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Correlation-length bounds, and estimates for intermittent islands in parabolic SPDEs*

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Abstract

We consider the nonlinear stochastic heat equation in one dimension. Under some conditions on the nonlinearity, we show that the "peaks" of the solution are rare, almost fractal like. We also provide an upper bound on the length of the "islands", the regions of large values. These results are obtained by analyzing the *correlation length* of the solution.

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1 Introduction

Let $\dot{W} := \{\dot{W}_t(x)\}_{t>0,x\in\mathbf{R}}$ denote space-time white noise, and consider the nonlinear stochastic heat equation,

$$\frac{\partial}{\partial t}u_t(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u_t(x) + \sigma(u_t(x))\dot{W}_t(x), \tag{1.1}$$

for $(t\,,x) \in (0\,,\infty)\times\mathbf{R}$, subject to $u_0(x) := 1$ for all $x \in \mathbf{R}$. *Throughout we consider only the case that $\sigma : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous.* In that case, the theory of Walsh [13] explains the meaning of (1.1) and shows that (1.1) has a unique strong solution that is continuous for all $(t\,,x) \in [0\,,\infty)\times\mathbf{R}$. The goal of this article is to make some observations about the geometric structure of the random function $x \mapsto u_t(x)$ for $t > 0$ fixed.

Remark 1.1. *Since $u_0(x)$ is constant, it is possible to prove that the law of $u_t(x)$ does not depend on x [4].*

Remark 1.2. *We have chosen the initial condition $u_0(x) := 1$ to simplify the exposition. All of the following results will continue to hold—after we make a few minor modifications on the assumptions—for measurable functions u_0 that are bounded away from 0 and ∞; that is $0 < \inf_{x\in\mathbf{R}} u_0(x) < \sup_{x\in\mathbf{R}} u_0(x) < \infty$.*

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Although we will have some results that are valid for (1.1) in general, we are mainly motivated by the following two special cases of Eq. (1.1):

Case 1. There exists $q > 0$ such that $\sigma(z) = qz$ for all $z \in \mathbf{R}$. In this case, (1.1) is known as the *parabolic Anderson model*;

Case 2. $0 < \inf_{z\in\mathbf{R}} \sigma(z) \leqslant \sup_{z\in\mathbf{R}} \sigma(z) < \infty$. An important special case of this case occurs when σ is a constant; then (1.1) is the linear SPDE whose solution is a stationary *Gaussian* process.

Before we proceed, we mention that the *parabolic Anderson model* of **Case 1** has been well studied in literature, in part because of its connection to the *KPZ equation* [1, 9]. In fact, $\log u$ is the "*Cole–Hopf* solution" to the KPZ equation.

Let $\log_+(x) := \log(x \vee \mathrm{e})$ and define, for all $R, \alpha > 0$,

$$g_\alpha(R) := \begin{cases} \exp\left(\alpha(\log_+ R)^{2/3}\right) & \text{in Case 1,} \\ \alpha(\log_+ R)^{1/2} & \text{in Case 2.} \end{cases} \tag{1.2}$$

["g" stands for "gauge."] Our recent effort [3] implies that, for both Cases 1 and 2, for all $t > 0$ fixed there exist $\alpha_*, \alpha^* > 0$ such that with probability one,

$$\limsup_{R\to\infty} \frac{u_t(R)}{g_\alpha(R)} = \begin{cases} 0 & \text{if } \alpha > \alpha^*, \\ \infty & \text{if } \alpha \in (0\,, \alpha_*). \end{cases} \tag{1.3}$$

In other words, the "exceedence set,"

$$E_\alpha(R) := \{x \in [0\,, R] :\; u_t(x) \geqslant g_\alpha(R)\}\,, \tag{1.4}$$

is a.s. empty for all $R \gg 1$ if $\alpha > \alpha^*$; and $E_\alpha(R)$ is a.s. unbounded for all $R > 1$ if $\alpha \in (0\,, \alpha_*)$.

Next, let us observe that the rescaled version $R^{-1}E_\alpha(R)$ of $E_\alpha(R)$ is a random subset of $[0\,, 1]$. One of our original aims was to show that $R^{-1}E_\alpha(R)$ "converges" to a random fractal of Hausdorff dimension $d(\alpha) \in (0\,, 1)$ as $R \to \infty$ when α is sufficiently small. So far we have not been able to do this, though as we will soon see we are able to furnish strong evidence in favor of this claim.

If $R^{-1}E_\alpha(R)$ *did* look like a random fractal subset of $[0\,, 1]$ with Hausdorff dimension $d(\alpha) \in (0\,, 1)$, then we would expect its Lebesgue measure to behave as $R^{-d(\alpha)+o(1)}$ as $R \to \infty$. Or stated in more precise terms, we would expect that if α is sufficiently small, then

$$\lim_{R\to\infty} \frac{\log |E_\alpha(R)|}{\log R} = 1 - d(\alpha) \qquad \text{a.s.} \tag{1.5}$$

[This is an example of the so-called "codimension argument" in fractal analysis.] The first theorem of this paper comes close to proving this last assertion.

Theorem 1.3. *If either Case 1 or Case 2 holds, then there exists $\alpha_0 > 0$ such that for all $\alpha \in (0\,, \alpha_0)$,*

$$0 < \liminf_{R\to\infty} \frac{\log |E_\alpha(R)|}{\log R} \leqslant \limsup_{R\to\infty} \frac{\log |E_\alpha(R)|}{\log R} < 1 \qquad \textit{a.s.} \tag{1.6}$$

The results of [3] imply that $E_\alpha(R)$ is eventually empty a.s. when $\alpha > \alpha^*$. Therefore, α_0 cannot be made to be arbitrarily large.

Choose and fix a time $t > 0$. Given two numbers $0 < a < b$, we say that a closed interval $I \subset \mathbf{R}_+$ is an $(a\,, b)$-*island* [at time t] if:

1. $u_t(\inf I) = u_t(\sup I) = a$;
2. $u_t(x) > a$ for all $x \in \text{int}(I)$; and
3. $\sup_{x\in I} u_t(x) > b$.

Define

$$J_t(a\,,b\,;R) := \text{the length of the largest } (a\,,b)\text{-island } I \subset [0\,,R]. \tag{1.7}$$

The following result shows that the relative length of the largest "tall island" in $[0\,,R]$—also known as "intermittency islands"—is vanishingly small as $R \to \infty$. Let us expand on the idea of intermittency further.

Speaking informally and merely phenomenologically, we can think of *intermittency* in the present context as the appearance of rare and very tall peaks in the space-time profile of the solution u to the stochastic heat equation. This picture is made more precise using the concept of *mathematical intermittency* [1, 7] which is a certain growth condition on the rate of temporal growth of the moments of the solution. An application of the ergodic theorem then shows that mathematical intermittency implies that most of the contribution to successive moments is from decreasingly smaller regions (the so called *intermittency islands*) in space [1]. Intermittency is an asymptotic (in time) property, but the following result implies that (relatively small) islands begin to form at quite early stages in time. [In physical terms, our work shows that the stochastic heat equation exhibits a great deal of "hysteresis."]

Theorem 1.4. *Assume that $\sigma(1) \neq 0$. Then for every $t > 0$ and all $(a\,,b)$ such that $1 < a < b$ and $\mathrm{P}\{u_t(0) > b\} > 0$,*

$$\limsup_{R\to\infty} \frac{J_t(a\,,b\,;R)}{|\log R|^2} < \infty \quad \textit{a.s.} \tag{1.8}$$

If Case 2 occurs, then the preceding can be improved to the following:

$$\limsup_{R\to\infty} \frac{J_t(a\,,b\,;R)}{\log R \cdot |\log\log R|^{3/2}} < \infty \quad \textit{a.s.} \tag{1.9}$$

Let us make a few remarks before we continue our introduction.

Remark 1.5.
1. *During the course of the proof of this theorem, we will establish the existence of numbers $b > 1$ that satisfy $\mathrm{P}\{u_t(0) > b\} > 0$; therefore, the result always has content.*
2. *The condition $\sigma(1) \neq 0$ is necessary. Indeed, if $\sigma(1)$ were zero, then $u_t(x) = 1$ for all $t > 0$ and $x \in \mathbf{R}$ [this is because $u_0 \equiv 1$].* □

Theorems 1.3 and 1.4 both rely on a fairly good estimation of "correlation length" for the random field $x \mapsto u_t(x)$. There are many ways one can understand the loose term, "correlation length." The following is a rigorous definition that suits the purposes of the present work. As far as we know, this definition is new.

Let $\{X_x\}_{x\in\mathbf{R}}$ be a random field on $(\Omega\,,\mathcal{F},\mathrm{P})$, and let $\mathcal{L}(\ell)$ denote the collection of all weakly stationary random fields $\{Y_x\}_{x\in\mathbf{R}}$ on $(\Omega\,,\mathcal{F},\mathrm{P})$ such that Y has "lag" ℓ; that is, Y_z is independent of $(Y_{x_i})_{i=1}^N$ for all $z, x_1, \ldots, x_N \in \mathbf{R}$ that satisfy $\min_{1\leqslant j\leqslant N} |z - x_j| \geqslant \ell$. Then, the *correlation length* of X is the function

$$L_X(\epsilon\,;\delta) := \inf\left\{\ell > 0 : \inf_{Y\in\mathcal{L}(\ell)} \sup_{x\in\mathbf{R}} \mathrm{P}\{|X_x - Y_x| > \delta\} < \epsilon\right\}, \tag{1.10}$$

where $\epsilon, \delta > 0$ can be thought of as *fidelity* parameters. Informally speaking, when we find $L_X(\epsilon\,;\delta)$, we seek to find the smallest lag-length ℓ for which there exists a coupling of X with a lag-ℓ process Y, such that the coupling is good to within δ units with probability at least $1-\epsilon$.

The following is the main technical result of this paper. It states that the correlation length of the solution to (1.1) is logarithmic in the fidelity parameter ϵ; and the fidelity parameter δ can be as small as $\exp(-K\,|\log\epsilon|^{2/3})$ for a universal constant $K = K(t) \in (0\,,\infty)$.

Theorem 1.6. *For every $t>0$, there exists a positive and finite constant $K := K(t)$, such that as $\epsilon \downarrow 0$,*

$$L_{u_t}\left(\epsilon\,;\mathrm{e}^{-K|\log\epsilon|^{2/3}}\right) = O\left(|\log\epsilon|\right). \tag{1.11}$$

If σ is a bounded function, then in fact there exists $\theta \in (0\,,1)$ such that

$$L_{u_t}\left(\epsilon\,;\left[\frac{\log|\log\epsilon|}{|\log\epsilon|}\right]^{\theta}\right) = O\left([\log|\log\epsilon|]^{3/2}\right) \qquad (\epsilon\downarrow 0). \tag{1.12}$$

Our notion of correlation length can be stronger than other, somewhat simpler, notions of this general type. For instance, consider the following: Let $\{X_x\}_{x\in\mathbf{R}}$ be a random field, and define $L^*_X(\epsilon\,;\delta)$ to be the smallest $\ell>0$ for which we can find—on *some* probability space—a coupling $(X^*\,,Y^*)$, where X^* has the same law as X and Y^* has lag ℓ, and $\sup_{x\in\mathbf{R}}\mathrm{P}\{|X^*_x - Y^*_x| > \delta\} < \epsilon$. Since $L^*_X(\epsilon\,;\delta) \leqslant L_X(\epsilon\,;\delta)$, Theorem 1.6 readily implies that

$$L^*_{u_t}\left(\epsilon\,;\mathrm{e}^{-K|\log\epsilon|^{2/3}}\right) = O(|\log\epsilon|) \qquad (\epsilon\downarrow 0). \tag{1.13}$$

Open Problem. Is it true that $L^*_{u_t}(\epsilon\,;0) = O(|\log\epsilon|)$? It is easy to see that this is equivalent to asking whether or not $x \mapsto u_t(x)$ is exponentially mixing.

Although we do not know how to prove that $x\mapsto u_t(x)$ is exponentially mixing, we are able to prove that the coupling in Theorem 1.6 is "good on all scales." In order to interpret this, note that if $\ell := L_{u_t}(\epsilon;\,\delta)$ then we can basically approximate u_t well enough by a random field Y in $\mathcal{L}(\ell)$ such that Y replicates u_t to within δ units. According to (1.11) this can be done—with $\ell = O(|\log\epsilon|)$—with a value of δ that has the form $\exp\{-K\,|\log\epsilon|^{2/3}\}$ for some $K := K(t)$. Thus, for example, if we wanted to know how small $x\mapsto u_t(x)$ can possibly get, then we could study instead Y provided that "how small" means "$\exp\{-K\,|\log\epsilon|^{2/3}\}$ or more." Our next result shows that this notion of "how small" is generic [and not at all a restriction]. Our proof borrows several important ideas from a paper by Mueller and Nualart [12].

Theorem 1.7. *If $\sigma(0)=0$, then for every $t,a>0$ and $x\in\mathbf{R}$,*

$$\lim_{\epsilon\downarrow 0}\frac{1}{|\log\epsilon|}\log\mathrm{P}\left\{u_t(x)\leqslant \mathrm{e}^{-a|\log\epsilon|^{2/3}}\right\} = -\infty. \tag{1.14}$$

Throughout this paper, "log" denotes the natural logarithm, $p_t(x)$ denotes the standard heat kernel for $(1/2)\Delta$,

$$p_t(x) := \frac{\mathrm{e}^{-x^2/(2t)}}{(2\pi t)^{1/2}} \qquad (t>0\,,\ x\in\mathbf{R}), \tag{1.15}$$

and $\|Z\|_k := \{\mathrm{E}(|Z|^k)\}^{1/k}$ denotes the $L^k(\mathrm{P})$-norm of a random variable $Z\in L^k(\mathrm{P})$ $(k\in[1\,,\infty))$.

Let us conclude the Introduction with a brief outline of the paper. In Section 2 we prove Theorem 1.6, whose corollaries, Theorems 1.3 and 1.4, are proved respectively in §3 and §4. In a final Section 5 we state and prove an improved version of Theorem 1.7, which might turn out to be a first step in answering the mentioned Open Problem.

2 Proof of Theorem 1.6

First of all, recall that the solution to the stochastic PDE (1.1) is the unique continuous solution to the following random evolution equation [13]:

$$u_t(x) = 1 + \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)\sigma\left(u_s(y)\right)\, W(\mathrm{d}s\,\mathrm{d}y). \tag{2.1}$$

For all $\beta > 0$, let $U^{(\beta)}$ solve the following closely-related stochastic evolution equation:

$$U_t^{(\beta)}(x) = 1 + \int_{(0,t)\times[x-\sqrt{\beta t},x+\sqrt{\beta t}]} p_{t-s}(y-x)\sigma\left(U_s^{(\beta)}(y)\right)\, W(\mathrm{d}s\,\mathrm{d}y). \tag{2.2}$$

It has been observed in [3] that the same methods as in [13] can be used to show that there exists a unique continuous random field $U^{(\beta)}$ that solves the preceding. The following result of [3] shows that $U^{(\beta)} \approx u$ if β is large.

Lemma 2.1 ([3, Lemma 4.2])**.** *For every $T > 0$ there exists finite and positive constants a_i $[i = 1, 2]$ such that for all $\beta > 0$, and for all real numbers $k \in [1\,, \infty)$,*

$$\sup_{\substack{t\in(0,T)\\ x\in\mathbf{R}}} \mathrm{E}\left(\left|u_t(x) - U_t^{(\beta)}(x)\right|^k\right) \leqslant a_1^k \mathrm{e}^{a_1 k\left[k^2 - a_2\beta\right]}. \tag{2.3}$$

It is easy to adapt the arguments of [3] to improve the preceding in the case that σ is bounded. Because all of the key steps are already in Ref. [3], we state the end result without proof.

Lemma 2.2. *Suppose that σ is bounded. Then for every $T > 0$ there exists finite and positive constants $\bar{a}_i$ $[i = 1, 2]$ such that for all $\beta > 0$, and for all real numbers $k \in [1\,, \infty)$,*

$$\sup_{\substack{t\in(0,T)\\ x\in\mathbf{R}}} \mathrm{E}\left(\left|u_t(x) - U_t^{(\beta)}(x)\right|^k\right) \leqslant \bar{a}_1^k \mathrm{e}^{\bar{a}_1 k[\log k - \bar{a}_2\beta]}. \tag{2.4}$$

The process $U^{(\beta)}$ is useful only as a first step in a better coupling, which we describe next. Define $U_t^{(\beta,0)}(x) := 1$. Then, once $U^{(\beta,l)}$ is defined [for some $l \geqslant 0$] we define $U^{(\beta,l+1)}$ as follows:

$$U_t^{(\beta,\,l+1)}(x) := 1 + \int_{(0,t)\times[x-\sqrt{\beta t},x+\sqrt{\beta t}]} p_{t-s}(y-x)\sigma\left(U_s^{(\beta,\,l)}(y)\right)\, W(\mathrm{d}s\,\mathrm{d}y). \tag{2.5}$$

In other words, $U^{(\beta,l)}$ is the l^{th} step in the Picard-iteration approximation to $U^{(\beta)}$. The following result of [3] tells us that if l is large then $U^{(\beta,l)} \approx U^{(\beta)}$.

Lemma 2.3 ([3, Eq. (4.22) & Lemma 4.4])**.** *For every $T > 0$ there exists finite and positive constants b_1 and b_2 such that for all $\beta > 0$, all integers $n \geqslant 0$, and for all real* *numbers* $k \in [1\,,\infty)$*,*

$$\sup_{\substack{t\in(0,T)\\x\in\mathbf{R}}} \mathrm{E}\left(\left|U_t^{(\beta)}(x) - U_t^{(\beta,n)}(x)\right|^k\right) \leqslant b_1^k \mathrm{e}^{b_1 k\left[k^2 - b_2 n\right]}. \tag{2.6}$$

Furthermore, $U_t^{(\beta,n)} \in \mathcal{L}(2n\sqrt{\beta t})$ *for all* $\beta, t > 0$ *and* $n \geqslant 0$*.*

Once again, we state—without proof—an improvement in the case that σ is bounded.

Lemma 2.4. *Suppose that* σ *is bounded. Then, for every* $T > 0$ *there exists finite and positive constants* $\bar{b}_1$ *and* $\bar{b}_2$ *such that for all* $\beta > 0$*, all integers* $n \geqslant 0$*, and for all real numbers* $k \in [1\,,\infty)$*,*

$$\sup_{\substack{t\in(0,T)\\x\in\mathbf{R}}} \mathrm{E}\left(\left|U_t^{(\beta)}(x) - U_t^{(\beta,n)}(x)\right|^k\right) \leqslant \bar{b}_1^k \mathrm{e}^{\bar{b}_1 k\left[\log k - \bar{b}_2 n\right]}. \tag{2.7}$$

Now we are ready to establish Theorem 1.6.

Proof of Theorem 1.6. Choose and fix $t > 0$. The final assertion of Lemma 2.3 implies that the process $x \mapsto Y_x := U_t^{(\beta,n)}(x)$ is in $\mathcal{L}(2n\sqrt{\beta t})$ for every $\beta > 0$ and $n \geqslant 0$. Therefore, we may apply Lemmas 2.1 and 2.3 in conjunction with Chebyshev's inequality to see that for all $k \in [1\,,\infty)$ and $\delta > 0$,

$$\inf_{Y\in\mathcal{L}(2n\sqrt{\beta t})} \sup_{x\in\mathbf{R}} \mathrm{P}\left\{|u_t(x) - Y_x| > \delta\right\} \leqslant (2c_1/\delta)^k \mathrm{e}^{c_1 k[k^2 - c_2(\beta\wedge n)]}, \tag{2.8}$$

where $c_1 := \max\{a_1\,,b_1\}$, $c_2 := \min\{(a_1a_2)/c_1,\, (b_1b_2)/c_1\}$ do not depend on $(\beta\,,n\,,k\,,\delta)$. Now we choose $\beta = n := 1 + \lfloor(2/c_2)k^2\rfloor$ in order to find that there exists $\bar{c} \in (1\,,\infty)$ such that for all k sufficiently large,

$$\inf_{Y\in\mathcal{L}(\bar{c}k^3)} \sup_{x\in\mathbf{R}} \mathrm{P}\left\{|u_t(x) - Y_x| > \delta\right\} \leqslant \delta^{-k}\mathrm{e}^{-2k^3/\bar{c}}. \tag{2.9}$$

Because $\bar{c}$ does not depend on δ, we can set $\delta := \exp(-k^2/\bar{c})$ to deduce from the preceding that for every $\nu \in (0\,,1)$ fixed,

$$L_{u_t}\left(\mathrm{e}^{-k^3/\bar{c}}\,;\mathrm{e}^{-k^2/\bar{c}}\right) \leqslant \bar{c}k^3, \tag{2.10}$$

uniformly for all k sufficiently large. It follows that if $\epsilon := \exp(-k^3/\bar{c})$, then

$$L_{u_t}\left(\epsilon\,;\exp\left\{-\frac{|\log\epsilon|^{2/3}}{\bar{c}^{1/3}}\right\}\right) \leqslant \bar{c}^2\,|\log\epsilon|\,. \tag{2.11}$$

In the case that ϵ is a general positive number, (1.11) follows from the preceding and a simple monotonicity argument.

In the case that σ is bounded, we proceed similarly as in the general case, but apply Lemmas 2.2 and 2.4 in place of Lemmas 2.1 and 2.3, and then select the various parameters accordingly. In this way, we find the following improvement to (2.8) in the case that σ is bounded:

$$\inf_{Y\in\mathcal{L}(2n\sqrt{\beta t})} \sup_{x\in\mathbf{R}} \mathrm{P}\left\{|u_t(x) - Y_x| > \delta\right\} \leqslant (2c_1'/\delta)^k \mathrm{e}^{c_1' k[\log k - c_2'(\beta\wedge n)]}, \tag{2.12}$$

where c_1', c_2' do not depend on $(\beta\,,n\,,k\,,\delta)$. Now we choose $\beta = n := 1 + \lfloor(2/c_2')\log k\rfloor$ in order to deduce the existence of a constant $c'' \in (1\,,\infty)$ such that for all suffiently large

k,

$$\inf_{Y\in\mathcal{L}\left(c''[\log k]^{3/2}\right)} \sup_{x\in\mathbf{R}} \mathrm{P}\left\{|u_t(x)-Y_x|>\delta\right\} \leqslant \delta^{-k}\mathrm{e}^{-2k\log k/c''}. \tag{2.13}$$

This is our improvement to (2.9) in the case that σ is bounded. In particular, for all k large,

$$\inf_{Y\in\mathcal{L}(c''[\log k]^{3/2})} \sup_{x\in\mathbf{R}} \mathrm{P}\left\{|u_t(x)-Y_x|>k^{-1/c''}\right\} \leqslant \mathrm{e}^{-k\log k/c''}. \tag{2.14}$$

If $\epsilon := \exp\{-(1/c'')k\log k\}$ is small, then $k\approx c''\,|\log\epsilon|\,/\log|\log\epsilon|$ and (1.12) follows from (2.14) for every $\theta\in(0\,,1/c'')$. We apply monotonicity in order to deduce (1.12) for general [small] ϵ. □

3 Proof of Theorem 1.3

Before we proceed with the proof we need a few technical results. Suppose $Y\in\mathcal{L}(\ell)$ for some $\ell>0$, and define, for all integers $n\geqslant 1$ and real numbers $\alpha>0$,

$$\mathfrak{Y}_\alpha(n) := \int_0^{n\ell} \mathbf{1}_{\{Y_x\geqslant \bar G((n\ell)^{-\alpha})\}}\,\mathrm{d}x, \tag{3.1}$$

where

$$\bar G(a) := \sup\left\{b>0:\ \mathrm{P}\{Y_0\geqslant b\}\geqslant a\right\}. \tag{3.2}$$

Lemma 3.1. *Assume that $Y\in\mathcal{L}(\ell)$ for some $\ell\geqslant 1$. Then for every integer $k\geqslant 3$ there exists a universal constant $C_k\in(0\,,\infty)$ such that for all $\alpha\in(0\,,1/2)$ and $n\geqslant 2$,*

$$\left\|\frac{\mathfrak{Y}_\alpha(n)}{\mathrm{E}\mathfrak{Y}_\alpha(n)}-1\right\|_k \leqslant C_k\cdot\frac{\ell^\alpha}{n^{\frac12-\alpha}}. \tag{3.3}$$

Proof. We can write

$$\mathfrak{Y}_\alpha(n) := \sum_{j=0}^{n-1} Z_j, \quad\text{where}\quad Z_j := \int_{j\ell}^{(j+1)\ell} \mathbf{1}_{\{Y_x\geqslant \bar G((n\ell)^{-\alpha})\}}\,\mathrm{d}x. \tag{3.4}$$

Define

$$S_n^{(\mathrm{o})} := \sum_{0\leqslant 2j+1\leqslant n-1}\left(Z_{2j+1}-\mathrm{E}Z_{2j+1}\right),\quad S_n^{(\mathrm{e})} := \sum_{0\leqslant 2j\leqslant n-1}\left(Z_{2j}-\mathrm{E}Z_{2j}\right). \tag{3.5}$$

It follows that

$$\mathfrak{Y}_\alpha(n)-\mathrm{E}\mathfrak{Y}_\alpha(n) = S_n^{(\mathrm{o})}+S_n^{(\mathrm{e})}. \tag{3.6}$$

The processes $S^{(\mathrm{o})}$ and $S^{(\mathrm{e})}$ are mean-zero random walks, and hence martingales [in their respective filtrations]. Define

$$X_k^{(\mathrm{x})} := S_k^{(\mathrm{x})}-S_{k-1}^{(\mathrm{x})} \qquad (k\geqslant 1) \tag{3.7}$$

to be the increments of $S^{(\mathrm{x})}$ for $\mathrm{x}\in\{\mathrm{o}\,,\mathrm{e}\}$, and $\mathcal{G}_k^{(\mathrm{x})}$ the sigma-algebra generated by $\{X_j^{(\mathrm{x})}\}_{j=1}^k$. Because $\mathrm{Var}(Z_1)\leqslant \mathrm{E}(Z_1^2)\leqslant \ell\mathrm{E}(Z_1)\leqslant\ell^2$, $|X_j^{(\mathrm{x})}|<\ell$ for every j, and since $\ell\geqslant 1$, an application of Burkholder's inequality [2] (specifically, see Hall and Heyde [8, Theorem 2.10, p. 23]) implies that for every $k\geqslant 1$ there exists a universal constant $c_k\in(0\,,1)$ such that for every $k\geqslant 2$ and $n\geqslant 2$ and $\mathrm{x}\in\{\mathrm{o}\,,\mathrm{e}\}$,

$$c_k^k\mathrm{E}\left(\left|S_n^{(\mathrm{x})}\right|^k\right) \leqslant n^{k/2}\ell^k+\ell^k \leqslant 2n^{k/2}\ell^k. \tag{3.8}$$

The lemma follows from the above, (3.6), and Minkowski's inequality together with the observation that $\mathrm{E}\mathfrak{Y}_\alpha(n) \geqslant (n\ell)^{1-\alpha}$. $\square$

Proof of Theorem 1.3. Throughout the demonstration, we choose and fix a time $t > 0$.

Consider first Case 1. According to [3], there exist constants $A_1, \ldots, A_4 \in (0\,,\infty)$ such that for all $\lambda \geqslant 1$ and $x \in \mathbf{R}$,

$$A_1 \mathrm{e}^{-A_2(\log\lambda)^{3/2}} \leqslant \mathrm{P}\left\{u_t(x) > \lambda\right\} \leqslant A_3 \mathrm{e}^{-A_4(\log\lambda)^{3/2}}. \tag{3.9}$$

The preceding probability does not depend on $x \in \mathbf{R}$, thanks to translation invariance [3].

According to Theorem 1.6, for every $m > 1$ there exists $c \in (0\,,\infty)$ such that for all R large enough, we can find a process $Y \in \mathcal{L}(c \log R)$ such that

$$\mathrm{P}\left\{|u_t(x) - Y_x| \geqslant 1\right\} \leqslant \text{const} \cdot R^{-m}. \tag{3.10}$$

In particular, we can choose $Y_x = U_t^{(\beta,n)}(x)$ for appropriate β and n so that the preceding probability does not depend on $x \in \mathbf{R}$ (see the details of the proof of Theorem 1.6). Note, in particular, that

$$\begin{aligned} \mathrm{P}\left\{\int_0^R \mathbf{1}_{\{|u_t(x)-Y_x|\geqslant 1\}}\,\mathrm{d}x \geqslant 1\right\} &\leqslant \mathrm{E}\left(\int_0^R \mathbf{1}_{\{|u_t(x)-Y_x|\geqslant 1\}}\,\mathrm{d}x\right) \\ &\leqslant \text{const} \cdot R^{1-m}. \end{aligned} \tag{3.11}$$

For all $\alpha \in (0\,,1)$ and R large enough,

$$\begin{aligned} \mathrm{P}\left\{Y_x \geqslant \mathrm{e}^{\alpha(\log R)^{2/3}}\right\} &\leqslant \mathrm{P}\left\{u_t(x) \geqslant \mathrm{e}^{(\alpha/2)(\log R)^{2/3}}\right\} + \text{const} \cdot R^{-m} \\ &\leqslant A_3 R^{-A_4(\alpha/2)^{3/2}} + \text{const} \cdot R^{-m} \\ &\leqslant \text{const} \cdot R^{-A_4(\alpha/2)^{3/2}}, \end{aligned} \tag{3.12}$$

provided that $m > A_4$. Similarly,

$$\begin{aligned} \mathrm{P}\left\{Y_x \geqslant \mathrm{e}^{\alpha(\log R)^{2/3}}\right\} &\geqslant A_1 R^{-A_2(2\alpha)^{3/2}} - \text{const} \cdot R^{-m} \\ &\geqslant \text{const} \cdot R^{-A_2(2\alpha)^{3/2}}. \end{aligned} \tag{3.13}$$

provided that $m > (2A_2)^{3/2}$. We combine the preceding two bounds, and then relabel α to see that there exist $B_1, B_2, B_3, B_4 \in (0\,,\infty)$ and $\alpha_0 \in (0\,,1/4)$ such that for all $\alpha \in (0\,,\alpha_0)$,

$$B_1 \mathrm{e}^{B_2(\alpha\log R)^{2/3}} \leqslant -1 + \bar{G}(R^{-\alpha}) \leqslant 1 + \bar{G}(R^{-\alpha}) \leqslant B_3 \mathrm{e}^{B_4(\alpha\log R)^{2/3}}, \tag{3.14}$$

where $\bar{G}$ was defined in (3.2). According to (3.11),

$$\begin{aligned} \mathrm{P}\left\{\int_0^R \mathbf{1}_{\{u_t(x)\geqslant 1+\bar{G}(R^{-\alpha})\}}\,\mathrm{d}x \geqslant 1 + \int_0^R \mathbf{1}_{\{Y_x \geqslant \bar{G}(R^{-\alpha})\}}\,\mathrm{d}x\right\} \\ \leqslant \text{const} \cdot R^{1-m}, \end{aligned} \tag{3.15}$$

and

$$\mathrm{P}\left\{\int_0^R \mathbf{1}_{\{u_t(x)\geqslant -1+\bar{G}(R^{-\alpha})\}}\,\mathrm{d}x \leqslant -1+\int_0^R \mathbf{1}_{\{Y_x\geqslant \bar{G}(R^{-\alpha})\}}\,\mathrm{d}x\right\} \leqslant \text{const}\cdot R^{1-m}. \tag{3.16}$$

We emphasize that“const” does not depend on R in the previous two displays. We may apply Lemma 3.1 with $\ell := c\log R$, $n := R/\ell$ and $\mathfrak{Y}_\alpha(n) = \int_0^R \mathbf{1}_{\{Y_x\geqslant\bar{G}(R^{-\alpha})\}}\,\mathrm{d}x$ to see that for all $k \geqslant 2$, R sufficiently large,

$$\mathrm{P}\left\{\left|\int_0^R \frac{\mathbf{1}_{\{Y_x\geqslant\bar{G}(R^{-\alpha})\}}}{\mathrm{E}\mathfrak{Y}_\alpha(n)}\,\mathrm{d}x - 1\right| \geqslant R^{-\alpha}\right\} \leqslant \text{const}\cdot\frac{(\log R)^{k/2}}{R^{k(1-4\alpha)/2}}. \tag{3.17}$$

Let us pause and recall that $\alpha < \alpha_0 < 1/4$, so that the right-hand side is at most R^{-2} provided that we have chosen k sufficiently large. We also note that there exists $\gamma \in (0\,,1)$ such that

$$R^{1-\alpha} \leqslant \mathrm{E}\mathfrak{Y}_\alpha(n) \leqslant R^{1-\alpha\gamma}, \tag{3.18}$$

for all sufficiently large R; see (3.9) and (3.12). Therefore, we can combine (3.14), (3.15), and (3.16), together with the Borel–Cantelli lemma to see that as long as α_0 were selected sufficiently small,

$$0 < \liminf_{\substack{R\to\infty\\ R\in\mathbf{Z}}} \frac{\log|E_\alpha(R)|}{\log R} \leqslant \limsup_{\substack{R\to\infty\\ R\in\mathbf{Z}}} \frac{\log|E_\alpha(R)|}{\log R} < 1 \qquad \text{a.s.} \tag{3.19}$$

A monotonicity argument finishes the proof for Case 1.

Case 2 is proved similarly, but we apply the following estimate [3] in place of (3.9): $C_1\exp(-C_2\lambda^2) \leqslant \mathrm{P}\{u_t(x) > \lambda\} \leqslant C_3\exp(-C_4\lambda^2)$ (for $\lambda \geqslant 1$). We omit the details. □

4 Proof of Theorem 1.4

First of all, let us note that the conditions of the theorem are non vacuous. In other words, we need to prove that there exists $b > 1$ such that $\mathrm{P}\{u_t(0) \geqslant b\} > 0$. Because $\mathrm{E}u_t(0) = 1$, it follows that there exists $b \geqslant 1$ such that $\mathrm{P}\{u_t(0) \geqslant b\} > 0$. Suppose to the contrary that $\mathrm{P}\{u_t(0) > 1\} = 0$. Then, $u_t(0)$ is a.s. equal to 1. It follows that that the stochastic integral in (2.1) vanishes a.s. for $x = 0$. The corresponding quadratic variation must too; that is,

$$\int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ \left[p_{t-s}(y)\sigma\left(u_s(y)\right)\right]^2 = 0 \quad \text{a.s.} \tag{4.1}$$

Since the heat kernel never vanishes, we find that $\sigma\left(u_s(y)\right) = 0$ for almost all $(s\,,y) \in (0\,,t)\times\mathbf{R}$, whence for all $(s\,,y)\in(0\,,t)\times\mathbf{R}$ by continuity. This is a contradiction since $u_0 \equiv 1$. Therefore, there exists $b > 1$ such that $\mathrm{P}\{u_t(0)\geqslant b\} > 0$. Now we proceed with our proof of the bulk of Theorem 1.4.

Theorem 1.4 is a simple consequence of Theorem 1.6 together with ideas that are borrowed from a classical paper by Erdős and Rényi [6] on the length of the longest run of heads in an infinitely-long sequence of independent coin tosses.

Choose and *fix* two integers $R, m \gg 1$ and a real $\delta \in (0\,,1)$ small enough that $a - 2\delta > 1$ and $\mathrm{P}\{u_t(0) > b + 2\delta\} > 0$. According to Theorem 1.6 we can find a constant

$c \in (0\,,\infty)$—independent of R—and a random field $Y \in \mathcal{L}(c\log R)$ such that

$$\mathrm{P}\left\{|u_t(x) - Y_x| > \delta\right\} \leqslant \frac{\text{const}}{R^m}. \tag{4.2}$$

In fact, the field Y can be chosen so that the above probability does not depend on x (see the proof of Theorem 1.6).

Define $x_j := cj\log R$ for all non negative integers j, and observe that

$$\mathrm{P}\left\{\max_{0\leqslant j\leqslant\lfloor R/(c\log R)\rfloor}\left|u_t(x_j) - Y_{x_j}\right| > \delta\right\} \leqslant \text{const}\cdot R^{1-m}. \tag{4.3}$$

Let us call the index j "good" if $Y_{x_j}, Y_{x_{j+2}} < a-\delta$ and $Y_{x_{j+1}} > b+\delta$. Otherwise j is deemed "bad." Clearly,

$$\begin{aligned} p &:= \mathrm{P}\left\{j \text{ is good}\right\} \\ &= \left(\mathrm{P}\left\{Y_0 < a-\delta\right\}\right)^2 \cdot \mathrm{P}\left\{Y_0 > b+\delta\right\} \\ &\geqslant \left(\mathrm{P}\{u_t(0) < a-2\delta\} - \frac{c}{R^m}\right)^2 \cdot \left(\mathrm{P}\{u_t(0) > b+2\delta\} - \frac{c}{R^m}\right). \end{aligned} \tag{4.4}$$

We may observe that p does not depend on j. Moreover, $\mathrm{P}\{u_t(0) < a-2\delta\} \wedge \mathrm{P}\{u_t(0) > b+2\delta\} > 0$ because of the choice of $(b\,,\delta)$ and the fact that $\mathrm{E}u_t(0) = 1 < a-2\delta$. Therefore, we may choose R large enough to ensure that $p > 0$. note that we may also choose m independently of $R \gg 1$.

Because

$$\mathrm{P}\left\{j\,,j+3\,,\ldots,j+3n \text{ are all bad}\right\} = (1-p)^n, \tag{4.5}$$

it follows that

$$\begin{aligned} \mathrm{P}\left\{\exists\, 0 \leqslant j \leqslant \left\lfloor\frac{R}{c\log R}\right\rfloor : j\,,j+3\,,\ldots,j+3\lfloor\gamma\log R\rfloor \text{ are all bad}\right\} \\ \leqslant \text{const}\cdot R^{-2}, \end{aligned} \tag{4.6}$$

provided that γ is a sufficiently-large universal constant. This and the Borel–Cantelli lemma together imply that a.s. for all sufficiently-large integers R, the maximum distance between two good points is at most $6\gamma\log R\cdot c\log R$. Combined with (4.3), we can conclude that the size of the largest island is at most $6c\gamma(\log R)^2$. This proves the theorem for Case 1.

If Case 2 holds, then we proceed exactly as we did above, but can find our random field $Y \in \mathcal{L}(c[\log\log R]^{3/2})$ instead of $\mathcal{L}(c\log R)$. The remaining details are omitted. □

5 Proof of Theorem 1.7

We conclude by proving Theorem 1.7. Throughout we assume that

$$\sigma(0) = 0. \tag{5.1}$$

[This of course includes Case 1.] In that case Mueller's comparison principle [3, 11] guarantees that $u_t(x) \geqslant 0$ for all $t > 0$ and $x \in \mathbf{R}$ a.s. We offer the following quantitative improvement, which clearly implies Theorem 1.7:

Theorem 5.1. *For every $t > 0$ there exist $A, B \in (0\,,\infty)$ such that uniformly for all $\epsilon \in (0\,,1)$ and $x \in \mathbf{R}$,*

$$\mathrm{P}\{u_t(x) < \epsilon\} \leqslant A\exp\left(-B\left\{|\log\epsilon|\cdot\log|\log\epsilon|\right\}^{3/2}\right). \tag{5.2}$$

Before we prove this result, let us state and prove two corollories to Theorem 5.1. The corollaries are of independent interest, but also showcase the usefulness of quantitative estimates in this area. The first corollary identifies an upper bound for the exponential growth of the high negative moments of $u_t(x)$ when $\sigma(0) = 0$. We believe that the rate provided below is sharp.

Corollary 5.2. *For all $t > 0$ and $x \in \mathbf{R}$,*

$$\limsup_{k\to\infty}\left[\left(\frac{\log k}{k}\right)^3 \log \mathrm{E}\left(|u_t(x)|^{-k}\right)\right] < \infty. \tag{5.3}$$

Proof. Theorem 5.1 implies that $u_t(x) > 0$ a.s., whence $X := 1/u_t(x)$ is well defined. Because $\mathrm{E}(X^k) = k\int_0^\infty \lambda^{k-1}\mathrm{P}\{X > \lambda\}\,\mathrm{d}\lambda$, we can divide the integral into two pieces where: (i) $\lambda < \mathrm{e}$; and (ii) $\lambda \geqslant \mathrm{e}$. In this way we find that

$$\mathrm{E}(X^k) \leqslant \mathrm{e}^k + Ak\cdot\int_1^\infty \mathrm{e}^{f_k(s)}\,\mathrm{d}s, \tag{5.4}$$

where

$$f_k(s) := ks - B(s\log s)^{3/2}. \tag{5.5}$$

Laplace's method [and/or the method of stationary phase] tells us that

$$\log \mathrm{E}(X^k) \leqslant (1+o(1))\sup_{s\geqslant 1} f_k(s) \qquad \text{as } k\to\infty. \tag{5.6}$$

This is a simple maximization problem whose solution can be sketched as follows: $f_k(s)$ is maximized at $s = s_0$, where s_0 solves $(3B/2)(s_0\log s_0)^{1/2}\{\log s_0 + 1\} = k$. When k is large, $(3B/2)s_0^{1/2}(\log s_0)^{3/2} \approx k$ and consequently $s_0 \approx (2/(3B))k^2(\log s_0)^{-2}$. It then follows immediately that $\sup_{s\geqslant 1} f_k(s) = f_k(s_0)$ is of order $(k/\log k)^3$, as claimed. □

We mention [and verify] the second corollary to Theorem 5.1 next. This corollary describes a bound for how close $u_t(x)$ can come to zero, as $x \to \infty$.

Corollary 5.3. *For all $t > 0$ and all $\zeta > \zeta_0$ for some $\zeta_0 > 0$,*

$$\lim_{x\to\infty}\left[\mathrm{e}^{\zeta(\log x)^{2/3}} u_t(x)\right] = \infty \qquad \textit{a.s.} \tag{5.7}$$

Proof. Let $\gamma > 0$ be fixed, and define, for every $n \geqslant 1$, a set $A(n)$ as the following finite collection of points in the interval $[n\,,2n]$:

$$A(n) := \left\{n + jn^{-\gamma}\right\}_{j=0}^{1+\lfloor n^{1-\gamma}\rfloor}. \tag{5.8}$$

According to Theorem 5.1, for all $\zeta > 0$ large enough,

$$\mathrm{P}\left\{\inf_{x\in A(n)} u_t(x) < 3\mathrm{e}^{-\zeta(\log n)^{2/3}}\right\} = O\left(n^{-2}\right) \qquad \text{as } n\to\infty. \tag{5.9}$$

According to [7, Lemma A.3], there exists an $c \in (0\,,\infty)$ such that for all $t > 0$, $k \in [2\,,\infty)$, and $x, y \in \mathbf{R}$,

$$\mathrm{E}\left(|u_t(x) - u_t(y)|^k\right) \leqslant \mathrm{e}^{ck^3 t}|x-y|^{k/2}. \tag{5.10}$$

Therefore, a suitable form of the Kolmogorov continuity theorem [5, Theorem 4.3, p.

10] implies that for all $k \in [2\,,\infty)$, $\eta \in (0\,,1)$, and $t > 0$

$$B_k := B_k(t\,,\eta) := \sup_I \mathrm{E}\left(\sup_{\substack{x,y\in I\\ x\neq y}} \frac{|u_t(x)-u_t(y)|^k}{|x-y|^{k\eta/2}}\right) < \infty, \tag{5.11}$$

where "$\sup_I$" denotes the supremum over all closed intervals $I \subset \mathbf{R}$ of length one. We emphasize that we need a suitable quantitative form of Kolmogorov's continuity theorem because we will need the fact that because the constant $\exp(ck^3t)$ in (5.10) is independent of the interval I, we can add in the quantifier "$\sup_I$" to the expectation on the right-hand side of the preceding display.

Next we apply Chebyshev's inequality to see that for every $t > 0$, $\eta \in (0\,,1)$, and $k \in [2\,,\infty)$,

$$\begin{aligned}
&\mathrm{P}\left\{\sup_{\substack{n\leqslant x,y\leqslant 2n\\ |x-y|\leqslant n^{-\gamma}}} |u_t(x)-u_t(y)| \geqslant 2\mathrm{e}^{-\zeta(\log n)^{2/3}}\right\}\\
&\qquad\leqslant \sum_{j=0}^{n-1} \mathrm{P}\left\{\sup_{\substack{j\leqslant x,y\leqslant j+1\\ |x-y|\leqslant n^{-\gamma}}} |u_t(x)-u_t(y)| \geqslant \mathrm{e}^{-\zeta(\log n)^{2/3}}\right\}\\
&\qquad\leqslant B_k n^{1-(k\gamma\eta/2)} \mathrm{e}^{\zeta k(\log n)^{2/3}} = O\left(n^{1-(k\gamma\eta/2)+o(1)}\right).
\end{aligned} \tag{5.12}$$

Now we choose and fix $k > 4/(\eta\gamma)$ so that the left-hand side of (5.12) sums [in n]. It follows from (5.9), (5.12), and the triangle inequality that, for every $\zeta > 0$ large enough,

$$\sum_{n=1}^{\infty} \mathrm{P}\left\{\inf_{x\in(n,2n)} u_t(x) < \mathrm{e}^{-\zeta(\log n)^{2/3}}\right\} < \infty. \tag{5.13}$$

The Borel–Cantelli lemma completes the proof. □

Proof of Theorem 5.1. We are going to prove that for all $n \geqslant 1$,

$$\mathrm{P}\left\{\inf_{x\in(-1,1)} \inf_{s\in(0,t)} u_s(x) \leqslant \mathrm{e}^{-n}\right\} \leqslant A\exp\left(-B(n\log n)^{3/2}\right). \tag{5.14}$$

This is a stronger result than the one advertised by the statement of the theorem.

Let $v_t(x)$ denote the unique continuous solution to (1.1) subject to $v_0(x) = \mathbf{1}_{(-1,1)}(x)$. Because $v_0(x) \leqslant 1 = u_0(x)$ for all x, Mueller's comparison principle [11] tells us that there exists a null set off which $u_t(x) \geqslant v_t(x)$. Therefore, it suffices to prove that for all $n \geqslant 1$,

$$\mathrm{P}\left\{\inf_{x\in(-1,1)} \inf_{s\in(0,t)} v_s(x) \leqslant \mathrm{e}^{-n}\right\} \leqslant A\exp\left(-B(n\log n)^{3/2}\right). \tag{5.15}$$

Set $T_0 := 0$, and then define iteratively

$$T_{k+1} := \inf\left\{s > T_k : \inf_{x\in(-1,1)} v_s(x) \leqslant \mathrm{e}^{-k-1}\right\}, \tag{5.16}$$

where $\inf\varnothing := \infty$. Evidently, the T_k's are $\{\mathcal{F}_t\}_{t>0}$-stopping times, where $\mathcal{F}_t$ denotes the filtration generated by time t by all the values of the white noise. Without loss of any generality we may assume that $\{\mathcal{F}_t\}_{t>0}$ is augmented in the usual way, so that $t \mapsto v_t$ is a $C(\mathbf{R})$-valued strong Markov process.

Next we observe that for every $k \geqslant 1$,

$$\mathrm{e}^k v_{T_k}(x) \geqslant \mathbf{1}_{(-1,1)}(x) \quad \text{for all } x \in \mathbf{R}\text{, a.s. on } \{T_k < \infty\}. \tag{5.17}$$

Therefore, we apply first the strong Markov property, and then Mueller's comparison principle, in order to see that the following holds a.s. on $\{T_k < t\}$:

$$\begin{aligned} &\mathrm{P}\left(T_{k+1} - T_k \leqslant \frac{2t}{n} \,\middle|\, \mathcal{F}_{T_k} \right) \\ &\qquad\qquad \leqslant \mathrm{P}\left\{ \inf_{s \in (0,2t/n)} \inf_{x \in (-1,1)} v_s^{(k+1)}(x) \leqslant \mathrm{e}^{-k-1} \right\}, \end{aligned} \tag{5.18}$$

where $v^{(k+1)}$ designates the unique continuous solution to (1.1) [for a different white noise, pathwise], starting at $v_0^{(k+1)}(x) := \exp(-k)\mathbf{1}_{(-1,1)}(x)$. Note that

$$w_t^{(k+1)}(x) := \mathrm{e}^k v_t^{(k+1)}(x) \tag{5.19}$$

solves the SPDE

$$\frac{\partial}{\partial t} w_t^{(k+1)}(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} w_t^{(k+1)}(x) + \sigma_k\left(w_t^{(k+1)}(x)\right) \eta_t^{(k+1)}(x), \tag{5.20}$$

subject to $w_0^{(k+1)}(x) = \mathbf{1}_{(-1,1)}(x)$, where $\eta^{(k+1)}$ is a space-time white noise for every k and

$$\sigma_k(x) := \mathrm{e}^k \sigma\big(\mathrm{e}^{-k} x\big)\,. \tag{5.21}$$

Therefore, the following holds a.s. on $\{T_k < t\}$:

$$\begin{aligned} &\mathrm{P}\left(T_{k+1} - T_k \leqslant \frac{2t}{n} \,\middle|\, \mathcal{F}_{T_k} \right) \\ &\qquad\qquad \leqslant \mathrm{P}\left\{ \sup_{\substack{x \in (-1,1) \\ s \in (0,2t/n)}} \left| w_s^{(k+1)}(x) - w_0^{(k+1)}(x) \right| \geqslant 1 - \frac{1}{\mathrm{e}} \right\}. \end{aligned} \tag{5.22}$$

Let Lip_σ denote the optimal Lipschitz constant of σ. Because $\sigma(0) = 0$, it follows that

$$\sup_{k \geqslant 1} |\sigma_k(z)| \leqslant \mathrm{Lip}_\sigma |z| \qquad \text{for all } z \in \mathbf{R}. \tag{5.23}$$

It is this important property that allows us to appeal to the estimates of [7, Appendix] , and deduce the following: For all $\eta \in (0\,,1)$, there exists a constant $Q := Q(\eta) \in (0\,,\infty)$ such that for all $k \geqslant 0$, $m \in [2\,,\infty)$, and $\tau \in (0\,,1)$,

$$\sup_{k \geqslant 0} \mathrm{E}\left(\sup_{\substack{x \in (-1,1) \\ s \in (0,\tau)}} \left| \frac{w_s^{(k+1)}(x) - w_0^{(k+1)}(x)}{s^{\eta/4}} \right|^m \right) \leqslant Q\mathrm{e}^{Qm^3\tau}. \tag{5.24}$$

In other words,

$$\sup_{k \geqslant 0} \mathrm{E}\left(\sup_{\substack{x \in (-1,1) \\ s \in (0,\tau)}} \left| w_s^{(k+1)}(x) - w_0^{(k+1)}(x) \right|^m \right) \leqslant Q\mathrm{e}^{Qm^3\tau}\tau^{\eta m/4}. \tag{5.25}$$

We apply this inequality with $\tau := 2t/n$ and optimize over m in order to deduce from (5.22) that there exists a constant $L := L(\eta, t) \in (0\,, \infty)$ such that for all integers $n > 2t$ the following holds a.s. on $\{T_k < t\}$:

$$\mathrm{P}\left(T_{k+1} - T_k \leqslant \frac{2t}{n} \;\middle|\; \mathcal{F}_{T_k} \right) \leqslant L \exp\left(-Ln^{1/2} (\log n)^{3/2} \right). \tag{5.26}$$

Finally, we notice that if $T_n < t$, then certainly there are at least $\lfloor n/2 \rfloor$-many distinct values of $k \in \{0\,, \ldots, n-1\}$ such that $T_{k+1} - T_k \leqslant 2t/n$. [This is just an application of the so-called "pigeonhole principle," itself a contrapositive formulation of the triangle inequality.] Therefore, (5.26) implies that for all $n > t/2$,

$$\mathrm{P}\left\{ T_n < t \right\} \leqslant \binom{n}{\lfloor n/2 \rfloor} L^{\lfloor n/2 \rfloor} \exp\left(-L \left\lfloor \frac{n}{2} \right\rfloor n^{1/2} (\log n)^{3/2} \right). \tag{5.27}$$

This, Stirling's formula, and monotonicity together imply (5.15). □

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