Electronic Journal of Probability

Electron. J. Probab. **18** (2013), no. 68, 1–17.
ISSN: 1083-6489 DOI: 10.1214/EJP.v18-2403

CLT for crossings of random trigonometric polynomials

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Abstract

We establish a central limit theorem for the number of roots of the equation $X_N(t) = u$ when $X_N(t)$ is a Gaussian trigonometric polynomial of degree N. The case $u = 0$ was studied by Granville and Wigman. We show that for some size of the considered interval, the asymptotic behavior is different depending on whether u vanishes or not. Our mains tools are: a) a chaining argument with the stationary Gaussian process with covariance $\frac{\sin t}{t}$, b) the use of Wiener chaos decomposition that explains some singularities that appear in the limit when $u \neq 0$.

Keywords: Crossings of random trigonometric polynomials; Rice formula; Chaos expansion.
AMS MSC 2010: 60G15.
Submitted to EJP on October 10, 2012, final version accepted on July 6, 2013.
Supersedes HAL:hal-00747030.

1 Introduction

Let us consider the random trigonometric polynomial:

$$X_N(t) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} (a_n \sin nt + b_n \cos nt), \tag{1.1}$$

where the coefficients a_n and b_n are independent standard Gaussian random variables and N is some integer.

The number of zeroes of such a process on the interval $[0, 2\pi)$ has been studied in the paper by Granville and Wigman [5] where a central limit theorem, as $N \to +\infty$ is proved for the first time using the method of Malevich [8].

The aim of this paper is twofold: firstly we extend their result to the number of crossings of every level and secondly we propose a simpler proof. The key point consist in proving that after a convenient scaling the process $X_N(t)$ converges in a certain sense to the stationary process $X(t)$ with covariance $r(t) = \frac{\sin t}{t}$. The central limit theorem for

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the crossings of process $X_N(t)$ is then a consequence of the central limit theorem for the crossings in large time for $X(t)$.

The above idea is outlined in Granville and Wigman [5] but the authors could not implement this procedure. Let us quoted their words: "While computing the asymptotic of the variance of the crossings of process $X_N(t)$, we determined that the covariance function r_{X_N} of X_N has a scaling limit $r(t)$, which proved useful for the purpose of computing the asymptotics. Rather than scaling r_{X_N}, one might consider scaling X_N. We realize, that the above should mean, that the distribution of the zeros of X_N is intimately related to the distribution of the number of the zeros on (roughly) $[0, N]$ of a certain Gaussian stationary process $X(t)$, defined on the real line $\mathbb{R}$, with covariance function r....Unfortunately, this approach seems to be difficult to make rigorous, due to the different scales of the processes involved".

Our method can roughly be described as follows. In the first time in Section 3 we defined the two process X_N (or rather its normalization Y_N, see its definition in the next section) and X in the same probability space. This fact allows us to compute the covariance between these two processes. Afterwards we get a representation of the crossings of both processes in the Wiener's Chaos. These representations and the Mehler formula for non-linear functions of four dimensional Gaussian vectors, permit us to compute the L^2 distance between the crossings of Y_N and the crossings of X. The central limit theorem for the crossings of X can be obtained easily by a modification of the method of m-dependence approximation, developed firstly by Malevich [8] and Berman [3] and improved by Cuzick [4]. The hypotheses in this last work are more in accord with ours. Finally the closeness in L^2 (in quadratic mean) of the two numbers of crossings: those of $X(t)$ and those of the m-dependent approximation gives us the central limit theorem for the crossings of X_N.

The organization of the paper is the following: in Section 2 we present basic calculations; Section 3 is devoted to the presentation of the Wiener chaos decomposition and to the study of the variance. Section 4 states the central limit theorem. Additional proofs are given in Section 5 and 6. A table of notation is given in Section 7.

2 Basic results and notation

$r_{X_N}(\tau)$ will be the covariance of the process $X_N(t)$ given by

$$r_{X_N}(\tau) := \mathbb{E}[X_N(0)X_N(\tau)] = \frac{1}{N}\sum_{n=1}^{N}\cos n\tau = \frac{1}{N}\cos(\frac{(N+1)\tau}{2})\frac{\sin(\frac{N\tau}{2})}{\sin\frac{\tau}{2}}. \tag{2.1}$$

We define the process

$$Y_N(t) = X_N(t/N),$$

with covariance

$$r_{Y_N}(\tau) = r_{X_N}(\tau/N).$$

We have

$$r'_{Y_N}(\tau) = \frac{1}{2N\sin\frac{\tau}{2N}}\cos\big(\frac{2N+1}{2N}\tau\big) - \frac{\sin\tau}{4N^2\sin^2\frac{\tau}{2N}}, \tag{2.2}$$

$$r''_{X_N}(\tau) = -\frac{\sin\frac{\tau}{2}}{2N\sin^2\frac{\tau}{2N}}[\sin\frac{(N+1)\tau}{2N}\sin\frac{\tau}{2N} + \cos\frac{(N+1)\tau}{2N}\cos\frac{\tau}{2N}] \tag{2.3}$$

$$\begin{aligned} r''_{Y_N}(\tau) &= \frac{1}{N^2}r''_{Y_N}(\frac{\tau}{N}) \\ &= \frac{\cos\frac{\tau}{2N}\cos\frac{(2N+1)}{2N}\tau - 2\frac{(2N+1)}{2}\sin\frac{\tau}{2N}\sin\frac{(2N+1)}{2N}\tau - \cos\tau}{4N^2\sin^2\frac{\tau}{2N}} \\ &\quad - \frac{(2N\sin\frac{\tau}{2N}\cos(\frac{2N+1}{2N}\tau) - \sin\tau)\cos\frac{\tau}{2N}}{4N^3\sin^3\frac{\tau}{2N}}. \end{aligned} \tag{2.4}$$

The convergence of Riemann sums to the integral implies simply that

$$r_{Y_N}(\tau) \to r(\tau) := \sin(\tau)/\tau, \tag{2.5}$$

$$r'_{Y_N}(\tau) \to r'(\tau) = \cos(\tau)/\tau - \tau^{-2}\sin(\tau), \tag{2.6}$$

$$r''_{Y_N}(\tau) = \frac{1}{N^2}r''_N(\frac{\tau}{N}) \to r''(\tau) = -\frac{\sin(\tau)}{\tau} - 2\frac{\cos(\tau)}{\tau^2} + 2\frac{\sin(\tau)}{\tau^3}. \tag{2.7}$$

And these convergences are uniform in every compact interval that does not contains zero. We will need also the following upper-bounds that are easy
When $\tau \in [0, N\pi]$:

$$|r_{Y_N}(\tau)| \le \pi/\tau \quad ; \quad |r'_{Y_N}(\tau)| \le \frac{\pi}{2\tau} + \frac{\pi^2}{4\tau^2} \quad ; \quad |r''_{Y_N}(\tau)| \le (const)\big(\tau^{-1} + \tau^{-2} + \tau^{-3}\big). \tag{2.8}$$

We now compute the ingredients of the Rice formula [2]

$$\mathbb{E}X_N^2(t) = 1, \quad \text{and} \quad \mathbb{E}(X'_N(t))^2 = \frac{1}{N}\sum_{n=1}^{N} n^2 = \frac{(N+1)(2N+1)}{6}.$$

Denoting by $N^{X_N}_{[0,2\pi)}(u)$ the numbers of crossings of the level u of X_N on the interval $[0, 2\pi)$, the Rice formula gives

$$\mathbb{E}[N^{X_N}_{[0,2\pi)}(u)] = 2\pi.\sqrt{\mathbb{E}(X'_N(t))^2}\sqrt{2/\pi}\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} = \frac{2}{\sqrt{3}}\sqrt{\frac{(N+1)(2N+1)}{2}}\,e^{-\frac{u^2}{2}}.$$

Hence

$$\lim_{N\to\infty}\frac{\mathbb{E}[N^{X_N}_{[0,2\pi)}(u)]}{N} = \frac{2}{\sqrt{3}}\,e^{-\frac{u^2}{2}}.$$

When not specified, all limits are taken when $N \to \infty$.

3 Spectral representation and Wiener Chaos

This section has as main goal to build both processes $X(t)$ and $Y_N(t)$ in the same probability space. This chaining argument is one of our main tools. It makes it possible to show that the two processes are close in L^2 distance and by consequence the same result holds true for the crossings of both processes.

We have

$$X(t) = \int_0^1 \cos(t\lambda)\,dB_1(\lambda) + \int_0^1 \sin(t\lambda)\,dB_2(\lambda), \tag{3.1}$$

where B_1 and B_2 are two independent Brownian motions. Using the same Brownian motions we can write

$$Y_N(t) = \int_0^1 \sum_{n=1}^{N} \cos(\frac{nt}{N})\mathbb{1}_{[\frac{n-1}{N},\frac{n}{N})}(\lambda)dB_1(\lambda) + \int_0^1 \sum_{n=1}^{N} \sin(\frac{nt}{N})\mathbb{1}_{[\frac{n-1}{N},\frac{n}{N})}(\lambda)dB_2(\lambda).$$

It is easy to check, using isometry properties of stochastic integrals that $Y_N(t)$ has the desired covariance.
By defining the functions

$$\gamma_N^1(t,\lambda) = \sum_{n=1}^{N} \cos(\frac{nt}{N})1_{[\frac{n-1}{N},\frac{n}{N})}(\lambda) \quad \text{and} \quad \gamma_N^2(t,\lambda) = \sum_{n=1}^{N} \sin(\frac{nt}{N})1_{[\frac{n-1}{N},\frac{n}{N})}(\lambda),$$

we can write

$$Y_N(t) = \int_0^1 \gamma_N^1(t,\lambda)dB_1(\lambda) + \int_0^1 \gamma_N^2(t,\lambda)dB_2(\lambda). \tag{3.2}$$

In the sequel we are going to express the representation (3.1) and (3.2) in an isonormal process framework. Let define $\mathfrak{H}^2$ the Hilbert vector space defined as

$$\{\mathbf{h} = (h_1,h_2) : \int_{\mathbb{R}} h_1^2(\lambda)d\lambda + \int_{\mathbb{R}} h_2^2(\lambda)d\lambda < \infty\},$$

with scalar product

$$< \mathbf{h},\mathbf{g} >= \int_{\mathbb{R}} h_1(\lambda)g_1(\lambda)d\lambda + \int_{\mathbb{R}} h_2(\lambda)g_2(\lambda)d\lambda.$$

The transformation

$$\mathbf{h} \to W(\mathbf{h}) := \int_{\mathbb{R}} h_1(\lambda)dB_1(\lambda) + \int_{\mathbb{R}} h_2(\lambda)dB_2(\lambda),$$

defines an isometry between $\mathfrak{H}^2$ and a Gaussian subspace of $L^2(\Omega,\mathcal{A},P)$ where $\mathcal{A}$ is the $\sigma-$field generated by $B_1(\lambda)$ and $B_2(\lambda)$.

Thus $W(\mathbf{h})_{\mathbf{h}\in\mathfrak{H}^2}$ is the isonormal process associated to $\mathfrak{H}^2$. By using the representations (3.1) and (3.2), readily we get

$$\begin{aligned}
X(t) &= W(\mathbb{1}_{[0,1]}(\cdot,\cdot)(\cos t\cdot,\sin t\cdot)),\\
Y_N(t) &= W(\mathbb{1}_{[0,1]}(\cdot,\cdot)(\gamma_N^1(\cdot,t),\gamma_N^2(\cdot,t))),\\
\tilde{X}'(t) &:= \frac{X'(t)}{\sqrt{1/3}} = W(\frac{\mathbb{1}_{[0,1]}}{\sqrt{1/3}}(\cdot,\cdot)(-\sin t\cdot,\cos t\cdot)),\\
\tilde{Y}_N'(t) &:= \frac{Y_N'(t)}{\sqrt{-r''_{Y_N}(0)}} = W(\frac{\mathbb{1}_{[0,1]}}{\sqrt{-r''_{Y_N}(0)}}(\cdot,\cdot)((\gamma_N^1(\cdot,t))',(\gamma_N^2(\cdot,t))').
\end{aligned}$$

We are going to present now the Wiener's chaos, it will be our second main tool. For a general reference about this topic see [9]. Let H_k be the Hermite polynomial of degree k defined by

$$H_k(x) = (-1)^k e^{\frac{x^2}{2}} \frac{d^k}{dx^k}(e^{-\frac{x^2}{2}}).$$

It is normalized such that for Y a standard Gaussian random variable we have $\mathbb{E}(H_k(Y)H_m(Y)) = \delta_{k,m}k!$. Consider $\{e_i\}_{i\in\mathbb{N}}$ an ortonormal basis for $\mathfrak{H}^2$. Let Λ be the set the sequences

$a = (a_1, a_2, \dots)$ $a_i \in \mathbb{N}$ such that all the terms except a finite number vanish. For $a \in \Lambda$ we set $a! = \prod_{i=1}^{\infty} a_i!$ and $|a| = \sum_{i=1}^{\infty} a_i$. For any multiindex $a \in \Lambda$ we define

$$\Phi_a = \frac{1}{\sqrt{a!}} \prod_{i=1}^{\infty} H_{a_i}(W(e_i)).$$

For each $n \geq 1$, we will denote by $\mathcal{H}_n$ the closed subspace of $L^2(\Omega, \mathcal{A}, P)$ spanned by the random variables $\{\Phi_a,\, a \in \Lambda,\, |a| = n\}$. The space $\mathcal{H}_n$ is the nth Wiener chaos associated with $B_1(\lambda)$ and $B_2(\lambda)$. If $\mathcal{H}_0$ denotes the space of constants we have the ortogonal decomposition

$$L^2(\Omega, \mathcal{A}, P) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n.$$

For any Hermite's polynomial H_q, it holds

$$\begin{aligned} H_q(W(\mathbf{h})) = & I_q(\mathbf{h}) := \int_0^{+\infty} \dots \int_0^{+\infty} h^1(\lambda_1) \dots h^1(\lambda_q) dB_1(\lambda_1) \dots dB_1(\lambda_q) \\ & + \int_0^{+\infty} \dots \int_0^{+\infty} h^2(\lambda_1) \dots h^2(\lambda_q) dB_2(\lambda_1) \dots dB_2(\lambda_q), \end{aligned}$$

with $\mathbf{h} = (h^1, h^2)$. For instance as $Y_N(t) = W(\mathbb{1}_{[0,1]}(\cdot,\cdot)(\gamma_N^1(\cdot,t), \gamma_N^2(\cdot,t)))$, we obtain

$$\begin{aligned} H_2(Y_N(t)) = & \int_0^1 \int_0^1 \gamma_N^1(\lambda_1, t) \gamma_{1,N}^1(\lambda_2, t) dB_1(\lambda_1) dB_1(\lambda_2) \\ & + \int_0^1 \int_0^1 \gamma_N^2(\lambda_1, t) \gamma_N^2(\lambda_2, t) dB_2(\lambda_1) dB_2(\lambda_2). \end{aligned}$$

We now write the Wiener Chaos expansion for the number of crossings. As the absolute value function belongs to $\mathbb{L}^2(\mathbb{R}, \varphi(x)dx)$, where φ is the standard Gaussian density, we have $|x| = \sum_{k=0}^{\infty} a_{2k} H_{2k}(x)$ with

$$a_{2k} = 2 \frac{(-1)^{k+1}}{\sqrt{2\pi} 2^k k! (2k-1)}.$$

It is shorter to study first $X_N(t)$ on $[0, \pi]$ (resp. $Y_N(t)$ and $X(t)$ on $[0, N\pi]$), the generalization to $[0, 2\pi]$ (resp. to $[0, 2N\pi]$) will be done in Section 4. The result of Kratz & León [6] or Th 10.10 in [2] imply

$$\begin{aligned} & \frac{1}{\sqrt{N\pi}} (N_{[0,\pi N]}^X(u) - \mathbb{E} N_{[0,\pi N]}^X(u)) \\ & \qquad = \sqrt{1/3} \varphi(u) \sum_{q=1}^{\infty} \sum_{k=0}^{[\frac{q}{2}]} \frac{H_{q-2k}(u)}{(q-2k)!} \frac{a_{2k}}{\sqrt{N\pi}} \int_0^{\pi N} H_{q-2k}(X(s)) H_{2k}(\tilde{X}'(s))\, ds, \end{aligned} \tag{3.3}$$

where $[x]$ is the integer part. We introduce the notation

$$f_q(u, x_1, x_2) = \varphi(u) \sum_{k=0}^{[\frac{q}{2}]} \frac{H_{q-2k}(u)}{(q-2k)!} a_{2k} H_{q-2k}(x_1) H_{2k}(x_2). \tag{3.4}$$

For each s, the random variable

$$\begin{aligned} f_q(u, X(s), \tilde{X}'(s)) = & \varphi(u) \sum_{k=0}^{[\frac{q}{2}]} \frac{H_{q-2k}(u)}{(q-2k)!} a_{2k} \\ & \times H_{q-2k}(W(\mathbb{1}_{[0,1]}(\cdot,\cdot)(\cos s\cdot, \sin s\cdot))) H_{2k}(W(\frac{\mathbb{1}_{[0,1]}}{\sqrt{1/3}}(\cdot,\cdot)(-\sin s\cdot, \cos s\cdot))) \end{aligned}$$

belongs the q-th chaos as a consequence of linearity and the property of multiplication of two functionals belonging to different chaos, cf. [9] Proposition 1.1.3. Furthermore also by linearity the same is true for

$$I_q\big([0,t]\big) = \frac{\sqrt{1/3}}{\sqrt{t}} \int_0^t f_q(u, X(s), \tilde{X}'(s))ds. \tag{3.5}$$

So that

$$\frac{1}{\sqrt{N\pi}}(N^X_{[0,\pi N]}(u) - \mathbb{E}N^X_{[0,\pi N]}(u)) = \sum_{q=1}^{\infty} I_q\big([0,\pi N]\big),$$

gives the decomposition into the Wiener's chaos. The same type of expansion is also true for $N^{Y_N}_{[0,\pi N]}(u)$

$$\frac{1}{\sqrt{N}}\Big(N^{Y_N}_{[0,\pi N]}(u) - \mathbb{E}N^{Y_N}_{[0,\pi N]}(u)\Big) = \sum_{q=1}^{\infty} I_{q,N}\big([0,\pi N]\big), \tag{3.6}$$

where

$$I_{q,N}\big([0,\pi N]\big) = \frac{\sqrt{-r''_{Y_N}(0)}}{\sqrt{\pi N}} \int_0^{\pi N} f_q(u, Y_N(s), \tilde{Y}'_N(s))ds. \tag{3.7}$$

Our first goal is to compute the limit variance of (3.6). Our main tool will be the Arcones inequality. We define the norm

$$||f_q||^2 := \mathbb{E}f_q^2(u, Z_1, Z_2),$$

where (Z_1, Z_2) is a bidimensional standard Gaussian vector. We have

$$||f_q||^2 = \varphi^2(u) \sum_{k=0}^{[\frac{q}{2}]} \frac{H^2_{q-2k}(u)}{(q-2k)!}\, a^2_{2k}(2k)! \le (const) \sum_{k=0}^{[\frac{q}{2}]} a^2_{2k}(2k)! \le (const),$$

where $(const)$ is some constant that does not depend on q. Now we must introduce a dependence coefficient that is defined in Arcones [1] (pag. 2245)

$$\psi_N(\tau) = \sup\Big(\big|r_{Y_N}(\tau)\big| + \big|\frac{r'_{Y_N}(\tau)}{\sqrt{-r''_{Y_N}(0)}}\big|, \big|\frac{r'_{Y_N}(\tau)}{\sqrt{-r''_{Y_N}(0)}}\big| + \big|\frac{r''_{Y_N}(\tau)}{r''_{Y_N}(0)}\big|\Big).$$

The Arcones inequality says that if $\psi_N(s'-s) < 1$, it holds

$$\big|\mathbb{E}[f_q(u, Y_N(s), \tilde{Y}'_N(s)) f_q(u, Y_N(s'), \tilde{Y}'_N(s'))]\big| \le \psi^q_N(s'-s)||f_q||^2.$$

We will use also the following Lemma the proof of which is given in Section 5

Lemma 3.1. *For every $a > 0$, there exists a constant K_a such that*

$$\sup_N \quad \mathrm{Var}\big(N^{Y_N}_{[0,a]}(u)\big) \le K_a < \infty. \tag{3.8}$$

Choose some $\rho < 1$, using the inequality (2.8), we can choose a big enough such that for $\tau > a$ we have $\psi_N(\tau) < \frac{K}{\tau} \le \rho$.

Then we partition $[0, N\pi]$ into $L = [\frac{N\pi}{a}]$ intervals $J_1, \dots, J_L$ of length larger than a, and we set for short

$$N_\ell = N^{Y_N}_{J_\ell}(u).$$

We have

$$\mathrm{Var}(N^{Y_N}_{[0,N\pi]}(u)) = \mathrm{Var}(N_1 + \dots + N_L) = \sum_{\ell,\ell',|\ell-\ell'|\le 1} \mathrm{Cov}\,(N_\ell, N'_\ell) + \sum_{\ell,\ell',|\ell-\ell'|>1} \mathrm{Cov}\,(N_\ell, N'_\ell).$$

The first sum is easily shown to be $O(N)$ by applying Lemma 3.1 and the Cauchy-Schwarz inequality.

Let us look at a term of the second sum. Using the expansion (3.6) we set

$$\frac{N_\ell - \mathbb{E}(N_\ell)}{\sqrt{\pi N}} = \sum_{q=1}^{\infty} I_{q,N}(J_\ell),$$

where $I_{q,N}(J_\ell) = \frac{\sqrt{-r''_{Y_N}(0)}}{\sqrt{\pi N}} \int_{J_\ell} f_q(u, Y_N(s), \tilde{Y}'_N(s))ds$. Let us consider the terms corresponding to $q > 1$. The Arcones inequality implies that

$$\begin{aligned} \left|\text{Cov}\left(I_{q,N}(J_\ell), I_{q,N}(J_{\ell'})\right)\right| &\leq (const) \int_{J_\ell \times J_{\ell'}} \frac{1}{N\pi}(-r''_{Y_N}(0))(K/\tau)^q dsdt \\ &\leq \frac{(const)}{N} \int_{J_\ell \times J_{\ell'}} \rho^{q-2}\tau^{-2} dsdt, \end{aligned} \tag{3.9}$$

where $\tau = s - t$. Summing over all pairs of intervals and over $q \geq 2$ it is easy to check that this sum is bounded.

It remains to study the case $q = 1$. Since $H_1(x) = x$

$$I_{1,N}(J_\ell) = (N\pi)^{-1/2}\sqrt{-r''_{Y_N}(0)}u\phi(u) \int_{J_\ell} Y_N(s)ds.$$

So that

$$\Big| \sum_{\ell,\ell',|\ell-\ell'|>1} \text{Cov}\left(I_{1,N}(J_\ell), I_{1,N}(J_{\ell'})\right)\Big| \leq (const)\Big|\frac{1}{N}\int_0^{\pi N}\int_0^{\pi N} r_{Y_N}(s-s')dsds'\Big|,$$

which is bounded because of the following result

$$\begin{aligned} \frac{1}{N}\int_0^{\pi N}\int_0^{\pi N} r_{Y_N}(s-s')dsds' &= \frac{2}{N}\int_0^{\pi N}(\pi N - \tau)r_{Y_N}(\tau)d\tau \\ &= 2\sum_{n=1}^{N}\int_0^{\pi N}(\pi - \frac{\tau}{N})\frac{1}{N}\cos n\frac{\tau}{N}\, d\tau \\ &= 2\sum_{n=1}^{N}\frac{1-\cos n\pi}{n^2} \\ &= 4\sum_{j=0}^{N}\frac{1}{(2j+1)^2} \\ &\to 4\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2} = 4\frac{\pi^2}{8} = \frac{\pi^2}{2}. \end{aligned} \tag{3.10}$$

∎

Define $\sigma_q^2 := \lim_{N\to\infty} \text{Var}\left(I_q([0,\pi N])\right) < \infty$.

Proposition 3.2. *For $q > 1$ we have*

$$\text{Var}\left(I_{q,N}([0,\pi N])\right) \to \sigma_q^2 \quad \textit{as } N \to +\infty.$$

For $q = 1$

$$\text{Var}\left(I_{1,N}([0,\pi N])\right) \to \frac{1}{3}u^2\phi^2(u)\pi.$$

In the case $u \neq 0$ this limit is different from

$$\lim_{N\to\infty} \operatorname{Var}\left(I_1([0,\pi N])\right) = \frac{2}{3}u^2\phi^2(u)\pi.$$

Remark 3.3. *This different behavior depending on whether or not the variable belongs to the chaos of order one, is explicit thanks to the decomposition of the crossings into Wiener's chaos.*

Proof. Firstly we consider the case $q > 2$:

$$\mathbb{E}\left(I_{q,N}^2([0,N\pi]\right) = -r''_{Y_N}(0)\varphi^2(u)\sum_{k_1=0}^{[\frac{q}{2}]}\sum_{k_2=0}^{[\frac{q}{2}]}\frac{H_{q-2k_1}(u)}{(q-2k_1)!}\,a_{2k_1}\,\frac{H_{q-2k_2}(u)}{(q-2k_2)!}\,a_{2k_2}$$

$$\frac{1}{N\pi}\int_0^{N\pi}\int_0^{N\pi}\mathbb{E}[H_{q-2k}(Y_N(s))H_{2k}(\frac{Y'_N(s')}{\sqrt{-r''_{Y_N}(0)}})H_{q-2k}(Y_N(s'))H_{2k}(\frac{Y'_N(s)}{\sqrt{-r''_{Y_N}(0)}})]\,ds'ds$$

$$= -r''_{Y_N}(0)\varphi^2(u)\sum_{k_1=0}^{[\frac{q}{2}]}\sum_{k_2=0}^{[\frac{q}{2}]}\frac{H_{q-2k_1}(u)}{(q-2k_1)!}\,a_{2k_1}\,\frac{H_{q-2k_2}(u)}{(q-2k_2)!}\,a_{2k_2}$$

$$2\int_0^{\pi N}(1-\frac{s}{N\pi})\mathbb{E}[H_{q-2k_1}(Y_N(0))H_{2k_1}(\frac{Y'_N(0)}{\sqrt{-r''_{Y_N}(0)}})H_{q-2k_2}(Y_N(s))H_{2k_2}(\frac{Y'_N(s)}{\sqrt{-r''_{Y_N}(0)}})]\,ds.$$

We now use the generalized Mehler formula (Lemma 10.7 page 270 of [2]).

Lemma 3.4. *Let (X_1, X_2, X_3, X_4) be a centered Gaussian vector with variance matrix*

$$\Sigma = \begin{pmatrix} 1 & 0 & \rho_{13} & \rho_{14} \\ 0 & 1 & \rho_{23} & \rho_{24} \\ \rho_{13} & \rho_{23} & 1 & 0 \\ \rho_{14} & \rho_{24} & 0 & 1 \end{pmatrix}$$

Then, if $r_1 + r_2 = r_3 + r_4$,

$$\mathbb{E}\left(H_{r_1}(X_1)H_{r_2}(X_2)H_{r_3}(X_3)H_{r_4}(X_4)\right) = \sum_{(d_1,d_2,d_3,d_4)\in J}\frac{r_1!r_2!r_3!r_4!}{d_1!d_2!d_3!d_4!}\rho_{13}^{d_1}\rho_{14}^{d_2}\rho_{23}^{d_3}\rho_{24}^{d_4},$$

where J is the set of d_i's satisfying : $d_i \geq 0$;

$$d_1 + d_2 = r_1\ ;\ d_3 + d_4 = r_2\ ;\ d_1 + d_3 = r_3\ ;\ d_2 + d_4 = r_4. \tag{3.11}$$

If $r_1 + r_2 \neq r_3 + r_4$ the expectation is equal to zero.

Using this lemma, there exist a finite set $\mathcal{J}_q$ and constants C_{q,k_1,k_2} such that

$$\begin{aligned}&\mathbb{E}[H_{q-2k_1}(Y_N(0))H_{2k_1}(\tilde{Y}'_N(0))H_{q-2k_2}(Y_N(\tau))H_{2k_2}(\tilde{Y}'_N(\tau))]\\ &= \sum_{\mathcal{J}_q} C_{q,k_1,k_2}|r_{Y_N}(\tau)|^{2q-(2k_1+2k_1)-h_1}\Big|\frac{r'_{Y_N}(\frac{\tau}{N})}{\sqrt{-r''_{Y_N}(0)}}\Big|^{2h_1}\Big|\frac{r''_{Y_N}(\tau)}{\sqrt{-r''_{Y_N}(0)}}\Big|^{2k_1+2k_2-h_1}\\ &:= \tilde{G}_{q,k_1,k_2,N}(\tau).\end{aligned} \tag{3.12}$$

This clearly proves that

$$\begin{aligned}&\mathbb{E}[H_{q-2k_1}(Y_N(0))H_{2k_1}(\tilde{Y}'_N(0))H_{q-2k_2}(Y_N(\tau))H_{2k_2}(\tilde{Y}'_N(\tau))]\\ &\qquad\to \mathbb{E}[H_{q-2k_1}(X(0))H_{2k_1}(\tilde{X}'(0))H_{q-2k_2}(X(\tau))H_{2k_2}(\tilde{X}'(\tau))],\end{aligned}$$

and Formula (3.9) gives a domination proving the convergence of the integral and the fact that σ_q^2 is finite.

Let us look to the case $q = 1$. In one hand by using (3.10), it holds

$$\begin{aligned}\mathbb{E}\big(I_{1,N}^2([0,N\pi]\big) &= -r''_{Y_N}(0)\varphi^2(u)(ua_0)^2\frac{1}{N\pi}\int_0^{N\pi}\int_0^{N\pi}\mathbb{E}(Y_N(s)Y_N(s'))ds'ds \\ &\to 1/3\varphi^2(u)\frac{2u^2}{\pi^2}\pi^2/2 = \frac{1}{3}u^2\phi^2(u). \end{aligned} \tag{3.13}$$

On the other hand we have

$$\begin{aligned}\mathbb{E}\big(I_1^2([0,N\pi]\big) &= \frac{1}{3}\varphi^2(u)(ua_0)^2\frac{1}{N\pi}\int_0^{N\pi}\int_0^{N\pi}\frac{\sin(s-s')}{s-s'}ds'ds \\ &= \frac{1}{3}\varphi^2(u)\frac{2u^2}{\pi^2}2\int_0^{N\pi}(\pi-\tau/N)\frac{\sin(\tau)}{\tau}d\tau \to \frac{2}{3}u^2\phi^2(u). \end{aligned} \tag{3.14}$$

□

4 Central limit Theorem with a chaining argument

In this section we first establish a central limit theorem for the crossings of the process $X(t)$, Theorem 4.1. Secondly, we show that this theorem implies our main result: Theorem 4.2, central limit theorem for the crossings of the process $X_N(t)$.

The covariance $r(t)$ of the limit process $X(t)$ is not a summable in the sense that

$$\int_0^{+\infty}|r(t)|dt = +\infty,$$

but it satisfies

$$\int_0^N r(t)dt \text{ converges as } N \to \infty,$$

for $q > 1$

$$\int_0^{+\infty}|r(t)|^q\,dt < +\infty.$$

The following theorem is a direct adaptation of Theorem 1 in [7] or of Theorem 10.11 of [2]. Its proof is given in Section 6 for completeness.

Theorem 4.1. *As $t \to +\infty$,*

$$\frac{1}{\sqrt{t}}\big(N_{[0,t]}^X(u) - \mathbb{E}(N_{[0,t]}^X(u))\big) \Rightarrow N(0,\frac{2}{3}u^2\phi^2(u) + \sum_{q=2}^{\infty}\sigma_q^2(u)),$$

where $\Rightarrow$ is the convergence in distribution.

The main idea is to use this result to extend it to the crossings of $Y_N(t)$. Our main result is the following:

Theorem 4.2. *As $N \to +\infty$,*

1. $\dfrac{1}{\sqrt{N\pi}}\big(N_{[0,N\pi]}^{Y_N}(u) - \mathbb{E}(N_{[0,N\pi]}^{Y_N}(u))\big) \Rightarrow N(0,\dfrac{1}{3}u^2\phi^2(u) + \sum_{q=2}^{\infty}\sigma_q^2(u)),$
2. $\dfrac{1}{\sqrt{2N\pi}}\big(N_{[0,2N\pi]}^{Y_N}(u) - \mathbb{E}(N_{[0,2N\pi]}^{Y_N}(u))\big) \Rightarrow N(0,\dfrac{2}{3}u^2\phi^2(u) + \sum_{q=2}^{\infty}\sigma_q^2(u)),$

Remark 4.3. *We point out that in the case $u = 0$ the two limit variances are the same and this is the result of Granville and Wigman [5], but in the other cases this is a new result. The chaos method permits an easy interpretation of the difference between these two behaviors.*

Proof. Let us introduce the cross correlation:

$$\begin{aligned}\rho_N(s,t) = \mathbb{E}(X(s)Y_N(t)) &= \sum_{n=1}^{N} \int_{\frac{n-1}{N}}^{\frac{n}{N}} \cos(s\lambda - t\frac{n}{N})\, d\lambda \\ &= \sum_{n=1}^{N} \int_0^{\frac{1}{N}} \cos((s-t)\frac{n}{N} - sv)\, dv = \Re\{\int_0^{\frac{1}{N}} e^{-isv} dv \sum_{n=1}^{N} e^{i(s-t)\frac{n}{N}}\} \\ &= \frac{\sin\frac{s}{N}}{\frac{s}{N}} \frac{1}{N} \sum_{n=1}^{N} \cos(s-t)\frac{n}{N} + \frac{1-\cos\frac{s}{N}}{\frac{s^2}{2N^2}} \frac{s}{2N^2} \sum_{n=1}^{N} \sin(s-t)\frac{n}{N},\end{aligned}$$

where $\Re$ is the real part. So we can write

$$\rho_N(s,t) = \frac{\sin(s/N)}{s/N} r_{Y_N}(t-s) + \frac{1-\cos(s/N)}{s^2/(2N^2)} \frac{s}{2N} \frac{1}{N} \sum_{n=1}^{N} \sin(s-t)\frac{n}{N}.$$

The two functions $\frac{\sin(z)}{z}$ and $\frac{1-\cos(z)}{z^2/2}$ are bounded, with bounded derivatives and $\frac{\sin(z)}{z}$ tend to 1 as z tends to 0. We have also

$$\left|\frac{1}{N} \sum_{n=1}^{N} \sin(s-t)\frac{n}{N}\right| = \left|\frac{2}{s-t} \frac{\sin\frac{(s-t)}{2}}{\frac{2N}{s-t}} \frac{\sin(\frac{N+1}{2N}(s-t))}{\sin\frac{(s-t)}{2N}}\right| \le (const)|s-t|^{-1},$$

whenever $|s-t| < \pi N$.

We have already proved that $r_{Y_N}(s-t) = \frac{1}{N}\sum_{n=1}^{N} \cos\left((s-t)\frac{n}{N}\right)$, converges to $r(s-t)$ uniformly on every compact that does not contains zero. The same result is true for the first two derivatives that converge respectively to the corresponding derivative of $r(s-t)$. In addition for large values of $|s-t|$ these functions are bounded by $K|s-t|^{-1}$ and for each fixed s, $\frac{s}{2N^2}\sum_{n=1}^{N} \sin(s-t)\frac{n}{N} \to 0$. Using the derivation rules it is easy to see that this is enough to have

$$\begin{aligned}\rho_N(s,t) &\to r(s-t) \\ \frac{\partial \rho_N(s,t)}{\partial s} &= \mathbb{E}(X'(s)Y_N(t)) \to r'(s-t) \\ \frac{\partial \rho_N(s,t)}{\partial t} &= \mathbb{E}(X(s)Y_N'(t)) \to -r'(s-t) \\ \frac{\partial^2 \rho_N(s,t)}{\partial s \partial t} &= \mathbb{E}(X'(s)Y_N'(t)) \to -r''(s-t),\end{aligned}$$

again the convergence being uniform on every compact that does not contains zero. In additions these function are bounded by $(const)(s-t)^{-1}$.

Before beginning the proofs, we present two results that were established in Peccati & Tudor [10] (Theorem 1 and Proposition 2) and we state as a theorem for later reference.

We will denote as $\zeta_{q,r}$ a generic element of the q-th chaos depending of a parameter r that tends to infinity. For instance in our cases we will have $\zeta_{q,t} = I_q([0,t])$ and $\zeta_{q,N} = I_{q,N}([0,\pi N])$ respectively.

Theorem 4.4.

(i) *Assume that for every* $q_1 \leq q_2, \ldots \leq q_m$*, it holds that* $\lim_{t\to\infty} \mathbb{E}[\zeta_{q_i,t}]^2 = \sigma_{ii}^2$ *and that for* $i \neq j$ $\lim_{t\to\infty} \mathbb{E}[\zeta_{q_i,t}\zeta_{q_j,t}] = 0$.
Then, if D_m *is the diagonal matrix with entries* σ_{ii}^2*, Theorem 1 of [10] says that the random vector*

$$(\zeta_{q_1,t}, \ldots, \zeta_{q_m,t}) \Rightarrow N(0, D_m),$$

if and only if each $\zeta_{q_i,t}$ *converges in distribution towards* $N(0, \sigma_{ii}^2)$ *when* $t \to \infty$*.*

(ii) *Considering now* d *functionals of the q-th chaos* $\{\zeta_{q,r}^l\}_{l=1}^d$*, Proposition 2 of [10] says that*

$$(\zeta_{q,r}^1, \zeta_{q,r}^2, \ldots, \zeta_{q,r}^d) \Rightarrow N(0, C)$$

if an only if $\zeta_{q,t}^i \Rightarrow N(0, c_{ii})$ *and* $\mathbb{E}[\zeta_{q,t}^i \zeta_{q,t}^j] \to c_{ij}$ *when* $t \to \infty$*, where* c_{ij} *is the entry* i, j *of the matrix* C*.*

We are now ready to prove the following lemma.

Lemma 4.5. *For* $q \geq 2$

$$\lim_{N\to\infty} \mathbb{E}\big[I_{q,N}([0, N\pi]) - I_q([0, N\pi])\big]^2 = 0.$$

Proof.

$$\mathbb{E}\big[I_{q,N}([0, N\pi]) - I_q([0, N\pi])\big]^2 = \mathbb{E}\big[I_{q,N}([0, N\pi])\big]^2 + \mathbb{E}\big[I_q([0, N\pi])\big]^2 - 2\mathbb{E}\big[I_{q,N}([0, N\pi]) I_q([0, N\pi])\big].$$

We have already shown that the first two terms tend to $\sigma_q^2(u)$. It only remains to prove that the third also does. But, since the cross correlation $\rho_N(s,t)$ shares all the properties of $r_{Y_N}(s-t)$, the same proof as in Section 3 shows that the limit is again $\sigma_q^2(u)$. □

We now finish the proof of Theorem 4.2.

Proof of Theorem 4.2.
Proof of 1. The case of $I_{1,N}([0, N\pi])$ is easy to handle since it is already a Gaussian variable and that its limit variance is easy to compute using (3.10). By Lemma 4.5, for $q \geq 2$, $I_{q,N}([0, N\pi])$ inherits the asymptotic Gaussian behavior of $I_q([0, N\pi])$. By using (i) of Theorem 4.4, this is enough to obtain the normality of the sum.
Proof of 2. We have already proved that

$$\chi_N(1) := \frac{1}{\sqrt{N\pi}} \big(N_{[0,N\pi]}^{Y_N}(u) - \mathbb{E}(N_{[0,N\pi]}^{Y_N}(u))\big) \Rightarrow N(0, \frac{1}{3}u^2\phi^2(u) + \sum_{q=2}^{\infty} \sigma_q^2(u)),$$

the same result holds by stationarity for the sequence

$$\chi_N(2) := \frac{1}{\sqrt{N\pi}} \big(N_{[N\pi,2N\pi]}^{Y_N}(u) - \mathbb{E}(N_{[N\pi,2N\pi]}^{Y_N}(u))\big),$$

and given that

$$\frac{1}{\sqrt{2N\pi}} \big(N_{[0,2N\pi]}^{Y_N}(u) - \mathbb{E}(N_{[0,2N\pi]}^{Y_N}(u))\big) = \frac{1}{\sqrt{2}} (\chi_N(1) + \chi_N(2)).$$

It only remains to show that the limit of the vector $(\chi_N(1), \chi_N(2))$ is jointly Gaussian and that the variance of the sum converges to the corresponding one. Defining

$$I_{q,N}([\pi N, 2\pi N]) = \frac{\sqrt{-r''_{Y_N}(0)}}{\sqrt{\pi N}} \int_{\pi N}^{2\pi N} f_q(u, Y_N(s), \widetilde{Y}'_N(s)ds,$$

we can write the sum above as

$$\frac{1}{\sqrt{2}}(\chi_N(1) + \chi_N(2)) = \frac{1}{\sqrt{2}}(\sum_{q=1}^{\infty} I_{q,N}([0, \pi N]) + \sum_{q=1}^{\infty} I_{q,N}([\pi N, 2\pi N])),$$

and given that the limit variance is finite we have

$$\frac{1}{\sqrt{2}}(\chi_N(1) + \chi_N(2)) = \frac{1}{\sqrt{2}}(\sum_{q=1}^{Q} I_{q,N}([0, \pi N]) + \sum_{q=1}^{Q} I_{q,N}([\pi N, 2\pi N])) + o_{\mathbb{P}}(1),$$

where $o_{\mathbb{P}}(1)$ denotes a term that tends to zero in probability when $Q \to \infty$ uniformly in N. Let us consider first the term corresponding to the first chaos ($q = 1$). We have

$$\begin{aligned}
\mathcal{E} &:= \mathbb{E}\big(I_{1,N}([0, N\pi]) I_{1,N}([N\pi, 2N\pi])\big) \\
&= -r''_{Y_N}(0)\varphi^2(u)(ua_0)^2 \frac{1}{N\pi} \int_0^{N\pi} \int_{N\pi}^{2N\pi} \mathbb{E}(Y_N(s)Y_N(s'))ds'ds \\
&= -r''_{Y_N}(0)\varphi^2(u)(ua_0)^2 \frac{1}{N\pi} \int_0^{N\pi} \int_{N\pi}^{2N\pi} r_{Y_N}(s'-s)ds'ds,
\end{aligned}$$

making the change of variable $s' - s = \tau$ we get

$$= -r''_{Y_N}(0)\varphi^2(u)(ua_0)^2 \frac{1}{N\pi}(\int_0^{\pi N} \tau r_{Y_N}(\tau)d\tau + \int_{\pi N}^{2\pi N} (2\pi N - \tau) r_{Y_N}(\tau)d\tau).$$

Since r_{Y_N} is periodic with period $2\pi N$:

$$\begin{aligned}
\mathcal{E} &= -r''_{Y_N}(0)\varphi^2(u)(ua_0)^2 \frac{1}{N\pi}(\int_0^{\pi N} \tau r_{Y_N}(\tau)d\tau - \int_{-\pi N}^{0} \tau r_{Y_N}(\tau)d\tau) \\
&= -r''_{Y_N}(0)\varphi^2(u)(ua_0)^2 \frac{2}{N\pi} \int_0^{\pi N} \tau r_{Y_N}(\tau)d\tau \to \frac{1}{3}\varphi^2(u)u^2,
\end{aligned}$$

using the same computation as for getting (3.10).

This implies that $\frac{1}{2}\mathbb{E}\big(I_{1,N}([0, N\pi]) + I_{1,N}([N\pi, 2N\pi])\big)^2 \to \frac{2}{3}\varphi^2(u)u^2$. Since the two random variables $I_{1,N}([0, N\pi])$ and $I_{1,N}([N\pi, 2N\pi])$ are jointly Gaussian this implies the convergence of $\frac{1}{\sqrt{2}}(I_{1,N}([0, N\pi]) + I_{1,N}([N\pi, 2N\pi]))$ in distribution.

Let us consider the term in the other chaos ($q \geq 2$).

$$\mathbb{E}\big(I_{q,N}([0, N\pi]) I_{q,N}([N\pi, 2N\pi])\big)$$

$$= -r''_{Y_N}(0)\varphi^2(u) \sum_{k_1=0}^{[\frac{q}{2}]} \sum_{k_2=0}^{[\frac{q}{2}]} \frac{H_{q-2k_1}(u)}{(q-2k_1)!} a_{2k_1} \frac{H_{q-2k_2}(u)}{(q-2k_2)!} a_{2k_2} \frac{1}{\pi N} \int_0^{\pi N} \int_{\pi N}^{2\pi N} G_{q,k_1,k_2,N}(s-s')dsds',$$

where we have put

$$G_{q,k_1,k_2,N}(s-s') = \mathbb{E}[H_{q-2k_1}(Y_N(0)) H_{2k_1}(\widetilde{Y}'_N(0)) H_{q-2k_2}(Y_N(s-s')) H_{2k_2}(\widetilde{Y}'_N(s-s'))].$$

A change of variables and Fubini's Theorem give

$$\frac{1}{\pi N}\int_0^{\pi N}\int_{\pi N}^{2\pi N} G_{q,k_1,k_2,N}(s-s')dsds'$$
$$= \frac{1}{N\pi}\Big(\int_0^{\pi N}\tau G_{q,k_1,k_2,N}(\tau)d\tau - \int_{\pi N}^{2\pi N}(2\pi N-\tau)G_{q,k_1,k_2,N}(\tau)d\tau\Big)$$
$$= \frac{1}{N\pi}\Big(\int_0^{\pi N}\tau G_{q,k_1,k_2,N}(\tau)d\tau + \int_0^{\pi N}\tau G_{q,k_1,k_2,N}(-\tau)d\tau\Big),$$

where this last equality is a consequence of periodicity and the change of variable $\tau = v + 2\pi N$ in the second integral. In this form we get

$$\Big|\frac{1}{\pi N}\int_0^{\pi N}\int_{\pi N}^{\pi N} G_{q,k_1,k_2,N}(s-s')dsds'\Big| \leq \frac{2}{N\pi}\int_0^{\pi N}\tau\tilde{G}_{q,k_1,k_2,N}(\tau)d\tau.$$

$\tilde{G}_{q,k_1,k_2,N}(\tau)$ has been defined in (3.12) and we also recall that this function is even. Moreover, it is plain that over any compact interval $[0,a]$ it holds

$$\lim_{N\to\infty}\frac{2}{N\pi}\int_0^a \tau\tilde{G}_{q,k_1,k_2,N}(\tau)d\tau = 0,$$

for the integral over $[a,\pi N]$ we use the bound (2.8) and Arcones' inequality. Thereby

$$\lim_{N\to\infty}\Big|\frac{2}{N\pi}\int_0^{\pi N}\tau G_{q,k_1,k_2,N}(\tau)d\tau\Big| = 0.$$

By using (ii) of Theorem 4.4, we get for $q\geq 2$

$$(I_{q,N}([0,N\pi]), I_{q,N}([N\pi,2N\pi])) \Rightarrow N(0,\sigma_q^2\mathcal{I}),$$

where $\mathcal{I}$ is the identity matrix in $\mathbb{R}^2$.

Defining

$$I_{q,N}([0,2N\pi]) = \frac{1}{\sqrt{2}}(I_{q,N}([0,N\pi]) + I_{q,N}([N\pi,2N\pi]),$$

it holds for each q that $I_{q,N}([0,2N\pi]) \Rightarrow N(0,\sigma_q^2)$, this asymptotic normality holds true also for $q=1$. The theorem now follows applying again (i) of Theorem 4.4 and the expansion (3.3). $\square$

5 Proof of Lemma 3.1

It suffices to prove that $N^{Y_N}_{[0,a]}(u)$ has a second moment which is bounded uniformly in N. Let $U^{Y_N}_{[0,a]}(u)$ be the number of up-crossings of the level u by $Y_N(t)$ in the interval $[0,a]$ i.e. the number of instants t such that $Y_N(t)=u; Y_N'(t)>0$. The Rolle theorem implies

$$N^{Y_N}_{[0,a]}(u) \leq 2U^{Y_N}_{[0,a]}(u) + 1.$$

So it suffices to give a bound for the second moment of the number up-crossings. Writing U for $U^{Y_N}_{[0,a]}(u)$ for short, we have

$$\mathbb{E}(U^2) = \mathbb{E}(U(U-1)) + \mathbb{E}(U).$$

We have already proven that the last term gives a finite contribution after normalization. For studying the first one we define the function $\theta_N(t)$ by

$$r_{Y_N}(\tau) = 1 + \frac{r''_{Y_N}(0)}{2}\tau^2 + \theta_N(\tau).$$

and we use the order two Rice formula and relation (4.14) of [2] to get

$$\begin{aligned}\mathbb{E}(U(U-1)) &= 2\int_0^a (a-\tau)\mathbb{E}[|Y_N'(0)Y_N'(\tau)|\,|\,Y_N(0)=Y_N(\tau)=u]p_{Y_N(0),Y_N(\tau)}(u,u)d\tau\\ &\le (const)a\int_0^a \frac{\theta_N'(\tau)}{\tau^2}d\tau.\end{aligned}$$

By a Taylor-Lagange expansion we obtain

$$r'_{Y_N}(\tau) = r'_{Y_N}(0) + \frac{1}{6N^5}\sum_{n=1}^{N} n^4\tau^3\cos(\theta(n,N)),$$

with $\theta(n,N)\le \tau/N$. We obtain that $|\theta_N'(\tau)|\le (const)\tau^3$, the constant being uniform in N. This gives the result. □

6 Proof of Theorem 4.1

Let D_m be a diagonal matrix with diagonal terms $d_{ii} = \lim_{t\to\infty}\mathrm{Var}(I_{q_i}([0,t]))$, where $I_q([0,t])$ has been defined in (3.5). Theorem 4.4 part (i), says that the random vector

$$(I_{q_1}([0,t]),\dots,I_{q_m}([0,t])) \Rightarrow N(0,D_m), \quad \text{when } t\to\infty$$

if an only if each $I_{q_i}([0,t])$ converges in distribution towards $N(0,d_{ii})$. We will prove this last assertion.

Let us begin with the term corresponding to the first chaos ($q=1$).

$$I_1([0,t]) = \sqrt{1/3}\varphi(u)u\,a_0\frac{1}{\sqrt{t}}\int_0^t X(s)ds = \sqrt{1/3}\,\frac{e^{-\frac{u^2}{2}}u}{\pi}\frac{1}{\sqrt{t}}\int_0^t X(s)ds,$$

the random variable $I_1([0,t])$ is Gaussian and we have already proven that its variance converge thus we get

$$\begin{aligned} I_1([0,t]) &\Rightarrow N\Big(0,\frac{e^{-u^2}u^2}{3\pi^2}\,2\int_0^\infty \frac{\sin\tau}{\tau}d\tau\Big) = N\Big(0,e^{-u^2}\frac{u^2}{3}\Big)\\ &= N\Big(0,\frac{2}{3}u^2\phi^2(u)\pi\Big), \end{aligned} \tag{6.1}$$

when $t\to\infty$.

For the other chaos ($q>1$) we can adapt the proof of the cited references, [2] and [7], those proofs are inspired in the seminal work of Malevich [8] see also [3] and [4]. Furthermore the hypothesis of this last work consist in demanding the convergence of integrals of the covariances, thus they are similar to those used in our work.

The process $X(t)$ has $f(\lambda)=\frac{1}{2}\mathbb{1}_{[-1,1]}(\lambda)$ as spectral density, if we symmetrize the spectrum. Let β be an even function with $\int_{-\infty}^{\infty}|\lambda|^j|\beta(\lambda)|d\lambda<\infty, j=1,2$ and such that its Fourier Transform has support in $[-1,1]$. By defining $\beta_\varepsilon=\frac{1}{\varepsilon}\beta(\frac{\cdot}{\varepsilon})$ and putting $f_\varepsilon(\lambda)=f*\beta_\varepsilon(\lambda)$ the following process

$$X_\varepsilon(t) = \int_0^\infty \cos(t\lambda)\sqrt{f_\varepsilon(\lambda)}dB_1(\lambda) + \int_0^\infty \sin(t\lambda)\sqrt{f_\varepsilon(\lambda)}dB_2(\lambda), \tag{6.2}$$

satisfies

$$\begin{aligned} r_\varepsilon(\tau) := \mathbb{E}\big[X_\varepsilon(\tau)X_\varepsilon(0)\big] &= \int_0^\infty \cos(\tau\lambda)f_\varepsilon(\lambda)d\lambda\\ &= \frac{1}{2}\int_{-\infty}^\infty \cos(t\lambda)f_\varepsilon(\lambda)d\lambda = \frac{1}{2}r(\tau)\hat{\beta}(\varepsilon\tau). \end{aligned} \tag{6.3}$$

Thus X_ε is a Gaussian $\frac{1}{\varepsilon}$-dependent process, this process has variance one if $\hat{\beta}(0) = 2$, which is always possible. Moreover,

$$r'_\varepsilon(\tau) = \frac{1}{2}(r'(\tau)\hat{\beta}(\varepsilon\tau) + r(\tau)\varepsilon\hat{\beta}'(\varepsilon\tau))$$

and

$$r''_\varepsilon(\tau) = \frac{1}{2}(r''(\tau)\hat{\beta}(\varepsilon\tau) + 2r'(\tau)\varepsilon\hat{\beta}'(\varepsilon\tau) + r(\tau)\varepsilon^2\hat{\beta}''(\varepsilon\tau)).$$

These functions have bounded support and converge to r, r' and r'' respectively, functions that belong to $L^2(\mathbb{R})$. Recalling the set of indexes $\mathcal{J}_q$ of Lemma 3.4, we get by using Dominate Convergence Theorem

$$\lim_{\varepsilon\to 0}\sum_{\mathcal{J}_q}\int_0^{\frac{1}{\varepsilon}}\Big|r_\varepsilon(\tau)\Big|^{2q-(2k_1+2k_1)-h_1}\Big|\frac{r'_\varepsilon(\tau)}{\sqrt{-r''_\varepsilon(0)}}\Big|^{2h_1}\Big|\frac{r''_\varepsilon(\tau)}{r''_\varepsilon(0)}\Big|^{2k_1+2k_2-h_1}d\tau$$

$$= \sum_{\mathcal{J}_q}\int_0^{\infty}\Big|r(\tau)\Big|^{2q-(2k_1+2k_1)-h_1}\Big|\frac{r'(\tau)}{\sqrt{-r''(0)}}\Big|^{2h_1}\Big|\frac{r''(\tau)}{r''(0)}\Big|^{2k_1+2k_2-h_1}d\tau.$$

The same result holds dropping the absolute value in the integrant. Let us define

$$I_{q,\varepsilon}([0,t]) = \sqrt{-r''_{X_\varepsilon}(0)}\frac{1}{\sqrt{t}}\int_0^t f_q(u, X_\varepsilon(s), \frac{X'_\varepsilon(s)}{\sqrt{-r''_{X_\varepsilon}(0)}})ds. \tag{6.4}$$

The above result and Lemma 3.4, allow us to conclude that

$$\lim_{t\to\infty}\mathbb{E}[I_{q,\varepsilon}([0,t])]^2 = \frac{1}{\pi}\sigma_q^2(u).$$

We shall now to consider the convergence for the covariances.

$$\rho_\varepsilon(\tau) = \mathbb{E}[X_\varepsilon(\tau)X(0)] = \int_0^1 \cos(\tau\lambda)\sqrt{f_\varepsilon(\lambda)}d\lambda \to r(\tau), \tag{6.5}$$

$$\rho'_\varepsilon(\tau) = \mathbb{E}[X'_\varepsilon(\tau)X(0)] = -\int_0^1 \lambda\sin(\tau\lambda)\sqrt{f_\varepsilon(\lambda)}d\lambda \to -r'(\tau), \tag{6.6}$$

$$\rho''_\varepsilon(\tau) = \mathbb{E}[X'_\varepsilon(\tau)X'(0)] = -\int_0^1 \lambda^2\cos(\tau\lambda)\sqrt{f_\varepsilon(\lambda)}d\lambda \to -r''(\tau), \tag{6.7}$$

when $\varepsilon\to 0$. Moreover,

$$\begin{aligned}\rho_\varepsilon(\tau) &= \frac{1}{2}\int_{-\infty}^{\infty}\cos(\tau\lambda)\sqrt{f_\varepsilon(\lambda)}\mathbb{1}_{[-1,1]}(\lambda)d\lambda\\ &= \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty}\cos(\tau\lambda)\sqrt{f_\varepsilon(\lambda)}\sqrt{\frac{1}{2}\mathbb{1}_{[-1,1]}(\lambda)}d\lambda.\end{aligned}$$

By using Fatou, Parseval equality and, the fact that $f_\varepsilon(\lambda)\to 2f(\lambda)$ in $L^2(\mathbb{R})$, we obtain easily

$$\int_0^\infty |r(\tau)|^2d\tau \le \limsup_{\varepsilon\to 0}\int_0^\infty|\rho_\varepsilon(\tau)|^2d\tau = \limsup_{\varepsilon\to 0}\frac{1}{2}\int_{-\infty}^{\infty}|\rho_\varepsilon(\tau)|^2d\tau$$

$$= \limsup_{\varepsilon\to 0}\frac{1}{4\pi}\int_{-\infty}^{\infty}|\sqrt{f_\varepsilon(\lambda)}\sqrt{\frac{1}{4}\mathbb{1}_{[-1,1]}(\lambda)}|^2d\lambda = \int_0^\infty|r(\tau)|^2d\tau.$$

Thus

$$\lim_{\varepsilon\to 0}\int_0^\infty |\rho_\varepsilon(\tau)|^2 d\tau = \int_0^\infty |r(\tau)|^2 d\tau. \tag{6.8}$$

In the same form we get

$$\lim_{\varepsilon\to 0}\int_0^\infty |\rho_\varepsilon'(\tau)|^2 d\tau = \int_0^\infty |r(\tau)|^2 d\tau \text{ and } \lim_{\varepsilon\to 0}\int_0^\infty |\rho_\varepsilon''(\tau)|^2 d\tau = \int_0^\infty |r''(\tau)|^2 d\tau. \tag{6.9}$$

We will compute now

$$\limsup_{\varepsilon\to 0}\limsup_{t\to\infty}\mathbb{E}[I_{q,\varepsilon}([0,t]) - I_q([0,t])]^2 = \frac{2}{\pi}\,\sigma_q^2(u) - 2\lim_{\varepsilon\to 0}\lim_{t\to\infty}\mathbb{E}[I_{q,\varepsilon}([0,t])I_q([0,t])].$$

This limit vanish if we can prove that the third term tends to $\frac{2}{\pi}\,\sigma_q^2(u)$ also. But this is a consequence again of Lemma 3.4, (6.8) and (6.9) cf. [7]. Let us sketch the proof. Defining

$$d_{q,2k}(u) = \frac{H_{q-2k}(u)}{(q-2k)!}\,a_{2k},$$

we have

$$\begin{aligned}\mathbb{E}\big[I_{q,\varepsilon}([0,t]), I_q([0,t])\big] &= -r_X''(0)\varphi^2(u)\sum_{k_1,k_2} d_{q,2k_1}(u)d_{q,2k_1}(u)\\ &\times\frac{1}{t}\int_0^t\int_0^t \mathbb{E}[H_{q-2k_1}(X^\varepsilon(s))H_{2k_1}(\frac{X'^\varepsilon(s)}{\sqrt{-r_{X^\varepsilon}''(0)}})H_{q-2k_1}(X(s'))H_{2k_1}(\widetilde{X}'(s'))]dsds'.\end{aligned}$$

The integral is by Lemma 3.4 equal to

$$\begin{aligned}\sum_{L_q} V_{q,2k_1,2k_2}\frac{1}{t}\int_0^t\int_0^t (\mathbb{1}_{\{s>s'\}}[\frac{-\rho_\varepsilon'(s-s')}{\sqrt{-\rho_\varepsilon''(0)}}]^{2h_1} + \mathbb{1}_{\{s'>s\}}[\frac{-\rho_\varepsilon'(s'-s)}{\sqrt{-r_X''(0)}}]^{2h_1})\\ \times\rho_\varepsilon(|s-s'|)^{2q-(2k_1+2k_1)-h_1}[\frac{-\rho_\varepsilon''(|s-s'|)}{\sqrt{-\rho_\varepsilon''(0)}\sqrt{-r''(0)}}]^{2k_1+2k_2-h_1}dsds',\end{aligned} \tag{6.10}$$

where L_q is a set of indexes and $V_{q,2k_1,2k_2}$ are fixed constant. Given that $q>1$ and by using that the functions ρ_ε and its derivatives converge in L^2 towards their pointwise limit, it yields that this sum converges towards

$$2\int_0^\infty \mathbb{E}[H_{q-2k_1}(X(0))H_{2k_1}(\frac{X'(0)}{\sqrt{-r_X''(0)}})H_{q-2k_1}(X(\tau))H_{2k_1}(\frac{X'(\tau)}{\sqrt{-r_X''(0)}})]d\tau.$$

Thus the result follows.

The $\frac{1}{\varepsilon}$-dependence entails that $I_{q,\varepsilon}([0,t])$ is asymptotically Gaussian and the proved proximity in L^2 allows concluding the same for $I_{q,\varepsilon}([0,t])$, with asymptotic variance $\frac{1}{\pi}\,\sigma_q^2(u)$. The CLT for the crossings of X follows from the expansion

$$N_{[0,t]}^X(u) - \mathbb{E}[N_{[0,t]}^X(u)] = \sum_{q=1}^\infty I_q\big([0,t]\big),$$

the asymptotic independence of the Gaussian limit in each chaos and the convergence of the variance.

7 Notation table

$X_N(t)$	see (1.1)
r_{X_N}	covariance of $X_N(t)$
$X(t)$	stat. process with cov. $r(t) = \sin(t)/t$
$Y_N(t)$	$X_N(t/N)$
r_{Y_N}	covariance of $Y_N(t)$
$N^{X_N}_{[0,t]}(u)$	Number of crossings of level u by $X_N(t)$ on $[0,t]$.
$U^{X_N}_{[0,t]}(u)$	Number of up-crossings of level u by $X_N(t)$ on $[0,t]$
$\widetilde{X}(t)$	$X'(t)/(\sqrt{1/3})$
$\widetilde{Y}_N(t)$	$Y'_N(t)/(\sqrt{-r''_{Y_N}(0)})$
$f_q([0,t])$	see (3.4)
$I_q([0,t])$	see (3.5)
$I_{q,N}([0,t])$	see (3.7)
$X_\varepsilon(t)$	see (6.2)
$r_\varepsilon(\tau)$	see (6.3)
$I_{q,\varepsilon}([0,t])$	see (6.4)
$\rho_\varepsilon(\tau)$; $\rho'_\varepsilon(\tau)$; $\rho''_\varepsilon(\tau)$	see (6.5) ; (6.6) ; (6.7)

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Acknowledgments. We thank an anonymous referee for a very careful reading of the first version of this paper..