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Fixation for coarsening dynamics in 2D slabs

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Abstract

We study zero-temperature Ising Glauber Dynamics, on $2D$ slabs of thickness $k \geq 2$. In this model, ± 1-valued spins at integer sites update according to majority vote dynamics with two opinions. We show that all spins reaches a final state (that is, the system fixates) for $k = 2$ under free boundary conditions and for $k = 2$ or 3 under periodic boundary conditions. For thicker slabs there are sites that fixate and sites that do not.

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1 Introduction

In this paper, we study some natural questions concerning coarsening on two-dimensional slabs and how the answers to those questions depend on the width k of the slab. Coarsening is a particular continuous time Markov process (which is the zero-temperature limit of processes for which the Ising model Gibbs distribution is stationary). The coarsening process, which will be defined precisely below, is a particular type of majority vote model, in which the state space is assignments of ± 1 to the vertices of a (generally infinite) graph. For the nearest neighbor graph $\mathbb{Z}^1$, it is exactly the standard voter model. We will be interested in the case when the initial distribution on the state space is i.i.d. product measure with probability p for a site to be $+1$.

Focusing first on the symmetric case, $p = 1/2$, we note that it is known that on $\mathbb{Z}^d$ with $d = 1, 2$, no sites fixate (almost surely). For $d = 1$ this is a result about the standard one-dimensional voter model and holds for all $p \in (0, 1)$ [1], while for $d = 2$, the

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result is somewhat more recent [6]. For $d \geq 3$, it is a wide open problem to determine whether or not (and for which values of d) some sites fixate; there are some hints from the computational physics literature that fixation may indeed occur for large enough d [9], [8]. See also [7] for interesting numerical results about non-fixation for the $d = 3$ periodic cube.

Motivated by non-fixation for $d = 2$ and the open $d = 3$ problem, in this paper we study graphs that interpolate between $\mathbb{Z}^2$ and $\mathbb{Z}^3$ by considering width-k slabs, $S_k = \mathbb{Z}^2 \times \{0, 1, \dots, k-1\}$, with free or periodic boundary conditions in the third coordinate. As we shall see, there is an interesting, and somewhat unexpected, dependence on the value of k. The main results of this paper are that all sites fixate if and only if $2 \leq k \leq k_c^*$ (where the superscript $*$ is either f or p denoting free or periodic boundary conditions); for $k_c^* < k < \infty$, some sites fixate and some do not. The critical widths are $k_c^f = 2$ and $k_c^p = 3$.

An announcement of these results and proofs for the simpler regions of k-values appear in [3]. The most difficult cases of $k = 3$ (periodic), where all sites fixate, and $k = 4$ (free), where some sites do not fixate, are treated in this paper. The proof for $k = 3$ (periodic) is of particular interest, because the percolation theoretic arguments may be of use in other settings. Two other contributions of this paper are (a) to show that the same results are valid for all k for any $p \in (0, 1)$, and (b) to provide, in an appendix, a simpler analysis of the $k = 2$ (periodic) case than that given in [3].

The fact that the results for slabs do not depend on $p \in (0, 1)$ is also true on Z^1 where there is never fixation. But this is not so in general, as it has been proved in [4] that on $\mathbb{Z}^d$ with $d \geq 2$, all sites fixate at $+1$ (resp., at -1) when p is close enough to 1 (resp., to 0). It is conjectured, and an important open problem to prove, that this is in fact the case as long as $p \neq 1/2$.

We conclude our introductory remarks by mentioning two open problems about $2D$ slabs. One is whether for those slabs with non-fixated sites, these sites percolate rather than forming only finite connected components surrounded by fixated sites? For small values of k that seems unlikely, but perhaps percolation can occur for large k. A second interesting question is how the density of fixated sites behaves as $k \to \infty$. In the free boundary condition setting, one could consider the probability that the site at $(0, 0, [(k-1)/2])$ fixates. If it vanishes in this limit, that might supply a mechanism for proving that no fixation occurs in $\mathbb{Z}^3$.

1.1 Definitions

The slab $S_k,\ k \geq 2$, is the graph with vertex set $\mathbb{Z}^2 \times \{0, 1, \dots, k-1\}$ and edge set $\mathcal{E}_k = \{\{x, y\} : \|x - y\|_1 = 1\}$. As is usual, we take an initial spin configuration $\sigma(0) = (\sigma_x(0))_{x \in S_k}$ on $\Omega_k = \{-1, 1\}^{S_k}$ distributed using the product measure of $\mu_p,\ p \in (0, 1)$, where

$$\mu_p(\sigma_x(0) = +1) = p = 1 - \mu_p(\sigma_x(0) = -1) .$$

The configuration $\sigma(t)$ evolves as t increases according to the zero-temperature limit of Glauber dynamics (the majority rule). To describe this, define the energy (or local cost function) of a site x at time t as

$$e_x(t) = - \sum_{y : \{x, y\} \in \mathcal{E}_k} \sigma_x(t) \sigma_y(t) . \tag{1.1}$$

Note that $e_x(t)$ is just the number of neighbors of x that disagree with x minus the number of neighbors that agree with x.

Each site has an exponential clock with rate 1 and clocks at different sites are independent of each other. When a site's clock rings, it makes an update according to the rules

$$\sigma_x(t) = \begin{cases} -\sigma_x(t^-) & \text{if } e_x(t^-) > 0 \\ \pm 1 & \text{with probability } 1/2 \text{ if } e_x(t^-) = 0 \\ \sigma_x(t^-) & \text{if } e_x(t^-) < 0 \end{cases} .$$

Hence each spin flips with probability 1, (or 0, or $1/2$) when it disagrees, (or agrees, or there is a tie) with most of its neighbors. This coincides with the majority vote dynamics.

Write $\mathbb{P}_p$ for the joint distribution of $(\sigma(0), \omega)$, the initial spins and the dynamics realizations.

The main questions we will address involve fixation. We say that the slab S_k fixates for some value of p if for all $x \in S_k$

$$\mathbb{P}_p(\text{there exists } T = T(x, \sigma(0), \omega) < \infty \text{ such that } \sigma_x(t) = \sigma_x(T) \text{ for all } t \geq T) = 1 .$$

By translation invariance, to show S_k fixates, it suffices to show that the origin fixates; that is, that the above equation holds for $x = 0$. Furthermore, the probability that site 0 fixates is strictly between zero and one if and only if a proper subset of the set of vertices fixates. All of our results will hold for all $p \in (0,1)$, so we will write $\mathbb{P}$ for the measure $\mathbb{P}_p$. The setup thus far corresponds to the model with free boundary conditions; in the case of periodic boundary conditions, we consider sites of the form $(x, y, k-1)$ and $(x, y, 0)$ to be neighbors in S_k. If $k = 2$ then this enforces two edges between $(x, y, 1)$ and $(x, y, 0)$, so that in the computation of energy of a site, that neighbor counts twice.

1.2 Main results

Let $p \in (0,1)$ be arbitrary.

Theorem 1.1. *With periodic boundary conditions, all sites in S_3 fixate.*

The proof of Theorem 1.1 is given in Section 5, using the results of Sections 2 and 4. In the appendix we give a simplified proof for fixation of all sites in S_2 with periodic boundary conditions. It does not use a comparison to bootstrap percolation (as in [3]) and therefore should allow for more general initial measures for $\sigma(0)$.

Theorem 1.2. *With periodic boundary conditions some sites in S_4 do not fixate.*

The proof of Theorem 1.2 is given in Section 6. Results for free boundary conditions (for all k) or periodic conditions for $k = 2$ and $k \geq 5$ were proved in [3] for $p = 1/2$. It is straightforward to see that the proofs in [3] extend to all $p \in (0,1)$. Combined with the preceding two theorems, we have the following complete characterization of slabs, where we say that S_k fixates if all sites in S_k fixate.

Theorem 1.3. *With free boundary conditions, S_k fixates if and only if $k = 2$. With periodic boundary conditions and $k \geq 2$, S_k fixates if and only if $k \in \{2, 3\}$.*

Remark. It is an elementary fact that for $k \geq 2$ and either free or periodic boundary conditions, some sites fixate. Thus in all cases where S_k does not fixate, there are sites that fixate and sites that do not.

2 Preliminary results

We will need to develop some terminology and recall some results before proceeding. All results hold for slabs with periodic boundary conditions unless stated otherwise. We only consider $k \geq 2$. We say a vertex v *flips* at time t if $\sigma_v(t^-) \neq \sigma_v(t)$.

Definition 2.1. *A vertex is called variable in the realization $(\sigma(0),\omega)$ if it flips infinitely many times. We call a flip at time t of a vertex v energy lowering if $e_v(t^-) > 0$.*

Note that if a vertex flips infinitely many times then the set of times at which it flips is almost surely unbounded. This follows from the fact that the waiting time for clocks has a non-degenerate distribution. The following lemma is proved in [6] in some generality and applies to the slab S_k for any k.

Lemma 2.2. *For $k \geq 2$, any vertex in S_k has almost surely only finitely many energy lowering flips.*

For the proofs of the main results, we need the notion of stability.

Definition 2.3. *The vertex v is called unstable in $\sigma \in \Omega_k$ if $e_v(t) \geq 0$. Otherwise v is stable. A vertex is stable (unstable) at time t if it is stable (unstable) in $\sigma(t)$.*

We make the following observation regarding stability. For the statement, we say that an event $A \subset \Omega_k$ *occurs infinitely often* if the set of times $\{t : \sigma(t) \in A\}$ is unbounded.

Lemma 2.4. *Let $v \in S_k$ for $k \geq 2$. With probability one the following statements hold.*

1. *There exists $T = T(v,\sigma(0),\omega) < \infty$ such that $e_v(t) \leq 0$ for all $t \geq T$.*
2. *v is variable if and only if it is unstable infinitely often if and only if it has exactly three same sign neighbors infinitely often.*

Proof. For the proof, we use the following simple lemma from [3]:

Lemma 2.5. *Let A, B be cylinder events in Ω_k for $k \geq 2$. If*

$$\inf_{\sigma \in A} \mathbb{P}\left(\sigma(t) \in B \text{ for some } t \in [0,1) \;\middle|\; \sigma(0) = \sigma\right) > 0\,,$$

then

$$\mathbb{P}\left(B \text{ occurs infinitely often} \;\middle|\; A \text{ occurs infinitely often}\right) = 1\,.$$

Note that the events $\{e_v(t) > 0\}$ and $\{e_v(t) < 0\}$ are cylinder events. So, if with positive probability $e_v(t) > 0$ infinitely often, then an application of Lemma 2.5 shows that v has infinitely many energy-lowering flips with positive probability, contradicting Lemma 2.2. This proves the first statement.

Next, if v is variable, it must flip infinitely often and, letting t be one time at which v flips, $\sigma_v(s) \geq 0$ for all $s < t$ sufficiently close to t. This means v is unstable infinitely often. Conversely, if v is unstable infinitely often, $e_v(t) \geq 0$ infinitely often, and using Lemma 2.5 with $A = \{v \text{ is unstable}\}$ and $B = \{\sigma_v = +1\}$ or $B = \{\sigma_v = -1\}$ we see that v flips infinitely often.

If v is unstable infinitely often then $e_v(t) \geq 0$ infinitely often. By the first statement of this lemma, $e_v(t) = 0$ infinitely often, and v has exactly three same sign neighbors infinitely often. Conversely, if v has exactly three same sign neighbors infinitely often then it is clearly unstable infinitely often. □

3 Fixed columns are monochromatic

For $(x,y) \in \mathbb{Z}^2$ we write $C_{x,y}$ for the column of vertices at coordinate (x,y):

$$C_{x,y} = \{(x,y,i) : i = 0,\ldots,k-1\}\,.$$

A column C is *monochromatic* in $\sigma \in \Omega_k$ if $\sigma_v = \sigma_w$ for all $v,w \in C$. A realization $(\sigma(0),\omega)$ of initial configuration and dynamics is said to be *eventually in* $A \subset \Omega_k$ if there is some $T = T(\sigma(0),\omega,A) < \infty$ such that if $t \geq T$ then $\sigma(t) \in A$. We say a column *flips finitely often* if each of its vertices flips finitely often.

In this section we prove the following.

Proposition 3.1. *With probability one, if a column in S_3 with periodic boundary conditions flips finitely often then it is eventually monochromatic.*

We note that this result is parallel to the one for S_2 shown in [3]: all columns in a slab are eventually monochromatic if and only if all sites in the slab flip finitely often. However we will see in Section 6 that it fails for S_4.

The idea of the proof: It is easy to see that four positive columns arranged in a square (seen from above) form a 'satisfied cluster' as each of their sites have at least four positive neighbors. Taking two such clusters one can construct a connecting beam, at least 2×2 in cross-section, running perpendicular to the short dimension of the slab while the surrounding sites can remain negative. An attempt to construct a fixed positive cluster containing a beam of smaller cross-section will require connecting this beam to fixed column clusters on multiple sides. This will create 'corners' where negative spins can flip and thicken the beam. The proof proceeds by analyzing the percolation structure in $\mathbb{Z}^2$ of non-monochromatic columns in order to identify such corners.

Proof. The proof will proceed by contradiction, so assume that with positive probability there is a column that flips finitely often but is not eventually monochromatic. It must then have a terminal state; that is, the spins at vertices in this column have a limit as $t \to \infty$. This limit is assumed to be non-monochromatic, so we begin the analysis by defining a site percolation process on $\mathbb{Z}^2$ corresponding to certain non-monochromatic columns. Given a configuration $\sigma \in \Omega_k$ and $(x, y) \in \mathbb{Z}^2$, we say that the column $C_{x,y}$ is type-1 in σ if

$$\sigma_{(x,y,0)} = \sigma_{(x,y,1)} = +1 \text{ but } \sigma_{(x,y,2)} = -1 \ .$$

and type-2 if $\sigma_{(x,y,0)} = \sigma_{(x,y,2)} = +1$ but $\sigma_{(x,y,1)} = -1$. We define $\eta = \eta(\sigma) \in \{0, 1, 2\}^{\mathbb{Z}^2}$ by

$$\eta_{(x,y)} = \begin{cases} 1 & \text{if } C_{x,y} \text{ is type-1 in } \sigma \\ 2 & \text{if } C_{x,y} \text{ is type-2 in } \sigma \\ 0 & \text{otherwise} \end{cases} .$$

If $\eta_{(x,y)} = r$ then we say (x, y) is type-r in η. The pair $(\sigma(0), \omega)$ induces a configuration $\eta(t) = \eta(\sigma(t))$. Let

$$A_r(x, y) = \{C_{x,y} \text{ is eventually type-}r\} \text{ for } r = 1, 2 \ .$$

By the assumption that there exist columns that flip finitely often but are not eventually monochromatic, we must have either $\mathbb{P}(A_r(x, y)) > 0$ for some r and all (x, y) or the corresponding statement with a global flip; that is $\mathbb{P}(B_r(x, y)) > 0$ for some r and all (x, y), where $B_r(x, y)$ is the event that $C_{x,y}$ is eventually type-r in the configuration $\eta(-\sigma(t))$, induced by the global flip $-\sigma(t)$. Both cases are handled identically, so we will assume here that $\mathbb{P}(A_r(x, y)) > 0$ for some r.

By spatial symmetry of the slab layers (due to periodic boundary conditions), we must then have $\mathbb{P}(A_r(x, y)) > 0$ for all r and all (x, y). By translation-ergodicity of the model there are almost surely infinitely many values of $n \in \mathbb{N}$ such that each $A_r(n, 0)$ occurs. It follows that there exist $M_0, N_0 \in \mathbb{N}$ such that

$$\mathbb{P}(A_1(0, 0) \cap A_2(M_0, 0) \cap A_2(0, N_0)) > 0 \ . \tag{3.1}$$

Next we recall the notion of $*$-connectedness: two vertices $w, z \in \mathbb{Z}^2$ are *neighbors* if $\|w - z\|_1 = 1$ and are $*$-*neighbors* if $\|w - z\|_\infty = 1$. A *path* is a sequence of vertices $(w_1, \ldots, w_k)$ such that w_i and w_{i+1} are neighbors for $i = 1, \ldots, k - 1$ and a $*$-*path* is a sequence such that w_i and w_{i+1} are $*$-neighbors for $i = 1, \ldots, k - 1$. Given a realization

$\eta \in \{0,1,2\}^{\mathbb{Z}^2}$ and $r \in \{0,1,2\}$, the r-*cluster* ($r*$-*cluster*) of a vertex z is the set of vertices of type-r which are connected to z by a path ($*$-path) all of whose vertices are type-r. Note that if z is not of type-r then both its r-cluster and $r*$-cluster are empty. Two sets $V, U \subset \mathbb{Z}^2$ are r-connected ($r*$-connected), written $V \to_r U$ ($V \to_{r*} U$) if there are vertices $v \in V$ and $u \in U$ such that the r-cluster ($r*$-cluster) of v contains u. If this connection can be made using only vertices in some set D, we say U and V are r-connected ($r*$-connected) in D and write $U \xrightarrow{D}_r V$ ($U \xrightarrow{D}_{r*} V$).

In the box $B = \{0,\dots,M_0\} \times \{0,\dots,N_0\}$, write $L = \{0\} \times \{0,\dots,N_0\}$, $R = \{M_0\} \times \{0,\dots,N_0\}$, $D = \{0,\dots,M_0\} \times \{0\}$ and $U = \{0,\dots,M_0\} \times \{N_0\}$ for the left, right, lower and upper sides respectively (see Figure 1). We note the following property of $r*$-clusters of $\eta(t)$ in this box:

Lemma 3.2. *Assume* (3.1) *and let E be the event (in $\{0,1,2\}^{\mathbb{Z}^2}$) that $(0,0)$ is not $1*$-connected in B to $R \cup U$. Then*

$$\mathbb{P}(A_1(0,0) \text{ but } \eta(t) \in E \text{ infinitely often}) > 0 .$$

Proof. Let η be a configuration such that the following three conditions hold:

$$(0,0) \xrightarrow{B}_{1*} R \cup U,\ (M_0,0) \xrightarrow{B}_{2*} L \cup U \text{ and } (0,N_0) \xrightarrow{B}_{2*} R \cup D \tag{3.2}$$

and write $\tilde{B}$ for the set of type-2 vertices in B (see Figure 1 for an example). By planarity, the last two conditions ensure that the connected (as opposed to $*$-connected) component C of $(0,0)$ in $B \setminus \tilde{B}$ does not intersect $R \cup U$. (Here C is just a maximal connected set in $B \setminus \tilde{B}$ and does not need to be a cluster, so it may have both vertices of type-0 and type-1.) So write B as a disjoint union $C \cup \tilde{B} \cup \hat{C}$, where $\hat{C}$ is defined as $B \setminus (C \cup \tilde{B})$. Note that because C is a maximal connected subset of $B \setminus \tilde{B}$, C does not contain a vertex that is a neighbor of $\hat{C}$.

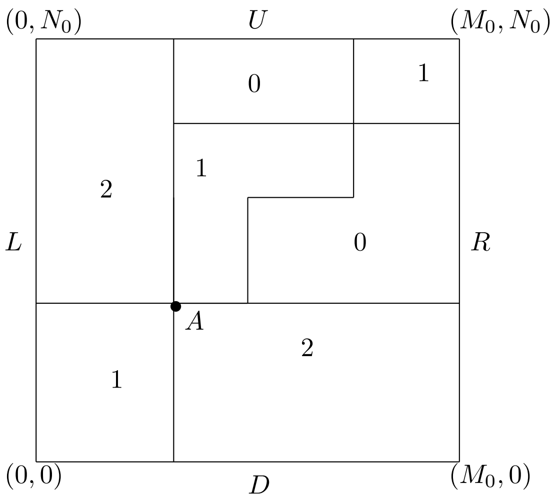

Figure 1: One example of a configuration satisfying (3.2). Note the "intersection" of the *-clusters at the point $A \in \mathbb{Z}^2 + (1/2,1/2)$.

The first condition in (3.2) gives a $*$-connected path P (with vertices written in order as $x_0,\dots,x_n$) from $(0,0)$ to $R \cup U$ in B all of whose vertices are type-1. Because $x_0 \in C$ and $x_n \notin C$ there exists a first i such that $x_i \in C$ but $x_{i+1} \notin C$. Both of these vertices are type-1, so they cannot be in $\tilde{B}$. This means $x_i \in C$ but $x_{i+1} \in \hat{C}$, so they are not neighbors, they are $*$-neighbors. Both of these vertices share a 2×2 block with two other vertices a and b. If either of a or b were in C (or $\hat{C}$), they would be neighbors to a vertex in $\hat{C}$ (or C), a contradiction, so they must be in $\tilde{B}$ and thus type-2. Summarizing, we can find a vertex $v \in \{0, \ldots, M_0 - 1\} \times \{0, \ldots, N_0 - 1\}$ such that either

1. $\eta_v = \eta_{v+(1,1)} = 1$ and $\eta_{v+(1,0)} = \eta_{v+(0,1)} = 2$ or
2. $\eta_v = \eta_{v+(1,1)} = 2$ and $\eta_{v+(1,0)} = \eta_{v+(0,1)} = 1$.

If $\eta(t)$ satisfies (3.2) then in the first case above, writing $v = (x, y)$, the vertex $(x, y, 2)$ has a negative spin but at least 4 neighbors with positive spin, giving it an opportunity for an energy-lowering flip. A similar statement holds in the second case. Therefore Lemmas 2.5 and 2.2 show that almost surely, $\eta(t)$ satisfies (3.2) only finitely often.

On the event $A_1(0,0) \cap A_2(M_0, 0) \cap A_2(0, N_0)$ in (3.1), almost every configuration must fail to satisfy at least one of the conditions in (3.2) infinitely often. Therefore at least one of the following three events has positive probability:

1. $A_1(0,0) \cap \left\{ \eta(t) \in \{(0,0) \xrightarrow{B}_{1*} R \cup U\}^c \text{ infinitely often} \right\}$
2. $A_2(M_0,0) \cap \left\{ \eta(t) \in \{(M_0,0) \xrightarrow{B}_{2*} L \cup U\}^c \text{ infinitely often} \right\}$
3. $A_2(0,N_0) \cap \left\{ \eta(t) \in \{(0,N_0) \xrightarrow{B}_{2*} R \cup D\}^c \text{ infinitely often} \right\}$.

However spatial symmetry of $\mathbb{P}$ implies that these events have the same probability. Therefore the first has positive probability and we are done. □

The previous lemma imposes a certain restriction on the geometry of the $1*$ cluster of $(0,0)$ in B. To take advantage of this, we consider minimal clusters. Given $r \in \{0,1,2\}$, we say that a set $V \subset B$ is a *recurrent r-cluster* in B for the realization $(\sigma(0), \omega)$ if, infinitely often, V is the intersection of an r-cluster of $\eta(t)$ with B. $V \subset B$ is a *minimal recurrent r-cluster* if it is a recurrent r-cluster but no proper subset of V is a recurrent r-cluster. In other words, a minimal recurrent r-cluster is a smallest vertex subset of B which is an r-cluster of B infinitely often *in time* in $\eta(t)$. On the event in Lemma 3.2, there is a recurrent cluster in B that contains $(0,0)$ but no point of $R \cup U$, so there is a minimal such cluster. Because there are only finitely many clusters in this box, we can fix $V \subset B$ such that V contains $(0,0)$, V does not intersect $R \cup U$ and

$$\mathbb{P}(V \text{ is a minimal recurrent } 1\text{-cluster}) > 0 . \tag{3.3}$$

V must contain a vertex v such that

$$v \in V \text{ but } v + (1,0),\ v + (0,1) \text{ and } v + (1,1) \notin V . \tag{3.4}$$

To see this, choose any vertex $v \in V$ with maximal ℓ_1 norm. Since V is finite, such a v exists. Because V does not intersect $R \cup U$, the vertices $v + (1,0)$, $v + (0,1)$ and $v + (1,1)$ must be in B. Since they have ℓ_1 norm larger than that of v, they also cannot be in V.

The following lemma will contradict (3.3) and complete the proof of Proposition 3.1. Note that this lemma makes precise the notion of 'corners' that induce the flipping of non-monochromatic columns, as mentioned above.

Lemma 3.3. *Let $V \subset B$ be such that* (3.4) *holds for some v. Then*

$$\mathbb{P}(V \textit{ is a minimal recurrent } 1\textit{-cluster}) = 0 .$$

Proof. Define E_V to be the event that V is the intersection of a 1-cluster of η with B. First assume that the event

$$\hat{E} = E_V \cap \{C_v \text{ has an unstable spin}\}$$

occurs infinitely often with positive probability. Then by Lemma 2.5, for almost every realization in $\hat{E}$, a spin in C_v could flip (before any others in B), forcing $E_{V\setminus\{C_v\}}$ to occur infinitely often. This means that almost surely on $\{\hat{E}$ occurs infinitely often$\}$, V cannot be minimal, so we conclude that almost surely, if V is a minimal recurrent 1-cluster, then at all large times at which E_V occurs, the vertices in C_v are stable.

Now assume the lemma is false and let $(\sigma(0), \omega)$ be a realization in which V is a minimal recurrent 1-cluster. Writing $v = (x, y)$, the stable vertex $(x, y, 2)$, which has a negative spin and two neighbors in C_v with positive spin, must have all other neighbors with negative spin at all large times at which E_V occurs. This implies that for all large t, the spins at $(x+1, y, 2)$ and $(x, y+1, 2)$ are negative when $\sigma(t) \in E_V$. Furthermore, these vertices must be stable for all large t, lest at least one flips to +1 and makes $(x, y, 2)$ unstable. By (3.4), the remaining spins of $C_{(x+1,y)}$ and $C_{(x,y+1)}$ can not both be positive. Hence for all large t at which $\sigma(t) \in E_V$, at least one other vertex in each of $C_{(x+1,y)}$ and $C_{(x,y+1)}$ must have negative spin.

Stability of vertices $(x, y, 0)$ and $(x, y, 1)$ (both of which have positive spin) gives now that only one vertex of each of $C_{(x+1,y)}$ and $C_{(x,y+1)}$ with third coordinate not equal to 2 can have negative spin. In addition, they must have different third coordinate. Using symmetry, we have now argued that if the lemma fails, then with positive probability, the configuration pictured in Figure 2 occurs in $\sigma(t)$ infinitely often.

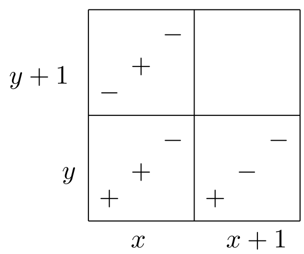

Figure 2: Depiction of the configuration near a corner in a minimal recurrent 1-cluster at a large time. The bottom left box represents the column C_v and the spins are listed for vertices in this column in increasing third coordinate.

For large t at which the above configuration occurs, we again invoke stability, but of the vertices $(x + 1, y, 0)$ and $(x, y + 1, 1)$ with positive spin. This implies that the vertices $(x + 1, y + 1, 0)$ and $(x + 1, y + 1, 1)$ must have positive spin. However, $(x + 1, y + 1) \notin V$, so at these times, the column $C_{(x+1,y+1)}$ is not type-1, so the spin at $(x + 1, y + 1, 2)$ is $+1$, and the column is monochromatic. We have now reached a contradiction: at each of these times, a finite sequence of flips can force the minimal cluster to shrink. The spin at $(x + 1, y, 1)$ can flip to +1, followed by the spin at $(x + 1, y, 2)$ and then the spin at $(x, y, 2)$. Applying Lemma 2.5 completes the proof. □

Under assumption (3.1), we derived inequality (3.3). The contradiction given by Lemma 3.3 implies that (3.1) must have been false and we are done. □

4 Fixed columns proliferate in S_3

In this section we continue the analysis of the slab S_3 with periodic boundary conditions and prove that the neighbors of fixed columns are fixed.

Proposition 4.1. *Let $u, v \in \mathbb{Z}^2$ be neighbors. With probability one, if C_u flips finitely often in S_3 (with periodic boundary conditions), then so does C_v.*

The idea of the proof: The presence of a fixed monochromatic column at (x, y) imposes a restriction on possible sign assignments of the surrounding spins (for instance, for all large times each spin in the column must be stable). Likewise, the presence of a flipping site next to such a column infinitely often will impose restrictions on the signs of its surrounding sites. We will analyze these restrictions and show that the two can not coexist in the slab of thickness three. The proof, while somewhat tedious, involves only elementary arguments.

An event $A \subset \Omega_k$ is *eventually absent* (or e-absent) if $\mathbb{P}(\sigma(t) \in A \text{ infinitely often}) = 0$.

Lemma 4.2. *Let $v, w \in S_k$ be neighbors for $k \geq 2$. The event $\{v \text{ and } w \text{ are unstable}\} \cap \{\sigma_v = \sigma_w\}$ is e-absent.*

Proof. For a contradiction, suppose that with positive probability, the event that both v and w are unstable and $\sigma_v = \sigma_w$ occurs infinitely often. At any one of these times, v has a chance to flip. If v flips but no clocks assigned to any other vertex within distance 2 of v ring beforehand, then w would have at least 4 opposite sign neighbors. Therefore we can apply Lemma 2.5 to deduce that with positive probability, $e_w(t) > 0$ infinitely often. This contradicts part 1 of Lemma 2.4. □

The following two lemmas are used repeatedly in the proof. A column C is called *positive* if the spins of all its vertices equal $+1$ and called *negative* if the spins are -1.

Lemma 4.3. *For $x, y \in \mathbb{Z}$, let $A_{(x,y)} \subset \Omega_3$ be the event defined by the conditions*

1. *$C_{(x,y)}$ is positive but at least one of $C_{(x+1,y)}, C_{(x,y+1)}, C_{(x+1,y+1)}$ is not,*
2. *for some $i, j \in \{0, 1, 2\}, \sigma_{(x+1,y,i)} = \sigma_{(x,y+1,j)} = +1$ and*
3. *$\sigma_{(x+1,y+1,m)} = +1$ for some $m \in \{0, 1, 2\} \setminus \{i, j\}$.*

(See Figure 3.) Then $A_{(x,y)}$ is e-absent.

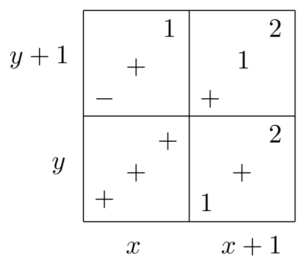

Figure 3: Illustration of the event $A_{(x,y)}$ in Lemma 4.3 with $i = j = 1, m = 0$. The bottom left column is $C_{(x,y)}$. All unmarked spins are unspecified. The negative spin flips to $+1$, then the spins flip in order according to their numbering, unless already positive.

Remark 4.4. *By identical reasoning, Lemma 4.3 holds after exchanging positive and negative spins.*

Proof. By way of contradiction, suppose that with positive probability, $A_{(x,y)}$ occurs infinitely often. When $A_{(x,y)}$ occurs, the vertices $(x + 1, y, m)$ and $(x, y + 1, m)$ have at least three positive neighbors each, so either they are unstable or positive. If the spins at these two vertices flip to $+1$ (without any other clocks ringing for vertices in $C_{(x,y)}, C_{(x+1,y)}, C_{(x,y+1)}$ or $C_{(x+1,y+1)}$) then all other spins of vertices in $C_{(x+1,y)}$ and $C_{(x,y+1)}$ have at least three positive neighbors, and can flip. Continuing, we see that there is a finite sequence of clock rings and spin flips that force $B_{(x,y)}$ to occur, where $B_{(x,y)}$ is the event such that $C_{(l,n)}$ is positive for $l \in \{x, x+1\}$ and $n \in \{y, y+1\}$. Therefore,

$$\inf_{\sigma \in A_{(x,y)}} \mathbb{P}(\sigma(t) \in B_{(x,y)} \text{ for some } t \in [0,1) \mid \sigma(0) = \sigma) > 0 .$$

By Lemma 2.5, $B_{(x,y)}$ occurs infinitely often with positive probability. Since this event is also absorbing (that is, $\mathbb{P}(\sigma(t) \in B_{(x,y)} \text{ for all } t \geq 0 \mid \sigma(0) \in B_{(x,y)}) = 1$), the event $A_{(x,y)}$ is e-absent, a contradiction. □

Lemma 4.5. *Let A be the event that $C_{(1,1)}$ is positive, $C_{(2,2)}$ is negative, $C_{(1,2)}$ is monochromatic and $C_{(2,1)}$ contains an unstable vertex. (See Figure 4.) Then A is e-absent.*

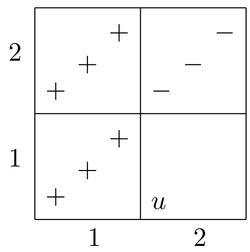

Figure 4: Illustration of the event A in Lemma 4.5. The bottom left column is $C_{(1,1)}$ and $C_{(1,2)}$ is monochromatic (pictured here as $+1$). The column $C_{(2,1)}$ has an unstable spin (marked u) at vertex $(2,1,0)$.

Proof. Suppose that with positive probability, A occurs infinitely often. Since $C_{(1,2)}$ has at least two spins of the same sign at any time, at least one of $A_0 := A \cap \{C_{(1,2)} \text{ is positive}\}$ and $A \cap \{C_{(1,2)} \text{ is negative}\}$ occurs infinitely often with positive probability. We will assume that A_0 does, as the following reasoning is identical in the other case. By part 1 of Lemma 2.4, with probability one, any given vertex must have at least three same sign neighbors at any sufficiently large time. Using permutation invariance of the different levels of the slab, the event $A_1 := A_0 \cap \{(2,1,0) \text{ has three positive and three negative neighbors}\}$ occurs infinitely often with positive probability.

We will consider two cases depending on the status of spins in $C_{(2,1)}$. To do so, we define

$$A_2^+ := A_1 \cap \{C_{(2,1)} \text{ is positive}\} \text{ and } A_2^- := A_1 \cap \{C_{(2,1)} \text{ is negative}\} .$$

We claim that at least one of A_2^+ or A_2^- occurs infinitely often with positive probability. By way of contradiction, assume this is false, so that almost surely, for all large times, if A_2 occurs, then two vertices of $C_{(2,1)}$ have spins opposite of that of the third. As illustrated in Figure 5, the third spin must be unstable and at large times has exactly three positive and three negative neighbors. It has a positive probability to flip, so Lemma 2.5 implies that almost surely, it will flip infinitely often, leaving $C_{(2,1)}$ monochromatic and this spin still unstable. Permutation invariance of the levels of the slab implies that $A_2^+ \cup A_2^-$ occurs infinitely often with positive probability.

Case 1. A_2^+ occurs infinitely often with positive probability. When A_2^+ occurs, $C_{(2,1)}$ is positive and $(2,1,0)$ has equal number of positive and negative neighbors, so the spins at $(3,1,0)$ and $(2,0,0)$ must be negative. If one of the vertices of $C_{(2,2)}$ is unstable then it may flip, leading to a configuration in $C_{(1,1)}, C_{(2,1)}, C_{(1,2)}$ and $C_{(2,2)}$ from

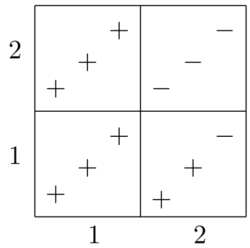

Figure 5: Illustration of the case that neither A_2^+ nor A_2^- occurs infinitely often. The spin at $(2,1,2)$ will flip to $+1$, remaining unstable. Alternatively, the unstable spin at $(2,1,0)$ will flip to -1, followed by $(2,1,1)$, resulting in $C_{(2,1)}$ turning negative and $(2,1,1)$ still unstable.

the event $A_{(1,1)}$ in Lemma 4.3. Therefore, by Lemma 2.5, the event $A_3^+ := A_2^+ \cap \{\text{all vertices of } C_{(2,2)} \text{ are stable}\}$ occurs infinitely often with positive probability. This means that when A_3^+ occurs, the column $C_{(3,2)}$ is negative. This results in the sign distribution pictured in Figure 6.

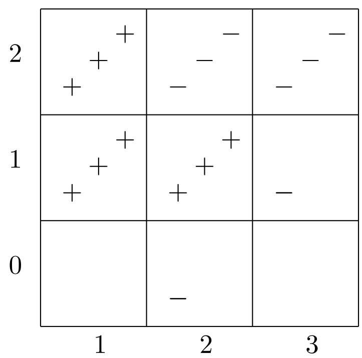

Figure 6: Illustration of the event A_3^+ in Lemma 4.5. All unmarked spins are unspecified. The middle left box represents the column $C_{(1,1)}$ and the bottom vertex of the middle box is unstable.

Now we claim that almost surely, for all large times at which A_3^+ occurs, the spins at $(3,1,1)$ and $(3,1,2)$ must be $+1$. For suppose that this is false; that is, with positive probability, infinitely often both A_3^+ occurs and at least one of these spins is -1. Because $(2,1,0)$ is unstable, it can flip to -1, and an application of Lemma 2.5 shows that with positive probability, the columns $C_{(2,1)}, C_{(3,1)}, C_{(2,2)}$ and $C_{(3,2)}$ would have a configuration described in Remark 4.4 infinitely often, a contradiction. This means that $A_4^+ := A_3^+ \cap \{\sigma_{(3,1,1)} = \sigma_{(3,1,2)} = +1\}$ occurs infinitely often with positive probability.

The spins at $(3,1,1)$ and $(3,1,2)$ now have at least two negative neighbors. If they have at least three, then they can flip, forcing one of $(2,1,1)$ or $(2,1,2)$ to be $+1$ but unstable. If this occurred infinitely often with positive probability, then it would contradict Lemma 4.2, since $(2,1,0)$ is a neighboring $+1$ unstable vertex. Therefore $A_5^+ = A_4^+ \cap \{\sigma_{(3,0,1)} = \sigma_{(3,0,2)} = +1\}$ occurs infinitely often with positive probability (see Figure 7).

We now invoke Lemma 4.3. If, with positive probability, for infinitely many of the times at which A_5^+ occurs, either the spin at $(2,0,1)$ or $(2,0,2)$ were equal to $+1$ then

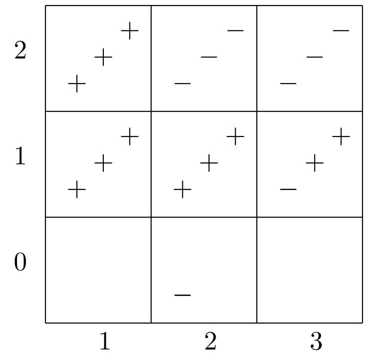

Figure 7: Illustration of the event A_5^+ in Lemma 4.5. All unmarked spins are unspecified. The middle left box represents the column $C_{(1,1)}$ and the bottom vertex of the middle box is unstable.

the columns $C_{(2,1)}, C_{(3,1)}, C_{(2,0)}$ and $C_{(3,0)}$ would have a configuration described in that lemma. Therefore $A_6^+ = A_5^+ \cap \{C_{(2,0)}$ is negative$\}$ occurs infinitely often with positive probability (see Figure 8).

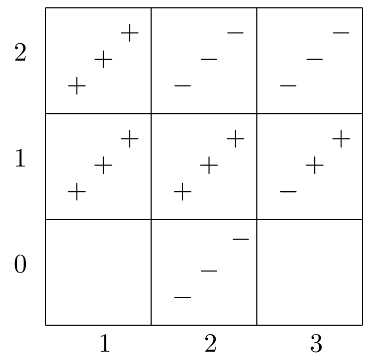

Figure 8: Illustration of the event A_6^+ in Lemma 4.5. All unmarked spins are unspecified. The middle left box represents the column $C_{(1,1)}$ and the bottom vertex of the middle box is unstable.

But since $\sigma_{(2,1,0)}$ is unstable, it can flip to -1, with the other two spins in $C_{(2,1)}$ positive and unstable. Lemma 2.5 says this occurs infinitely often with positive probability, contradicting Lemma 4.2 with $v = (2,1,1)$ and $w = (2,1,2)$.

Case 2. $A_2^- \setminus A_2^+$ (A_2^- but not A_2^+) occurs infinitely often with positive probability. We first claim that almost surely, for all large times at which A_2^- occurs, the spins at $(2,1,1)$ and $(2,1,2)$ must each have at least three negative neighbors not contained in $C_{(2,1)}$. To see this, suppose for a contradiction that with positive probability, infinitely often both A_2^- occurs and one of these spins (by symmetry, we can say $\sigma_{(2,1,1)}$) has at most two negative neighbors not contained in $C_{(2,1)}$. Because $\sigma_{(2,1,0)}$ is unstable, it can flip to $+1$ and by Lemma 2.5, this will occur infinitely often almost surely. After this flip, $\sigma_{(2,1,1)}$ is then unstable and negative, so can flip to $+1$. This leaves $\sigma_{(2,1,2)}$ negative and unstable (and therefore, for all large times, with exactly three positive and three negative neighbors). After $\sigma_{(2,1,2)}$ flips to $+1$, then $C_{(2,1)}$ is positive with $\sigma_{(2,1,2)}$ unstable. Using Lemma 2.5 and permuting levels 0 and 2 in the slab shows that

A_2^+ occurs infinitely often with positive probability, contradicting the assumption.

Therefore $A_3^- := A_2^- \cap \{C_{(3,1)}$ and $C_{(2,0)}$ are negative in levels 1 and $2\}$ occurs infinitely often with positive probability. Further, as $\sigma_{(2,1,0)}$ is unstable on A_3^- and already has three negative neighbors, the spins at $(3,1,0)$ and $(2,0,0)$ must be $+1$. This results in the sign distribution displayed in Figure 9.

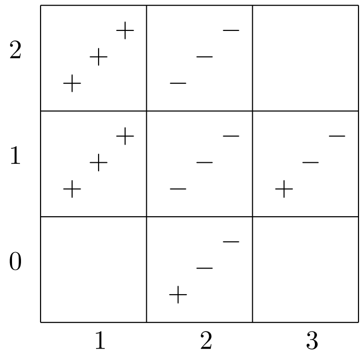

Figure 9: Illustration of the event A_3^- in Lemma 4.5. All unmarked spins are unspecified. The middle left box represents the column $C_{(1,1)}$ and the bottom vertex of the middle box is unstable.

On the event A_3^-, if there is any negative spin in $C_{(3,2)}$, then the columns $C_{(2,1)}, C_{(2,2)}, C_{(3,1)}$ and $C_{(3,2)}$ would have a configuration described in Remark 4.4, so almost surely, for all large times at which A_3^- occurs, the column $C_{(3,2)}$ must be positive (see Figure 10). However, $\sigma_{(2,1,0)}$ can flip to $+1$, leaving $\sigma_{(2,2,0)}$ unstable, and flipping to $+1$, leaving both $\sigma_{(2,2,1)}$ and $\sigma_{(2,2,2)}$ unstable and negative. Lemma 2.5 implies this will occur infinitely often with positive probability and this contradicts Lemma 4.2.

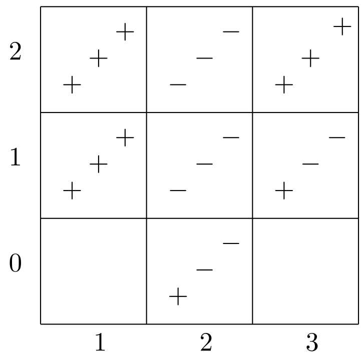

Figure 10: Illustration of the event $A_3^- \cap C_{(3,2)}$ is positive in Lemma 4.5. All unmarked spins are unspecified. The middle left box represents the column $C_{(1,1)}$ and the bottom vertex of the middle box is unstable.

□

Proof of Proposition 4.1. By translation invariance and symmetry we can take $u = (1,1)$ and $v = (2,1)$. We will show that almost surely, when $C_{(1,1)}$ fixates then $C_{(2,1)}$ also fixates (either to $+1$ or to -1). By Proposition 3.1, $C_{(1,1)}$ fixates to either $+1$ or -1. We will prove neighbor fixation for the positive case. The proof in case when $C_{(1,1)}$ fixates to -1 is identical.

We first prove that almost surely, if $C_{(1,1)}$ fixates to $+1$ then for all large times, each vertex in this column must have at least 2 stable neighbors outside $C_{(1,1)}$ with spin $+1$. If this were false, then with positive probability there would be infinitely many times at which some spin (say at $(1,1,0)$) has at least three neighbors outside $C_{(1,1)}$ which are either not positive or not stable. Note that none of these neighbors are neighbors of each other, so the unstable ones will all be unstable even if any of them flip. Since they have a positive probability to flip, Lemma 2.5 implies that with positive probability, on the event that $C_{(1,1)}$ fixates to $+1$, the spin at $(1,1,0)$ will have at least three negative neighbors. Another application of Lemma 2.5 implies that this spin will flip infinitely often, a contradiction since it fixates.

Therefore B occurs infinitely often with positive probability, where

$$B = \left\{ \begin{array}{c} C_{(1,1)} \text{ is positive and each of its spins has at least} \\ \text{two positive stable neighbors outside } C_{(1,1)} \end{array} \right\}.$$

We next claim that almost surely, if $C_{(1,1)}$ fixates to $+1$ then $C_{(2,1)}$ is positive infinitely often or negative infinitely often. If this is not the case, then with positive probability, $C_{(1,1)}$ fixates to $+1$ and $C_{(2,1)}$ has exactly two like spins for all large times. By Proposition 3.1, $C_{(2,1)}$ cannot fixate, so it has exactly two positive spins infinitely often. But then the negative spin is unstable, and Lemma 2.5 implies that $C_{(2,1)}$ is positive infinitely often, a contradiction.

If $C_{(2,1)}$ does not fixate (with positive probability) then it must contain the spin that flips infinitely often, so the previous paragraph implies that almost surely, if $C_{(1,1)}$ fixates then infinitely often $C_{(2,1)}$ will both be monochromatic and have an unstable spin. By spatial symmetry we may assume that $\sigma_{(2,1,0)}$ is unstable. So far we have shown that at least one of B^+ or B^- occurs infinitely often with positive probability, where $B^+ = B \cap \{C_{(2,1)}$ is positive and $\sigma_{(2,1,0)}$ is unstable$\}$ and B^- is the same event with positive replaced by negative.

Case 1. First suppose that B^+ occurs infinitely often with positive probability. Because $\sigma_{(2,1,0)}$ is unstable and already has three positive neighbors, almost surely for all large times at which B^+ occurs, the spins at $(2,2,0), (3,1,0)$ and $(2,0,0)$ must be negative. (See Figure 11).

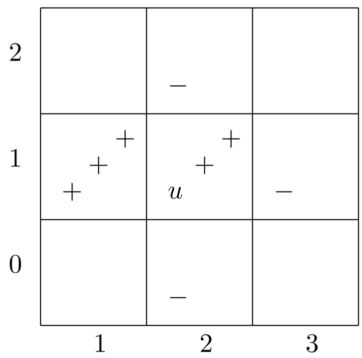

Figure 11: Illustration of the event B^+ at large times, where the middle left box represents $C_{(1,1)}$. The neighbors of the unstable spin at $(2,1,0)$ (labeled u) outside of $C_{(1,1)}$ and $C_{(2,1)}$ are negative.

Two neighbors of $(1,1,0)$ outside of $C_{(1,1)}$ must be positive and stable, so at least one must be in the set $\{\sigma_{(1,2,0)}, \sigma_{(1,0,0)}\}$. By spatial symmetry we may assume that $\sigma_{(1,2,0)}$ is positive and stable on B^+ infinitely often with positive probability. This means that at least one other spin at a vertex in $C_{(1,2)}$ is positive. The remaining spin has at least 3 positive neighbors and by Lemma 2.5 will be $+1$ infinitely often (see Figure 12). This means that with positive probability, infinitely often on B^+, the column $C_{(1,2)}$ is positive. If there is a positive spin in $C_{(2,2)}$, then $C_{(1,1)}, C_{(1,2)}, C_{(2,1)}$ and $C_{(2,2)}$ contain a configuration described in Lemma 4.3, so for all large times $C_{(2,2)}$ is negative. Lemma 4.5 then implies that B^+ is e-absent, a contradiction.

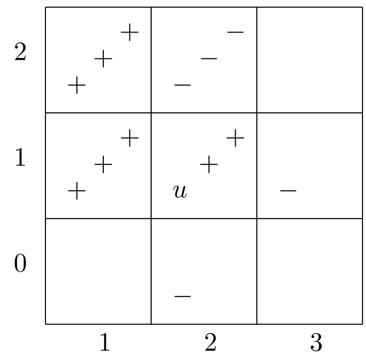

Figure 12: Illustration of the event $B^+ \cap \{C_{(1,2)}$ is positive$\}$ at large times. $C_{(2,2)}$ is marked negative, contradicting Lemma 4.5.

Case 2. B^- occurs infinitely often with positive probability. When B^- occurs, $(2,1,0)$ has three positive neighbors, so either $(2,0,0)$ or $(2,2,0)$ (or both) are positive. By symmetry we may assume that $B^- \cap \{\sigma_{(2,2,0)} = +1\}$ occurs infinitely often with positive probability. Because each spin in $C_{(1,1)}$ has at least two positive stable neighbors outside of $C_{(1,1)}$, either $C_{(1,2)}$ or $C_{(1,0)}$ must contain at least two positive spins. Just as before, if $C_{(1,2)}$ contains two positive spins, the other has at least three positive neighbors and Lemma 2.5 implies the columns will be positive infinitely often. The same holds for $C_{(1,0)}$. These two cases will complete the proof below.

Case 2a. We first consider the case that $B^- \cap \{\sigma_{(2,2,0)} = +1, C_{(1,2)}$ is positive$\}$ occurs infinitely often, and the configuration is shown in Figure 13. By lemma 4.5 the positive

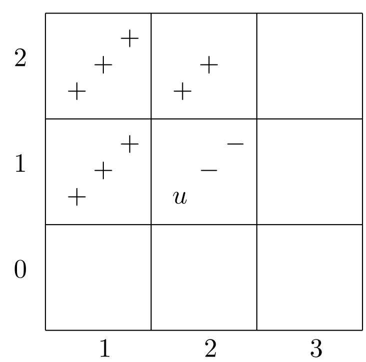

Figure 13: Illustration of the event B^- in the case that $C_{(1,2)}$ is positive (case 2a). $C_{(2,2)}$ has two positive spins.

vertex $(2,2,0)$ must be stable, hence $C_{(2,2)}$ must contain at least two positive spins. But then the unstable spin $\sigma_{(2,1,0)}$ can flip to $+1$, giving a configuration in these four columns described in Lemma 4.3. This occurs infinitely often with positive probability, a contradiction.

Case 2b. The other possibility is that $B^- \cap \{\sigma_{(2,2,0)} = +1, C_{(1,0)}$ is positive$\}$ occurs infinitely often with positive probability. If the spin $\sigma_{(2,0,0)}$ is positive at infinitely many of these times, then we have a configuration symmetrical to that in the previous paragraph, leading to a contradiction. Otherwise $\sigma_{(2,0,0)}$ is negative at all such large times. The configuration is displayed in Figure 14. Again, if $C_{(2,0,0)}$ has another vertex with a

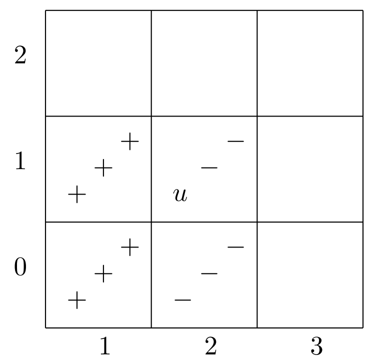

Figure 14: Illustration of the event B^- in the case that $C_{(1,0)}$ is positive (case 2b). $C_{(2,0)}$ is negative.

positive spin infinitely often, Lemma 4.3 gives a contradiction after $\sigma_{(2,1,0)}$ flips to $+1$, so $C_{(2,0,0)}$ is negative for all large times. Applying Lemma 4.5 to $C_{(1,0)}, C_{(1,1)}, C_{(2,1)}$ and $C_{(2,0)}$, we obtain a contradiction.

□

5 Proof of Theorem 1.1

For any $z \in \mathbb{Z}^2$,

$$\mathbb{P}(C_z \text{ fixates}) \geq [\max\{p, (1-p)\}]^{12} > 0 ,$$

since C_z fixates whenever all 12 spins in $C_z, C_{z+(1,0)}, C_{z+(0,1)}$ and $C_{z+(1,1)}$ are initially of the same sign. By translation-ergodicity, almost surely there exist columns that fixate. If not all columns fixate, we may almost surely find neighboring columns, one which fixates and one which doesn't. By countability, there exist neighboring columns C_u and C_v that have positive probability for C_u to fixate but for C_v not to and this contradicts Proposition 4.1. □

6 Proof of Theorem 1.2

Here we will show that S_4 does not fixate. This is a proof by example and is illustrated in Figure 15. The notation used in the figure is as follows. Each unit square represents a column $C_{(x,y)}$ in S_4. The spin of vertex $(x,y,3)$ is shown at the top-left of the square, with $(x,y,2),(x,y,1)$ and $(x,y,0)$ proceeding counter-clockwise. With $C_{(0,0)}$ the column at the bottom left of the figure, let $A_{(0,0)}$ be the event (in Ω_4) that all spins in the box $[0,13] \times [0,10] \times \{0,1,2,3\}$ have values as shown in Figure 15 (with blank spins unspecified). The reader may check that (a) all sites within the medium-line box (outside the bold box) are fixed with one positive and three negative spins, (b) all specified

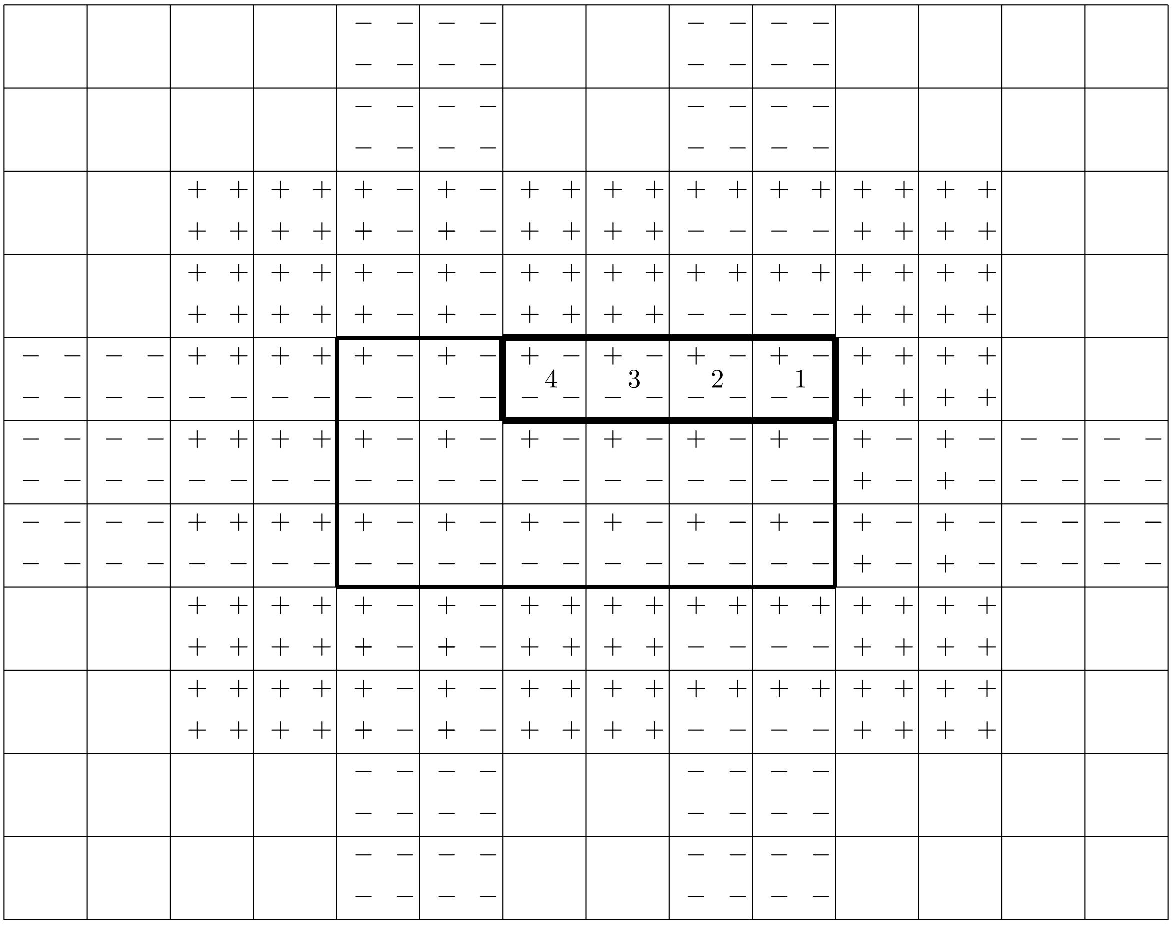

Figure 15: Construction showing existence of non-fixating sites within the bold lines. Each unit box represents a column and the spins in a box begin with level 3 in the top left and proceed counter-clockwise to level 0 in the top right. All specified spins outside thick bold lines are fixed. The level 0 spins in the thick bold lines begin -1 and flip infinitely often.

spins outside the medium-line box are fixed and (c) the spins in the bold box are fixed except for those with third coordinate 0 (that is, those pictured in the upper right of the unit boxes) flip infinitely often. The flipping spins begin as all with value -1 and flip right to left from -1 to $+1$ as denoted by the numbering. Once they have flipped from -1 to $+1$ they flip back in the reverse order.

The event $A_{(0,0)}$ has positive probability and by translation ergodicity, almost surely some translate of it occurs. So with probability one, there exist spins which flip infinitely often.

A S_2 fixation under periodic boundary conditions

In this appendix we give an alternative proof (to the one in [3]) of fixation in S_2 with periodic boundary conditions. As the arguments follow the same lines as those presented in this paper, we keep the proofs concise. We use a similar notation to the S_3 slab; the column $C_{(x,y)}$ consists of the pair of spins with the first two coordinates (x, y). $C_{(x,y)}$ is positive (negative) if both of its spins are positive (negative). Note that, due to the boundary condition, the edge between $(x, y, 0)$ and $(x, y, 1)$ counts twice in the energy computation (1.1) of either site.

The proof structure is identical to the S_3 case and directly combines analogous Propositions A.1 and A.2. As before, the first proposition shows that fixed columns are monochromatic. The proof is simple and is contained in [3].

Proposition A.1. *With probability one, if a column in S_2 flips finitely often then it is eventually monochromatic.*

The second proposition shows that neighbors of fixed columns are fixed.

Proposition A.2. *Let $u, v \in \mathbb{Z}^2$ be neighbors. With probability one, if C_u flips finitely often in S_2, then so does C_v.*

We will use the following lemma repeatedly in the proof:

Lemma A.3. *Let A be the event in S_2 that $C_{(1,1)}$ is positive, each of the columns $C_{(1,2)}$, $C_{(2,1)}$ and $C_{(2,2)}$ contains at least one positive spin, and at least one of these columns contains a negative spin. A is e-absent.*

Remark A.4. *By identical reasoning, the lemma holds after exchanging positive and negative spins.*

Proof. If A occurs infinitely often with positive probability then at each occurrence of A, all non-positive spins among the four columns have at least 3 positive neighbors (counting spins in the same column twice) and thus have positive probability to flip to $+1$. (See Figure 16.) By Lemma 2.5, all will flip to $+1$ and fixate, and this is a contradiction. $\square$

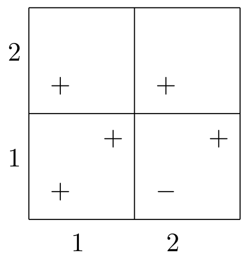

Figure 16: One possible assignment of spins to the four columns is illustrated. The assignment results in an energy lowering flip.

Proof of Proposition A.2. By translation invariance and symmetry, we can take $u = (1, 1)$ and $v = (2, 1)$. We will show that almost surely, when $C_{(1,1)}$ fixates to $+1$, then $C_{(2,1)}$ also fixates (either to $+1$ or -1). The case in which $C_{(1,1)}$ fixates to -1 is handled similarly.

We will assume the contrary, that with positive probability $C_{(1,1)}$ fixates to $+1$ but a vertex in $C_{(2,1)}$ flips infinitely often. It must then be unstable infinitely often. By spatial symmetry and Lemma 2.5, we will assume this unstable spin to be $\sigma_{(2,1,0)}$ and take it to be positive. As in the proof of Proposition 4.1, almost surely, if $C_{(1,1)}$ fixates to $+1$ then for all large times, each vertex in this column must have at least 2 stable neighbors outside $C_{(1,1)}$ with spin $+1$. The proof is exactly as before – if not, then a spin of $C_{(1,1)}$ has at least three unstable neighbors which can flip to -1 and then force it, by Lemma 2.5, to flip. By spatial symmetry then, the event

$$B = \{C_{(1,1)} \text{ is positive, } \sigma_{(2,1,0)} \text{ is positive unstable and } \sigma_{(1,0,0)} = +1\}$$

occurs infinitely often with positive probability. Define $B^+ = B \cap \{\sigma_{(2,1,1)} = +1\}$ and $B^- = B \cap \{\sigma_{(2,1,1)} = -1\}$. We give two cases.

Case 1. B^+ occurs infinitely often with positive probability. Almost surely for all large times at which B^+ occurs, $C_{(2,0)}$ must be negative; this follows from Lemma A.3. Furthermore at all such large times $\sigma_{(2,1,0)}$ has exactly three positive and three negative neighbors. This implies that $\sigma_{(3,1,0)} = \sigma_{(2,2,0)} = -1$. Further, both of $\sigma_{(3,1,1)}$ and $\sigma_{(2,2,1)}$ must be positive, for if either were negative, $\sigma_{(2,1,0)}$ could flip to -1, leaving $\sigma_{(2,1,1)}$ with 4 negative neighbors and an energy lowering flip. We can now apply Lemma A.3 again to both blocks of columns $C_{(1,1)}, C_{(2,1)}, C_{(1,2)}, C_{(2,2)}$ and $C_{(2,1)}, C_{(2,2)}, C_{(3,1)}, C_{(3,2)}$ respectively to deduce that $C_{(1,2)}$ and $C_{(3,2)}$ are negative (see Figure 17) . But this leaves $\sigma_{(2,2,1)}$ positive with at least 4 negative neighbors, so it can make an energy lowering flip, a contradiction for large times.

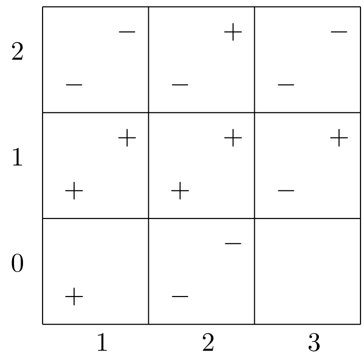

Figure 17: The event B^+ from Case 1 of the proof of Proposition A.2.

Case 2. B^- occurs infinitely often with positive probability. Again by Lemma A.3, $C_{(2,0)}$ must almost surely be negative at all large times at which B^- occurs. The spin $\sigma_{(2,1,0)}$ must have three positive neighbors at large times, so $\sigma_{(2,2,0)} = \sigma_{(3,1,0)} = +1$. By Lemma A.3 again applied to $C_{(1,1)}, C_{(2,1)}, C_{(1,2)}$ and $C_{(2,2)}$, the column $C_{(1,2)}$ is negative for all large times at which B^- occurs. As above, both spins $\sigma_{(3,1,1)}$ and $\sigma_{(2,2,1)}$ must be negative, lest $\sigma_{(2,1,0)}$ flips to $+1$ and giving $\sigma_{(2,1,1)}$ an energy lowering flip. But now if $\sigma(2,1,0)$ flips to -1, the positive spin at $(2,2,0)$ has at least four negative neighbors, so it can make an energy lowering flip, a contradiction for large times (see Figure 18). □

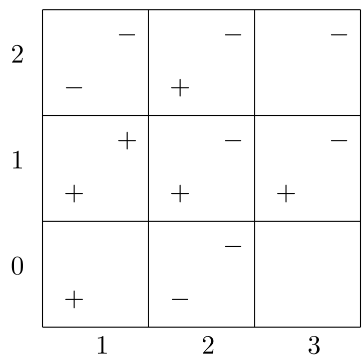

Figure 18: The event B^- from Case 2 of the proof of Proposition A.2.

References

[1] R. Arratia. (1983). Site recurrence for annihilating random walks on $\mathbb{Z}^d$. *Ann. Probab.* **11** 706–713. MR-0704557

[2] F. Camia, E. De Santis and C. M. Newman. (2002). Clusters and recurrence in the two-dimensional zero-temperature stochastic Ising model. *Ann. Appl. Probab.* **12** 565–580. MR-1910640

[3] M. Damron. H. Kogan, C. M. Newman and V. Sidoravicius. (2013). Coarsening in 2D slabs. To appear in *Topics in Percolative and Disordered Systems*, Springer, 2014.

[4] L.R. Fontes, R.H. Schonmann and V. Sidoravicius. (2002). Stretched exponential fixation in stochastic Ising models at zero temperature. *Commun. Math. Phys.* **228** 495–518. MR-1918786

[5] T. E. Harris. (1977). A correlation inequality for Markov processes in partially ordered state space. *Ann. Probab.* **5** 451–454. MR-0433650

[6] S. Nanda, C. M. Newman and D. L. Stein. (2000). Dynamics of Ising spin systems at zero temperature. In *On Dobrushin's Way (from Probability Theory to Statistical Mechnics)*, R. Minlos, S. Shlosman and Y. Suhov, eds., Amer. Math. Soc. Transl. (2) **198**:183–193. MR-1766351

[7] J. Olejarz, P.L. Krapivsky and S. Redner. (2011) Zero-temperature relaxation of three-dimensional Ising ferromagnet. *Phys. Rev. E.* **83** 051104-1–051104-11.

[8] V. Spirin, P.L. Krapivsky and S. Redner. (2001). Freezing in Ising ferromagnet. *Phys. Rev. E.* **65** 016119-1–016119-9.

[9] D. Stauffer. (1994). Ising spinodal decomposition at $T = 0$ in one to five dimensions. *J. Phys. A* **27** 5029–55032.

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