the Lévy measure of a gamma distribution. While (3.7) with $\gamma = 1$ is verified easily by applying Schwarz's inequality to $F(z) - F(0) = \int_0^z F'(w)dw$, showing an inequality of the form (3.5) is more difficult and we postpone it until the next section. However, as will be seen below, the reduction itself is proved in a similar way to [12].

Proof of Theorem 3.2. Consider a function Ψ expressed as a finite sum $\Psi = \sum_i c_i \Psi_{f_i}$, where $c_i \in \mathbf{R}$ and $f_i \in B_+(E)$. Putting $d_i = c_i e^{-\psi(f_i)}$, observe from (3.1) that

$$\begin{aligned} \text{var}(\Psi) &= \sum_{i,j} c_i c_j \left(e^{-\psi(f_i+f_j)} - e^{-\psi(f_i)} e^{-\psi(f_j)} \right) \\ &= \sum_{i,j} d_i d_j \left(e^{\psi(f_i)+\psi(f_j)-\psi(f_i+f_j)} - 1 \right) \\ &= \sum_{i,j} d_i d_j \left[\exp \left(\int_E \int m_a(dr) \Lambda(dz) (1 - e^{-f_i^*(r)z}) (1 - e^{-f_j^*(r)z}) \right) - 1 \right] \\ &= \sum_{i,j} d_i d_j \sum_{N=1}^\infty \frac{1}{N!} \left(\int_E \int m_a(dr) \Lambda(dz) (1 - e^{-f_i^*(r)z}) (1 - e^{-f_j^*(r)z}) \right)^N. \end{aligned} \tag{3.8}$$

Rewrite in terms of the N -fold product measures $m_a^{\otimes N}$ and $\Lambda^{\otimes N}$ to obtain the following

disintegration formula for the variance functional:

$$\begin{aligned} \text{var}(\Psi) &= \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{i,j} d_i d_j \int_{E^N} \int_{\mathbf{R}_+^N} m_a^{\otimes N}(d\mathbf{r}_N) \Lambda^{\otimes N}(d\mathbf{z}_N) \\ &\quad \prod_{k=1}^N (1 - e^{-f_i^*(r_k)z_k}) \prod_{l=1}^N (1 - e^{-f_j^*(r_l)z_l}) \\ &= \sum_{N=1}^{\infty} \frac{1}{N!} \int_{E^N} \int_{\mathbf{R}_+^N} m_a^{\otimes N}(d\mathbf{r}_N) \Lambda^{\otimes N}(d\mathbf{z}_N) \left[\sum_i d_i \prod_{k=1}^N (1 - e^{-f_i^*(r_k)z_k}) \right]^2 \end{aligned} \quad (3.9)$$

where $\mathbf{r}_N = (r_1, \dots, r_N)$ and $\mathbf{z}_N = (z_1, \dots, z_N)$. Given $\mathbf{r}_N = (r_1, \dots, r_N) \in E^N$ and $z_1, \dots, z_{N-1} \in \mathbf{R}_+$ arbitrarily, apply (3.5) to the function

$$z_N \mapsto \sum_i d_i \left\{ \prod_{k=1}^{N-1} (1 - e^{-f_i^*(r_k)z_k}) \right\} e^{-f_i^*(r_N)z_N}$$

to get

$$\begin{aligned} &\frac{2}{\gamma} \int \Lambda(dz_N) \left[\sum_i d_i \prod_{k=1}^N (1 - e^{-f_i^*(r_k)z_k}) \right]^2 \\ &\leq \int \Lambda(dz) z \int n_B(dy) \left[\sum_i d_i \prod_{k=1}^{N-1} (1 - e^{-f_i^*(r_k)z_k}) (e^{-f_i^*(r_N)(z+y)} - e^{-f_i^*(r_N)z}) \right]^2 \\ &\quad + \int n_I(dy) \left[\sum_i d_i \prod_{k=1}^{N-1} (1 - e^{-f_i^*(r_k)z_k}) (e^{-f_i^*(r_N)y} - 1) \right]^2 \\ &= \frac{a(r_N)^{1+1/\alpha}}{b(r_N)^{1+1/\alpha}} \int \Lambda(dz) z \int n_B(dy) \left[\sum_i d_i \prod_{k=1}^{N-1} (1 - e^{-f_i^*(r_k)z_k}) e^{-f_i^*(r_N)z} (1 - e^{-f_i(r_N)y}) \right]^2 \\ &\quad + \frac{a(r_N)}{b(r_N)} \int n_I(dy) \left[\sum_i d_i \prod_{k=1}^{N-1} (1 - e^{-f_i^*(r_k)z_k}) (1 - e^{-f_i(r_N)y}) \right]^2. \end{aligned} \quad (3.10)$$

Here, a suitable change of variable has been made for each integral with respect to $n_B(dy)$ and $n_I(dy)$ in order to replace $f_i^*(r_N)y$ by $f_i(r_N)y$.

Set $C = \text{esssup}_{(E,m)}(b^{-1})$ so that

$$m_a(dr_N) = a(r_N)^{-1} m(dr_N) \leq C \cdot b(r_N) a(r_N)^{-1} m(dr_N)$$

in distributional sense. Combining (3.9) with (3.10), we can dominate $2\text{var}(\Psi)/\gamma$ by

$$\begin{aligned}
 & \sum_{N=1}^{\infty} \frac{C}{N!} \int_{E^{N-1}} \int_{\mathbf{R}_+^{N-1}} m_a^{\otimes N-1}(d\mathbf{r}_{N-1}) \Lambda^{\otimes N-1}(d\mathbf{z}_{N-1}) \int_E m(dr) \frac{a(r)^{1/\alpha}}{b(r)^{1/\alpha}} \\
 & \quad \int \Lambda(dz)z \int n_B(dy) \left[\sum_i d_i \prod_{k=1}^{N-1} (1 - e^{-f_i^*(r_k)z_k}) e^{-f_i^*(r)z} (1 - e^{-f_i(r)y}) \right]^2 \\
 & \quad + \sum_{N=1}^{\infty} \frac{C}{N!} \int_{E^{N-1}} \int_{\mathbf{R}_+^{N-1}} m_a^{\otimes N-1}(d\mathbf{r}_{N-1}) \Lambda^{\otimes N-1}(d\mathbf{z}_{N-1}) \int_E m(dr) \\
 & \quad \int n_I(dy) \left[\sum_i d_i \prod_{k=1}^{N-1} (1 - e^{-f_i^*(r_k)z_k}) (1 - e^{-f_i(r)y}) \right]^2 \\
 & \leq \sum_{N=0}^{\infty} \frac{C}{N!} \int_{E^N} \int_{\mathbf{R}_+^N} m_a^{\otimes N}(d\mathbf{r}_N) \Lambda^{\otimes N}(d\mathbf{z}_N) \int_E m(dr) \frac{a(r)^{1/\alpha}}{b(r)^{1/\alpha}} \\
 & \quad \int \Lambda(dz)z \int n_B(dy) \left[\sum_i d_i \prod_{k=1}^N (1 - e^{-f_i^*(r_k)z_k}) e^{-f_i^*(r)z} (1 - e^{-f_i(r)y}) \right]^2 \\
 & \quad + \sum_{N=0}^{\infty} \frac{C}{N!} \int_{E^N} \int_{\mathbf{R}_+^N} m_a^{\otimes N}(d\mathbf{r}_N) \Lambda^{\otimes N}(d\mathbf{z}_N) \int_E m(dr) \\
 & \quad \int n_I(dy) \left[\sum_i d_i \prod_{k=1}^N (1 - e^{-f_i^*(r_k)z_k}) (1 - e^{-f_i(r)y}) \right]^2 \\
 & = \sum_{i,j} d_i d_j \sum_{N=0}^{\infty} \frac{C}{N!} \left(\int_E \int m_a(dr_1) \Lambda(dz_1) (1 - e^{-f_i^*(r_1)z_1}) (1 - e^{-f_j^*(r_1)z_1}) \right)^N \\
 & \quad \int_E m(dr) \frac{a(r)^{1/\alpha}}{b(r)^{1/\alpha}} \int n_B(dy) (1 - e^{-f_i(r)y}) (1 - e^{-f_j(r)y}) \int \Lambda(dz) z e^{-(f_i^*(r)+f_j^*(r))z} \\
 & \quad + \sum_{i,j} d_i d_j \sum_{N=0}^{\infty} \frac{C}{N!} \left(\int_E \int m_a(dr_1) \Lambda(dz_1) (1 - e^{-f_i^*(r_1)z_1}) (1 - e^{-f_j^*(r_1)z_1}) \right)^N \\
 & \quad \int_E m(dr) \int n_I(dy) (1 - e^{-f_i(r)y}) (1 - e^{-f_j(r)y}) \\
 & = C \sum_{i,j} c_i c_j e^{-\psi(f_i+f_j)} \int_E m(dr) \frac{a(r)^{1/\alpha}}{b(r)^{1/\alpha}} \int n_B(dy) (1 - e^{-f_i(r)y}) (1 - e^{-f_j(r)y}) \\
 & \quad \int \Lambda(dz) z e^{-(f_i^*(r)+f_j^*(r))z} \\
 & \quad + C \sum_{i,j} c_i c_j e^{-\psi(f_i+f_j)} \int_E m(dr) \int n_I(dy) (1 - e^{-f_i(r)y}) (1 - e^{-f_j(r)y}),
 \end{aligned}$$

where the last two equalities are seen by similar calculations to (3.8) and (3.9). Since the symmetric part $\tilde{\mathcal{E}}$ of Dirichlet form is bilinear, (3.6) for $\Psi \in \mathcal{D}$ follows from Lemma 3.1.

It remains to prove that (3.6) extends to $\Psi \in D(\overline{\mathcal{L}_\alpha}^{(2)})$. Since $(\overline{\mathcal{L}_\alpha}^{(2)}, D(\overline{\mathcal{L}_\alpha}^{(2)}))$ is the closure of $(\mathcal{L}_\alpha, \mathcal{F})$ in $L^2(Q_\alpha)$, we need only to show that (3.6) extends to $\Psi \in \mathcal{F}$. Given $\Psi \in \mathcal{F}$, we see from Theorem 2.2 (i) that there exists a sequence $\{\Psi_N\}_{N=1}^\infty \subset \mathcal{D}$ such that

$$\|\Psi_N - \Psi\|_\infty + \|\mathcal{L}_\alpha \Psi_N - \mathcal{L}_\alpha \Psi\|_\infty \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Hence

$$\|\Psi_N - \Psi\|_{L^2(Q_\alpha)} + \mathcal{E}(\Psi_N - \Psi) \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

This implies that (3.6) holds for any $\Psi \in \mathcal{F}$ and we complete the proof of Theorem 3.2. \square

Remark 3.3. *In view of calculations in the proof of Theorem 3.2, it is clear that under the same assumption an appropriate version of the inequality (3.6) holds for a more general class of functions a and b . To be more precise, suppose that $a, b \in B_+(E)$ are uniformly positive and that Q_α has Laplace transform (2.17). Then, assuming that (3.5) is valid for all $F \in D$, we have*

$$E^{Q_\alpha} \left[(\Psi - E^{Q_\alpha}[\Psi])^2 \right] \leq \gamma \operatorname{ess\,sup}_{(E,m)}(b^{-1}) E^{Q_\alpha} [\Gamma(\Psi, \Psi)]$$

for any $\Psi \in \mathcal{D}$, where

$$\begin{aligned} \Gamma(\Psi, \Psi)(\eta) &= \frac{1}{2} \int n_B(dz) \int_E \eta(dr) a(r) [\Psi(\eta + z\delta_r) - \Psi(\eta)]^2 \\ &\quad + \frac{1}{2} \int n_I(dz) \int_E m(dr) [\Psi(\eta + z\delta_r) - \Psi(\eta)]^2. \end{aligned}$$

This fact reflects the convolution property with respect to m mentioned in the Introduction. (Note that the condition (3.5) is independent of m .)

4 Spectral gap for the α -CIR model

This section is devoted to the proof of (3.5) for some $0 < \gamma < \infty$. The strategy should be different from the one already mentioned for (3.7) with $\Lambda_1(dz) = z^{-1}e^{-z}dz$ at least because no informative expression for the density of Λ in (3.1) appears to be available. Let us illustrate another approach we will take and call ‘the method of intrinsic kernel’ by revisiting (3.7). Suppose that $F \in D$ is a finite sum $F = \sum_i c_i F_{\lambda_i}$. We will use the notation $\mathbf{1}_S$ standing for the indicator function of a set S and $\partial_t = \partial/\partial t$ for simplicity. Letting $\psi_1(\lambda) = \log(1 + \lambda) = \int \Lambda_1(dz)(1 - e^{-\lambda z})$, observe that

$$\begin{aligned} \mathbf{U}_1(F) &:= \int \Lambda_1(dz)(F(z) - F(0))^2 = \sum_{i,j} c_i c_j \int \Lambda_1(dz)(1 - e^{-\lambda_i z})(1 - e^{-\lambda_j z}) \\ &= \sum_{i,j} c_i c_j (-\psi_1(\lambda_i + \lambda_j) + \psi_1(\lambda_i) + \psi_1(\lambda_j)) \\ &= \sum_{i,j} c_i c_j \int_0^{\lambda_i} ds \int_0^{\lambda_j} dt (-\psi_1''(s+t)) \\ &= \int ds \int dt \bar{F}(s) \bar{F}(t) (-\psi_1''(s+t)), \end{aligned} \tag{4.1}$$

where $\bar{F}(s) = \sum_i c_i \mathbf{1}_{[0, \lambda_i]}(s)$. On the other hand, by putting $K_1(s, t) = st\psi_1'(s+t)$

$$\begin{aligned} \mathbf{V}_1(F) &:= \int \Lambda_1(dz) z (F'(z))^2 = \sum_{i,j} c_i c_j \lambda_i \lambda_j \int \Lambda_1(dz) z e^{-\lambda_i z} e^{-\lambda_j z} \\ &= \sum_{i,j} c_i c_j K_1(\lambda_i, \lambda_j) \\ &= \sum_{i,j} c_i c_j \int_0^{\lambda_i} ds \int_0^{\lambda_j} dt \partial_s \partial_t K_1(s, t) \\ &= \int ds \int dt \bar{F}(s) \bar{F}(t) \partial_s \partial_t K_1(s, t). \end{aligned} \tag{4.2}$$

It is reasonable to call $\partial_s \partial_t K_1$ the intrinsic kernel of the quadratic form \mathbf{V}_1 . Similarly, (4.1) shows that the intrinsic kernel of \mathbf{U}_1 is the function $(s, t) \mapsto -\psi_1''(s + t)$.

Given two symmetric measurable functions J and K on \mathbf{R}_+^2 , we write $K \gg J$ if $K - J$ is nonnegative definite in the sense that

$$\int ds \int dt G(s)G(t)(K(s, t) - J(s, t)) \geq 0$$

for any bounded Borel function G on \mathbf{R}_+ with compact support. By virtue of Fubini's theorem, $K \gg 0$ if K is of 'canonical form'

$$K(s, t) = \int_S M(d\omega) \sigma(s, \omega) \sigma(t, \omega), \quad s, t \in \mathbf{R}_+$$

for some measure space (S, M) and measurable function σ on $\mathbf{R}_+ \times S$. In view of (4.1) and (4.2), it is clear that the inequality $\gamma \mathbf{V}_1(F) \geq \mathbf{U}_1(F)$ is implied by

$$\gamma \partial_s \partial_t K_1(s, t) + \psi_1''(s + t) \gg 0.$$

For $\gamma = 1$, this holds true since by direct calculations

$$\partial_s \partial_t K_1(s, t) + \psi_1''(s + t) = \frac{2st}{(1 + s + t)^3} = \int dz z^2 e^{-z} s e^{-sz} t e^{-tz},$$

which is of canonical form. Furthermore, this expression makes it possible to identify the associated 'remainder form':

$$\begin{aligned} \mathbf{V}_1(F) - \mathbf{U}_1(F) &= \int ds \int dt \bar{F}(s) \bar{F}(t) \int dz z^2 e^{-z} s e^{-sz} t e^{-tz} \\ &= \int dz e^{-z} \left(\int ds \bar{F}(s) s z e^{-sz} \right)^2 \\ &= \int dz e^{-z} \left(\sum_i c_i \int_0^{\lambda_i} ds s z e^{-sz} \right)^2 \\ &= \int dz e^{-z} \left(\sum_i c_i \frac{e^{-\lambda_i z} - 1 + \lambda_i z e^{-\lambda_i z}}{z} \right)^2 \\ &= \int \Lambda_1(dz) z^{-1} (F(z) - F(0) - zF'(z))^2. \end{aligned}$$

It should be emphasized that the above calculations require only an explicit form of the Laplace exponent ψ_1 .

Turning to the case $0 < \alpha < 1$, we adopt the method of intrinsic kernels to show (3.5) for Λ such that

$$\psi(\lambda) := \log(1 + \lambda^\alpha) = \int \Lambda(dz) (1 - e^{-\lambda z}), \quad \lambda \geq 0. \tag{4.3}$$

(We continue to adopt this notation as it is a one-dimensional version of (3.1).) Namely, we shall

(I) calculate the intrinsic kernels of $\mathbf{U}(F) := \int \Lambda(dz) (F(z) - F(0))^2$ and of

$$\mathbf{V}(F) := \frac{1}{2} \left[\int \Lambda(dz) z \int n_B(dy) (F(z + y) - F(z))^2 + \int n_I(dy) (F(y) - F(0))^2 \right],$$

and then

(II) compare the two kernels as nonnegative definite functions.

The following lemma concerns the step (I).

Lemma 4.1. *The intrinsic kernels of \mathbf{U} and of \mathbf{V} are respectively given by $J(s, t) := -\psi''(s + t)$ and $\tilde{K}(s, t) := \partial_s \partial_t K(s, t)$, where*

$$2K(s, t) = \frac{(s + t)^{\alpha-1}}{1 + (s + t)^\alpha} [(s + t)^{\alpha+1} - s^{\alpha+1} - t^{\alpha+1}] + [s^\alpha + t^\alpha - (s + t)^\alpha]. \quad (4.4)$$

Proof. The intrinsic kernel of \mathbf{U} is deduced in the same manner as (4.1). The derivation of (4.4) is similar to the calculations at the beginning of the proof of Lemma 3.1. Indeed, for $F = \sum_i c_i F_{\lambda_i}$

$$\begin{aligned} \mathbf{V}(F) &= \frac{1}{2} \sum_{i,j} c_i c_j \int \Lambda(dz) z e^{-(\lambda_i + \lambda_j)z} \int n_B(dy) (1 - e^{-\lambda_i y})(1 - e^{-\lambda_j y}) \\ &\quad + \frac{1}{2} \sum_{i,j} c_i c_j \int n_I(dy) (1 - e^{-\lambda_i y})(1 - e^{-\lambda_j y}) \\ &= \sum_{i,j} c_i c_j K(\lambda_i, \lambda_j), \end{aligned}$$

where the last equality is seen from $\int \Lambda(dz) z e^{-\lambda z} = \psi'(\lambda)$, (4.3), (3.2) and (3.3). The rest of the proof is the same as (4.2) with K in place of K_1 . \square

Remark that, as $\alpha \uparrow 1$, the right side of (4.4) tends to $\frac{2st}{1+s+t} = 2K_1(s, t)$. The main result of this section is obtained by accomplishing not only the step (II) but also identification of the remainder form.

Theorem 4.2. *For each $F \in D$,*

$$\begin{aligned} 2\mathbf{V}(F) - \mathbf{U}(F) &= \frac{\alpha}{\alpha + 1} \int \Lambda(dz) z^3 \int n_B(dy) \left[\frac{F(z) - F(0)}{z} - \frac{F(z + y) - F(0)}{z + y} \right]^2 \\ &\quad + \int \Lambda(dz) z^2 \int n_I(dy) \left[\frac{F(z) - F(0)}{z} - \frac{F(z + y) - F(0)}{z + y} \right]^2 \\ &\quad + \frac{1}{\alpha + 1} \int \Lambda(dz) z \int n_B(dy) (F(z + y) - F(z))^2. \end{aligned} \quad (4.5)$$

In particular, (3.5) with $\gamma = 2$ holds true.

Proof. Let J, K and \tilde{K} be as in Lemma 4.1. Recalling (3.2) and (3.3), we will exploit the following expression for $2K$ in (4.4):

$$2K(s, t) = \rho(s + t)K_B(s, t) + K_I(s, t), \quad (4.6)$$

where

$$\rho(\lambda) = \frac{\alpha \lambda^{\alpha-1}}{1 + \lambda^\alpha} = \psi'(\lambda) = \int \Lambda(dz) z e^{-\lambda z}.$$

Differentiating (4.6) in s and t yields

$$2\tilde{K}(s, t) = \partial_s \partial_t [\rho(s + t)K_B(s, t)] + \alpha(1 - \alpha)(s + t)^{\alpha-2} \quad (4.7)$$

$$\begin{aligned} &= \rho''(s + t)K_B(s, t) + \rho'(s + t)(\partial_s + \partial_t)K_B(s, t) \\ &\quad + \rho(s + t)(\alpha + 1)(s + t)^{\alpha-1} + \alpha(1 - \alpha)(s + t)^{\alpha-2}. \end{aligned} \quad (4.8)$$

Since

$$(\partial_s + \partial_t)K_B(s, t) = \frac{\alpha + 1}{\alpha} [-K_I(s, t) + (s + t)^\alpha],$$

(4.8) becomes

$$2\tilde{K}(s, t) = \rho''(s+t)K_B(s, t) - \frac{\alpha+1}{\alpha}\rho'(s+t)K_I(s, t) + \frac{\alpha+1}{\alpha}\rho'(s+t)(s+t)^\alpha + \rho(s+t)(\alpha+1)(s+t)^{\alpha-1} + \alpha(1-\alpha)(s+t)^{\alpha-2}. \tag{4.9}$$

Here, it is direct to see that the sum of the last three terms on the right side equals

$$-\frac{\alpha+1}{\alpha}\rho'(s+t) - (1-\alpha)(s+t)^{\alpha-2}$$

or equivalently that

$$\frac{1}{\alpha}\rho'(\lambda)(\lambda^\alpha + 1) + \rho(\lambda)\lambda^{\alpha-1} = -(1-\alpha)\lambda^{\alpha-2}.$$

By the above identity and (4.9) together

$$\begin{aligned} 2\tilde{K}(s, t) &= \rho''(s+t)K_B(s, t) - \frac{\alpha+1}{\alpha}\rho'(s+t)K_I(s, t) \\ &\quad - \frac{\alpha+1}{\alpha}\rho'(s+t) - (1-\alpha)(s+t)^{\alpha-2} \\ &= \rho''(s+t)K_B(s, t) - \frac{\alpha+1}{\alpha}\rho'(s+t)K_I(s, t) \\ &\quad + \frac{\alpha+1}{\alpha}J(s, t) - (1-\alpha)(s+t)^{\alpha-2} \\ &= \rho''(s+t)K_B(s, t) - \frac{\alpha+1}{\alpha}\rho'(s+t)K_I(s, t) \\ &\quad + \frac{\alpha+1}{\alpha}J(s, t) - \frac{2}{\alpha}\tilde{K}(s, t) + \frac{1}{\alpha}\partial_s\partial_t[\rho(s+t)K_B(s, t)], \end{aligned}$$

where the last equality follows from (4.7). Consequently

$$\begin{aligned} 2\tilde{K}(s, t) - J(s, t) &= \frac{\alpha}{\alpha+1}\rho''(s+t)K_B(s, t) - \rho'(s+t)K_I(s, t) + \frac{1}{\alpha+1}\partial_s\partial_t[\rho(s+t)K_B(s, t)]. \end{aligned}$$

Each of the terms on the right side is nonnegative definite because of

$$\begin{aligned} \rho''(s+t)K_B(s, t) &= \int \Lambda(dz)z^3 \int n_B(dy)e^{-sz}(1-e^{-sy})e^{-tz}(1-e^{-ty}), \\ -\rho'(s+t)K_I(s, t) &= \int \Lambda(dz)z^2 \int n_I(dy)e^{-sz}(1-e^{-sy})e^{-tz}(1-e^{-ty}) \end{aligned}$$

and

$$\partial_s\partial_t[\rho(s+t)K_B(s, t)] = \int \Lambda(dz)z \int n_B(dy)\partial_s[e^{-sz}(1-e^{-sy})] \partial_t[e^{-tz}(1-e^{-ty})].$$

Therefore, $2\tilde{K} \gg J$ and so $2\mathbf{V}(F) \geq \mathbf{U}(F)$ for any $F \in D$. We further proceed to identify the remainder form $F \mapsto 2\mathbf{V}(F) - \mathbf{U}(F)$. With the help of the canonical representations in the above, we deduce $2\mathbf{V}(F) - \mathbf{U}(F) = \sum_{i,j} c_i c_j K'(\lambda_i, \lambda_j)$ for $F = \sum_i c_i F_{\lambda_i}$, where

$$\begin{aligned} K'(\lambda_i, \lambda_j) &= \int_0^{\lambda_i} ds \int_0^{\lambda_j} dt \left(2\tilde{K}(s, t) - J(s, t) \right) \\ &= \frac{\alpha}{\alpha+1} \int \Lambda(dz)z^3 \int n_B(dy) \left[\frac{1-e^{-\lambda_i z}}{z} - \frac{1-e^{-\lambda_i(z+y)}}{z+y} \right] \left[\frac{1-e^{-\lambda_j z}}{z} - \frac{1-e^{-\lambda_j(z+y)}}{z+y} \right] \\ &\quad + \int \Lambda(dz)z^2 \int n_I(dy) \left[\frac{1-e^{-\lambda_i z}}{z} - \frac{1-e^{-\lambda_i(z+y)}}{z+y} \right] \left[\frac{1-e^{-\lambda_j z}}{z} - \frac{1-e^{-\lambda_j(z+y)}}{z+y} \right] \\ &\quad + \frac{1}{\alpha+1} \int \Lambda(dz)z \int n_B(dy) [e^{-\lambda_i z}(1-e^{-\lambda_i y})] [e^{-\lambda_j z}(1-e^{-\lambda_j y})]. \end{aligned}$$

Accordingly

$$\begin{aligned} 2\mathbf{V}(F) - \mathbf{U}(F) &= \frac{\alpha}{\alpha + 1} \int \Lambda(dz) z^3 \int n_B(dy) \left[\sum_i c_i \left(\frac{1 - e^{-\lambda_i z}}{z} - \frac{1 - e^{-\lambda_i(z+y)}}{z+y} \right) \right]^2 \\ &\quad + \int \Lambda(dz) z^2 \int n_I(dy) \left[\sum_i c_i \left(\frac{1 - e^{-\lambda_i z}}{z} - \frac{1 - e^{-\lambda_i(z+y)}}{z+y} \right) \right]^2 \\ &\quad + \frac{1}{\alpha + 1} \int \Lambda(dz) z \int n_B(dy) \left[\sum_i c_i \left(e^{-\lambda_i z} - e^{-\lambda_i(z+y)} \right) \right]^2. \end{aligned}$$

This coincides with the right side of (4.5) for $F = \sum_i c_i F_{\lambda_i}$, and the proof of Theorem 4.2 is complete. \square

Combining Theorem 4.2 with Theorem 3.2 gives immediately a lower estimate

$$\text{gap}(\overline{\mathcal{L}}_\alpha^{(2)}) \geq \frac{1}{2} \text{ess inf}_{(E,m)} b$$

for ergodic measure-valued α -CIR models discussed in the previous section. In fact, this bound is optimal as seen in

Theorem 4.3. *Let \mathcal{L}_α be of the form (2.1) for some $a, b \in C_{++}(E)$ and $m \in \mathcal{M}(E)^\circ$. Then*

$$\text{gap}(\overline{\mathcal{L}}_\alpha^{(2)}) = \frac{1}{2} \text{ess inf}_{(E,m)} b.$$

Proof. For the aforementioned reason, it suffices to show the upper estimate

$$\text{gap}(\overline{\mathcal{L}}_\alpha^{(2)}) \leq \frac{1}{2} \text{ess inf}_{(E,m)} b.$$

To this end, we use (a variant of) a characterization due to Liggett ([7], (2.5)):

$$\text{gap}(\overline{\mathcal{L}}_\alpha^{(2)}) = \inf_{t>0} \frac{1}{2t} \inf \left\{ -\log \frac{\text{var}(T^2(t)\Psi)}{\text{var}(\Psi)} : 0 < \text{var}(\Psi) < \infty \right\}.$$

This implies that

$$\text{gap}(\overline{\mathcal{L}}_\alpha^{(2)}) \leq \liminf_{t \rightarrow \infty} \frac{1}{2t} (-\log \text{var}(T^2(t)\Psi)) \tag{4.10}$$

for any Ψ such that $0 < \text{var}(\Psi) < \infty$. We now take $\Psi(\eta) = \exp(-\eta(E))$, for which $T^2(t)\Psi(\eta)$ is given by the right side of (2.15) with $f = \mathbf{1}_E =: \mathbf{1}$. Recalling that the log-Laplace functional of Q_α is $\psi(f) = \langle m_\alpha, \log(1 + ab^{-1}f^\alpha) \rangle$, one can derive by (2.15)

$$\begin{aligned} \text{var}(T^2(t)\Psi) &= E^{Q_\alpha} \left[\exp \left(-2\langle \eta, V_t \mathbf{1} \rangle - 2 \int_0^t \langle m, (V_s \mathbf{1})^\alpha \rangle ds \right) \right] - e^{-2\psi(\mathbf{1})} \\ &= e^{-2\psi(\mathbf{1})} (\exp[-\psi(2V_t \mathbf{1}) + 2\psi(V_t \mathbf{1})] - 1), \end{aligned}$$

where the last equality is deduced from

$$\psi(\mathbf{1}) - \int_0^t \langle m, (V_s \mathbf{1})^\alpha \rangle ds = \psi(V_t \mathbf{1}).$$

Since by (2.16) $(V_t \mathbf{1}(r))^\alpha \leq e^{-tb(r)}$, $\Delta(t) := 2\psi(V_t \mathbf{1}) - \psi(2V_t \mathbf{1}) \rightarrow 0$ as $t \rightarrow \infty$ and so

$$\liminf_{t \rightarrow \infty} \frac{1}{t} (-\log \text{var}(T^2(t)\Psi)) = -\limsup_{t \rightarrow \infty} \frac{1}{t} \log \Delta(t).$$

By virtue of (4.10), the proof of Theorem 4.3 reduces to showing that

$$-\limsup_{t \rightarrow \infty} \frac{1}{t} \log \Delta(t) \leq \operatorname{ess\,inf}_{(E,m)} b. \tag{4.11}$$

By straightforward calculations

$$\begin{aligned} \Delta(t) &= \left\langle m_a, \log \left(1 + \frac{\frac{a}{b}(2-2^\alpha)(V_t \mathbf{1})^\alpha + \left(\frac{a}{b}\right)^2 (V_t \mathbf{1})^{2\alpha}}{1 + \frac{a}{b} 2^\alpha (V_t \mathbf{1})^\alpha} \right) \right\rangle \\ &\geq \left\langle m_a, \log \left(1 + \frac{\frac{a}{b}(2-2^\alpha)(V_t \mathbf{1})^\alpha}{1 + \frac{a}{b} 2^\alpha (V_t \mathbf{1})^\alpha} \right) \right\rangle. \end{aligned} \tag{4.12}$$

Further, with the help of the inequality $\log(1+z) \geq z/(1+z)$ for $z \geq 0$, we get

$$\Delta(t) \geq \left\langle m_a, \frac{\frac{a}{b}(2-2^\alpha)(V_t \mathbf{1})^\alpha}{1 + \frac{a}{b} 2^\alpha (V_t \mathbf{1})^\alpha} \right\rangle \geq \left\langle m, (V_t \mathbf{1})^\alpha \right\rangle \operatorname{ess\,inf}_{(E,m)} \frac{2-2^\alpha}{b+2a}.$$

Here, again by (2.16)

$$(V_t \mathbf{1}(r))^\alpha \geq e^{-tb(r)} \operatorname{ess\,inf}_{(E,m)} \frac{b}{b+a},$$

and therefore

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \Delta(t) \geq \lim_{t \rightarrow \infty} \frac{1}{t} \log \langle m, e^{-tb} \rangle = \operatorname{ess\,sup}_{(E,m)}(-b) = -\operatorname{ess\,inf}_{(E,m)} b.$$

This establishes (4.11) and completes the proof of Theorem 4.3. □

Remark 4.4. (i) The same argument may apply to the case $\alpha = 1$. But the resulting bound exhibits discontinuity at $\alpha = 1$. Indeed, for $\alpha = 1$ (4.12) becomes

$$\Delta(t) = \left\langle m_a, \log \left(1 + \frac{\left(\frac{a}{b}\right)^2 (V_t \mathbf{1})^2}{1 + \frac{a}{b} 2 V_t \mathbf{1}} \right) \right\rangle,$$

where $V_t \mathbf{1}(r)$ is given by the right side of (2.16) with $\alpha = 1$ and $f(r) = 1$. (See the formula for $\psi_t(f)(x)$ on p.1380 in [11].) As a result we can show that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \Delta(t) \geq -2 \operatorname{ess\,inf}_{(E,m)} b$$

and accordingly $\operatorname{gap}(\overline{\mathcal{L}}_1^{(2)}) \leq \operatorname{ess\,inf}_{(E,m)} b$. In fact, the equality is valid because the opposite inequality is implied by (3.7) with $\gamma = 1$ with the help of Theorem 1.2 in [12]. (See also Remark 2.15 in [10].) Thus $\operatorname{gap}(\overline{\mathcal{L}}_\alpha^{(2)})$ is discontinuous at $\alpha = 1$.

(ii) Theorem 4.3 would be regarded as a sort of continuous analogue of Theorem 2.6 in [7], which concerns a vector Markov process whose (countably many) components are independent Markov processes.

5 An application to generalized Fleming-Viot processes

Throughout this section, we assume that E is a compact metric space containing at least two distinct points. As mentioned in the Introduction, the previous results will be applied to study convergence to equilibrium for a class of generalized Fleming-Viot processes, whose state space is $\mathcal{M}_1(E)$. These models have been discussed in [4], where their stationary distributions were identified by exploiting connection with suitable measure-valued α -CIR models. To be more precise, given $0 < \alpha < 1$ and

$m \in \mathcal{M}(E)^\circ$, we consider in this section the process associated with \mathcal{L}_α in (2.1) with $a \equiv 1 \equiv b$ and the generalized Fleming-Viot process associated with

$$\begin{aligned} \mathcal{A}_\alpha \Phi(\mu) &:= \int_0^1 \frac{B_{1-\alpha, 1+\alpha}(du)}{u^2} \int_E \mu(dr) [\Phi((1-u)\mu + u\delta_r) - \Phi(\mu)] \\ &\quad + \int_0^1 \frac{B_{1-\alpha, \alpha}(du)}{(\alpha+1)u} \int_E m(dr) [\Phi((1-u)\mu + u\delta_r) - \Phi(\mu)], \quad \mu \in \mathcal{M}_1(E), \end{aligned} \tag{5.1}$$

where B_{c_1, c_2} denotes the beta distribution with parameter (c_1, c_2) and Φ belongs to the class \mathcal{F}_1 of functions of the form $\Phi_f(\mu) := \langle \mu^{\otimes n}, f \rangle$ for some positive integer n and $f \in C(E^n)$. It has been proved in [4] (Proposition 3.1) that the closure of $(\mathcal{A}_\alpha, \mathcal{F}_1)$ generates a Feller semigroup $(S(t))_{t \geq 0}$ on $C(\mathcal{M}_1(E))$. Also, as observed in [4] (Proposition 3.3), the generators \mathcal{L}_α and \mathcal{A}_α together enjoy the following identity:

$$\mathcal{L}_\alpha \Psi(\eta) = \Gamma(\alpha + 2)\eta(E)^{-\alpha} \mathcal{A}_\alpha \Phi(\eta(E)^{-1}\eta), \quad \eta \in \mathcal{M}(E)^\circ, \tag{5.2}$$

where $\Psi(\eta) = \Phi(\eta(E)^{-1}\eta)$ and Φ is in the linear span \mathcal{F}_0 of functions of the form $\mu \mapsto \langle \mu, f_1 \rangle \cdots \langle \mu, f_n \rangle$ with $f_i \in C_{++}(E)$ and n being a positive integer. Notice that in one dimension (5.2) takes the form (1.3). Under the assumption that $m(E) > 1$, it was proved in Proposition 3.4 of [4] that

$$P_\alpha(\cdot) := \Gamma(\alpha + 1)(m(E) - 1)E^{Q_\alpha} [\eta(E)^{-\alpha}; \eta(E)^{-1}\eta \in \cdot] \tag{5.3}$$

is a unique stationary distribution of the process associated with \mathcal{A}_α , where Q_α is the stationary distribution of processes associated with (2.1) with $a \equiv 1 \equiv b$. The pre-factor $\Gamma(\alpha + 1)(m(E) - 1)$ on the right side arises as the normalizing constant. More generally, the following moment formula holds for the random variable $\eta(E)$ under Q_α .

Lemma 5.1. *Let Q_α be as in (2.17) with $a \equiv 1 \equiv b$. Then it holds that*

$$E^{Q_\alpha} [\eta(E)^\beta] = \begin{cases} \frac{\Gamma(1 - \frac{\beta}{\alpha})\Gamma(m(E) + \frac{\beta}{\alpha})}{\Gamma(1 - \beta)\Gamma(m(E))}, & (-\alpha m(E) < \beta < \alpha) \\ \infty & (\text{otherwise}). \end{cases} \tag{5.4}$$

Proof. Observe that, for each $u > 0$, (2.17) with $a \equiv 1 \equiv b$ and $f \equiv u$ reads

$$E^{Q_\alpha} [e^{-u\eta(E)}] = e^{-m(E) \log(1+u^\alpha)} = (1 + u^\alpha)^{-m(E)}.$$

We only need to consider the case $\beta < 1$ because $E^{Q_\alpha}[\eta(E)^\beta] = \infty$ for each $\beta \geq 1$ is implied by $E^{Q_\alpha}[\eta(E)^\alpha] = \infty$. Since in this case $\Gamma(1 - \beta)t^{-(1-\beta)} = \int_0^\infty duu^{-\beta}e^{-ut}$ for $t > 0$, we get with the help of Fubini's theorem

$$\begin{aligned} E^{Q_\alpha} [\eta(E)^\beta] &= E^{Q_\alpha} [\eta(E)\eta(E)^{-(1-\beta)}] \\ &= \frac{1}{\Gamma(1 - \beta)} \int_0^\infty duu^{-\beta} E^{Q_\alpha} [\eta(E)e^{-u\eta(E)}] \\ &= \frac{1}{\Gamma(1 - \beta)} \int_0^\infty duu^{-\beta} \left(-\frac{d}{du} E^{Q_\alpha} [e^{-u\eta(E)}] \right) \\ &= \frac{\alpha m(E)}{\Gamma(1 - \beta)} \int_0^\infty duu^{-\beta} u^{\alpha-1} (1 + u^\alpha)^{-(m(E)+1)} \\ &= \frac{m(E)}{\Gamma(1 - \beta)} \int_0^\infty dvv^{-\frac{\beta}{\alpha}} (1 + v)^{-(m(E)+1)}, \end{aligned}$$

which is easily seen to diverge for $\beta \geq \alpha$ and $\beta \leq -\alpha m(E)$. For $\beta \in (-\alpha m(E), \alpha)$, the above integral is equal to the right side of (5.4). \square

In the rest of the paper, we assume that $m(E) > 1$. A key observation for the subsequent argument is that (5.2) yields the following connection between Dirichlet forms:

$$E^{Q_\alpha} [(-\mathcal{L}_\alpha)\Psi \cdot \Psi] = \frac{\Gamma(\alpha + 2)}{\Gamma(\alpha + 1)(m(E) - 1)} E^{P_\alpha} [(-\mathcal{A}_\alpha)\Phi \cdot \Phi]. \tag{5.5}$$

Here, the left side calls for some explanation since it can not necessarily be written as $\mathcal{E}(\Psi)$. Instead, one can approximate such special Ψ 's suitably by functions in $D(\overline{\mathcal{L}_\alpha}^{(2)})$. This rather technical point is the contents of the next lemma.

Lemma 5.2. *Suppose that $m(E) > 1$ and let \mathcal{L}_α and Q_α be as above. Let $\Phi \in \mathcal{F}_0$ be arbitrary and define $\Psi(\eta) = \Phi(\eta(E)^{-1}\eta)$ for $\eta \in \mathcal{M}(E)^\circ$. Then there exists a sequence $\{\Psi_N\} \subset D(\overline{\mathcal{L}_\alpha}^{(2)})$ such that $\Psi_N \rightarrow \Psi$ in $L^2(Q_\alpha)$ and $\mathcal{E}(\Psi_N) \rightarrow E^{Q_\alpha} [(-\mathcal{L}_\alpha)\Psi \cdot \Psi]$ as $N \rightarrow \infty$. Moreover, $\text{var}_{Q_\alpha}(\Psi) \leq 2E^{Q_\alpha} [(-\mathcal{L}_\alpha)\Psi \cdot \Psi]$.*

Proof. For simplicity we assume that Φ is of the form $\Phi(\mu) = \langle \mu, f_1 \rangle \cdots \langle \mu, f_n \rangle$ for some $f_i \in C_{++}(E)$ and positive integer n . Accordingly $\Psi(\eta) = \langle \eta, f_1 \rangle \cdots \langle \eta, f_n \rangle \eta(E)^{-n}$. Define $\{\Psi_N\} \subset \mathcal{F}$ by

$$\Psi_N(\eta) = \langle \eta, f_1 \rangle \cdots \langle \eta, f_n \rangle (\eta(E) + 1/N)^{-n}.$$

In fact, $\Psi_N \in D(\overline{\mathcal{L}_\alpha}^{(2)})$ and $\overline{\mathcal{L}_\alpha}^{(2)}\Psi_N = \mathcal{L}_\alpha\Psi_N$ as will be shown in the last half of the proof, and we temporarily suppose the validity of them. Obviously $\Psi_N \rightarrow \Psi$ boundedly and pointwise on $\mathcal{M}(E)^\circ$ and so $\Psi_N \rightarrow \Psi$ in $L^2(Q_\alpha)$. In view of the paragraph preceding to Lemma 2.1 and the calculations in the proof of it, one can verify that $\mathcal{L}_\alpha\Psi_N(\eta) \rightarrow \mathcal{L}_\alpha\Psi(\eta)$ for each $\eta \in \mathcal{M}(E)^\circ$. Moreover, by virtue of Lemma 2.1 (ii)

$$|\mathcal{L}_\alpha\Psi_N(\eta)| \leq C_1(\eta(E) + 1/N)^{-\alpha} + C_2$$

for some constants C_1 and C_2 independent of $\eta \in \mathcal{M}(E)^\circ$ and N . Therefore, by Lebesgue's dominated convergence theorem

$$\mathcal{E}(\Psi_N) = E^{Q_\alpha} [(-\overline{\mathcal{L}_\alpha}^{(2)})\Psi_N \cdot \Psi_N] = E^{Q_\alpha} [(-\mathcal{L}_\alpha)\Psi_N \cdot \Psi_N] \rightarrow E^{Q_\alpha} [(-\mathcal{L}_\alpha)\Psi \cdot \Psi]$$

since $\eta(E)^{-\alpha}$ is integrable with respect to Q_α by Lemma 5.1 together with $m(E) > 1$. In addition, $\text{var}_{Q_\alpha}(\Psi_N) \leq 2\mathcal{E}(\Psi_N) = 2E^{Q_\alpha} [(-\mathcal{L}_\alpha)\Psi_N \cdot \Psi_N]$ by $\text{gap}(\overline{\mathcal{L}_\alpha}^{(2)}) = 1/2$. Letting $N \rightarrow \infty$ gives the required inequality for Ψ .

The rest of the proof is devoted to showing that $\Psi_N \in D(\overline{\mathcal{L}_\alpha}^{(2)})$ and $\overline{\mathcal{L}_\alpha}^{(2)}\Psi_N = \mathcal{L}_\alpha\Psi_N$ for arbitrarily fixed N . For this purpose, it suffices to construct $\{\Psi^{(k)}\}_{k=1}^\infty \subset \mathcal{F}$ such that $\Psi^{(k)} \rightarrow \Psi_N$ and $\mathcal{L}_\alpha\Psi^{(k)} \rightarrow \mathcal{L}_\alpha\Psi_N$ boundedly and pointwise on $\mathcal{M}(E)$ as $k \rightarrow \infty$. Clearly this reduces to finding a sequence $\{\varphi^{(k)}\}_{k=1}^\infty \subset C_0^2(\mathbf{R}^{n+1})$ which approximates to $\varphi(x_1, \dots, x_n) := x_1 \cdots x_n (x_{n+1} + 1/N)^{-n}$ in some appropriate manner. To this end, take a sequence $\{\chi_k\}$ in $C_0^2(\mathbf{R})$ with the following properties; $\{\chi'_k\}$ and $\{\chi''_k\}$ are uniformly bounded,

$$0 \leq \chi_k(x) \leq x \text{ for all } x \in \mathbf{R}_+ \quad \text{and} \quad \chi_k(x) = \begin{cases} x, & (0 \leq x \leq k) \\ 0, & (x \geq 2k). \end{cases}$$

Define $\tilde{\varphi}^{(k)} \in C^2(\mathbf{R}_+^{n+1})$ by $\tilde{\varphi}^{(k)}(x_1, \dots, x_{n+1}) = \chi_k(x_1) \cdots \chi_k(x_n) (x_{n+1} + 1/N)^{-n}$ and accordingly set

$$\Psi^{(k)}(\eta) = \tilde{\varphi}^{(k)}(\langle \eta, f_1 \rangle, \dots, \langle \eta, f_n \rangle, \eta(E)). \tag{5.6}$$

It follows that $\|\Psi^{(k)}\|_\infty \leq \|\Psi_N\|_\infty$. While the support of $\tilde{\varphi}^{(k)}$ is not compact, $\Psi^{(k)}(\eta) = 0$ whenever $\eta(E) \geq 2k(\min_{1 \leq i \leq n} \inf_{x \in E} f_i(x))^{-1} =: 2kR$. Therefore, letting $\tau_k \in C_0^2(\mathbf{R}_+)$

be such that $\tau_k(x) = (x + 1/N)^{-n}$ for any $x \in [0, 2kR]$ and defining $\varphi^{(k)} \in C_0^2(\mathbf{R}_+^{n+1})$ by $\varphi^{(k)}(x_1, \dots, x_{n+1}) = \chi_k(x_1) \cdots \chi_k(x_n) \tau_k(x_{n+1})$, we get

$$\Psi^{(k)}(\eta) = \varphi^{(k)}(\langle \eta, f_1 \rangle, \dots, \langle \eta, f_n \rangle, \eta(E)) \quad \text{for all } \eta \in \mathcal{M}(E).$$

Thus $\Psi^{(k)} \in \mathcal{F}$ and $\Psi^{(k)}(\eta) = \Psi_N(\eta)$ whenever $\eta(E) < k(\max_{1 \leq i \leq n} \sup_{x \in E} f_i(x))^{-1} (\leq kR)$. Hence $\Psi^{(k)} \rightarrow \Psi_N$ boundedly and pointwise on $\mathcal{M}(E)$ and $\mathcal{L}_\alpha \Psi^{(k)} \rightarrow \mathcal{L}_\alpha \Psi_N$ pointwise on $\mathcal{M}(E)$. In addition, by Lemma 2.1 (ii) and analogous calculations for (5.6) to the paragraph preceding to Lemma 2.1 one can show that $\{\mathcal{L}_\alpha \Psi^{(k)}\}_{k=1}^\infty$ is uniformly bounded. This completes the proof of Lemma 5.2. \square

It follows from (5.5) and Lemma 5.2 that for any $\Phi \in \mathcal{F}_0$

$$\text{var}_{Q_\alpha}(\Psi) \leq \frac{2\Gamma(\alpha + 2)}{\Gamma(\alpha + 1)(m(E) - 1)} E^{P_\alpha} [(-\mathcal{A}_\alpha)\Phi \cdot \Phi], \tag{5.7}$$

where $\Psi(\eta) = \Phi(\eta(E)^{-1}\eta)$. Moreover, noting that by the Stone-Weierstrass theorem the linear span of functions f on E^n of the form $f(r_1, \dots, r_n) = f_1(r_1) \cdots f_n(r_n)$ with $f_i \in C_+(E)$ is dense in $C(E^n)$, one can show, with the help of the expression (3.2) in [4] for $\mathcal{A}_\alpha \Phi_f$ with $f \in C(E^n)$, that (5.7) extends to any $\Phi \in \mathcal{F}_1$. Indeed, that expression takes the form

$$\mathcal{A}_\alpha \Phi_f(\mu) = \langle \mu^{\otimes n}, \Theta^{(n)} f \rangle + \langle \mu^{\otimes n}, \Xi^{(n)} f \rangle - c_n \Phi_f(\mu)$$

for some nonnegative bounded operators $\Theta^{(n)}, \Xi^{(n)} : C(E^n) \rightarrow C(E^n)$ and some positive constant c_n independent of f and μ , and so if $\{g_k\} \subset C(E^n)$, $g_k(r_1, \dots, r_n) \rightarrow f(r_1, \dots, r_n)$ uniformly on E^n as $k \rightarrow \infty$, then $\mathcal{A}_\alpha \Phi_{g_k} \rightarrow \mathcal{A}_\alpha \Phi_f$ and $\Phi_{g_k} \rightarrow \Phi_f$ uniformly on $\mathcal{M}_1(E)$ as $k \rightarrow \infty$.

In contrast, the variance functionals of P_α and of Q_α , denoted by var_{P_α} and var_{Q_α} respectively, do not seem to enjoy any nice relation with each other. Although it is not clear if exponential convergence to equilibrium occurs for the process associated with \mathcal{A}_α , we are going to discuss a weaker ergodic property by introducing another functional $\text{osc}^2(\Phi)$ for $\Phi \in L^2(P_\alpha)$, which is defined to be the essential supremum of the function

$$\mathcal{M}_1(E) \times \mathcal{M}_1(E) \ni (\mu, \mu') \mapsto |\Phi(\mu) - \Phi(\mu')|^2$$

with respect to $P_\alpha \otimes P_\alpha$. Let $Z : \mathcal{M}(E)^\circ \rightarrow \mathcal{M}_1(E)$ be given by $Z(\eta) = \eta(E)^{-1}\eta$. Since by (5.3) P_α and $Q_\alpha \circ Z^{-1}$ are mutually absolutely continuous, $\text{osc}^2(\Phi)$ coincides with the essential supremum of the function

$$\mathcal{M}(E)^\circ \times \mathcal{M}(E)^\circ \ni (\eta, \eta') \mapsto |(\Phi \circ Z)(\eta) - (\Phi \circ Z)(\eta')|^2$$

with respect to $Q_\alpha \otimes Q_\alpha$.

Lemma 5.3. *Suppose that $m(E) > 1$ and let Q_α and P_α be as in (5.3). Let $\Phi \in L^2(P_\alpha)$ be arbitrary and define $\Psi(\eta) = \Phi(\eta(E)^{-1}\eta)$ for $\eta \in \mathcal{M}(E)^\circ$. Then for any $q > 1$*

$$\text{var}_{P_\alpha}(\Phi) \leq \Gamma(\alpha + 1)(m(E) - 1) (E^{Q_\alpha} [\eta(E)^{-\alpha q}] \text{osc}^2(\Phi))^{1/q} (\text{var}_{Q_\alpha}(\Psi))^{1/p}, \tag{5.8}$$

where $p > 1$ is such that $1/p + 1/q = 1$.

Proof. It follows from (5.3) that

$$\begin{aligned} \text{var}_{P_\alpha}(\Phi) &= \inf_{c \in \mathbf{R}} E^{P_\alpha} [(\Phi - c)^2] \leq E^{P_\alpha} [(\Phi - E^{Q_\alpha}[\Psi])^2] \\ &= \Gamma(\alpha + 1)(m(E) - 1) E^{Q_\alpha} [\eta(E)^{-\alpha} (\Psi(\eta) - E^{Q_\alpha}[\Psi])^2]. \end{aligned}$$

By Hölder’s inequality

$$\begin{aligned} & E^{Q_\alpha} \left[\eta(E)^{-\alpha} (\Psi(\eta) - E^{Q_\alpha}[\Psi])^2 \right] \\ & \leq \left(E^{Q_\alpha} \left[\eta(E)^{-\alpha q} |\Psi(\eta) - E^{Q_\alpha}[\Psi]|^2 \right] \right)^{1/q} \left(E^{Q_\alpha} \left[|\Psi(\eta) - E^{Q_\alpha}[\Psi]|^2 \right] \right)^{1/p} \\ & \leq \left(E^{Q_\alpha} \left[\eta(E)^{-\alpha q} \text{osc}^2(\Phi) \right] \right)^{1/q} (\text{var}_{Q_\alpha}(\Psi))^{1/p}, \end{aligned}$$

where the last inequality is seen from

$$\begin{aligned} & E^{Q_\alpha} \left[\eta(E)^{-\alpha q} |\Psi(\eta) - E^{Q_\alpha}[\Psi]|^2 \right] \\ & \leq E^{Q_\alpha \otimes Q_\alpha} \left[\eta(E)^{-\alpha q} |\Psi(\eta) - \Psi(\eta')|^2 \right] \\ & = E^{Q_\alpha \otimes Q_\alpha} \left[\eta(E)^{-\alpha q} |(\Phi \circ Z)(\eta) - (\Phi \circ Z)(\eta')|^2 \right]. \end{aligned}$$

These inequalities together prove (5.8). □

Our strategy is to carry out the well-known procedure to show algebraic convergence to equilibrium. More specifically, our ingredient for this is Theorem 2.2 in [8]. (See also [9].) Let $\{S^2(t)\}_{t \geq 0}$ be the strongly continuous semigroup on $L^2(P_\alpha)$ of the process associated with \mathcal{A}_α . To be more precise, $\{S^2(t)\}_{t \geq 0}$ is defined to be the C_0 -semigroup on $L^2(P_\alpha)$ generated by the closure $(\overline{\mathcal{A}_\alpha}^{(2)}, D(\overline{\mathcal{A}_\alpha}^{(2)}))$ of $(\mathcal{A}_\alpha, \mathcal{F}_1)$ as an operator on $L^2(P_\alpha)$. As in the case of $(T^2(t))_{t \geq 0}$, $(S^2(t))_{t \geq 0}$ on $L^2(P_\alpha)$ coincides with $(S(t))_{t \geq 0}$ when restricted to $C(\mathcal{M}_1(E))$. The following property of $(S^2(t))_{t \geq 0}$ is needed for the abovementioned strategy.

Lemma 5.4. *Let $(S^2(t))_{t \geq 0}$ be as above. Then it holds that $\text{osc}^2(S^2(t)\Phi) \leq \text{osc}^2(\Phi)$ for all $t > 0$ and $\Phi \in L^2(P_\alpha)$.*

Proof. We may and do assume additionally that $\text{osc}^2(\Phi) < \infty$. Let \mathcal{S} be an arbitrary Borel set of $\mathcal{M}_1(E) \times \mathcal{M}_1(E)$ and $P_t(\mu, \cdot)$ ($t > 0, \mu \in \mathcal{M}_1(E)$) denote the transition function of the process associated with \mathcal{A}_α . It then follows that for any $q > 1$

$$\begin{aligned} & E^{P_\alpha \otimes P_\alpha} \left[|S^2(t)\Phi(\mu) - S^2(t)\Phi(\mu')|^2 \mathbf{1}_\mathcal{S}(\mu, \mu') \right] \\ & \leq E^{P_\alpha \otimes P_\alpha} \left[\int_{\mathcal{M}_1(E)} P_t(\mu, d\nu) \int_{\mathcal{M}_1(E)} P_t(\mu', d\nu') |\Phi(\nu) - \Phi(\nu')|^2 \mathbf{1}_\mathcal{S}(\mu, \mu') \right] \\ & \leq \left(E^{P_\alpha \otimes P_\alpha} \left[\int_{\mathcal{M}_1(E)} P_t(\mu, d\nu) \int_{\mathcal{M}_1(E)} P_t(\mu', d\nu') |\Phi(\nu) - \Phi(\nu')|^{2q} \right] \right)^{1/q} \\ & \quad \times ((P_\alpha \otimes P_\alpha)(\mathcal{S}))^{1/p} \\ & = \left(E^{P_\alpha \otimes P_\alpha} \left[|\Phi(\mu) - \Phi(\mu')|^{2q} \right] \right)^{1/q} ((P_\alpha \otimes P_\alpha)(\mathcal{S}))^{1/p} \\ & \leq \text{osc}^2(\Phi) ((P_\alpha \otimes P_\alpha)(\mathcal{S}))^{1/p}, \end{aligned}$$

where $p > 1$ is such that $1/p + 1/q = 1$ and the equality is implied by the stationarity of P_α . By letting $q \rightarrow \infty$ or $p \downarrow 1$

$$E^{P_\alpha \otimes P_\alpha} \left[|S^2(t)\Phi(\mu) - S^2(t)\Phi(\mu')|^2 \mathbf{1}_\mathcal{S}(\mu, \mu') \right] \leq \text{osc}^2(\Phi) (P_\alpha \otimes P_\alpha)(\mathcal{S}).$$

Since \mathcal{S} is arbitrary, we conclude that $\text{osc}^2(S^2(t)\Phi) \leq \text{osc}^2(\Phi)$. □

We can at last state the main result of this section.

Theorem 5.5. *Suppose that $m(E) > 1$. Let P_α be as in (5.3) and $(S^2(t))_{t \geq 0}$ be as above. Then for any $\Phi \in L^2(P_\alpha)$*

$$\begin{aligned} \limsup_{t \rightarrow \infty} \text{var}_{P_\alpha} (S^2(t)\Phi) \frac{t^{m(E)-1}}{\log t} \\ \leq \frac{e\Gamma(\alpha)(m(E)-1)}{\Gamma(\alpha m(E))} (\Gamma(\alpha+2)(m(E)-1))^{m(E)-1} \text{osc}^2(\Phi). \end{aligned} \tag{5.9}$$

Proof. Given $\Phi \in L^2(Q_\alpha)$, let $\Psi(\eta) = \Phi(\eta(E)^{-1}\eta)$ for $\eta \in \mathcal{M}(E)^\circ$. We claim that (5.7) is extended as

$$\text{var}_{Q_\alpha}(\Psi) \leq \frac{2\Gamma(\alpha+2)}{\Gamma(\alpha+1)(m(E)-1)} E^{P_\alpha} \left[(-\overline{\mathcal{A}_\alpha}^{(2)})\Phi \cdot \Phi \right] \tag{5.10}$$

for any $\Phi \in D(\overline{\mathcal{A}_\alpha}^{(2)})$. Indeed, there exists a sequence $\{\Phi_N\} \subset \mathcal{F}_1$ such that $\Phi_N \rightarrow \Phi$ and $\mathcal{A}_\alpha \Phi_N \rightarrow \overline{\mathcal{A}_\alpha}^{(2)} \Phi$ in $L^2(P_\alpha)$ as $N \rightarrow \infty$, and according to (5.7) for elements of \mathcal{F}_1 we have for each $N = 1, 2, \dots$

$$E^{Q_\alpha}[(\Psi_N)^2] - (E^{Q_\alpha}[\Psi_N])^2 \leq \frac{2\Gamma(\alpha+2)}{\Gamma(\alpha+1)(m(E)-1)} E^{P_\alpha} [(-\mathcal{A}_\alpha)\Phi_N \cdot \Phi_N], \tag{5.11}$$

where $\Psi_N(\eta) = \Phi_N(\eta(E)^{-1}\eta)$. Taking a subsequence if necessary, we can assume that $\Phi_N \rightarrow \Phi$ P_α -a.s. and so $\Psi_N \rightarrow \Psi$ Q_α -a.s. Since (5.11) implies that $\{\Psi_N\}$ is bounded in $L^2(Q_\alpha)$, $\Psi \in L^2(Q_\alpha)$ and $\Psi_N \rightarrow \Psi$ in $L^1(Q_\alpha)$. Letting $N \rightarrow \infty$ in (5.11) yields

$$E^{Q_\alpha}[\Psi^2] - (E^{Q_\alpha}[\Psi])^2 \leq \frac{2\Gamma(\alpha+2)}{\Gamma(\alpha+1)(m(E)-1)} E^{P_\alpha} [(-\overline{\mathcal{A}_\alpha}^{(2)})\Phi \cdot \Phi]$$

with the help of Fatou's lemma. Thus (5.10) holds for $\Phi \in D(\overline{\mathcal{A}_\alpha}^{(2)})$.

Let $q \in (1, m(E))$ be arbitrary. Combining (5.10) with (5.8) leads to

$$\text{var}_{P_\alpha}(\Phi) \leq (\mathcal{V}(\Phi))^{1/q} \left(2\Gamma(\alpha+2) E^{P_\alpha} \left[(-\overline{\mathcal{A}_\alpha}^{(2)})\Phi \cdot \Phi \right] \right)^{1/p}, \quad \Phi \in D(\overline{\mathcal{A}_\alpha}^{(2)}),$$

where $\mathcal{V}(\Phi) = \Gamma(\alpha+1)(m(E)-1) E^{Q_\alpha} [\eta(E)^{-\alpha q}] \text{osc}^2(\Phi)$. Thus the condition (2.3) of Theorem 2.2 in [8] is fulfilled with a quadratic functional $\Phi \mapsto \mathcal{V}(\Phi)$ and $C := (2\Gamma(\alpha+2))^{1/p}$. In addition, by Lemma 5.4 $\mathcal{V}(S^2(t)\Phi) \leq \mathcal{V}(\Phi)$ for all $t \geq 0$ and $\Phi \in L^2(P_\alpha)$. Therefore, by the assertion (i) of Theorem 2.2 in [8] together with the calculation in the final part of its proof, we obtain

$$\text{var}_{P_\alpha}(S^2(t)\Phi) \leq \mathcal{V}(\Phi) C^q \left(\frac{q/p}{2t} \right)^{q/p}$$

for all $\Phi \in L^2(P_\alpha)$ and $t > 0$. Because of $q/p = q - 1$, this is rewritten as

$$\begin{aligned} \text{var}_{P_\alpha}(S^2(t)\Phi) \\ \leq \Gamma(\alpha+1)(m(E)-1) E^{Q_\alpha} [\eta(E)^{-\alpha q}] \text{osc}^2(\Phi) \left(\frac{\Gamma(\alpha+2)(q-1)}{t} \right)^{q-1}. \end{aligned} \tag{5.12}$$

As the final step, we will optimize the value of q in (5.12) for each $t > 0$ large enough. Observe from (5.4) that the right side of (5.12) becomes

$$\frac{\Gamma(\alpha)\Gamma(q)\Gamma(m(E)-q)}{\Gamma(\alpha q)\Gamma(m(E)-1)} \left(\frac{\Gamma(\alpha+2)(q-1)}{t} \right)^{q-1} \text{osc}^2(\Phi). \tag{5.13}$$

For t sufficiently large, take $q = q(t) := m(E) - \delta(t) \in (1, m(E))$, where $\delta(t) := 1/(\log t)$ is verified to minimize the function $0 < \delta \mapsto t^\delta/\delta$. Then, noting that

$$\Gamma(m(E) - q(t)) = \frac{\Gamma(\delta(t) + 1)}{\delta(t)} = \Gamma(\delta(t) + 1) \cdot \log t$$

and $t^{\delta(t)} = e$, we see that (5.13) with this choice of q equals

$$\frac{\Gamma(\alpha)\Gamma(q(t))\Gamma(\delta(t) + 1)}{\Gamma(\alpha q(t))\Gamma(m(E) - 1)} \cdot \frac{\log t}{t^{m(E)-1}} \cdot e (\Gamma(\alpha + 2)(q(t) - 1))^{q(t)-1} \text{osc}^2(\Phi).$$

This upper bound for $\text{var}_{P_\alpha}(S^2(t)\Phi)$ immediately gives (5.9). The proof of Theorem 5.5 is complete. \square

What is unpleasant to us is that we do not know whether $\text{gap}(\overline{\mathcal{A}_\alpha}^{(2)}) = 0$ or not. One difficulty is that any useful expression for the variance functional with respect to P_α nor Dirichlet form associated with \mathcal{A}_α does not seem available at least for conventional choice of test functions. Besides, our argument in this section does not work in the case where $0 < m(E) \leq 1$ although the process associated with \mathcal{A}_α still has a unique stationary distribution. (See Theorem 3.2 in [4].)

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